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Strategy-proofness, solidarity, and consistency for multiple assignment problems*

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Abstract. We consider a problem of allocating indivisible objects when agents may desire to consume more than one object and no monetary transfers are allowed. We are interested in allocation rules that satisfy desirable properties from an economic and social point of view. In addition to strategy-proofness and Pareto efficiency, we consider consistency and two solidarity properties (replacement-domination and population-monotonicity). In most of the cases, these properties are satisfied only by serially dictatorial rules.

Key words: Serial dictatorship, strategy-proofness, population-monotonicity, consistency

1. Introduction

We consider a problem of allocating indivisible objects when agents may desire more than one object and no monetary compensations are allowed. As an example, one may think of a heritage consisting of indivisible objects (*e.g.*, furniture and household items) that has to be distributed among the heirs (*e.g.*, the children of the deceased), respecting the wish that the objects should not be sold but allocated. Since an agent may receive more than one object, one could consider several interesting domains of preferences over sets of objects. We consider four domains: the general domain of strict preferences, the domain of strict and separable preferences, the domain of strict, separable, and responsive preferences, and the domain of strict and additive preferences.

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Pápai (1998, 2000a) studies essentially the same model, but considers the general domain of strict preferences and monotonic preference domains. Ehlers and Klaus (2000) study the same model and preference domains, but – as we will discuss later – analyze different properties for allocation rules.

Recent studies have shown that when each agent may receive at most one object, there exist some allocation rules, such as the core, that satisfy appealing properties; some of the most recent studies are Ehlers, Klaus, and Pápai (2000a,b), Ergin (2000), Miyagawa (2002), Pápai (2000b), and Svensson (1999). We demonstrate that the results differ considerably when we allow for the consumption of multiple objects; in most cases, our list of properties are satisfied only by serial dictatorships. By a serial dictatorship, we mean that one agent chooses his best set of objects, then the second agent chooses his best subset of the remaining set, then the third agent chooses, and so on, and the order in which agents choose is fixed in advance.

In situations when most or all of the objects are "good" for everyone, our results might be regarded as negative since the first dictator will consume most, or all, of the "good" objects and leave few, or no, objects to the other agents. However, when preferences are heterogeneous and objects are often "bad" for some agents (possibly due to capacity constraints or satiation), serial dictatorships are widely applied and accepted. For example, used household items sold or given away in garage sales are not "goods" for everyone. Often an item is a "good" for some and a "bad" for others. Indeed, the allocation rule used often in garage sales is the serial dictatorship where the order is determined on the first-come, first-served basis.

A practical advantage of serial dictatorships is that they are simple and can be implemented easily and quickly. Furthermore, they are *strategy-proof*, *Pareto efficient*, and satisfy other appealing properties discussed below. They can be considered "fair" as well when the ordering of the agents is determined fairly; for instance by queuing, seniority, or randomization (Abdulkadiroğlu and Sönmez, 1998, 1999; Bogomolnaia and Moulin, 1999).

We briefly discuss the organization of the paper and our results. The model is introduced in the next section. In Section 3, we first show that for the two-agent case, *strategy-proofness* and *Pareto efficiency* are satisfied only by serial dictatorships (Theorem 1). Next, we consider the notion of minimal rights: an agent with a minimal right for some object cannot be assigned a set of objects that is worse for him than the object he has a minimal right for. We prove that for the n-person case, *strategy-proofness* and *Pareto efficiency* exclude minimal rights for more than one agent (Corollary 1). If we extend the model by assuming that the objects are initially owned by (or assigned to) agents, then the latter result implies that *strategy-proofness*, *Pareto efficiency*, and *individual-rationality* are incompatible – a result also obtained by Sönmez (1999).[1]

Ehlers and Klaus (2000) consider the stronger non-manipulation property *coalitional strategy-proofness* (no group of agents ever gains by jointly misrepresenting their preferences). Their main result is that *coalitional strategy-proofness* and *Pareto efficiency* only allow for sequential dictatorships; that is, there exists a first dictator who always chooses his best set of objects. Depend-

[1] An earlier example of this type of impossibility is Hurwicz's (1972) seminal result for exchange economies: for two-agent two-good exchange economies, Hurwicz (1972) showed that, even on a restricted domain that contains all translated Cobb-Douglas preferences, no allocation rule satisfies *strategy-proofness*, *Pareto efficiency*, and *individual-rationality*.

ing on the first dictator's choice, a second dictator is determined who again chooses his best subset of the remaining objects. Depending on the choices of the previous dictators, a third dictator is determined, and so on. Pápai (1998) shows the same result on the domain of strict preferences.

