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Exposita Notes

An extremely simple proof of the K-K-M-S Theorem*

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Summary. An extremely simple proof of the K-K-M-S Theorem is given involving only Brouwer's fixed point theorem and some elementary calculus. A function is explicitly given such that a fixed point of it yields an intersection point of a balanced collection of sets together with balancing weights. Moreover, any intersection point of a balanced collection of sets together with balancing weights corresponds to a fixed point of the function. Furthermore, the proof can be used to show π-balanced versions of the K-K-M-S Theorem, with π-balancedness as introduced in Billera (1970). The proof makes clear that the conditions made with respect to π by Billera can be even weakened.

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1 Introduction

In intersection theorems conditions are given under which the members of a certain subset of a cover of some set have a non-empty intersection. Well-known intersection theorems on the unit simplex are given in Knaster, Kuratowski, and Mazurkiewicz (1929) (K-K-M Theorem), Scarf (1967a) (Scarf's Theorem), Shapley (1973) (K-K-M-S Theorem), Gale (1984) (Gale's Theorem), and Ichiishi (1988) (Ichiishi's Theorem). These theorems are very useful to prove the existence of solutions to mathematical programming problems, of solutions to problems in general equilibrium theory, and of solutions to game theoretic problems. The K-K-M-S Theorem is a very helpful tool to show that the core of any balanced non-transferable utility

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game is non-empty, a result first shown in Scarf (1967b) by means of a constructive method being related to the methods introduced in Scarf (1967a,1973). Shapley proved the K-K-M-S Theorem by means of another constructive method. An easy non-constructive proof is due to Ichiishi (1981) and is based on Fan's coincidence theorem as presented in Fan (1969).

Recently, a number of papers provided alternative elementary and simple proofs of the K-K-M-S Theorem. Shapley and Vohra (1991) have proofs involving either Kakutani's fixed point theorem or Fan's coincidence theorem. Komiya (1994) gives a proof of the K-K-M-S Theorem based on Kakutani's fixed point theorem, the separating hyperplane theorem, and the Berge maximum theorem. Krasa and Yannelis (1994) prove the K-K-M-S Theorem by means of Brouwer's fixed point theorem, the separating hyperplane theorem, and the existence of a continuous selection from a correspondence having open lower sections. In Zhou (1994) intersection theorems being close to the Ichiishi Theorem and the K-K-M-S Theorem are considered. Unlike the usual variants of these theorems, results with only open sets in the cover are given. The proof involves Brouwer's fixed point theorem and the existence of a collection of functions with some specific properties related to the cover.

In this paper a very elementary and simple proof of the K-K-M-S Theorem is given. In fact, only Brouwer's fixed point theorem and some elementary calculus is used to show the result. This shows that the K-K-M-S Theorem and Brouwer's fixed point theorem should be regarded as "equivalent" since it is elementary to show Brouwer's fixed point theorem using the K-K-M-S Theorem. The proofs of Komiya (1994) and Krasa and Yannelis (1994) are proofs by contradiction, i.e., it is assumed that the K-K-M-S Theorem does not hold and next a contradiction is obtained. A nice feature of the proof of this paper is that a function is explicitly given such that a fixed point of it yields an intersection point together with balancing weights. Moreover, any point in the intersection of a balanced collection of sets yields a fixed point of the function. This makes the proof very straightforward and intuitive. Furthermore, it implies that the function employed in the proof can be used for computational purposes. The proof can also be used to show π-balanced versions of the K-K-M-S Theorem, with π-balancedness as introduced in Billera (1970). In fact, from the proof it is obvious that the assumptions made with respect to π by Billera can be even weakened.

2 Preliminaries

For $n \in \mathbb{N}$, let I_n denote the set of integers $\{1, \ldots, n\}$, let $\mathbb{R}^n_+$ be the non-negative orthant of the n-dimensional Euclidean space, i.e., $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x_i \geq 0, \forall i \in I_n\}$, let Δ^n denote the $(n-1)$-dimensional unit simplex, $\Delta^n = \{x \in \mathbb{R}^n_+ | \sum_{i \in I_n} x_i = 1\}$, and let $\Gamma^n = \{x \in \mathbb{R}^n | \sum_{i \in I_n} x_i = 1$ and $x_i \geq -\frac{1}{n}, \forall i \in I_n\}$, an $(n-1)$-dimensional set containing the unit simplex in its relative interior. Let $p^n : \mathbb{R}^n \to \Delta^n$ denote the projection function defined by

$$p^n(x) = \arg\min_{y \in \Delta^n} \|y - x\|_2, \quad \forall x \in \mathbb{R}^n.$$

A useful property of p^n is given in the following completely straightforward lemma. It can be proved for instance using standard first order conditions.

