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Consistency of assessments in infinite signaling games

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Abstract

In this paper we investigate possible ways to define consistency of assessments in infinite signaling games, i.e. signaling games in which the sets of types, messages and answers are complete, separable metric spaces. Roughly speaking, a consistency concept is called appropriate if it implies Bayesian consistency and copies the original idea of consistency in finite extensive form games as introduced by Kreps and Wilson (*Econometrica*, 1982, 50, 863-894). We present a particular appropriate consistency concept, which we call strong consistency, and give a characterization of strongly consistent assessments. It turns out that all appropriate consistency concepts are refinements of strong consistency. Finally, we define and characterize structurally consistent assessments in infinite signaling games.

JEL classification: C72, C73

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1. Introduction

One of the most widely applied classes of games in economics is the class of signaling games. A signaling game is a game of incomplete information where two players are involved; player 1 – the sender – moves first and sends a message to player 2 – the receiver – who observes the message and chooses an answer. Player 1 has more information than player 2, which is modeled by assuming that

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player 1's type is drawn by a move of nature at the beginning of the game, player 1 is informed about the outcome of this draw, but player 2 is not; the distribution of nature's move, however, is common knowledge. Player 1's message may serve as a signal to convey or hide information about player 1's type. Think, for instance, of player 1 as the seller of a car who is informed about the quality of the car, and of player 2 as the uninformed potential buyer who is to say yes or no to a sales contract specifying the price as well as warranty conditions.

Signaling games – and, more generally, dynamic games of imperfect or incomplete information – are analyzed by considering not only the strategies of the players but also the beliefs that an uninformed player may have about his information sets. In a signaling game as described above, an assessment is a pair of strategies, together with a probability distribution (beliefs), assessed by player 2 over the possible types of player 1. Common to all kinds of Nash equilibrium refinements considered in the literature on signaling games is the *sequential rationality* requirement, which says that the players' strategies are best responses to each other, where player 2 maximizes his payoff, given his beliefs. Moreover, these beliefs must be consistent with Bayes' rule whenever possible – i.e. at all information sets reached with positive probability, where these probabilities depend on the distribution of nature's move and player 1's strategy. The latter property is called *Bayesian consistency*.

In a finite signaling game an assessment is called a *sequential equilibrium* if it satisfies sequential rationality and Bayesian consistency (see Cho and Kreps, 1987).

An essential part in the definition of sequential equilibria for general (finite) extensive form games, as introduced by Kreps and Wilson (1982), is the *consistency* condition. Roughly, this means that the assessment can be approximated by a sequence of Bayesian-consistent and completely mixed assessments. This condition is a kind of 'trembling hand' condition: the beliefs of a player should be consistent, in the limit, with the beliefs he would have according to Bayesian updating if all players would 'tremble', so that each of his information sets would be reached with positive probability. Just as is the case with perfect equilibria, the requirement is that beliefs be consistent with *some* trembles, not with *all* trembles.

Moreover, consistency of assessments can be viewed as a condition that requires the beliefs to reflect and respect the structure of information sets in the game. By the structure of the information sets we mean the positions of the information sets in the game tree and the way different information sets are connected via actions and chance moves. In infinite extensive form games, such as infinite signaling games, the collection of information sets may be (uncountably) infinite. However, similarly to finite extensive form games, information sets in infinite extensive form games have specific positions in the (possibly infinite) game tree and are connected in a specific way. This is the reason why we think that the idea of consistency is also meaningful in infinite extensive form games.

It is well known that consistency is equivalent to Bayesian consistency in finite signaling games. This is why the definition of sequential equilibria for finite signaling games uses Bayesian consistency instead of consistency.

In this paper we try to apply this idea of consistency in infinite signaling games where the sets of types, messages and answers are complete separable metric spaces. In the last few years much attention has been paid to infinite signaling games; see, for instance, Mailath (1987, 1988) and Manelli (1994), to name just three. In contrast to finite extensive form games, it is not obvious how to define completely mixed assessments and convergence of assessments in infinite signaling games. Since the concept of consistency depends on the way in which these two concepts are defined, a whole variety of definitions for consistency is possible. A natural requirement for a consistency concept is that it should imply Bayesian consistency. This requirement is called condition A. It turns out that this condition heavily restricts the number of convergence concepts for assessments that can be used in the consistency concept. An example will show that weak convergence of probability measures is too weak for this purpose. Therefore, we have to use a stronger kind of convergence, which we will call *pointwise convergence of probability measures*.

To stay close to the original definition of consistency for finite extensive form games, in an appropriate consistency concept almost all local strategies and beliefs in the supporting sequence of assessments should converge pointwise to the original local strategies and beliefs. We call this restriction condition C. In a finite extensive form game, a completely mixed assessment induces a positive probability on every node in the tree. If we consider an infinite signaling game as a tree where the number of nodes may be (uncountably) infinite, a completely mixed assessment should induce a positive probability on every non-empty, open set of nodes in the tree. This requirement is formalized in condition B. We call a consistency concept *appropriate* if it satisfies the three conditions A, B and C.

After introducing infinite signaling games in Section 2, we formulate conditions A, B and C in Section 3 and explain why these conditions are sensible. In Section 4, a particular consistency concept, called *strong consistency*, is introduced and is shown to be appropriate.

In addition, we provide a characterization of strongly consistent assessments in Section 5. This characterization states that an assessment is strongly consistent if and only if it is Bayesian-consistent and the local beliefs following messages that lie isolated in the message space are absolutely continuous with respect to the a priori probability distribution on the types. This characterization plays a crucial role in the remainder of the paper. A consequence of this characterization is, for example, the observation that strong consistency and Bayesian consistency are equivalent in signaling games where the message space contains no isolated points or the type space is discrete.

In Section 6 we show that conditions A, B and C imply the conditions in this characterization. As a consequence, every appropriate consistency concept in infinite signaling games is a refinement of strong consistency. Moreover, we provide an example that shows how strong consistency can exclude some sequential equilibria in signaling games where strong consistency is *not* equivalent to Bayesian consistency.

In Section 7 we give a definition of structurally consistent assessments in infinite signaling games that is, from our viewpoint, a natural implication of the original idea of structural consistency as given by Kreps and Wilson (1982). We conclude this paper with a characterization of the class of structurally consistent assessments.

Notation

For a metric space X, $\mathscr{M}(X)$ is the set of all finite measures on X and $\mathscr{P}(X)$ denotes the set of probability measures on X. By δ_x we denote the Dirac measure on the point $x \in X$. For an $\varepsilon > 0$ and an $x \in X$ the ε-neighborhood of x is denoted by $U_\varepsilon(x)$. Furthermore, by $a \vee b$ we denote the maximum of two numbers a and b.

A metric space X is called *separable* if it contains a countable dense subset. A Borel subset of X is denoted by X_B.

2. Preliminaries

In this section we present the model of an infinite signaling game. This model, which is very similar to that of Manelli (1994), is in our view a natural extension of the finite model.

2.1. Infinite signaling games

An *infinite signaling game* (from now on simply called *signaling game*) is a sextuple $(T, M, A, u_1, u_2, \tau)$, where T, M and A are complete, separable metric spaces, u_1 and u_2 are measurable real functions on $T \times M \times A$, and τ is a strictly positive probability measure on T (i.e. τ puts a positive weight on every non-empty, open subset of T).

The game is played according to the following rules: first, player 1's type is determined by nature according to the a priori distribution τ. After observing his type, player 1 sends a message, $m \in M$. Being ignorant of player 1's type, player 2 responds to this message with an answer, $a \in A$. Finally, the payoffs for both players are given by $u_1(t, m, a)$ and $u_2(t, m, a)$, respectively.

2.2. Strategies and beliefs

A *behavior strategy for player* 1 is a mapping $\sigma_1 : T \to \mathscr{P}(M)$, such that the function $t \mapsto \sigma_1(t)(M_B)$ is measurable on T for every M_B. So, for every type t, $\sigma_1(t)$ defines a probability measure on the message space M.