In Section 4, we study two "solidarity properties": *replacement-domination* and *population-monotonicity*. *Replacement-domination* says that if the preferences of an agent change, the welfare of the other agents is affected in the same direction; that is, either all of them (weakly) gain or all of them (weakly) lose. We show (Theorem 2) that this solidarity property is not compatible with *Pareto efficiency*. We also consider *population-monotonicity*, which is a similar condition applied to the case when the set of agents varies. It requires that if some agents leave the economy, the welfare of the remaining agents is affected in the same direction. On the domain of strict and separable (or strict and additive) preferences, *population-monotonicity* is compatible with both *strategy-proofness* and *Pareto efficiency*. However, the three properties are satisfied only by serial dictatorships (Theorem 3). On the other hand, these three properties are not compatible on the general domain of strict preferences (Corollary 2). Ehlers and Klaus (2000) prove that similar results can be obtained if instead of *population-monotonicity* we consider the solidarity property *resource-monotonicity* and *strategy-proofness* is strengthened to *coalitional strategy-proofness*.

In Section 5, we discuss *consistency*, which is usually regarded as a condition of stability. On all of the preference domains we consider, *consistency* is compatible with *strategy-proofness* and *Pareto efficiency*. Again, these properties are satisfied only by serial dictatorships (Theorem 4).

Finally, in Section 6, we prove the logical independence of the properties used in the characterization results.

2. The model

There are $k \geq 2$ objects, and the set of objects is denoted by $K = \{1, 2, \dots, k\}$. There are $n \geq 2$ agents, and the set of agents is denoted by $N = \{1, 2, \dots, n\}$. Let 2^K denote the set of all (possibly empty) subsets of K. For subsets of K consisting of one object, with some abuse of notation, we omit the brackets and write x instead of $\{x\}$. Each agent $i \in N$ has a complete and transitive preference relation R_i over 2^K. The associated strict preference relation is denoted by P_i. We assume that R_i is strict; that is, for all distinct subsets $S, S' \subseteq K$, we have either SP_iS' or $S'P_iS$. Thus, SR_iS' means that either SP_iS' or $S = S'$. We further assume that R_i is *separable*.

An agent's preferences are *separable* if he prefers x to nothing if and only if for any set S not containing x he prefers $S \cup x$ to S: for all $S \subseteq K$ and all $x \in K \backslash S$,

$$xP_i\varnothing \Leftrightarrow (S \cup x)P_iS.$$

Together with strictness and completeness of preferences, this implies that for all $S \subseteq K$ and all $x \in K \backslash S$,

$$\varnothing P_ix \Leftrightarrow SP_i(S \cup x).$$

For the notion of separability we use here, we refer to Barberà, Sonnenschein, and Zhou (1991).

Let $\mathscr{R}$ be the set of strict and separable preference relations over 2^K. At various points in the paper, we will also consider the following three domains of preferences: the domain of all (unrestricted) strict preferences $\mathscr{R}_u$; the domain of strict and additive[2] preferences $\mathscr{R}_a$; and the domain of strict, separable, and responsive[3] preferences $\mathscr{R}_{sr}$. Clearly, $\mathscr{R}_a \subsetneq \mathscr{R}_{sr} \subsetneq \mathscr{R} \subsetneq \mathscr{R}_u$. If not otherwise stated, we assume that preferences are strict and separable; that is, $\mathscr{R}$ is the default preference domain.[4]

A preference profile is denoted by $R = (R_1, R_2, \ldots, R_n)$ and the set of preference profiles is denoted by $\mathscr{R}^N$, $\mathscr{R}_a^N$, $\mathscr{R}_{sr}^N$, or $\mathscr{R}_u^N$.

Given an ordered collection of objects $\{x_1, x_2, \ldots, x_m\}$ where $m \leq k$, let $L(x_1, x_2, \ldots, x_m)$ be the class of preference relations $R_i \in \mathscr{R}$ such that $x_1 P_i x_2 P_i \cdots P_i x_m P_i \emptyset$, and $\emptyset P_i y$ for all $y \notin \{x_1, \ldots, x_m\}$. By separability, for example, $\{x_1, x_2, x_3\} P_i \{x_2, x_3\} P_i \emptyset P_i x_{m+1}$ and so on. Note that $R_i \in L(\emptyset)$ means that $\emptyset P_i y$ for all $y \in K$; that is, every object is a "bad" for agent i.