Lemma 2.1 *For every $x \in \mathbb{R}^n$, if $p^n(x) = y$, then there exists $\alpha \in \mathbb{R}$ and $\beta_i \in \mathbb{R}_+, \forall i \in I_n$, such that, for every $i \in I_n, y_i = x_i + \alpha + \beta_i$, whereas $\beta_i > 0$ implies $y_i = 0$.*

Fix some $n \in \mathbb{N}$. For $S \subset I_n$, define the set Δ^S by $\Delta^S = \{x \in \Delta^n | \sum_{i \in S} x_i = 1\}$. Notice that $\Delta^{I_n} = \Delta^n$. For $S \subset I_n$, let $m^S \in \Delta^n$ be the center of gravity of Δ^S, so $m_i^S = \frac{1}{|S|}$ if $i \in S$, and $m_i^S = 0$ if $i \in I_n \backslash S$, where $|S|$ denotes the cardinality of the set S. Let $\mathcal{N}$ be the collection of all non-empty subsets of I_n. Then $\mathcal{B} \subset \mathcal{N}$ is said to be balanced if there exists $\lambda_S \in \mathbb{R}_+, \forall S \in \mathcal{B}$, such that $\sum_{S \in \mathcal{B}} \lambda_S m^S = m^{I_n}$. It follows immediately that $\sum_{S \in \mathcal{B}} \lambda_S = 1$. The K-K-M-S Theorem is now defined as follows.

Theorem 2.2 (K-K-M-S). *Let $\{C^S \subset \Delta^n | S \in \mathcal{N}\}$ be a closed cover of Δ^n such that, for every $T \subset I_n, \Delta^T \subset \cup_{S \subset T} C^S$. Then there exists a balanced collection $\mathcal{B} \subset \mathcal{N}$ such that $\cap_{S \in \mathcal{B}} C^S \neq \oslash$.*

The following result is immediately implied by the K-K-M-S Theorem and is therefore called Weak K-K-M-S Theorem.

Theorem 2.3 (Weak K-K-M-S). *Let $\{\hat{C}^S \subset \Delta^n | S \in \mathcal{N}\}$ be a closed cover of Δ^n such that, for every $T \subset I_n$, not $S \subset T$ implies $\Delta^T \cap \hat{C}^S = \oslash$. Then there exists a balanced collection $\mathcal{B} \subset \mathcal{N}$ such that $\cap_{S \in \mathcal{B}} \hat{C}^S \neq \oslash$.*

The Weak K-K-M-S Theorem is also mentioned in Ichiishi (1981). In fact, as mentioned by Ichiishi, Shapley (1973) needs only the Weak K-K-M-S Theorem for the proof of the non-emptiness of the core of a balanced game.

In this paper the Weak K-K-M-S Theorem will be shown, which is sufficient to show the non-emptiness of the core. Moreover, it yields the K-K-M-S Theorem, since it is easy to show that the Weak K-K-M-S Theorem implies the K-K-M-S Theorem. For an illustration for the case $n = 3$, see Figure 1. A cover $\{C^S \subset \Delta^3 | S \in \mathcal{N}\}$ of Δ^3 satisfying the conditions of the K-K-M-S Theorem is given in Figure 1a. There are two intersection points, given by the intersection of the sets in the balanced collection $\{\{1,2\},\{1,3\},\{2,3\}\}$, and the intersection of the sets in the balanced collection $\{\{1\},\{2\},\{3\}\}$, respectively. It does not satisfy the conditions of the Weak K-K-M-S Theorem, since $C^{\{3\}} \cap \Delta^{\{1,2\}} \neq \oslash$. The cover $\{\hat{C}^S \subset \Delta^3 | S \in \mathcal{N}\}$ of Δ^3 in Figure 1b satisfying the conditions of the Weak K-K-M-S Theorem is now obtained by mapping the cover $\{C^S \subset \Delta^3 | S \in \mathcal{N}\}$ to the relative interior of Δ^3 and then extending it to Δ^3 such that it satisfies the conditions of the Weak K-K-M-S Theorem. Notice that $\hat{C}^{\{3\}} \cap \Delta^{\{1,2\}} = \oslash$.

