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TARGET RULES FOR PUBLIC CHOICE ECONOMIES ON TREE NETWORKS AND IN EUCLIDEAN SPACES *

ABSTRACT. We consider the problem of choosing the location of a public facility either (a) on a tree network or (b) in a Euclidean space. (a) (1996) characterize the class of target rules on a tree network by *Pareto efficiency* and *population-monotonicity*. Using Vohra's (1999) characterization of rules that satisfy *Pareto efficiency* and *replacement-domination*, we give a short proof of the previous characterization and show that it also holds on the domain of symmetric preferences. (b) The result obtained for model (a) proves to be crucial for the analysis of the problem of choosing the location of a public facility in a Euclidean space. Our main result is the characterization of the class of coordinatewise target rules by *unanimity*, *strategy-proofness*, and either *replacement-domination* or *population-monotonicity.*

KEY WORDS: Single-peaked preferences, Tree networks, Euclidean spaces, Target rules, Pareto efficiency, Population-monotonicity, Replacement-domination

1. INTRODUCTION

First, we consider the problem of choosing the location of a public facility on a tree network, or tree,[1] when agents have single-peaked preferences. For the special case where the tree equals a closed interval, the problem coincides with the problem of choosing a level of a public good when agents have single-peaked preferences (Moulin, 1980).[2] An example for the problems we consider is the problem of locating a public facility, e.g., a library, on a tree network that represents an infrastructure (the network of roads of a neighborhood). Several solutions for this class of problems have been proposed and characterized by desirable properties; see for instance Ching and Thomson (1996), Danilov (1994), Foster and Vohra (1998), Schummer and Vohra (2001), and Vohra (1999).

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For rules on tree networks, Ching and Thomson (1996) and Vohra (1999) consider the solidarity properties *population-monotonicity* and *replacement-domination*, respectively. *Replacement-domination* states that if an agent's preference relation is 'replaced' by some other admissible preference relation, then this unilateral change affects the remaining agents in the same direction, i.e., the remaining agents all (weakly) gain or they all (weakly) lose. *Population-monotonicity* requires that after the arrival of new agents all agents initially present are affected in the same direction. It turns out that the class of rules satisfying *Pareto efficiency* and *population-monotonicity* is the class of 'target rules' (Ching and Thomson, 1996).[3] Each target rule is determined by its target point. If the target point is *Pareto efficient*, then the target point is chosen by the rule. If the target point is not *Pareto efficient*, then the closest *Pareto efficient* point to the target point is chosen by the rule. This target oriented decision pattern is implicitly present in many decision processes in our daily lives and in many public choice decision processes, target oriented decisions prevail, particularly when the target point equals a status quo point.

Vohra (1999) proves for tree networks that if the set of agents is fixed, contains at least three agents, and has symmetric single-peaked preferences, then the class of rules satisfying *Pareto efficiency* and *replacement-domination* equals the class of target rules. We show that this result remains true for the larger domain of single-peaked preferences. In the first part of the paper, using Vohra's (1999) result and our result that *Pareto efficiency* and *population-monotonicity* imply *replacement-domination*, we give a short proof of Ching and Thomson's (1996) characterization. Furthermore, we prove that the characterization also holds on the smaller domain of symmetric single-peaked preferences. This latter result turns out to be crucial for the second part of the paper.

In the second part of the paper, we analyze the implications of the solidarity properties *population-monotonicity* and *replacement-domination* for the problem of choosing the location of a public facility in a Euclidean space or allocating several public issues, e.g., budget-constrained investment divisions among several public projects or bundles of public goods. We assume that every agent has an individual best point and his preferences decline according to the distance to this best point. Because agents might weigh coordinates differently, we assume that preferences are induced by separable-quadratic distance functions (Border and Jordan, 1983). Other papers that study solutions to the problem of choosing the location of a public facility in a Euclidean space and their properties are Barberà et al. (1993), Peremans et al. (1997), and Peters et al. (1992).

If we naturally extend target rules to Euclidean spaces, it is obvious that none of these rules satisfies either one of the solidarity properties. However the coordinatewise versions of the target rules, which are not *Pareto efficient*, do satisfy *replacement-domination*, *population-monotonicity*, and the weaker efficiency requirement of *unanimity*. It follows from a result of Border and Jordan (1983) and the previous results for rules on trees (or particularly the real line) that, essentially, the class of coordinatewise target rules is characterized by *unanimity*, *strategy-proofness*, and either *population-monotonicity* or *replacement-domination*.

2. PUBLIC GOOD ECONOMIES ON TREE NETWORKS

2.1. *The model*

As in Ching and Thomson (1996) and Vohra (1999) we consider the problem of choosing a location on a tree T. Since for our analysis it only matters that for any two locations on T there exists a unique path that connects these two locations, we omit a formal definition of a tree; see for instance Demange (1982). Let $x, y \in T$. Then, by $[x, y]$ we denote the path connecting x and y. Note that according to this notation, $[x, y] = [y, x]$.

There is a population of 'potential' agents, indexed by $\mathbb{P} \subseteq \mathbb{N}$ where $\mathbb{N}$ denotes the set of natural numbers. We assume that $\mathbb{P}$ contains at least three agents, i.e., $|\mathbb{P}| \geqslant 3$. Note that $\mathbb{P}$ can be either finite or infinite. Each agent $i \in \mathbb{P}$ is equipped with a continuous and 'single-peaked' preference relation R_i defined on T (Demange, 1982). As usual, $x\ R_i\ y$ is interpreted as 'x is weakly preferred to y', and $x\ P_i\ y$ as 'x is strictly preferred to y'. *Single-peakedness* of R_i means that R_i is single-peaked on every path of T; i.e., there exists a point $p(R_i) \in T$, called the *peak of agent i*, with the following property: for all $x, y \in T$, $x \neq y$, such that $[y, p(R_i)] \subset [x, p(R_i)]$, we have $y\ P_i\ x$. By $\mathcal{R}$ we denote the class of all continuous, single-peaked preference relations on T. By $\mathcal{P}$ we denote the class of non-empty and finite subsets of $\mathbb{P}$. For $N \in \mathcal{P}$, $\mathcal{R}^N$ denotes the set of *(preference) profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$. A *rule* φ is a function that assigns to every $N \in \mathcal{P}$ and every $R \in \mathcal{R}^N$ a location $\varphi(R) \in T$, i.e., $\varphi\colon \bigcup_{N \in \mathcal{P}} \mathcal{R}^N \to T$.