Let $B(R_i) = \{x \in K : x P_i \emptyset\}$ be the set of objects that are "goods" for agent i. By separability, $B(R_i)$ is the most preferred bundle of objects for agent i.

An *allocation* is a list $(S_1, \ldots, S_n)$ such that $S_i \subseteq K$ for all $i \in N$ and $S_i \cap S_j = \emptyset$ for all $i \neq j$. The set S_i is the (possibly empty) set of objects assigned to agent i. The second condition simply says that no two agents receive the same object. Note that we allow free disposal and therefore the union of all S_i's may be a strict subset of K.

An *(assignment) rule* is a function φ that associates with each preference profile $R \in \mathscr{R}^N$ an allocation $\varphi(R) = (S_i)_{i \in N}$. We denote by $\varphi_i(R)$ the set of objects assigned to agent i.

A rule φ is *Pareto efficient* if it assigns a *Pareto efficient* allocation to each preference profile; that is, for all $R \in \mathscr{R}^N$, there is no allocation $(S_i)_{i \in N}$ such that $S_i R_i \varphi_i(R)$ for all $i \in N$, with strict preference holding for some $j \in N$. Separability of preferences and free disposal imply the following:

Lemma 1. *If a rule φ is Pareto efficient, then for all $R \in \mathscr{R}^N$,*

1. $\varphi_i(R) \subseteq B(R_i)$ *for all $i \in N$ and*
2. $\bigcup_{i \in N} \varphi_i(R) = \bigcup_{i \in N} B(R_i)$.

Proof: Part 1 follows immediately from separability and free disposal. So, $\bigcup_{i \in N} \varphi_i(R) \subseteq \bigcup_{i \in N} B(R_i)$. To prove Part 2, suppose there exists an object

[2] A preference relation R is *additive* if there exists a function $u : K \to \mathbb{R}$ such that for all $S, S' \in 2^K$,

$$S R_i S' \Leftrightarrow \sum_{k \in S} u(k) \geq \sum_{k \in S'} u(k).$$

[3] An agent's preferences are *responsive* if, for any two sets that differ only in one object, the agent prefers the set containing the more preferred object: for all $S \subseteq K$ and all $x, y \in K \backslash S$,

$$x P_i y \Rightarrow (S \cup x) P_i (S \cup y).$$

Roth (1985) introduced the responsiveness of preferences for college admission problems.

[4] All results we establish for $\mathscr{R}$ remain true on $\mathscr{R}_a$ and $\mathscr{R}_{sr}$.

$x \in [\bigcup_{i \in N} B(R_i)] \backslash [\bigcup_{i \in N} \varphi_i(R)]$, which implies $x \in B(R_j)$ for some j. Then $\varphi(R)$ is Pareto dominated by the allocation $(S_i)_{i \in N}$ where $S_j = \varphi_j(R) \cup x$, and for all $i \neq j$, $S_i = \varphi_i(R)$.[5] □

The following notation will be useful later on. Given $R \in \mathscr{R}^N$ and $M \subseteq N$, we denote the profile $(R_i)_{i \in M}$ by R_M. This is the restriction of R to the set of agents M. We also use the notation $R_{-i} = R_{N \backslash \{i\}}$.

3. Strategy-proofness

A rule φ is *strategy-proof* if in its associated direct revelation game, it is a dominant strategy for each agent to report his preferences truthfully; that is, for all $R \in \mathscr{R}^N$, all $i \in N$, and all $R'_i \in \mathscr{R}$, $\varphi(R) R_i \varphi(R'_i, R_{-i})$.

We first show that for the two-agent case, *strategy-proofness* and *Pareto efficiency* are satisfied only by serial dictatorships. The result itself may be of limited importance since it applies only to the two-person case, but it turns out to be a key for the remainder of this paper. It will be used in the proofs of Corollary 1 and Theorems 3 and 4.