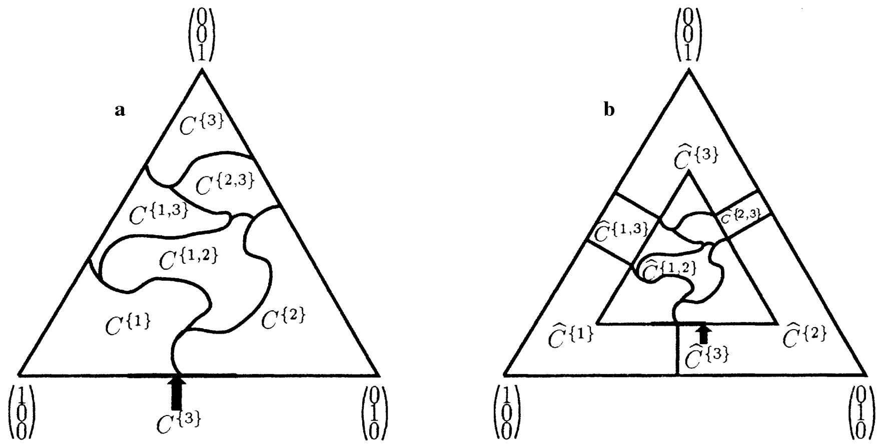

Figure 1 Illustration of the construction of the sets $\hat{C}^S$. **a** The sets C^S. **b** The sets $\hat{C}^S$.

Formally, the construction proceeds as follows. Let $\{C^S \subset \Delta^n | S \in \mathcal{N}\}$ be a cover of Δ^n satisfying the conditions of the K-K-M-S Theorem. For every x in the relative boundary of Δ^n fix a set S_x such that $x \in C^{S_x}$ and $x_i > 0, \forall i \in S_x$. Observe that the conditions of the K-K-M-S Theorem guarantee that such a set exists. For every $T \in \mathcal{N}$, define the set $\hat{C}^T$ by

$$\hat{C}^T = \tfrac{1}{2}[C^T \cup \mathrm{cl}(\{x \in \Gamma^n \backslash \Delta^n | S_{p^n(x)} = T\})] + \tfrac{1}{2}\{m^{I_n}\},$$

where cl denotes the closure of a set. Then $\{\hat{C}^S \subset \Delta^n | S \in \mathcal{N}\}$ satisfies the conditions of the Weak K-K-M-S Theorem. Moreover, if $\mathcal{B} \subset \mathcal{N}$ is a balanced collection and x a point in Γ^n, then $x \in \cap_{S \in \mathcal{B}} C^S$ if and only if $\frac{1}{2}x + m^{I_n} \in \cap_{S \in \mathcal{B}} \hat{C}^S$. The "only if" part is trivial. The "if" part is formally shown as follows. Let $\bar{x} \in \Gamma^n$ satisfy $\frac{1}{2}\bar{x} + \frac{1}{2}m^{I_n} \in \cap_{S \in \mathcal{B}} \hat{C}^S$. So, for every $T \in \mathcal{B}$ there is $x^T \in C^T \cup \mathrm{cl}(\{x \in \Gamma^n \backslash \Delta^n | S_{p^n(x)} = T\})$ such that $\frac{1}{2}\bar{x} + \frac{1}{2}m^{I_n} = \frac{1}{2}x^T + \frac{1}{2}m^{I_n}$, hence $x^T = \bar{x}$.