2.2. *Target rules and their properties*

The first property of a rule we introduce is *Pareto efficiency*. Let $N \in \mathcal{P}$ and $x, y \in T$. If for all $i \in N$, $x \, R_i \, y$ and for some $j \in N$, $x \, P_j \, y$, then we call x a *Pareto improvement of* y.

Pareto efficiency: For all $N \in \mathcal{P}$ and all $R \in \mathcal{R}^N$, there exists no Pareto improvement of $\varphi(R)$.

Let $N \in \mathcal{P}$ and $R \in \mathcal{R}^N$. Then, by $P(R)$ we denote the convex hull of all agents' peaks; i.e., the smallest connected subset of the tree that contains all agents' peaks.

Consider the 'degenerate' case where T equals an interval or the real line. Then, $P(R) = [\min_{i \in N} p(R_i), \max_{i \in N} p(R_i)]$ and *Pareto efficiency* is equivalent to $\varphi(R) \in P(R)$. It is easy to show that this condition also characterizes *Pareto efficiency* of rules on trees. We call $P(R)$ the *Pareto set of* R.

LEMMA 1. *A rule φ is Pareto efficient if and only if for all $N \in \mathcal{P}$ and all $R \in \mathcal{R}^N$, $\varphi(R) \in P(R)$.*

The following class of 'target rules' will play an important role in the sequel. Any target rule is determined by its target point. If the target point is *Pareto efficient*, then it is chosen by the rule. If the target point is not *Pareto efficient*, then the (unique) closest *Pareto efficient* point to it is chosen by the rule.

Target rules: Let $a \in T$.[4] Then, by φ^a we denote the following *target rule with target point a*: for all $N \in \mathcal{P}$ and all $R \in \mathcal{R}^N$,

$$\varphi^a(R) = \begin{cases} a & \text{if } a \in P(R), \\ x & \text{where } x \in P(R) \text{ is the closest point to } P(R) \\ & \text{otherwise.} \end{cases}$$

Next, we introduce the 'solidarity' property *replacement-domination.* It incorporates a notion of solidarity among agents when a single agent changes his preference relation, e.g., if an agent's preference relation is exchanged by another preference relation, then, after this change, either all remaining agents are (weakly) better off or they all are (weakly) worse off. In a recent paper, Thomson (1999) surveys the literature on *replacement-domination*.

Let $N, M \in \mathcal{P}$ with $N \subseteq M$ and $R \in \mathcal{R}^M$. We denote the *restriction* $(R_i)_{i \in N} \in \mathcal{R}^N$ *of* R *to* N by R_N. We also use the notation $R_{-i} = R_{N \setminus \{i\}}$. For example, $(\bar{R}_i, R_{-i})$ denotes the profile obtained from R by replacing R_i by $\bar{R}_i$.

Replacement-domination: For all $N \in \mathcal{P}$, all $R \in \mathcal{R}^N$, all $j \in N$, and all $\bar{R}_j \in \mathcal{R}$, either [for all $i \in N \setminus \{j\}$, $\varphi(R)\ R_i\ \varphi(\bar{R}_j, R_{-j})$] or [for all $i \in N \setminus \{j\}$, $\varphi(\bar{R}_j, R_{-j})\ R_i\ \varphi(R)$].

Thomson (1993) proved that if T is a closed interval and the set of agents is fixed and contains at least three agents, then the class of rules satisfying *Pareto efficiency* and *replacement-domination* equals the class of target rules. For tree networks T, Vohra (1999) proves the characterization for the subdomain of symmetric preferences $\mathcal{S} \subset \mathcal{R}$. We state his result for the variable population setting at hand.

THEOREM 1 (Vohra). *Suppose* $\varphi : \bigcup_{\substack{N \in \mathcal{P} \\ |N| \geqslant 3}} \mathcal{S}^N \to T$. *Then* φ *satisfies Pareto efficiency and replacement-domination if and only if for every* $N \in \mathcal{P}$, $|N| \geqslant 3$, *there exists* $a^N \in T$ *such that for all* $R \in \mathcal{S}^N$, $\varphi(R) = \varphi^{a^N}(R)$.

It is easy to show that the characterization of Theorem 1 also holds on the domain $\bigcup_{\substack{N \in \mathcal{P} \\ |N| \geqslant 3}} \mathcal{R}^N$.

COROLLARY 1. *Suppose* $\varphi : \bigcup_{\substack{N \in \mathcal{P} \\ |N| \geqslant 3}} \mathcal{R}^N \to T$. *Then* φ *satisfies Pareto efficiency and replacement-domination if and only if for every* $N \in \mathcal{P}$, $|N| \geqslant 3$, *there exists* $a^N \in T$ *such that for all* $R \in \mathcal{R}^N$, $\varphi(R) = \varphi^{a^N}(R)$.

Proof. It is easy to prove that if for every $N \in \mathcal{P}$, $|N| \geqslant 3$, there exists $a^N \in T$ such that for all $R \in \mathcal{R}^N$, $\varphi(R) = \varphi^{a^N}(R)$, then φ satisfies *Pareto efficiency* and *replacement-domination*.