Theorem 1. *If $n = 2$, then a rule φ is strategy-proof and Pareto efficient if and only if it is a serial dictatorship; that is, there exists $i \in N$ such that for all $R \in \mathscr{R}^N$, $\varphi_i(R) = B(R_i)$ and $\varphi_j(R) = B(R_j) \backslash B(R_i)$ for $j \neq i$.*

To get some intuition, consider the case of two goods. A natural way to achieve *Pareto efficiency* and some degree of equality is to split the set of objects between the two agents and then let them trade. To be concrete, suppose that we initially assign object x to agent 1 and object y to agent 2, and then make a Pareto improvement whenever possible. This is not compatible with *strategy-proofness*. To see this, suppose that $xP_1yP_1\varnothing$ and $xP_2\varnothing P_2y$. Since agent 1 won't give up x and agent 2 is not interested in y, the final assignment would be $(\{x, y\}, \varnothing)$. This implies that if we want to maintain *strategy-proofness*, we have to give both goods to agent 1 even when agent 1's preferences are $yP'_1xP'_1\varnothing$. Suppose now that agent 2's preferences are $xP'_2yP'_2\varnothing$. Then for (R'_1, R'_2), the only way to achieve a Pareto improvement over the initial assignment is to let the agents exchange their objects, and thus our final assignment is (y, x). But if agent 2 knows this outcome, he would report R'_2 to obtain x when his true preferences are R_2. We now turn to a formal proof.

Proof: It is easy to check that serial dictatorships are *strategy-proof* and *Pareto efficient*. The proof of the uniqueness part is divided into two steps. Step 1 establishes that, given a *strategy-proof* and *Pareto efficient* rule, each object "belongs" to an agent in the sense that he obtains the object whenever it is desirable for him. Step 2 shows that in fact all objects "belong" to the same agent. The desired result then follows immediately from *Pareto efficiency*.

Fix a rule φ that is *strategy-proof* and *Pareto efficient*. We say that object x *belongs to* agent i if for all $R \in \mathscr{R}^N$, $x \in B(R_i)$ implies $x \in \varphi_i(R)$.

[5] It is easy to see that the converse of Lemma 1 is false.

Step 1. Each object belongs to some agent.

Let $N = \{1, 2\}$ and fix $x \in K$. Let $R'_1 \in L(x)$ and $R'_2 \in L(x)$ be such that for all $i \in \{1, 2\}$,

$$(S \cup x) P'_i \varnothing \quad \text{for all } S \subseteq K. \tag{1}$$

This is the case if object x is sufficiently desirable for each agent.[6] By *Pareto efficiency*, $x \in \varphi_i(R')$ for some $i \in \{1, 2\}$. Without loss of generality, assume $i = 1$. Then, by *Pareto efficiency*, $\varphi(R') = (x, \varnothing)$. We show that object x belongs to agent 1.

Take any $R_2 \in \mathscr{R}$. *Strategy-proofness* implies $\varnothing R'_2 \varphi_2(R'_1, R_2)$. Then (1) implies that $x \notin \varphi_2(R'_1, R_2)$. By *Pareto efficiency*, it follows that

$$\varphi_1(R'_1, R_2) = x \quad \text{for all } R_2 \in \mathscr{R}. \tag{2}$$

Suppose, by contradiction, that there exists $R \in \mathscr{R}^N$ such that $x \in B(R_1)$ and $x \notin \varphi_1(R)$. Let $\varphi(R) = (S_1, S_2)$. Note that by *Pareto efficiency*, $x \in S_2$.

Let $\bar{R}_2 \in \mathscr{R}$ be such that $B(\bar{R}_2) = S_2$. *Strategy-proofness* implies $\varphi_2(R_1, \bar{R}_2) \bar{R}_2 S_2$. Since $B(\bar{R}_2) = S_2$, we have $\varphi_2(R_1, \bar{R}_2) = S_2$. Thus by *Pareto efficiency*,

$$\varphi(R_1, \bar{R}_2) = (S_1, S_2). \tag{3}$$

Let $\bar{R}_1 \in \mathscr{R}$ be such that $B(\bar{R}_1) = S_1 \cup x$ and $x \bar{P}_1 S_1$. This is the case if object x is sufficiently desirable for agent 1.[7] By *Pareto efficiency*, $S_1 \subseteq \varphi_1(\bar{R})$ since agent 2 is not interested in any object in S_1. Thus $\varphi_1(\bar{R})$ is either S_1 or $S_1 \cup x$. *Strategy-proofness* implies

$$\varphi_1(\bar{R}) \bar{R}_1 \varphi_1(R'_1, \bar{R}_2) = x \bar{P}_1 S_1,$$

where the equality follows from (2). This implies $\varphi_1(\bar{R}) = S_1 \cup x$. *Strategy-proofness* then implies

$$\varphi_1(R_1, \bar{R}_2) R_1 (S_1 \cup x).$$

By (3), this means $S_1 R_1 (S_1 \cup x)$, which is in contradiction with $x \in B(R_1)$.