Suppose $\bar{x} \in \Gamma^n \backslash \Delta^n$, then, for every $T \in \mathcal{B}, \bar{x} \in \mathrm{cl}(\{x \in \Gamma^n \backslash \Delta^n | S_{p^n(x)} = T\})$. So there exists a sequence $(x^{T^r})_{r \in \mathbb{N}}$ such that $x^{T^r} \to \bar{x}, x^{T^r} \in \Gamma^n \backslash \Delta^n$, and $S_{p^n}(x^{T^r}) = T$, hence $(p^n(x^{T^r}))_i > 0, \forall i \in T$. Clearly, $x_i^{T^r} \leq 0$ implies $(p^n(x^{T^r}))_i = 0$, so $x_i^{T^r} > 0, \forall i \in T$. Since $\bar{x} \in \Gamma^n \backslash \Delta^n$ there is $i' \in I_n$ such that $\bar{x}_{i'} < 0$, hence $x_{i'}^{T^r} < 0$ for r sufficiently large. Therefore, $i' \notin T, \forall T \in \mathcal{B}$, contradicting the balancedness of $\mathcal{B}$. Consequently, $\bar{x} \in \Delta^n$.

Now, for every $T \in \mathcal{B}, \bar{x} \in C^T$ or $\bar{x} \in \mathrm{cl}(\{x \in \Gamma^n \backslash \Delta^n | S_{p^n(x)} = T\})$. In the latter case there exists a sequence $(x^{T^r})_{r \in \mathbb{N}}$ as before. Since $x^{T^r} \to \bar{x}$ and p^n is continuous, it follows that $p^n(x^{T^r}) \to p^n(\bar{x}) = \bar{x}$. Using that $p^n(x^{T^r}) \in C^T$ and the closedness of C^T, it follows that also in the latter case $\bar{x} \in C^T$. Consequently, $\bar{x} \in \cap_{S \in \mathcal{B}} C^S$. Therefore, by the construction neither new intersection points are added nor old intersection points are deleted.

3 Proof of the K-K-M-S Theorem

In this section an extremely simple proof of the Weak K-K-M-S Theorem, and therefore of the K-K-M-S Theorem, is given.

Theorem 2.3 (Weak K-K-M-S). *Let $\{\hat{C}^S \subset \Delta^n | S \in \mathcal{N}\}$ be a closed cover of Δ^n such that, for every $T \subset I_n$, not $S \subset T$ implies $\Delta^T \cap \hat{C}^S = \oslash$. Then there exists a balanced collection $\mathcal{B} \subset \mathcal{N}$ such that $\cap_{S\in\mathcal{B}} \hat{C}^S \neq \oslash$.*

Proof. Let $\{S_1, \ldots, S_{2^n-1}\}$ be the collection of all non-empty subsets of I_n. Let A be the $n \times (2^n - 1)$ matrix defined by $A = [m^{I_n} - m^{S_1}, \ldots, m^{I_n} - m^{S_{2^n-1}}]$. Notice that $m^{I_n T} A = 0$. For every $j \in I_{2^n-1}$, let the continuous function $d_j : \Delta^n \to \mathbb{R}_+$ be defined by $d_j(x) = \min_{y\in\hat{C}^{S_j}} \|y - x\|_2$ if $\hat{C}^{S_j} \neq \oslash$, and $d_j(x) = 1$ otherwise. Let the continuous function $f : \Delta^n \times \Delta^{2^n-1} \to \Delta^n \times \Delta^{2^n-1}$ be defined by

$$f(x, \lambda) = [p^n(A\lambda + x), p^{2^n-1}(\lambda_1 - d_1(x), \ldots, \lambda_{2^n-1} - d_{2^n-1}(x))], \quad \forall (x, \lambda) \in \Delta^n \times \Delta^{2^n-1}.$$

By Brouwer's fixed point theorem, there is $(x^*, \lambda^*) \in \Delta^n \times \Delta^{2^n-1}$ such that $(x^*, \lambda^*) = f(x^*, \lambda^*)$. Now it will be shown that x^* is an intersection point of a balanced collection of sets with balancing weights yielded by λ^*. By Lemma 2.1 there is $\alpha^* \in \mathbb{R}, \beta_i^* \in \mathbb{R}_+, \forall i \in I_n, \varphi^* \in \mathbb{R}$, and $\psi_j^* \in \mathbb{R}_+, \forall j \in I_{2^n-1}$, such that