Let φ satisfy *Pareto efficiency* and *replacement-domination*. Let $N \in \mathcal{P}$ and $|N| \geqslant 3$. Then, by Theorem 1, there exists $a^N \in T$ such that for all $R \in \mathcal{S}^N$, $\varphi(R) = \varphi^{a^N}(R)$. We have to prove that for all $\bar{R} \in \mathcal{R}^N$, $\varphi(\bar{R}) = \varphi^{a^N}(\bar{R})$.

Let $\bar{R} \in \mathcal{R}^N$. If $\bar{R} \in \mathcal{S}^N$, then $\varphi(\bar{R}) = \varphi^{a^N}(\bar{R})$ and we are done. If $\bar{R} \notin \mathcal{S}^N$, then there exists some $j \in N$ such that $\bar{R}_j \notin \mathcal{S}$. Next, we replace agent j's preference relation by a symmetric preference relation with the original peak: let $\tilde{R}$ be such that $\tilde{R}_{-j} = \bar{R}_{-j}$, $p(\tilde{R}_j) = p(\bar{R}_j)$, and $\tilde{R}_j \in \mathcal{S}$. Hence, at profile $\tilde{R}$ we increased the number of agents with symmetric preferences by one. By *Pareto efficiency* and *replacement-domination*, $\varphi(\tilde{R}) = \varphi(\bar{R})$. Hence, if $\tilde{R} \in \mathcal{S}^N$, then $\varphi(\tilde{R}) = \varphi^{a^N}(\tilde{R}) = \varphi^{a^N}(\bar{R})$. Thus, $\varphi(\bar{R}) = \varphi^{a^N}(\bar{R})$. If $\tilde{R} \notin \mathcal{S}^N$, then there exists some $k \in N$ such that $\tilde{R}_k \notin \mathcal{S}$. Similarly as before, we can increase the number of agents with symmetric preferences by one, etc. Since the number of agents in N is finite, this procedure ends with a symmetric preference profile $\hat{R} \in \mathcal{S}^N$ such that for all $i \in N$, $p(\hat{R}_i) = p(\bar{R}_i)$ and $\varphi(\hat{R}) = \ldots = \varphi(\tilde{R}) = \varphi(\bar{R}) = \varphi^{a^N}(\bar{R})$. □

The next solidarity property we discuss is *population-monotonicity*. It incorporates a notion of solidarity among agents when changes in the population occur, e.g., if a group of agents leave, then, after this change, either all remaining agents are (weakly) better off or they all are (weakly) worse off. For a survey on *population-monotonicity* we refer to Thomson (1995).

Population-monotonicity: For all $N, M \in \mathcal{P}$ such that $N \subseteq M$ and all $R \in \mathcal{R}^M$, either [for all $i \in N$, $\varphi(R_N)\, R_i\, \varphi(R)$] or [for all $i \in N$, $\varphi(R)\, R_i\, \varphi(R_N)$].

The following lemma will be useful later on. We leave the simple proof to the reader.

LEMMA 2. *Let φ satisfy Pareto efficiency and population-monotonicity. Then, for all $N, M \in \mathcal{P}$ such that $N \subseteq M$, all $i \in N$, and all $R \in \mathcal{R}^M$, $\varphi(R_N)\, R_i\, \varphi(R)$. Furthermore, if $\varphi(R) \in P(R_N)$, then $\varphi(R_N) = \varphi(R)$. Particularly, if $P(R_N) = P(R)$, then $\varphi(R_N) = \varphi(R)$.*

Ching and Thomson (1996) proved that on the domain of single-peaked preferences the class of rules satisfying *Pareto efficiency* and *population-monotonicity* equals the class of target rules. We state this result for the domain of symmetric preferences.

THEOREM 2. *Suppose* $\varphi : \bigcup_{N \in \mathcal{P}} \mathcal{S}^N \rightarrow T$. *Then* φ *satisfies Pareto efficiency and population-monotonicity if and only if* $\varphi = \varphi^a$ *for some* $a \in T$.

The proof of Theorem 2 can be found in the Appendix. On one hand, the proof establishes the validity of Ching and Thomson's (1996) result on the domain of symmetric preferences, on the other hand, it is an alternative way to prove Ching and Thomson's (1996) characterization. Furthermore, Theorem 2 proves to be crucial for the analysis of the problem of choosing the location of a public facility in a Euclidean space in Section 3.

Next, we discuss the incentive property *strategy-proofness* for rules on tree networks. *Strategy-proofness* requires that no agent ever benefits from misrepresenting his preference relation.

Strategy-proofness: For all $N \in \mathcal{P}$, all $R \in \mathcal{R}^N$, all $j \in N$, and all $\bar{R}_j \in \mathcal{R}$, $\varphi(R)\ R_j\ \varphi(\bar{R}_j, R_{-j})$.

It is easy to prove that any target rule is *strategy-proof*. Since the names of the agents do not matter in the assignment of the location, these rules also satisfy the well-known property *anonymity*. Hence, the class of rules that satisfy *Pareto efficiency* and either *replacement-domination* or *population-monotonicity* are selections of the set of rules satisfying *Pareto efficiency*, *strategy-proofness*, and *anonymity*. In his seminal paper, Moulin (1980) characterized the latter class of rules for the case that T is a line or a closed interval as the class of 'generalized Condorcet-winner rules' or 'generalized median-voter rule': the outcome is the median of the peaks of the $|N|$ agents and $|N| - 1$ fixed ballots.[5] In the case of a target rule, all $|N| - 1$ fixed ballots are equal to the target point.

The last property for rules we consider is *unanimity*: If all agents have the same preference relation, then the unanimous best point for all, the common peak, is chosen by the rule.