A *behavior strategy for player* 2 is a mapping $\sigma_2 : M \to \mathscr{P}(A)$, such that the function $m \mapsto \sigma_2(m)(A_B)$ is measurable on M for every A_B. For every message m, $\sigma_2(m)$ defines a probability measure on the answer space A.

The probability measures $\sigma_1(t)$ and $\sigma_2(m)$ are called *local strategies.*

A pair $\sigma = (\sigma_1, \sigma_2)$ is called a *behavior strategy profile* (BSP).

A *belief system* is a mapping $\beta : M \to \mathscr{P}(T)$, such that the function $m \mapsto \beta(m)(T_B)$ is measurable for every T_B.

For a message m and Borel set T_B, $\beta(m)(T_B)$ can be interpreted as the (subjective) probability that player 2 assigns to the event that player 1 has a type in T_B if he observes the message m. The probability measures $\beta(m)$ are called *local beliefs.*

2.3. Sequential rationality and Bayesian consistency

A pair (σ, β) is called an *assessment.* An assessment (σ, β) is called *sequentially rational* if the local strategy $\sigma_1(t)$ maximizes player 1's expected payoff for every $t \in T$ and if at every message m, the local strategy $\sigma_2(m)$ maximizes player 2's expected payoff, given his beliefs about player 1's type. Formally, if for every $t \in T$ and $\mu \in \mathscr{P}(M)$:

$$\int_A \int_M u_1(t, m, a)\ \mathrm{d}\sigma_1(t)\ \mathrm{d}\sigma_2(m) \geq \int_A \int_M u_1(t, m, a)\ \mathrm{d}\mu\ \mathrm{d}\sigma_2(m),$$

and if for every $m \in M$ and $\alpha \in \mathscr{P}(A)$:

$$\int_A \int_T u_2(t, m, a)\ \mathrm{d}\beta(m)\ \mathrm{d}\sigma_2(m) \geq \int_A \int_T u_2(t, m, a)\ \mathrm{d}\beta(m)\ \mathrm{d}\alpha.$$

Note that $\sigma_1(t)$ in the first integral does *not* mean that we integrate over t. In this integral, t is fixed, and we integrate over m with respect to the probability measure $\sigma_1(t)$.

A BSP σ induces the probability measure π^σ on $T \times M$ defined by

$$\pi^\sigma(T_B \times M_B) := \int_{T_B} \sigma_1(t)(M_B)\ \mathrm{d}\tau.$$

Hence, $\pi^\sigma(T_B \times M_B)$ is the probability that player 1 has a type in T_B and sends a message in M_B, given the fact that σ is played.

We denote by P^σ the marginal distribution on M corresponding to π^σ. Hence,

$$P^\sigma(M_B) = \int_T \sigma_1(t)(M_B)\ \mathrm{d}\tau$$

is the probability that a message in M_B will be sent if σ is played.

An assessment (σ, β) is called *Bayesian-consistent* if β is a conditional distribution for π^{σ}. This means that for all $T_B \times M_B$:

$$\int_{M_B} \beta(m)(T_B)\, \mathrm{d}P^{\sigma} = \pi^{\sigma}(T_B \times M_B).$$

In finite extensive form games, Bayesian consistency completely determines the beliefs at all information sets that are reached with positive probability. The crucial difference between Bayesian consistency in finite extensive form games and Bayesian consistency in infinite signaling games lies in the fact that in infinite signaling games, it puts restrictions on the behavior of beliefs at *collections* of information sets rather than at *individual* information sets.

An assessment (σ, β) is called a *sequential equilibrium* if it is sequentially rational and Bayesian-consistent. However, a sequential equilibrium does not always exist, as is shown by an example of van Damme (1987).

3. Minimal requirements for consistency concepts

As mentioned earlier, we investigate several possibilities to define consistency of assessments in infinite signaling games. Of course, there are many different ways to do this, but not all of them are equally meaningful. To decide whether a given consistency concept is appropriate, we develop a system of minimal requirements that such a concept should satisfy.

Formally speaking, a consistency concept is a mapping φ that assigns to every signaling game Γ a set $\varphi(\Gamma)$ of assessments. An assessment in $\varphi(\Gamma)$ is called *consistent w.r.t.* φ.

An appropriate consistency concept should, from our point of view, reflect the idea of consistency as it was defined by Kreps and Wilson (1982) for finite extensive form games. This requirement can be formalized by saying that a consistency concept φ should have the following canonical form:

"An assessment (σ, β) is consistent w.r.t. φ if and only if there is a sequence $(\sigma^k, \beta^k)_{k \in \mathbb{N}}$ of Bayesian-consistent assessments such that
(1) (σ^k, β^k) is *completely mixed* for every k and
(2) $(\sigma^k, \beta^k)_{k \in \mathbb{N}}$ *converges to* (σ, β)."

This canonical form does not induce a unique consistency concept, since it depends on the way we define completely mixed assessments and convergence of assessments. The particular definitions of completely mixed assessments and convergence of assessments used in φ are called *complete mixedness w.r.t.* φ and *convergence w.r.t.* φ, respectively.

However, we will not regard every concept φ satisfying this canonical form as appropriate, for reasons to become clear later. More precisely, we will impose further requirements labeled as conditions A, B and C.

3.1. Consistency should imply Bayesian consistency

First of all, an appropriate consistency concept should always imply Bayesian consistency.

Condition A. Every assessment that is consistent w.r.t. φ should be Bayesian-consistent.

This condition may seem easy to fulfill, but, as we show later in this section, we need a rather strong convergence concept in order to satisfy this requirement. Weak convergence of probability measures, for example, is not strong enough for this purpose.

3.2. Restriction on completely mixed assessments

Next, we put a restriction on the definition of completely mixed assessments. In Selten's (1975) article about perfect equilibria, completely mixed behavior strategies are used because they induce a positive probability on every node in the tree of a finite extensive form game.

A signaling game can also be interpreted as an extensive form game with a possibly infinite number of nodes. The nodes that follow the actions of player 1 are given by pairs $(t, m) \in T \times M$. Of course, in general, it is not possible to require that player 1's strategy induces a positive probability on every single node in $T \times M$. This condition cannot be satisfied if, for example, $T \times M$ is uncountable. However, Simon and Stinchcombe (1995) discuss a very natural way to define completely mixed strategies in the infinite case. They call a mixed strategy in an infinite normal form game *of full support* if it puts positive weight on every non-empty, open subset of pure strategies.

Combining the ideas of both papers, we arrive at the following condition, which says that a completely mixed assessment in a signaling game should always induce a positive probability on every non-empty, open subset of nodes (t, m).

Condition B. Every assessment (σ, β) that is completely mixed w.r.t. φ should have the property that

$$\int_{T_B} \sigma_1(t)(M_B)\ \mathrm{d}\tau > 0,$$

for every non-empty, open subset $(T_B, M_B) \in T \times M$.

One possible way to define completely mixed assessments is to say that an assessment (σ, β) is completely mixed if the local strategy $\sigma_1(t)$ is a strictly positive probability measure on M for every t. We call such assessments *pointwise completely mixed*. Obviously, this definition satisfies condition B.

An important property of completely mixed assessments in finite extensive form games is the fact that they always induce strictly positive beliefs at every information set if the assessment satisfies Bayesian consistency. In an infinite signaling game, an assessment that is pointwise completely mixed and Bayesian-consistent does not necessarily have the property that the local belief $\beta(m)$ is strictly positive for every m. However, a similar but somewhat weaker property can be shown. It turns out that for every non-empty, open subset T_B the set $\{m \mid \beta(m)(T_B) > 0\}$ is dense in M.