$$(A\lambda^*)_i + \alpha^* + \beta_i^* = 0, \forall i \in I_n, \tag{1}$$

$$- d_j(x^*) + \varphi^* + \psi_j^* = 0, \forall j \in I_{2^n-1}, \tag{2}$$

$$\beta_i^* > 0 \Rightarrow x_i^* = 0, \forall i \in I_n, \tag{3}$$

$$\psi_j^* > 0 \Rightarrow \lambda_j^* = 0, \forall j \in I_{2^n-1}. \tag{4}$$

Since $\cup_{S\in\mathcal{N}} \hat{C}^S = \Delta^n$ there exists $j \in I_{2^n-1}$ such that $d_j(x^*) = 0$, hence by (2) $\varphi^* \leq 0$, so if $d_k(x^*) > 0$ for some $k \in I_{2^n-1}$ then by (2) $\psi_k^* > 0$ and hence by (4) $\lambda_k^* = 0$.

Suppose there is $i_1 \in I_n$ such that $(A\lambda^*)_{i_1} > 0$, hence by (1) $\alpha^* < 0$. Since $m^{I_n} \cdot A\lambda^* = 0$, there is $i_2 \in I_n$ such that $(A\lambda^*)_{i_2} < 0$, so by (1) $\beta_{i_2}^* > 0$ and hence by (3) $x_{i_2}^* = 0$. For every $j \in I_{2^n-1}$ with $i_2 \in S_j$, $\Delta^{I_n\setminus\{i_2\}} \cap \hat{C}^{S_j} = \oslash$, so $d_j(x^*) > 0$, and hence $\lambda_j^* = 0$ by the previous paragraph. So, $(A\lambda^*)_{i_2} = \frac{1}{n} > 0$, a contradiction. Consequently, $A\lambda^* \leq 0$ and since $m^{I_n} \cdot A\lambda^* = 0$, it follows that $A\lambda^* = 0$.

Let $J = \{j \in I_{2^n-1} | \lambda_j^* > 0\}$. Combining the previous two paragraphs yields $x^* \in \cap_{j\in J} \hat{C}^{S_j}$ and $\{S_j \in \mathcal{N} | j \in J\}$ is a balanced collection with balancing weights $\lambda_j^*, \forall j \in J$. Q.E.D.

Consider the function f used in the proof of Theorem 2.3. It has been shown that if (x^*, λ^*) is a fixed point of f and $J = \{j \in I_{2^n-1} | \lambda_j^* > 0\}$, then $x^* \in \cap_{j\in J} \hat{C}^{S_j}$ and $\{S_j \in \mathcal{N} | j \in J\}$ is a balanced collection of sets, where $\lambda_j^*, j \in J$, yield the balancing weights. Now let x^* be a point in $\cap_{j\in J} \hat{C}^{S_j}$, where $\{S_j \in \mathcal{N} | j \in J\}$ is a balanced collection with $\lambda_{S_j}^*, j \in J$, the balancing weights.

It will be shown that (x^*, λ^*) is a fixed point of f, where $\lambda_j^* = \lambda_{S_j}^*$ if $j \in J$, and $\lambda_j^* = 0$ if $j \in I_{2^n-1} \backslash J$. Clearly, $A\lambda^* + x^* = x^*$, so $p^n(A\lambda^* + x^*) = x^*$. Moreover, it holds that $d_j(x^*) = 0$ and $\lambda_j^* > 0$ for $j \in J$, while $d_j(x^*) \geq 0$ and $\lambda_j^* = 0$ for $j \in I_{2^n-1} \backslash J$. Therefore, it follows immediately that $p^{2^n-1}(\lambda_1^* - d_1(x^*), \ldots, \lambda_{2^n-1}^* - d_{2^n-1}(x^*)) = \lambda^*$.

Since the proof of Theorem 2.3 provides an explicit function whose fixed points are in a one-one correspondence to points in the intersection of balanced collection, of sets, it might provide possibilities for constructive methods of proof being different from the one used in Scarf (1967b) or Shapley (1973) and possibilities for alternative ways of computing an intersection point or a core element.