Unanimity: For all $N \in \mathcal{P}$ and all $R \in \mathcal{R}^N$ such that for all $i, j \in N$, $R_i = R_j$, $\varphi(R) = p(R_i)$.

It is easy to prove that *unanimity* and *strategy-proofness* together imply *Pareto efficiency*. Hence, in Moulin's (1980) characterization of generalized median-voter rules, *Pareto efficiency* can be weakened to *unanimity* (Ching, 1997).[6] This also implies that in Theorems 1 and 2, we can replace *Pareto efficiency* by *unanimity* and *strategy-proofness*.

COROLLARY 2. *Suppose* $\varphi : \bigcup_{\substack{N\in\mathcal{P}\\|N|\geqslant 3}} \mathcal{S}^N \to T$. *Then* φ *satisfies unanimity, strategy-proofness, and replacement-domination if and only if for every* $N \in \mathcal{P}$, $|N| \geqslant 3$, *there exists* $a^N \in T$ *such that for all* $R \in \mathcal{S}^N$, $\varphi(R) = \varphi^{a^N}(R)$.

COROLLARY 3. *Suppose* $\varphi : \bigcup_{N\in\mathcal{P}} \mathcal{S}^N \to T$. *Then* φ *satisfies unanimity, strategy-proofness, and population-monotonicity if and only if* $\varphi = \varphi^a$ *for some* $a \in T$.

Note that Corollaries 2 and 3 remain true on the larger domain of single-peaked preferences. The properties in Corollaries 2 and 3 are independent: Any constant rule satisfies *strategy-proofness*, *replacement-domination*, and *population-monotonicity*, but not *unanimity*. Any dictatorial rule satisfies *unanimity* and *strategy-proofness*, but neither *replacement-domination* nor *population-monotonicity*. The following rule ψ^a satisfies *unanimity*, *population-monotonicity*, and *replacement-domination*, but not *strategy-proofness*. Furthermore it is not *Pareto efficient*.

EXAMPLE 1. *Let* $a \in T$. *Then, for all* $N \in \mathcal{P}$ *and all* $R \in \mathcal{R}^N$,

$$\psi^a(R) = \begin{cases} p(R_i) & \text{if for all } i, j \in N,\ R_i = R_j, \\ a & \text{otherwise.} \end{cases}$$

3. PUBLIC GOOD ECONOMIES IN EUCLIDEAN SPACES

3.1. *The model*

As Border and Jordan (1983) we consider the problem of choosing a location in some Euclidean space E. We assume that $E = \mathbb{R}^m$ where $m \in \mathbb{N}$.

There is a population of 'potential' agents, indexed by $\mathbb{P} \subseteq \mathbb{N}$. We assume that $\mathbb{P}$ contains at least three agents, i.e., $|\mathbb{P}| \geqslant 3$. Note that $\mathbb{P}$ can be either finite or infinite. Each agent $i \in \mathbb{P}$ is equipped with a *separable-quadratic preference relation* R_i over E (Border and Jordan, 1983); that is: for each agent $i \in N$ there exists a strictly positive *weight vector* $\delta^i = (\delta^i_1, ..., \delta^i_m) \gg 0$ and a *peak* $p(R_i) \in E$ such that for all $x, y \in E$, $x\, R_i\, y$ if and only if $\sum_{j=1}^m \delta^i_j (x_j - p(R_i)_j)^2 \leqslant \sum_{j=1}^m \delta^i_j (y_j - p(R_i))^2$.

Note that every separable-quadratic preference relation R_i is completely determined by a pair $(\delta^i, p(R_i))$ where $\delta^i \gg 0$, $\sum_{j \in M} (\delta^i_j)^2 = 1$, and $p(R_i) \in E$. By $\mathcal{Q}$ we denote the class of all separable-quadratic preference relations on E. For each agent $i \in \mathbb{P}$, we identify the preference relation $R_i \in \mathcal{Q}$ with its characteristic pair $(\delta^i, p(R_i))$ and write $R_i = (\delta^i, p(R_i)) \in \mathcal{Q}$.

Single-peakedness of preference relations R_i on E means that R_i is single-peaked on every line in E that contains $p(R_i)$. It is easy to check that all separable-quadratic preference relations are *single-peaked*, i.e., for all $R_i \in \mathcal{Q}$, all $x \in E$, $x \neq p(R_i)$, and all $\lambda \in (0, 1)$, $p(R_i)\, P_i\, [\lambda p(R_i) + (1 - \lambda)x]\, P_i\, x$. A geometric implication of $R_i \in \mathcal{Q}$ being separable-quadratic is that the corresponding indifference sets are ellipsoids around the peak $p(R_i)$ with main diagonals parallel to the coordinate axes. The closer these ellipsoids are to $p(R_i)$ the better the points on it are.

For $N \in \mathcal{P}$, $\mathcal{Q}^N$ denotes the set of *(preference) profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{Q}$. A *rule* φ is a function that assigns to every $N \in \mathcal{P}$ and every $R \in \mathcal{Q}^N$ a location $\varphi(R) \in E$; i.e., $\varphi\colon \bigcup_{N \in \mathcal{P}} \mathcal{Q}^N \to E$.

3.2. *Coordinatewise target rules and their properties*

We are interested in the same properties as for rules on tree networks. The definitions of *Pareto efficiency*, *replacement-domination*, *population-monotonicity*, *strategy-proofness*, and *unanimity* are obtained from the previous definitions by simply replacing the domain of each agent's preferences $\mathcal{R}$ by $\mathcal{Q}$.

Similarly as before, a *target rule* is defined as follows. For any given target point in E and any preference profile the following holds. If the target point is *Pareto efficient*, then the target point is chosen by the rule. If the target is not *Pareto efficient*, then the (unique) closest *Pareto efficient* point to the target point is chosen by the rule.