3.3. Why is weak convergence not strong enough?

A possible convergence concept that can be used in a consistency concept is the so-called *weak convergence of assessments*. We say that a sequence $(\sigma^k, \beta^k)_{k \in \mathbb{N}}$ of assessments converges weakly to an assessment (σ, β) if $(\sigma_1^k(t))_{k \in \mathbb{N}}$ converges weakly to $\sigma_1(t)$ for every t, $(\sigma_2^k(m))_{k \in \mathbb{N}}$ converges weakly to $\sigma_2(m)$ for every m and $(\beta^k(m))_{k \in \mathbb{N}}$ converges weakly to $\beta(m)$ for every m.

Although weak convergence seems a very natural convergence concept in this situation, the following example shows that even the consistency concept that makes use of weak convergence and pointwise completely mixed assessments does not imply Bayesian consistency and can therefore not be appropriate.

Example 1. Let φ be the consistency concept induced by pointwise complete mixedness and weak convergence. Let Γ be a signaling game in which $T = M = [0, 1]$ and τ is the uniform distribution on T. We denote the uniform distribution on $[0, 1]$ by u, and for an interval $I \subset [0, 1]$ the uniform distribution on I is denoted by u_I.

Now, we construct an assessment (σ, β) that is consistent w.r.t. φ but not Bayesian-consistent.

Let the behavior strategy σ_1 and the belief system β be given by

$$\sigma_1(t) := u \quad \text{for every } t,$$

$$\beta(m) := \begin{cases} \frac{1}{3}u_{[0,1/2]} + \frac{2}{3}u_{[1/2,1]}, & \text{if } m \notin \mathbb{Q}, \\ u_{[0,1/2]}, & \text{if } m \in \mathbb{Q}. \end{cases}$$

Furthermore, we choose an arbitrary behavior strategy σ_2 for player 2.

The assessment (σ, β) is not Bayesian-consistent since

$$\int_{[0,1]} \beta(m)\left(\left[0, \tfrac{1}{2}\right]\right) \mathrm{d}u = \tfrac{1}{3}$$

but

$$\int_{[0,1/2]} \sigma_1(t)([0,1])\, du = \tfrac{1}{2}.$$

To show that (σ, β) is consistent w.r.t. φ, we construct a sequence $(\sigma^k, \beta^k)_{k \in \mathbb{N}}$ of assessments that are Bayesian-consistent and pointwise completely mixed such that $(\sigma^k, \beta^k)_{k \in \mathbb{N}}$ converges weakly to (σ, β).

For every $k \in \mathbb{N}$, let $M^k := \{0, 1/k, 2/k, \ldots, (k-1)/k\}$ and let μ^k be the probability measure on M that puts probability $1/k$ on every point in M^k. It can be shown that the sequence $(\mu^k)_{k \in \mathbb{N}}$ converges weakly to u. (See, for example, Example 25.3 in Billingsley, 1986.)
For every k, we define σ_1^k and β^k by

$$\sigma_1^k(t) := \begin{cases} \frac{1}{2}u + \frac{1}{2}\mu^k, & \text{if } t \le \frac{1}{2}, \\ u, & \text{if } t > \frac{1}{2}; \end{cases}$$

$$\beta^k(m) := \begin{cases} \frac{1}{3}u_{[0,1/2]} + \frac{2}{3}u_{[1/2,1]}, & \text{if } m \notin M^k, \\ u_{[0,1/2]}, & \text{if } m \in M^k. \end{cases}$$

Obviously, $\sigma_1^k(t)$ is strictly positive for every t, and the sequence $(\sigma_1^k(t))_{k \in \mathbb{N}}$ converges weakly to $\sigma_1(t)$ for every t. To show that $(\beta^k(m))_{k \in \mathbb{N}}$ converges weakly to $\beta(m)$ for all m, we distinguish two cases.

If $m \notin \mathbb{Q}$, then it follows that $m \notin M^k$ for every k. By construction, $\beta^k(m) = \beta(m)$ for all k, which implies that $\beta^k(m)$ converges weakly to $\beta(m)$.

For the case $m \in \mathbb{Q}$, we need the following lemma, which is formulated as Theorem 2.3 in Billingsley (1968).

Lemma 3.1. Let X be a metric space and $\mu, \mu^1, \mu^2, \ldots$ probability measures on X. Then, the sequence $(\mu^k)_{k \in \mathbb{N}}$ converges weakly to μ if and only if every subsequence of $(\mu^k)_{k \in \mathbb{N}}$ contains a further subsequence that converges weakly to μ.

Now, let $m \in \mathbb{Q}$ and $(\beta^{k'}(m))_{k \in \mathbb{N}}$ a subsequence of $(\beta^k(m))_{k \in \mathbb{N}}$. We can find a further subsequence $(\beta^{k''}(m))_{k \in \mathbb{N}}$ such that $m \in M^{k''}$ for every k'', which implies that $\beta^{k''}(m) = \beta(m)$ for every k''. It follows that $(\beta^{k''}(m))_{k \in \mathbb{N}}$ converges weakly to $\beta(m)$. By Lemma 3.1 $(\beta^k(m))_{k \in \mathbb{N}}$ converges weakly to $\beta(m)$.

Finally, it can be shown that (σ^k, β^k) is Bayesian-consistent for every k. This implies that (σ, β) is consistent w.r.t. φ. □

From the above example we learn that the 'weakness' of weak convergence lies in the fact that the expected value of a bounded function w.r.t. a weakly convergent sequence of probability measures converges only to the expected value w.r.t. the limit measure if this function is continuous. However, the functions that occur in signaling games are typically not continuous. Therefore, we introduce a sharpening of weak convergence, called *pointwise convergence of probability measures*, which preserves the expected value of every bounded (continuous or non-continuous) function in the limit.

3.4. Pointwise convergence of probability measures

Let X be a metric space and $(\mu^k)_{k \in \mathbb{N}}$ be a sequence of probability measures on X. We say that $(\mu^k)_{k \in \mathbb{N}}$ *converges pointwise* to a probability measure μ if

$$\lim_{k \to \infty} \mu^k(X_B) = \mu(X_B),$$

for every Borel set X_B.

Obviously, every pointwise convergent sequence is also weakly convergent, since weak convergence requires only the equation above to be true for Borel sets X_B in which the boundary has measure zero under μ.

If we consider the strong metric on probability measures given by

$$d(\mu, \nu) := \sup\{ | \mu(X_B) - \nu(X_B) | \mid X_B \text{ measurable}\},$$

it is clear that convergence w.r.t. the strong metric implies pointwise convergence. The following lemma, which is proved in the appendix, shows that pointwise convergence can also be defined by convergence of integrals of bounded and measurable functions. [1]

Lemma 3.2. Let X be a complete separable metric space and μ, μ^1, μ^2, ..., probablity measures on X. Then $(\mu^k)_{k \in \mathbb{N}}$ converges pointwise to μ if and only if

$$\lim_{k \to \infty} \int_X f \, d\mu^k = \int_X f \, d\mu,$$

for every bounded and measurable function f.

A similar characterization holds for weakly convergent sequences: the sequence $(\mu^k)_{k \in \mathbb{N}}$ converges weakly to μ if and only if the equation in the lemma is true for every bounded and *continuous* function f.

In the next section we show that the pointwise convergence concept enables us to find an appropriate consistency concept.

3.5. Restriction on convergence of assessments

In view of the fact that strategies and beliefs in signaling games typically induce non-continuous functions, we regard pointwise convergence as a natural

[1] We thank Peter Wakker for this proof.

convergence concept to define consistency. As a minimal condition for an appropriate consistency concept, we require that almost all local strategies and local beliefs in the supporting sequence of assessments should converge pointwise to the local strategies and local beliefs in the limit assessment.

Condition C. Convergence of assessments w.r.t. φ should be defined in such a way that, whenever $(\sigma^k, \beta^k)_{k \in \mathbb{N}}$ converges to (σ, β) w.r.t. φ, there is a dense subset in T such that $(\sigma_1^k(t))_{k \in \mathbb{N}}$ converges pointwise to $\sigma_1(t)$ for every t in this dense subset and $(\beta^k(m))_{k \in \mathbb{N}}$ converges pointwise to $\beta(m)$ for every m in some dense subset of M.