4 π- Balancedness

The way the K-K-M-S Theorem is proved in this paper might be applied to generalizations of this result. For instance, it is very easy to adjust the proof in order to show π-balanced versions of the K-K-M-S Theorem. The concept of π-balancedness has been introduced in Billera (1970) and is also considered in Shapley (1973). Let $\pi = \{\pi^S \in \mathbb{R}^n | S \in \mathcal{N}\}$ be an array of vectors such that π^S belongs to Δ^S and π^N belongs to the relative interior of Δ^N. Then a collection $\mathcal{B} \subset \mathcal{N}$ is said to be π-balanced if there exists $\lambda_S \in \mathbb{R}_+, \forall S \in \mathcal{B}$, such that $\sum_{S \in \mathcal{B}} \lambda_S \pi^S = m^{I_n}$. The π-balanced K-K-M-S Theorem is defined as follows.

Theorem 4.1 (π-balanced K-K-M-S). *Let $\{C^S \subset \Delta^n | S \in \mathcal{N}\}$ be a closed cover of Δ^n such that, for every $T \subset I_n, \Delta^T \subset \cup_{S \subset T} C^S$. Then there exists a π-balanced collection $\mathcal{B} \subset \mathcal{N}$ such that $\cap_{S \in \mathcal{B}} C^S \neq \emptyset$.*

The π-balanced Weak K-K-M-S Theorem is defined in the obvious way. Shapley and Vohra (1991), Komiya (1994), and Zhou (1994) mention that their proofs can be applied to the π-balanced K-K-M-S Theorem, where they take π^S in the relative interior of Δ^S. Also the proof of Theorem 2.3 can be applied to obtain this result. In fact, the only part of the proof that changes is the definition of the matrix A, which is now given by $A = [m^{I_n} - \pi^{S_1}, \ldots, m^{I_n} - \pi^{S_{2^n-1}}]$. Then the π-balanced Weak K-K-M-S Theorem is obtained, which immediately implies the π-balanced K-K-M-S Theorem as before.

The proof of Theorem 2.3 makes clear that the notion of π-balancedness can be even further generalized. Let $\rho = \{\rho^S \in \mathbb{R}^n | S \in \mathcal{N}\}$ be an array of vectors such that $\sum_{i \in I_n} \rho_i^S = 1$ and $\rho_i^S \leq 0$ if $i \notin S$. Notice that it is now no longer required that $\rho_i^S \geq 0$ if $i \in S$. Let r be an arbitrary element of Δ^{I_n}. Then a collection $\mathcal{B} \subset \mathcal{N}$ is said to be (ρ, r)-balanced if there exists $\lambda_S \in \mathbb{R}_+, \forall S \in \mathcal{B}$, such that $\sum_{S \in \mathcal{B}} \lambda_S \rho^S = r$. The (ρ, r)-balanced K-K-M-S Theorem is defined as follows.

Theorem 4.2 **((ρ, r)-balanced K-K-M-S).** *Let $\{C^S \subset \Delta^n | S \in \mathcal{N}\}$ be a closed cover of Δ^n such that, for every $T \subset I_n, \Delta^T \subset \cup_{S \subset T} C^S$. Then there exists a (ρ, r)-balanced collection $\mathcal{B} \subset \mathcal{N}$ such that $\cap_{S \in \mathcal{B}} C^S \neq \emptyset$.*

The (ρ, r)-balanced Weak K-K-M-S Theorem is defined in the obvious way. The (ρ, r)-balanced K-K-M-S Theorem can be modified as in Section 2 to yield a cover satisfying the conditions of the (ρ, r)-balanced Weak K-K-M-S Theorem. Again, it is easily shown that the (ρ, r)-balanced K-K-M-S Theorem follows immediately from the (ρ, r)-balanced Weak K-K-M-S Theorem. To prove the latter theorem, the proof of Theorem 2.3 has to be changed only at two instances. First, the matrix A should be defined as $A = [r - \rho^{S_1}, \ldots, r - \rho^{S_{2^n-1}}]$. Secondly, the contradiction in the second to last paragraph of the proof should be changed in "So, $(A\lambda^*)_{i_2} = r_{i_2} - \sum_{j \in \{k \in I_{2^n-1} | i_2 \notin S_k\}} \lambda_j^* \rho_{i_2}^{S_j} \geq 0$, a contradiction."

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