However, because now, agents report with their preference relation a separable-quadratic distance function, it is easy to show that none of the target rules satisfies *strategy-proofness*.[7] Furthermore, none of the target rules satisfies *replacement-domination* or *population-monotonicity*.

Therefore we consider the following variations of target rules: the coordinatewise target rules. We need some extra notation.

Let $j \in \{1, \ldots, m\}$, $i \in \mathbb{P}$ and $R_i \in \mathcal{Q}$. Then, by R_i^j, we denote the restriction, or projection, of the preference relation to the jth coordinate axes. Note that then R_i^j is a symmetric preference relation defined on $\mathbb{R}$ where $\mathbb{R}$ represents the jth coordinate axes. For $N \in \mathcal{P}$ and $R \in \mathcal{Q}^N$, $R^j = (R_i^j)_{i \in N}$ denotes the restriction of profile R to the jth coordinate axes. Let $\bar{E} \equiv (\mathbb{R} \cup \{-\infty, \infty\})^m$.

Coordinatewise target rules: Let $a = (a_1, \ldots, a_m) \in \bar{E}$. Then, by φ^a we denote the following *coordinatewise target rule with target point a*: for all $N \in \mathcal{P}$, all $R \in \mathcal{Q}^N$, and all $j \in \{1, \ldots, m\}$, $\varphi_j^a(R) = \varphi^{a_j}(R^j)$.

It is easy to prove that none of the coordinatewise target rules satisfies *Pareto efficiency*. However, any coordinatewise target rule satisfies *unanimity*, *anonymity*, *strategy-proofness*, *replacement-domination*, and *population-monotonicity*. Hence, the class of coordinatewise target rules is a selection of the set of rules satisfying *unanimity*, *anonymity*, and *strategy-proofness*. Recall that in the one-dimensional case, this latter class is equal to the class of generalized median-voter rules.

Border and Jordan (1983) showed that a rule that satisfies *unanimity* and *strategy-proofness* can be decomposed into coordinatewise rules that are again *unanimous* and *strategy-proof*.

For $N \in \mathcal{P}$ and $j \in \{1, \ldots, m\}$, the set of restricted preference profiles to the jth coordinate axes is denoted by $\mathcal{Q}_j^N = \{(R_i^j)_{i \in N} \mid R \in \mathcal{Q}^N\}$.

THEOREM 3 (Border and Jordan). *Let $N \in \mathcal{P}$. A rule $\varphi : \mathcal{Q}^N \to E$ satisfies unanimity and strategy-proofness if and only if there are m (coordinatewise) rules $\varphi^j : \mathcal{Q}_j^N \to \mathbb{R}$, $j \in \{1, \ldots, m\}$, which are unanimous and strategy-proof such that for all $N \in \mathcal{P}$, all $R \in \mathcal{Q}^N$, and all $j \in \{1, \ldots, m\}$, $\varphi_j(R) = \varphi^j(R^j)$.*

Hence, by Theorem 3, the class of rules that satisfy *unanimity strategy-proofness*, and *anonymity*, similarly as for the one-dimensional case (Moulin, 1980), consists of generalized median-voter rules: the outcome is the coordinatewise median of the peaks of the $|N|$ agents and $|N| - 1$ fixed ballots. Again, in the case of a coordinatewise target rule, all $|N| - 1$ fixed ballots are equal to the target point $a \in E$.

3.3. *Two characterizations*

Since none of the target rules satisfies *replacement-domination* or *population-monotonicity*, it is obvious that Theorems 1 and 2 do not extend to rules on the set of separable-quadratic profiles. The following theorems demonstrate that Corollaries 2 and 3 do extend to the model at hand.

THEOREM 4. *Suppose* $\varphi : \bigcup_{\substack{N \in \mathcal{P} \\ |N| \geqslant 3}} \mathcal{Q}^N \to E$. *Then* φ *satisfies unanimity, strategy-proofness, and replacement-domination if and only if for every* $N \in \mathcal{P}$, $|N| \geqslant 3$, *there exists* $a^N \in E$ *such that for all* $R \in \mathcal{Q}^N$, $\varphi(R) = \varphi^{a^N}(R)$.

Proof. Let $N \in \mathcal{P}$ and $|N| \geqslant 3$ and assume that φ satisfies *unanimity*, *strategy-proofness*, and *replacement-domination*. Then, by Theorem 3, there exist m (coordinatewise) rules $\varphi^j : \mathcal{Q}^N_j \to \mathbb{R}$, $j \in \{1, \dots, m\}$, which are *unanimous* and *strategy-proof*, such that for all $R \in \mathcal{Q}^N$, and all $j \in \{1, \dots, m\}$, $\varphi_j(R) = \varphi^j(R^j)$.

Since all rules φ^j satisfy *unanimity* and *strategy-proofness*, they also satisfy *Pareto efficiency* (with respect to dimension j). Furthermore, it is easy to see that *replacement-domination* of the rule φ implies *replacement-domination* (with respect to dimension j) for the rules φ^j.[8] Since any of the rules φ^j satisfies *Pareto efficiency* and *replacement-domination*, by Theorem 1,[9] for all $j \in \{1, \dots, m\}$ there exist $a^N_j \in \mathbb{R}$ such that for all $R^j \in \mathcal{Q}^N_j$, $\varphi^j(R^j) = \varphi^{a^N_j}(R^j)$. Let $a^N \equiv (a^N_1, \dots, a^N_m)$. Then, for all $R \in \mathcal{Q}^N$, $\varphi(R) = \varphi^{a^N}(R)$. □

THEOREM 5. *Suppose* $\varphi : \bigcup_{N \in \mathcal{P}} \mathcal{Q}^N \to E$. *Then* φ *satisfies unanimity, strategy-proofness, and population-monotonicity if and only if* $\varphi = \varphi^a$ *for some* $a \in E$.