3.6. Appropriate consistency concepts

The final condition completes the framework that we use in our search for appropriate consistency concepts.

Definition. We call a consistency concept φ *appropriate* if it has the canonical form and satisfies conditions A, B and C.

4. Strong consistency

In this section we present a particular consistency concept, which we call strong consistency.

We call an assessment (σ, β) *strongly consistent* if there is a sequence $(\sigma^k, \beta^k)_{k \in \mathbb{N}}$ of assessments that are Bayesian-consistent and pointwise completely mixed such that $(\sigma_1^k(t))_{k \in \mathbb{N}}$ converges pointwise to $\sigma_1(t)$ for every t and $(\beta^k(m))_{k \in \mathbb{N}}$ converges pointwise to $\beta(m)$ for every m.

To show that this is an appropriate consistency concept, we only have to prove that strong consistency implies Bayesian consistency since it is clear that it satisfies conditions B and C. The proof of this fact is based on the following lemma.

Lemma 4.1. Let X be a complete, separable metric space, $(\mu^k)_{k \in \mathbb{N}}$ a sequence of probability measures that converges pointwise to a probability measure μ, and $(f^k)_{k \in \mathbb{N}}$ a sequence of measurable functions from X to $[0, 1]$ that converges pointwise to a measurable function f. Then

$$\lim_{k \to \infty} \int_X f^k \, d\mu^k = \int_X f \, d\mu.$$

Proof. Let $\varepsilon > 0$ be given. We can find a compact subset K with $\mu(K) > 1 - \varepsilon$, $\mu^k(K) > 1 - \varepsilon$ for k large enough, and $|f^k(x) - f(x)| < \varepsilon$ for all $x \in K$ and k large enough. The latter follows from the fact that pointwise convergence of functions implies almost uniform convergence. Moreover, the pointwise convergence of $(\mu^k)_{k \in \mathbb{N}}$ implies that $|\int_X f \, \mathrm{d}\mu^k - \int_X f \, \mathrm{d}\mu| < \varepsilon$ for large k. But then, for large k,

$$\begin{aligned} \left| \int_X f^k \, \mathrm{d}\mu^k - \int_X f \, \mathrm{d}\mu \right| &\leq \left| \int_X f^k \, \mathrm{d}\mu^k - \int_X f \, \mathrm{d}\mu^k \right| + \left| \int_X f \, \mathrm{d}\mu^k - \int_X f \, \mathrm{d}\mu \right| \\ &\leq \int_K |f^k - f| \, \mathrm{d}\mu^k + \int_{X \setminus K} |f^k - f| \, \mathrm{d}\mu^k + \varepsilon \\ &\leq \varepsilon \cdot 1 + \varepsilon + \varepsilon, \end{aligned}$$

which leads to the conclusion that

$$\lim_{k \to \infty} \int_X f^k \, \mathrm{d}\mu^k = \int_X f \, \mathrm{d}\mu. \quad \square$$

To show that strong consistency implies Bayesian consistency, we need one further lemma, which can be found as Exercise 18.25(d) in Billingsley (1986).

Lemma 4.2. Let T, M be metric spaces, β a measurable function on M, τ a probability measure on T, $\sigma : T \to \mathscr{P}(M)$ such that the function $t \mapsto \sigma(t)(M_{\mathrm{B}})$ is measurable for every M_{B} and let the probability measure P on M be given by

$$P(M_{\mathrm{B}}) := \int_T \sigma(t)(M_{\mathrm{B}}) \, \mathrm{d}\tau, \quad \text{for every } M_{\mathrm{B}}.$$

Then we have

$$\int_M \beta(m) \, \mathrm{d}P = \int_T \left[\int_M \beta(m) \, \mathrm{d}\sigma(t) \right] \mathrm{d}\tau.$$

Lemma 4.3. Let $(\sigma^k, \beta^k)_{k \in \mathbb{N}}$ be a sequence of Bayesian-consistent assessments such that $(\sigma_1^k(t))_{k \in \mathbb{N}}$ converges pointwise to $\sigma_1(t)$ for every t and $(\beta^k(m))_{k \in \mathbb{N}}$ converges pointwise to $\beta(m)$ for every m. Then the assessment (σ, β) is Bayesian-consistent.

Proof. Let T_{B} and M_{B} be Borel sets in T and M, respectively. First, we show that

$$\lim_{k \to \infty} \int_{M_{\mathrm{B}}} \beta^k(m)(T_{\mathrm{B}}) \, \mathrm{d}P^{\sigma^k} = \int_{M_{\mathrm{B}}} \beta(m)(T_{\mathrm{B}}) \, \mathrm{d}P^{\sigma}.$$

Using Lemma 4.2 we obtain:

$$\int_{M_{\mathrm{B}}} \beta(m)(T_{\mathrm{B}}) \, \mathrm{d}P^{\sigma} = \int_T \left[\int_{M_{\mathrm{B}}} \beta(m)(T_{\mathrm{B}}) \, \mathrm{d}\sigma_1(t) \right] \mathrm{d}\tau,$$

and

$$\int_{M_B} \beta^k(m)(T_B)\,\mathrm{d}P^{\sigma^k} = \int_T \left[\int_{M_B} \beta^k(m)(T_B)\,\mathrm{d}\sigma_1^k(t) \right] \mathrm{d}\tau.$$

Since the functions $m \mapsto \beta^k(m)(T_B)$ are measurable functions from M to [0, 1] converging pointwise to the function $m \mapsto \beta(m)(T_B)$, Lemma 4.1 implies

$$\lim_{k\to\infty} \int_{M_B} \beta^k(m)(T_B)\,\mathrm{d}\sigma_1^k(t) = \int_{M_B} \beta(m)(T_B)\,\mathrm{d}\sigma_1(t).$$

Since this holds for every t, it follows with the dominated convergence theorem that

$$\lim_{k\to\infty} \int_{M_B} \beta^k(m)(T_B)\,\mathrm{d}P^{\sigma^k} = \int_T \left[\int_{M_B} \beta(m)(T_B)\,\mathrm{d}\sigma_1(t) \right] \mathrm{d}\tau$$

$$= \int_{M_B} \beta(m)(T_B)\,\mathrm{d}P^{\sigma}.$$

Furthermore, the functions $t \mapsto \sigma_1^k(t)(M_B)$ are measurable functions from T to [0, 1], which converge pointwise to the function $t \mapsto \sigma_1(t)(M_B)$. By the dominated convergence theorem, we obtain:

$$\lim_{k\to\infty} \int_{T_B} \sigma_1^k(t)(M_B)\,\mathrm{d}\tau = \int_{T_B} \sigma_1(t)(M_B)\,\mathrm{d}\tau.$$

Combining these two results and using the fact that (σ^k, β^k) is Bayesian-consistent leads to the conclusion that

$$\int_{T_B} \sigma_1(t)(M_B)\,\mathrm{d}\tau = \lim_{k\to\infty} \int_{T_B} \sigma_1^k(t)(M_B)\,\mathrm{d}\tau = \lim_{k\to\infty} \int_{M_B} \beta^k(m)(T_B)\,\mathrm{d}P^{\sigma^k}$$

$$= \int_{M_B} \beta(m)(T_B)\,\mathrm{d}P^{\sigma}.$$

Since this holds for arbitrary T_B and M_B it follows that (σ, β) is Bayesian-consistent. □

From this lemma it follows directly that strong consistency implies Bayesian consistency.

Corollary 4.4. Every strongly consistent assessment is Bayesian-consistent.

This leads to the following conclusion.

Corollary 4.5. The strong consistency concept is appropriate.

5. Characterization of strong consistency

In this section we give a characterization of strongly consistent assessments. Besides the fact that this characterization gives good insight into the structure of the set of strongly consistent assessments, it can be used later to show that every appropriate consistency concept is a refinement of strong consistency.