Proof. Let φ satisfy *unanimity*, *strategy-proofness*, and *population-monotonicity*. So, by Theorem 3, there exist m (coordinatewise) rules $\varphi^j : \bigcup_{N\in\mathcal{P}} \mathcal{Q}_j^N \to \mathbb{R}$, $j \in \{1, \ldots, m\}$, which are *unanimous* and *strategy-proof*, such that for all $N \in \mathcal{P}$, all $R \in \mathcal{Q}^N$, and all $j \in \{1, \ldots, m\}$, $\varphi_j(R) = \varphi^j(R^j)$.

Since all rules φ^j satisfy *unanimity* and *strategy-proofness*, they also satisfy *Pareto efficiency* (with respect to dimension j). Furthermore, it is easy to see that *population-monotonicity* of the rule φ implies *population-monotonicity* (with respect to dimension j) for the rules φ^j.[10] Since any of the rules φ^j satisfies *Pareto efficiency* and *population-monotonicity*, by Theorem 2,[11] for all $j \in \{1, \ldots, m\}$ there exist $a_j \in \mathbb{R}$ such that for all $N \in \mathcal{P}$, all $R^j \in \mathcal{Q}_j^N$, $\varphi^j(R^j) = \varphi^{a_j}(R^j)$. Let $a \equiv (a_1, \ldots, a_m)$. Then, $\varphi = \varphi^a$. □

The rules that prove the independence of the properties in Corollaries 2 and 3 can be easily adjusted to demonstrate the independence of the properties in Theorems 4 and 5.

As one of the referees pointed out: 'Theorems 4 and 5 can be viewed as answers to the following questions: Which (sequences of) generalized median voter schemes satisfy *population-monotonicity* (or *replacement-domination*)?'

4. CONCLUSION

Samuelson and Zeckhauser (1988) prove that in many situations individuals disproportionally stick to the status quo. In other words, a 'target bias' with the target equal to the status quo is present in many decisions. Our main results imply that in public good economies *Pareto efficiency* and solidarity imply such a target bias. Target rules with the target equal to the status quo are useful in economic situations when agents have veto power over changes in the status quo. A practical advantage of target rules is that they are simple and can be implemented easily and quickly. Furthermore, they are *strategy-proof* and to some extent fair if they use 'fair' target points. Similar results for probabilistic rules are obtained in Ehlers and Klaus (2001).

APPENDIX

Proof of Theorem 2

First we show that *Pareto efficiency* and *population-monotonicity* imply *replacement-domination*.

LEMMA 3. *Let* $\varphi : \bigcup_{N\in\mathcal{P}} \mathcal{S}^N \to T$ *be Pareto efficient and population-monotonic. Then* φ *satisfies replacement-domination.*

Proof. Let φ be a *Pareto efficient* and *population-monotonic* rule. Hence, we can apply Lemma 2 throughout the proof. Assume, by contradiction, that φ does not satisfy *replacement-domination*. Then there exist $N \in \mathcal{P}$, $j, k, l \in N$, $j \neq k, l$, and $R, \bar{R} \in \mathcal{S}^N$ such that $R_{-j} = \bar{R}_{-j}$ and

$$\varphi(R)\ P_k\ \varphi(\bar{R}) \text{ and } \varphi(\bar{R})\ P_l\ \varphi(R). \tag{1}$$

Lemma 2 applied to R and R_{-j} implies for all $i \in N\backslash\{j\}$,

$$\varphi(R_{-j})\ R_i\ \varphi(R). \tag{2}$$

Lemma 2 applied to $\bar{R}$ and $R_{-j} = \bar{R}_{-j}$ implies for all $i \in N\backslash\{j\}$,

$$\varphi(R_{-j})\ R_i\ \varphi(\bar{R}). \tag{3}$$

By (1) and (2), $\varphi(\bar{R}_{-j})\ P_k\ \varphi(\bar{R})$. Hence, by (3) and *Pareto efficiency*, $\varphi(\bar{R}) \notin P(\bar{R}_{-j})$. By (1) and (3), $\varphi(R_{-j})\ P_l\ \varphi(R)$. Hence, by (2) and *Pareto efficiency*, $\varphi(R) \notin P(R_{-j})$.

Assume that agent l leaves profile $\bar{R}$. Because $\varphi(\bar{R}) \notin P(\bar{R}_{-j})$ it follows that $\varphi(\bar{R}) \in [p(\bar{R}_j), p(\bar{R}_k)] \subseteq P(\bar{R}_{-l})$. Thus, by Lemma 2, $\varphi(\bar{R}_{-l}) = \varphi(\bar{R})$. Next, we add agent l with preference relation $\tilde{R}_l$ such that $p(\tilde{R}_l) = \varphi(\bar{R})$ to profile $\bar{R}_{-l}$. Since $P(\bar{R}_{-l}) = P(\bar{R}_{-l}, \tilde{R}_l)$, by Lemma 2, $\varphi(\bar{R}_{-l}, \tilde{R}_l) = \varphi(\bar{R}_{-l}) = \varphi(\bar{R})$.