Before formulating this result we first consider the topological structure of a separable metric space. We formulate several properties of such spaces in terms of the message set M.

A point $m \in M$ is called *isolated* if $\{m\}$ is an open subset of M. The set of all isolated points of M is denoted by M_{iso}. Note that $m \in M_{\mathrm{iso}}$ if and only if there is an $\varepsilon > 0$ such that $M \cap U_\varepsilon(m) = \{m\}$. A point in M that is not isolated is called an *accumulation* point of M and M_{accu} denotes the set of all accumulation points of M.

In the proof of Theorem 5.2 we make use of the following property of separable metric spaces.

Lemma 5.1. Let M be a separable metric space and let M^* be a countable dense subset of M. Then

(1) *If $m \in M^* \cap M_{\mathrm{accu}}$, then $M^* \setminus \{m\}$ is a dense subset of M.*
(2) $M_{\mathrm{iso}} \subset M^*$.

Theorem 5.2. Let Γ be a signaling game and let (σ, β) be a strongly consistent assessment. Then (σ, β) is Bayesian-consistent and $\beta(m)$ is absolutely continuous with respect to τ for every isolated point $m \in M$.

Proof. We already know that strong consistency implies Bayesian consistency. Now, let (σ, β) be a strongly consistent assessment with a supporting sequence $(\sigma^k, \beta^k)_{k \in \mathbb{N}}$ and let $m \in M_{\mathrm{iso}}$. Then, for a Borel set T_{B} with $\tau(T_{\mathrm{B}}) = 0$, the Bayesian consistency of (σ^k, β^k) implies that

$$\beta^k(m)(T_{\mathrm{B}}) \cdot P^{\sigma^k}(m) = \int_{\{m\}} \beta^k(m')(T_{\mathrm{B}})\, \mathrm{d}P^{\sigma^k} = \int_{T_{\mathrm{B}}} \sigma_1^k(t)(m)\, \mathrm{d}\tau = 0.$$

Because $\{m\}$ is an open subset of M, $\sigma_1^k(t)(m) > 0$ for all t. Hence,

$$P^{\sigma^k}(m) = \int_T \sigma_1^k(t)(m)\, \mathrm{d}\tau > 0,$$

and

$$\beta^k(m)(T_{\mathrm{B}}) = \frac{\int_{T_{\mathrm{B}}} \sigma_1^k(t)(m)\, \mathrm{d}\tau}{P^{\sigma^k}(m)} = 0.$$

Since this relation holds for any k, $\beta(m)(T_B)=0$. So $\beta(m)$ is absolutely continuous with respect to τ. □

The foregoing theorem describes a condition that is necessary for an assessment to be strongly consistent. In the next theorem we show that this condition is also sufficient.

Theorem 5.3. Let Γ be a signaling game and let (σ, β) be an assessment. If (σ, β) is Bayesian-consistent and $\beta(m)$ is absolutely continuous with respect to τ for every isolated point $m \in M$, then (σ, β) is strongly consistent.

The proof of this theorem will be based on three lemmas. First, we need some notation.

Let $M^* = \{m_1, m_2, \ldots\}$ be a countable dense subset of M and for every $m \in M_{\text{iso}}$, let $b(m): T \to \mathbb{R}$ be a density function of $\beta(m)$ with respect to τ (i.e. $\beta(m)(T_B) = \int_{T_B} b(m)(t)\,\mathrm{d}\tau$ for all T_B). Note that Radon–Nikodym's theorem guarantees the existence of this density function.

For $k \in \mathbb{N}$ we define the mapping $\hat{\sigma}_1^k : T \to \mathscr{M}(M)$ by

$$\hat{\sigma}_1^k(t)(M_B) := \sigma_1(t)(M_B) + \frac{1}{k^2}\left[\sum_{\substack{i \geq k \\ m_i \in M_B \cap M_{\text{accu}}}} \frac{1}{i^2} + \sum_{m_i \in M_B \cap M_{\text{iso}}} \frac{1}{i^2}\left[b(m_i)(t) \vee k + \frac{1}{k}\right]\right].$$

Then, $\lim_{k \to \infty} \hat{\sigma}_1^k(t)(M) = 1$, because for all k:

$$1 \leq \hat{\sigma}_1^k(t)(M) \leq 1 + \frac{1}{k^2}\sum_{m_i}\frac{1}{i^2}\left[k + \frac{1}{k}\right] = 1 + \left[\frac{1}{k} + \frac{1}{k^3}\right]\sum_{m_i}\frac{1}{i^2}.$$

Note that $\sum_{m_i} 1/i^2$ is finite since M^* is countable. For very $k \in \mathbb{N}$ and $t \in T$, let $R^k(t) := 1/\hat{\sigma}_1^k(t)(M)$. Obviously, $0 < R^k \leq 1$ is measurable on T for all k and $\lim_{k \to \infty} R^k(t) = 1$ for any t.

We consider the behavior strategy, $\sigma_1^k : T \to \mathscr{P}(M)$, with

$$\sigma_1^k(t)(M_B) := R^k(t) \cdot \hat{\sigma}_1^k(t)(M_B),$$

for all t and M_B. By Lemma 5.1, the set

$$M^*(k) := \{m_i \in M_{\text{accu}} \mid i \geq k\} \cup M_{\text{iso}} \subset M^*$$

is dense in M. So, if M_B is an open set, then $M^*(k) \cap M_B \neq \emptyset$. Hence, $\sigma_1^k(t)(M_B) > 0$ for every k and t, which implies that the probability measure $\sigma_1^k(t)$ is strictly positive.

Lemma 5.4. For every t, the sequence $(\sigma_1^k(t))_{k \in \mathbb{N}}$ converges pointwise to $\sigma_1(t)$.

The proof of this result is straightforward. For $k \in \mathbb{N}$ and Borel set T_{B}, we introduce:

$$\beta^k(m)(T_{\mathrm{B}}) := \begin{cases} \dfrac{\int_{T_{\mathrm{B}}} \sigma_1^k(t)(m)\ \mathrm{d}\tau}{\int_T \sigma_1^k(t)(m)\ \mathrm{d}\tau}, & \text{if } m \in M^*(k), \\ \\ \dfrac{\int_{T_{\mathrm{B}}} R^k(t)\ \mathrm{d}\beta(m)}{\int_T R^k(t)\ \mathrm{d}\beta(m)}, & \text{if } m \notin M^*(k) \end{cases} \tag{5.1}$$

Note that the two denominators in this definition are non-zero. If, for instance, $\int_T \sigma_1^k(t)(m)\ \mathrm{d}\tau = 0$ for an $m \in M^*(k)$, then $\sigma_1^k(t)(m) = 0$, τ almost everywhere. However, by construction, $\sigma_1^k(t)(m) > 0$ for all t.

Lemma 5.5. For every m, the sequence $(\beta^k(m))_{k \in \mathbb{N}}$ converges pointwise to $\beta(m)$.

Proof. (a) If $m \notin M_{\mathrm{iso}}$, then $m \notin M^*(k)$ for large k. Then the dominated convergence theorem implies that for all T_{B} it holds that

$$\lim_{k \to \infty} \beta^k(m)(T_{\mathrm{B}}) = \frac{\int_{T_{\mathrm{B}}} \mathrm{d}\beta(m)}{\int_T \mathrm{d}\beta(m)} = \beta(m)(T_{\mathrm{B}}).$$

(b) Let $m \in M_{\mathrm{iso}}$ and let T_{B} be fixed. Then $\{m\}$ is a Borel set and

$$\begin{aligned} &\int_{T_{\mathrm{B}}} \sigma_1^k(t)(m)\ \mathrm{d}\tau \\ &\quad = \int_{T_{\mathrm{B}}} R^k(t) \cdot \hat{\sigma}_1^k(t)(m)\ \mathrm{d}\tau \\ &\quad = \int_{T_{\mathrm{B}}} R^k(t) \cdot \sigma_1(t)(m)\ \mathrm{d}\tau + \frac{1}{k^2}\frac{1}{i^2}\int_{T_{\mathrm{B}}} R^k(t)\left[b(m)(t) \vee k + \frac{1}{k}\right]\mathrm{d}\tau. \end{aligned} \tag{5.2}$$

Next we distinguish two cases.

(b1) Suppose that $\int_T \sigma_1(t)(m)\ \mathrm{d}\tau > 0$. By the dominated convergence theorem:

$$\lim_{k \to \infty} \int_{T_{\mathrm{B}}} R^k(t)\sigma_1(t)(m)\ \mathrm{d}\tau = \int_{T_{\mathrm{B}}} \sigma_1(t)(m)\ \mathrm{d}\tau, \quad \text{for all } T_{\mathrm{B}}.$$

Hence, (5.1) in combination with (5.2) implies that

$$\lim_{k\to\infty} \beta^k(m)(T_B) = \frac{\int_{T_B} \sigma_1(t)(m)\ \mathrm{d}\tau}{\int_T \sigma_1(t)(m)\ \mathrm{d}\tau}.$$

The Bayesian consistency of (σ, β) implies that

$$\int_{T_B} \sigma_1(t)(m)\ \mathrm{d}\tau = \int_{\{m\}} \beta(m')(T_B)\ \mathrm{d}P^{\sigma} = \beta(m)(T_B) \int_T \sigma_1(t)(m)\ \mathrm{d}\tau.$$

So

$$\beta(m)(T_B) = \frac{\int_{T_B} \sigma_1(t)(m)\ \mathrm{d}\tau}{\int_T \sigma_1(t)(m)\ \mathrm{d}\tau} = \lim_{k\to\infty} \beta^k(m)(T_B).$$

(b2) Suppose that $\int_T \sigma_1(t)(m)\ \mathrm{d}\tau = 0$. With (5.1) and (5.2) it follows that

$$\beta^k(m)(T_B) = \frac{\int_{T_B} R^k(t)\left[b(m)(t) \vee k + \frac{1}{k}\right] \mathrm{d}\tau}{\int_T R^k(t)\left[b(m)(t) \vee k + \frac{1}{k}\right] \mathrm{d}\tau}.$$

Together with the dominated convergence theorem this leads to

$$\lim_{k\to\infty} \beta^k(m)(T_B) = \frac{\int_{T_B} b(m)(t)\ \mathrm{d}\tau}{\int_T b(m)(t)\ \mathrm{d}\tau} = \beta(m)(T_B).$$

This completes the proof. □

Lemma 5.6. For any k, (σ^k, β^k) is Bayesian-consistent.

Proof. In this proof $k \in \mathbb{N}$ is fixed. (a) For all T_B and any Borel set $M_B \subset M^*(k)$:

$$\begin{aligned}
\int_{M_B} \beta^k(m)(T_B)\ \mathrm{d}P^{\sigma^k} &= \sum_{m\in M_B} \beta^k(m)(T_B) \cdot P^{\sigma^k}(m) \\
&= \sum_{m\in M_B} \frac{\int_{T_B} \sigma_1^k(t)(m)\ \mathrm{d}\tau}{P^{\sigma k}(m)} P^{\sigma^k}(m) \\
&= \sum_{m\in M_B} \int_{T_B} \sigma_1^k(t)(m)\ \mathrm{d}\tau \\
&= \int_{T_B} \sigma_1^k(t)(M_B)\ \mathrm{d}\tau,
\end{aligned}$$

where the fourth equality is a consequence of the dominated convergence theorem.

(b) In this part of the proof we restrict ourselves to the set $M' := M \setminus M^*(k)$. First, we introduce for every Borel subset M'_B of M' the measure $\kappa(M'_B)$ on T as follows: for a Borel set T_B,

$$\kappa(M'_B)(T_B) := \int_{T_B} \sigma_1(t)(M'_B)\,\mathrm{d}\tau.$$

The Bayesian consistency of (σ, β) implies that

$$\kappa(M'_B)(T_B) = \int_{M'_B} \beta(m)(T_B)\,\mathrm{d}P^{\sigma}.$$

With the help of Lemma 4.2 this leads to

$$\int_{T_B} R^k(t)\,\mathrm{d}\kappa(M'_B) = \int_{M'_B} \left[\int_{T_B} R^k(t)\,\mathrm{d}\beta(m)\right] \mathrm{d}P^{\sigma}.$$

Then for every Borel set M'_B:

$$P^{\sigma^k}(M'_B) = \int_T R^k(t) \cdot \hat{\sigma}_1^k(t)(M'_B)\,\mathrm{d}\tau = \int_T R^k(t) \cdot \sigma_1(t)(M'_B)\,\mathrm{d}\tau$$

$$= \int_T R^k(t)\,\mathrm{d}\kappa(M'_B) = \int_{M'_B} \left[\int_T R^k(t)\,\mathrm{d}\beta(m)\right] \mathrm{d}P^{\sigma}.$$

So we may conclude that $m \mapsto \int_T R^k(t)\,\mathrm{d}\,\beta(m)$ is a density function of P^{σ^k} with respect to P^{σ} on M'. Hence, for all T_B:

$$\int_{M'_B} \beta^k(m)(T_B)\,\mathrm{d}P^{\sigma^k} = \int_{M'_B} \beta^k(m)(T_B) \left[\int_T R^k(t)\,\mathrm{d}\beta(m)\right] \mathrm{d}P^{\sigma}$$

$$= \int_{M'_B} \frac{\int_{T_B} R^k(t)\,\mathrm{d}\beta(m)}{\int_T R^k(t)\,\mathrm{d}\beta(m)} \left[\int_T R^k(t)\,\mathrm{d}\beta(m)\right] \mathrm{d}P^{\sigma}$$

$$= \int_{M'_B} \left[\int_{T_B} R^k(t)\,\mathrm{d}\beta(m)\right] \mathrm{d}P^{\sigma} = \int_{T_B} R^k(t)\,\mathrm{d}\kappa(M'_B)$$

$$= \int_{T_B} R^k(t) \cdot \sigma_1(t)(M'_B)\,\mathrm{d}\tau = \int_{T_B} \sigma_1^k(t)(M'_B)\,\mathrm{d}\tau.$$

(c) Parts (a) and (b) imply that for all T_B and M_B:

$$\begin{aligned}
&\int_{M_B} \beta^k(m)(T_B)\,\mathrm{d}P^{\sigma^k} \\
&\quad = \int_{M_B \cap M^*(k)} \beta^k(m)(T_B)\,\mathrm{d}P^{\sigma^k} + \int_{M_B \setminus M^*(k)} \beta^k(m)(T_B)\,\mathrm{d}P^{\sigma^k} \\
&\quad = \int_{T_B} \sigma_1^k(t)(M_B \cap M^*(k))\,\mathrm{d}\tau + \int_{T_B} \sigma_1^k(t)(M_B \setminus M^*(k))\,\mathrm{d}\tau \\
&\quad = \int_{T_B} \sigma_1^k(t)(M_B)\,\mathrm{d}\tau .
\end{aligned}$$

This completes the proof of the theorem. □

Corollary 5.7. Let Γ be a signaling game and let (σ, β) be an assessment. Then (σ, β) is strongly consistent if and only if (σ, β) is Bayesian-consistent and $\beta(m)$ is absolutely continuous with respect to τ for every isolated point $m \in M$.

Using this characterization, we can investigate the consequences of strong consistency in some special classes of signaling games.

Corollary 5.8. If the message space of a signaling game contains no isolated points or the type space is discrete, strong consistency is equivalent to Bayesian consistency.

In particular, this holds for finite and discrete signaling games.

Corollary 5.9. If the message space of a signaling game is discrete and the type space is not, then an assessment is strongly consistent if and only if it is Bayesian-consistent and $\beta(m)$ is absolutely continuous w.r.t. τ for every message m.