Now, agent j leaves profile $(\bar{R}_{-l}, \tilde{R}_l)$. Since $\varphi(\bar{R}_{-l}, \tilde{R}_l) \in P(\bar{R}_{-j,l}, \tilde{R}_l)$, by Lemma 2, $\varphi(\bar{R}_{-j,l}, \tilde{R}_l) = \varphi(\bar{R}_{-l}, \tilde{R}_l) = \varphi(\bar{R})$. Then, we add agent j with preference relation $\tilde{R}_j$ such that $p(\tilde{R}_j) = \varphi(R)$ to profile $(\bar{R}_{-j,l}, \tilde{R}_l)$. Recall that $\varphi(R)\ P_k\ \varphi(\bar{R}) = \varphi(\bar{R}_{-j,l}, \tilde{R}_l)$. Hence, if $\varphi(\bar{R}_{-j,l}, \tilde{R}_j, \tilde{R}_l) \notin P(\bar{R}_{-j,l}, \tilde{R}_l)$, then $\varphi(\bar{R}_{-j,l}, \tilde{R}_j, \tilde{R}_l) \in [\varphi(R), p(\bar{R}_k)]$. This implies that $\varphi(\bar{R}_{-j,l}, \tilde{R}_j, \tilde{R}_l)\ P_k\ \varphi(\bar{R}_{-j,l}, \tilde{R}_l)$. This is a contradiction to Lemma 2. So, $\varphi(\bar{R}_{-j,l}, \tilde{R}_j, \tilde{R}_l) \in P(\bar{R}_{-j,l}, \tilde{R}_l)$ and by Lemma 2, $\varphi(\bar{R}_{-j,l}, \tilde{R}_j, \tilde{R}_l) = \varphi(\bar{R}_{-j,l}, \tilde{R}_l) = \varphi(\bar{R})$.

Next, agent k leaves profile $(\bar{R}_{-j,l}, \tilde{R}_j, \tilde{R}_l)$. Since $\varphi(\bar{R}_{-j,l}, \tilde{R}_j, \tilde{R}_l) \in P(\bar{R}_{-j,k,l}, \tilde{R}_j, \tilde{R}_l)$, by Lemma 2,

$$\varphi(\bar{R}_{-j,k,l}, \tilde{R}_j, \tilde{R}_l) = \varphi(\bar{R}_{-j,l}, \tilde{R}_j, \tilde{R}_l) = \varphi(\bar{R}). \tag{4}$$

Next, assume that agent k leaves profile R. Because $\varphi(R) \notin P(R_{-j})$ it follows that $\varphi(R) \in [p(R_j), p(R_l)] \subseteq P(R_{-k})$. Thus, by Lemma 2, $\varphi(R_{-k}) = \varphi(R)$. Next, we add agent k with preference relation $\hat{R}_k$ such that $p(\hat{R}_k) = \varphi(R)$ to profile R_{-k}. Since $P(R_{-k}) = P(R_{-k}, \hat{R}_k)$, by Lemma 2, $\varphi(R_{-k}, \hat{R}_k) = \varphi(R_{-k}) = \varphi(R)$.

Now, agent j leaves $(R_{-k}, \hat{R}_k)$. Since $\varphi(R_{-k}, \hat{R}_k) \in P(R_{-j,k}, \hat{R}_k)$, by Lemma 2, $\varphi(R_{-j,k}, \hat{R}_k) = \varphi(R_{-k}, \hat{R}_k) = \varphi(R)$. Then, we add agent j with preference relation $\tilde{R}_j$ to profile $(R_{-j,k}, \hat{R}_k)$. Since $P(R_{-j,k}, \hat{R}_k) = P(R_{-k}, \tilde{R}_j, \hat{R}_k)$, by Lemma 2, $\varphi(R_{-k}, \tilde{R}_j, \hat{R}_k) = \varphi(R_{-j,k}, \hat{R}_k) = \varphi(R)$.

Next, agent k leaves profile $(R_{-k}, \tilde{R}_j, \hat{R}_k)$. Since $\varphi(R_{-k}, \tilde{R}_j, \hat{R}_k) \in P(R_{-k}, \tilde{R}_j)$, by Lemma 2, $\varphi(R_{-j,k}, \tilde{R}_j) = \varphi(R_{-k}, \tilde{R}_j, \hat{R}_k) = \varphi(R)$. Then, we add agent k with preference relation $\tilde{R}_k$ such that $p(\tilde{R}_k) = \varphi(\bar{R})$ to profile $(R_{-j,k}, \tilde{R}_j)$. Recall that $\varphi(\bar{R})\ P_l\ \varphi(R) = \varphi(R_{-j,k}, \tilde{R}_j)$. Therefore, if $\varphi(R_{-j,k}, \tilde{R}_j, \tilde{R}_k) \notin P(R_{-j,l}, \tilde{R}_j)$, then $\varphi(R_{-j,k}, \tilde{R}_j, \tilde{R}_k) \in [\varphi(\bar{R}), p(R_l)]$. Thus, $\varphi(R_{-j,k}, \tilde{R}_j, \tilde{R}_k)\ P_l\ \varphi(R_{-j,k}, \tilde{R}_j)$. This would be in contradiction to Lemma 2. So, $\varphi(R_{-j,k}, \tilde{R}_j, \tilde{R}_k) \in P(R_{-j,l}, \tilde{R}_j)$. Then, by Lemma 2, $\varphi(R_{-j,k}, \tilde{R}_j, \tilde{R}_k) = \varphi(R_{-j,l}, \tilde{R}_j) = \varphi(R)$.

Next, agent l leaves profile $(R_{-j,k}, \tilde{R}_j, \tilde{R}_k)$. Since $\varphi(R_{-j,k}, \tilde{R}_j, \tilde{R}_k) \in P(R_{-j,k,l}, \tilde{R}_j, \tilde{R}_k)$, by Lemma 2 it follows that $\varphi(R_{-j,k,l}, \tilde{R}_j, \tilde{R}_k) = \varphi(R_{-j,k}, \tilde{R}_j, \tilde{R}_k) = \varphi(R)$. Then, we add agent l with preference relation $\tilde{R}_l$ to $(R_{-j,k,l}, \tilde{R}_j, \tilde{R}_k)$. Since $P(R_{-j,k,l}, \tilde{R}_j, \tilde{R}_k) = P(R_{-j,k,l}, \tilde{R}_j, \tilde{R}_k, \tilde{R}_l)$, by Lemma 2, $\varphi(R_{-j,k,l}, \tilde{R}_j, \tilde{R}_k, \tilde{R}_l) = \varphi(R_{-j,k,l}, \tilde{R}_j, \tilde{R}_k) = \varphi(R)$.