6. Other consistency concepts

In the following theorem we show that every assessment that is consistent w.r.t. some appropriate definition φ automatically satisfies the conditions of the characterization in the previous section and is, therefore, strongly consistent. So, every appropriate consistency concept is a refinement of strong consistency.

Theorem 6.1. Let φ be an appropriate consistency concept. If an assessment (σ, β) is consistent w.r.t. φ then (σ, β) is Bayesian-consistent and $\beta(m)$ is absolutely continuous w.r.t. τ for every isolated point m.

Proof. Let φ be an appropriate consistency concept and (σ, β) a consistent assessment w.r.t. φ with a supporting sequence $(\sigma^k, \beta^k)_{k \in \mathbb{N}}$. Since φ is appropriate, (σ, β) must be Bayesian-consistent.

Now, take an arbitrary isolated point $m \in M$. Then, $\{m\}$ is open and therefore $T \times \{m\}$ is an open subset of $T \times M$. By condition B;

$$P^{\sigma^k}(m) = \int_T \sigma_1^k(t)(m)\ \mathrm{d}\tau > 0,$$

for every k. Since (σ^k, β^k) is Bayesian-consistent, it follows that

$$\beta^k(m)(T_{\mathrm{B}}) \cdot P^{\sigma^k}(m) = \int_{\{m\}} \beta^k(m')\ \mathrm{d}P^{\sigma^k} = \int_{T_{\mathrm{B}}} \sigma_1^k(t)(m)\ \mathrm{d}\tau,$$

for every T_{B}, which means that

$$\beta^k(m)(T_{\mathrm{B}}) = \frac{\int_{T_{\mathrm{B}}} \sigma_1^k(t)(m)\ \mathrm{d}\tau}{P^{\sigma^k}(m)},$$

for every T_{B}. If $\tau(T_{\mathrm{B}}) = 0$, we have $\beta^k(m)(T_{\mathrm{B}}) = 0$ for every k. Since m is in every dense subset of M, condition C implies that $\beta^k(m)$ converges pointwise to $\beta(m)$, which implies that $\beta(m)(T_{\mathrm{B}}) = \lim_{k \to \infty} \beta^k(m)(T_{\mathrm{B}}) = 0$. Hence, $\beta(m)$ is absolutely continuous w.r.t. τ. □

In view of Corollary 5.7, we arrive at the following conclusion.

Theorem 6.2. Every appropriate consistency concept is a refinement of strong consistency.

In the following example we consider a signaling game in which strong consistency excludes some sequential equilibria. By Theorem 6.2 it follows that every appropriate consistency concept excludes these equilibria.

Example 2. Let Γ be a signaling game in which $T = [0, 1]$, $M = \{y, n\}$, $A = \{b, c\}$ and τ is the uniform distribution on T. The payoffs are given by

$$u_1(t, m, a) := 0, \quad \text{for all } t, m, a,$$

$$u_2(t, m, a) := \begin{cases} t, & \text{if } a = b, \\ 0, & \text{if } a = c. \end{cases}$$

We define the assessment (σ, β) by

$$\sigma_1(t) := \begin{cases} \delta_y, & \text{if } t = 0, \\ \delta_n, & \text{if } t > 0, \end{cases}$$

$$\sigma_2(y) := \delta_c,$$

$$\sigma_2(n) := \delta_b,$$

$$\beta(y) := \delta_0,$$

$$\beta(n) := \tau.$$

It can be shown that (σ, β) is a sequential equilibrium. However, (σ, β) is not strongly consistent, since $\beta(y)$ is not absolutely continuous w.r.t. τ. □

7. Structurally consistent assessments

Finally, we consider a different form of consistency as introduced by Kreps and Wilson (1982). They call an assessment structurally consistent if for every information set there is a behavior strategy profile such that this information set will be reached with positive probability and the beliefs at this information set are completely determined by Bayes' rule. If we formulate this concept for signaling games, we obtain the following definition.

Definition. An assessment (σ, β) is called *structurally consistent* if for every $m \in M$ there is a BSP $\bar{\sigma}$ such that $P^{\bar{\sigma}}(m) > 0$, and for every T_B:

$$\beta(m)(T_B) = \frac{\int_{T_B} \bar{\sigma}_1(t)(m)\, d\tau}{P^{\bar{\sigma}}(m)}.$$

In the following theorem we characterize the class of structurally consistent assessments in signaling games with at least two messages.

Theorem 7.1. For a signaling game with at least two different messages, an assessment (σ, β) is structurally consistent if and only if, for every m, there is a constant $c_m > 0$ such that for all T_B:

$$\beta(m)(T_B) \leq c_m \cdot \tau(T_B).$$

Proof. '⇒' Let (σ, β) be a structurally consistent assessment and m a message in M. By definition, there is a BSP $\bar{\sigma}$ such that $P^{\sigma}(m) > 0$ and for any T_B:

$$\beta(m)(T_B) = \frac{\int_{T_B} \bar{\sigma}_1(t)(m)\, d\tau}{P^{\bar{\sigma}}(m)} \leq \frac{1}{P^{\bar{\sigma}}(m)} \tau(T_B).$$

Now choose $c_m := 1/P^{\bar{\sigma}}(m)$.

'⇐' Suppose, for every m, there is a constant $c_m > 0$ such that for all T_B we have $\beta(m)(T_B) \leq c_m \cdot \tau(T_B)$. Then, obviously, $\beta(m)$ is absolutely continuous w.r.t. τ for every m.

Let $m \in M$ and let $b(m): T \to \mathbb{R}$ be a density function of $\beta(m)$ with respect to τ. Then, for any T_B:

$$\int_{T_B} b(m)(t)\ \mathrm{d}\tau = \beta(m)(T_B) \le c_m \cdot \tau(T_B) = \int_{T_B} c_m\ \mathrm{d}\tau.$$

Hence, there is a Borel set T_B^* with $\tau(T_B^*) = 1$, such that $b(m)(t) \le c_m$ for all $t \in T_B^*$. To define a behavior strategy $\bar{\sigma}_1$ for player 1, we take a message $\hat{m} \neq m$ in M. If $t \notin T_B^*$, then let $\bar{\sigma}_1(t)$ be an arbitrary probability measure on M. Otherwise, for a Borel set M_B:

$$\bar{\sigma}_1(t)(M_B) := \begin{cases} 1, & \text{if } m \in M_B \text{ and } \hat{m} \in M_B, \\ \dfrac{b(m)(t)}{c_m}, & \text{if } m \in M_B \text{ and } \hat{m} \notin M_B, \\ 1 - \dfrac{b(m)(t)}{c_m}, & \text{if } m \notin M_B \text{ and } \hat{m} \in M_B, \\ 0, & \text{otherwise}. \end{cases}$$

Then,

$$\begin{aligned} P^{\bar{\sigma}}(m) &= \int_{T_B^*} \bar{\sigma}_1(t)(m)\ \mathrm{d}\tau = \int_{T_B^*} \frac{b(m)(t)}{c_m}\ \mathrm{d}\tau \\ &= \int_T \frac{b(m)(t)}{c_m}\ \mathrm{d}\tau = \frac{1}{c_m}\beta(m)(T) = \frac{1}{c_m} > 0. \end{aligned}$$

Furthermore, for any T_B:

$$\begin{aligned} \frac{\int_{T_B} \bar{\sigma}_1(t)(m)\ \mathrm{d}\tau}{P^{\bar{\sigma}}(m)} &= \frac{\int_{T_B \cap T_B^*} \bar{\sigma}_1(t)(m)\ \mathrm{d}\tau}{P^{\bar{\sigma}}(m)} = \frac{1/c_m \int_{T_B \cap T_B^*} b(m)(t)\ \mathrm{d}\tau}{1/c_m} \\ &= \int_{T_B} b(m)(t)\ \mathrm{d}\tau = \beta(m)(T_B). \end{aligned}$$

So (σ, β) is structurally consistent. □

From the above theorem it follows that every structurally consistent assessment (σ, β) must have the property that the local belief $\beta(m)$ is absolutely continuous w.r.t. τ for every m.