Next, agent k leaves $(R_{-j,k,l}, \tilde{R}_j, \tilde{R}_k, \tilde{R}_l)$. Since $\varphi(R_{-j,k,l}, \tilde{R}_j, \tilde{R}_k, \tilde{R}_l) \in P(R_{-j,k,l}, \tilde{R}_j, \tilde{R}_l) = p(\tilde{R}_l)$, by Lemma 2,

$$\varphi(R_{-j,k,l}, \tilde{R}_j, \tilde{R}_l) = \varphi(R_{-j,k,l}, \tilde{R}_j, \tilde{R}_k, \tilde{R}_l) = \varphi(R). \tag{5}$$

Since $\bar{R}_{-j,k,l} = R_{-j,k,l}$ and $\varphi(R) \neq \varphi(\bar{R})$, (4) and (5) constitute a contradiction. □

Note that Lemma 3 remains true on the larger domain of single-peaked preferences.

Applying Lemma 3 and Theorem 1, we provide a simple proof of Theorem 2. Note that this proof remains valid on the larger domain of single-peaked preferences.

Proof of Theorem 2. It is easy to prove that all target rules satisfy *Pareto efficiency* and *population-monotonicity*.

Let $|\mathbb{P}| \geqslant 3$ and let the rule φ satisfy *Pareto efficiency* and *population-monotonicity*. By Lemma 3, φ satisfies *replacement-domination*. Hence, by Theorem 1, for each $N \in \mathcal{P}$ such that $|N| \geqslant 3$, there exists a^N such that for all $R \in \mathcal{S}^N$, $\varphi(R) = \varphi^{a^N}(R)$.

First, we show that for all $N, M \in \mathcal{P}$ such that $|N|, |M| \geqslant 3$, $a^N = a^M \equiv a$. Let $x, y \in T$ and consider $R^1 \in \mathcal{S}^N$, $R^2 \in \mathcal{S}^M$, and $R^3 \in \mathcal{S}^{N \cup M}$ such that $R^3_N = R^1$, $R^3_M = R^2$, and $P(R^1) = P(R^2) = P(R^3) = [x, y]$.

Suppose $M \backslash N \neq \emptyset$. Adding all agents $j \in M \backslash N$ with R^3_j yields profile R^3. Since $P(R^1) = P(R^3)$, by Lemma 2, $\varphi^{a^N}(R^1) = \varphi^{a^{N \cup M}}(R^3)$. Since $x, y \in T$ were arbitrarily chosen and by the definition of φ^{a^N} and $\varphi^{a^{N \cup M}}$, $a^N = a^{N \cup M}$. Similarly, we can conclude that $a^M = a^{N \cup M}$. Hence, $a^N = a^M \equiv a$.

By *Pareto efficiency*, for all $N \in \mathcal{P}$ such that $|N| = 1$ and all $R \in \mathcal{S}^N$, $\varphi(R) = \varphi^a(R)$. Hence, it remains to be shown that for all $N \in \mathcal{P}$ such that $|N| = 2$ and all $R \in \mathcal{S}^N$, $\varphi(R) = \varphi^a(R)$.

Let $N \in \mathcal{P}$ be such that $|N| = 2$ and $R \in \mathcal{S}^N$. Let $j \in N$, $k \in \mathbb{P} \backslash N$, $R_k = R_j$, and consider $(R, R_k) \in \mathcal{S}^{N \cup \{k\}}$. Since $P(R) = P(R, R_k)$, by Lemma 2, $\varphi(R) = \varphi(R, R_k)$. Since $|N \cup \{k\}| = 3$, $\varphi(R, R_k) = \varphi^a(R, R_k) = \varphi^a(R)$. Hence, $\varphi(R) = \varphi^a(R)$. □

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NOTES

1. A *tree* is a connected graph that contains no cycles.
2. Preferences on an interval are single-peaked if up to a certain point, the peak level, preferences are strictly increasing, and strictly decreasing beyond that point. On a tree, preferences are single-peaked if preferences are single-peaked on all paths of the tree.
3. Target rules are sometimes called status quo rules or status quo solutions.
4. Note that if $T = \mathbb{R}$, then $a \in \mathbb{R} \cup \{-\infty, \infty\}$.
5. On tree networks, Schummer and Vohra (2001) characterize the class of *strategy-proof* and onto rules. Any such rule equals an 'extended generalized median voter rule' that coincides with a generalized median voter rule on any path of the tree.
6. Note that Moulin (1980) and Ching (1997) analyze the case where the tree is a closed interval or the real line. However, the proof that for rules on tree networks *unanimity* and *strategy-proofness* together imply *Pareto efficiency* is similar to the case where the tree is a closed interval or the real line.
7. For certain preference profiles, an agent, by lying over his distance function, can deform the Pareto set of the profile in such a way that the target rule assigns a point that he prefers to the outcome when he is honest.
8. To see this, note that we can construct φ^j from φ as follows. Let $R^j \in \mathcal{Q}_j^N$ and define $R \in \mathcal{Q}^N$ such that for all $k \neq j$ and all $i \in N$, $p(R_i^k) = 0$. Then, $\varphi^j(R^j) \equiv \varphi_j(R)$. Hence, if φ satisfies *replacement-domination*, then φ^j satisfies *replacement-domination*.
9. Here, it is important that Theorem 1 is valid on the domain of symmetric, single-peaked preferences.
10. Similarly as in the proof of Theorem 4, we can construct φ^j from φ and show that if φ satisfies *population-monotonicity*, then φ^j satisfies *population-monotonicity*.
11. Again, it is important that Theorem 2 is valid on the domain of symmetric, single-peaked preferences.

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