Consider, for instance, an infinite signaling game with $T = M = [0, 1]$, τ equal to the uniform distribution on T and an assessment (σ, β) in which $\beta(0) = \delta_0$. Such an assessment is ruled out by structural consistency, since $\beta(0)$ is not absolutely continuous w.r.t. τ.

Appendix

Proof of Lemma 3.2. It can be shown easily that every sequence $(\mu^k)_{k\in\mathbb{N}}$ satisfying the condition in the lemma is pointwise convergent to μ by choosing f equal to the indicator function on the Borel set X_B.

Now, let $(\mu^k)_{k\in\mathbb{N}}$ be pointwise convergent to μ and let f be a bounded measurable function on X. Without loss of generality, we assume that $f: X \to (0, 1)$. For every k, we define the sets:

$$A_i(k) := \left\{ x \mid f(x) \geq \frac{i}{k} \right\}, \quad i = 0, \ldots, k$$

and

$$B_i(k) := \left\{ x \mid \frac{i-1}{k} \leq f(x) < \frac{i}{k} \right\}, \quad i = 1, \ldots, k.$$

By the definition of the integral, we have for every k:

$$\sum_{i=1}^{k} \frac{i-1}{k} \mu(B_i(k)) \leq \int_X f \, \mathrm{d}\mu \leq \sum_{i=1}^{k} \frac{i}{k} \mu(B_i(k)).$$

Since

$$\sum_{i=1}^{k} \frac{i-1}{k} \mu(B_i(k)) = \frac{1}{k} \sum_{i=1}^{k} \mu(A_i(k)) \quad \text{and}$$

$$\sum_{i=1}^{k} \frac{i}{k} \mu(B_i(k)) = \frac{1}{k} + \frac{1}{k} \sum_{i=1}^{k} \mu(A_i(k)),$$

it follows that

$$\frac{1}{k} \sum_{i=1}^{k} \mu(A_i(k)) \leq \int_X f \, \mathrm{d}\mu \leq \frac{1}{k} + \frac{1}{k} \sum_{i=1}^{k} \mu(A_i(k)).$$

Similarly, we can show that

$$\frac{1}{k} \sum_{i=1}^{k} \mu^k(A_i(k)) \leq \int_X f \, \mathrm{d}\mu^k \leq \frac{1}{k} + \frac{1}{k} \sum_{i=1}^{k} \mu^k(A_i(k)).$$

These inequalities imply that

$$\int_X f \, \mathrm{d}\mu = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \mu(A_i(k)),$$

and

$$\lim_{k \to \infty} \int_X f \, \mathrm{d}\mu^k = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \mu^k(A_i(k)).$$

Therefore, it suffices to show that

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \mu^k(A_i(k)) = \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \mu(A_i(k)).$$

Assume that this were not true. Then, w.l.o.g., there is a $\delta > 0$ such that

$$\left| \frac{1}{k} \sum_{i=1}^{k} \mu^k(A_i(k)) - \frac{1}{k} \sum_{i=1}^{k} \mu(A_i(k)) \right| \geq \delta, \quad \text{for all } k.$$

So, for every k, there is an integer $i_k \leq k$ such that $|\mu^k(A_{i_k}(k)) - \mu(A_{i_k}(k))| \geq \delta$. Without loss of generality, we may assume that $\mu^k(A_{i_k}(k)) \geq \delta + \mu(A_{i_k}(k))$ for every k. Since $0 \leq (i_k/k) \leq 1$, the sequence i_k/k contains a monotone convergent subsequence. Without loss of generality, we assume that the sequence i_k/k is monotone and convergent.

Case 1. $i_k/k \uparrow r$ for some $r \in [0, 1]$. Then, by construction, $A_{i_{k+1}}(k+1) \subset A_{i_k}(k)$ for every k, and

$$\bigcap_k A_{i_k} = A := \{x \mid f(x) \geq r\}.$$

Since $\mu^l(A_{i_k}(k)) \geq \mu^l(A_{i_l}(l))$ for $l \geq k$, it follows that

$$\mu^l\big(A_{i_k}(k)\big) \geq \delta + \mu\big(A_{i_l}(l)\big), \quad \text{for } l \geq k,$$

and, therefore,

$$\lim_{l\to\infty} \mu^l\big(A_{i_k}(k)\big) \geq \delta + \lim_{l\to\infty} \mu\big(A_{i_l}(l)\big), \quad \text{for every } k.$$

By assumption, $\lim_{l\to\infty} \mu^l(X_B) = \mu(X_B)$ for every Borel set X_B, so $\lim_{l\to\infty} \mu^l(A_{i_k}(k)) = \mu(A_{i_k}(k))$. Furthermore, by the monotone convergence theorem, $\lim_{l\to\infty} \mu(A_{i_l}(l)) = \mu(A)$. Combining these facts leads to the conclusion that

$$\mu\big(A_{i_k}(k)\big) \geq \delta + \mu(A), \quad \text{for every } k.$$

However, this implies that

$$\lim_{k\to\infty} \mu\big(A_{i_k}(k)\big) \geq \delta + \mu(A),$$

which contradicts the fact that $\lim_{k\to\infty} \mu(A_{i_k}(k)) = \mu(A)$.

Case 2. $i_k/k \downarrow r$ for some $r \in [0, 1]$. Then, by construction, $A_{i_{k+1}}(k+1) \supset A_{i_k}(k)$ for every k and

$$\bigcup_k A_{i_k}(k) = A := \{x \mid f(x) \geq r\}.$$

Since $\mu^l(A_{i_k}(k)) \geq \mu^l(A_{i_l}(l))$ for $k \geq l$, it follows that

$$\mu^l\big(A_{i_k}(k)\big) \geq \delta + \mu\big(A_{i_l}(l)\big), \quad \text{for } k \geq l.$$

and

$$\lim_{k\to\infty} \mu^l\left(A_{i_k}(k)\right) \geq \delta + \mu\left(A_{i_l}(l)\right), \quad \text{for every } l.$$

By the monotone convergence theorem, $\lim_{k\to\infty} \mu^l(A_{i_k}(k)) = \mu^l(A)$. Together with the inequality above, we obtain $\mu^l(A) \geq \delta + \mu(A_{i_l}(l))$ for every l, which implies that $\lim_{l\to\infty} \mu^l(A) \geq \delta + \lim_{l\to\infty} \mu(A_{i_l}(l))$. However, this leads to a contradiction, since $\lim_{l\to\infty} \mu^l(A) = \mu(A)$ and $\lim_{l\to\infty} \mu(A_{i_l}(l)) = \mu(A)$. □

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References

Billingsley, P., 1968, Convergence of probability measures (John Wiley and Sons, New York).

Billingsley, P., 1986, Probability and Measure (John Wiley and Sons, New York).

Cho, I.-K. and D.M. Kreps, 1987, Signaling games and stable equilibria, The Quarterly Journal of Economics 102, 179–221.

Kreps, D.M. and R. Wilson, 1982, Sequential equilibria, Econometrica 50, 863–894.

Mailath, G.J., 1987, Incentive compatibility in signaling games with a continuum of types, Econometrica 55, 1349–1365.

Mailath, G.J., 1988, On the behavior of separating equilibria of signaling games with a finite set of types as the set of types becomes dense in an interval, Journal of Economic Theory 44, 413–424.

Manelli, A., 1994, Cheap talk and sequential equilibria in signaling games, Northwestern University.

Selten, R., 1975, Reexamination of the Nash equilibrium concept, International Journal of Game Theory 7, 73–80.

Simon, L.K. and M.B. Stinchcombe, 1995, Equilibrium refinement for infinite normal-form games, Econometrica 63, 1421–4143.

Van Damme, E., 1987, Equilibria in non-cooperative games, in: H. Peters and O. Vrieze, eds, Surveys in game theory and related topics, C.W.I. Tract 39, Amsterdam.