Step 2. All objects belong to the same agent.

Suppose, to the contrary, that object x belongs to agent 1 and object y belongs to agent 2. Let $R_1 \in L(y, x)$ and $R_2 \in L(x, y)$. Then $x \in \varphi_1(R)$ and $y \in \varphi_2(R)$, which is in violation of *Pareto efficiency*. □

Pápai (2000b) shows that the class of *strategy-proof* and *Pareto efficient* assignment rules that assign at most one object to each agent contains what she calls fixed endowment hierarchical exchange rules. One characteristic of these rules is that some agents receive some of the objects as initial endowments. The final allocation is determined by a sequential process of trading

[6] Let $K = \{x_1, \ldots, x_k\}$ and assume $x = x_k$. Then, R'_i can be represented by the following additive utility function: $u_i(x_k) = 10^k$ and $u_i(x_\ell) = -10^\ell$ for all $\ell \neq k$.

[7] Similarly as before, let $K = \{x_1, \ldots, x_k\}$ and assume $x = x_k$. Then, $\bar{R}_1$ can be represented by the following additive utility function: $u_1(x_\ell) = 10^\ell$ if $x_\ell \in S_1 \cup \{x_k\}$ and $u(x_\ell) = -10^\ell$ otherwise.

and inheriting endowments. Since trade is assumed to be voluntary, agents with initial endowments can only improve their allocation; that is, their "minimal right" to the endowments is respected. We prove that when an agent may desire multiple objects, *strategy-proofness* and *Pareto efficiency* exclude that minimal rights be assigned to more than one agent.

Let $x \in K$. We say that a rule φ *respects the minimal right of agent i for object x* if for all $R \in \mathscr{R}^N$, $\varphi_i(R) R_i x$. A rule φ *respects minimal rights for agent i* if there exists $x \in K$ such that φ respects the minimal right of i for x. The following result holds for any $n \geq 2$.

Corollary 1. *A strategy-proof and Pareto efficient rule respects minimal rights for at most one agent.*

Proof: Suppose, by contradiction, that φ respects minimal rights for agents 1 and 2. For $i \notin \{1,2\}$, let $\hat{R}_i \in L(\varnothing)$. Let g be the function on $\mathscr{R}^{\{1,2\}}$ defined by

$$g(R_1, R_2) = (\varphi_1(R_1, R_2, \hat{R}_3, \ldots, \hat{R}_n), \varphi_2(R_1, R_2, \hat{R}_3, \ldots, \hat{R}_n)). \tag{4}$$

So, $g_1(R_1, R_2)$ is the set of objects assigned to agent 1 when the profile is $(R_1, R_2, \hat{R}_3, \ldots, \hat{R}_n)$. Note that $\varphi_i(R_1, R_2, \hat{R}_3, \ldots, \hat{R}_n)$ is an empty set for all $i \notin \{1,2\}$. Thus g is a rule for two-agent economies, and it is *strategy-proof* and *Pareto efficient*. Thus by Theorem 1, g is a serial dictatorship. But then g does not respect minimal rights for the second dictator, and neither does φ. □

In the slightly different context where agents are initially endowed with the objects and a rule φ reallocates the objects among them, Corollary 1 can be interpreted as an impossibility result that is also obtained by Sönmez (1999): there exists no rule that satisfies *strategy-proofness*, *Pareto efficiency*, and *individual-rationality*[8].

Example 1. There exists a *strategy-proof* and *Pareto efficient* rule that does not respect minimal rights for any agent. To see this, let $N = \{1,2,3\}$ and $K = \{x, y\}$. Consider the rule

$$\varphi(R) = \begin{cases} (B(R_1), B(R_2) \backslash B(R_1), \varnothing) & \text{if } \varnothing P_3 x P_3 y, \\ (B(R_1) \backslash B(R_2), B(R_2), B(R_3) \backslash (B(R_1) \cup B(R_2))) & \text{otherwise.} \end{cases}$$

This rule is *strategy-proof* and *Pareto efficient* simply because agents choose objects sequentially and the agent who determines the sequence always chooses last. Note that it is *strategy-proof* for agent 3 since he obtains $B(R_3) \backslash (B(R_1) \cup B(R_2))$, regardless of the sequence. The rule does not respect minimal rights for anyone since no one can obtain an object for sure when it is a "good" for him.

Remark 1: Theorem 1 and Corollary 1 also hold if we restrict preferences to the domain of strict, separable, and responsive preferences ($\mathscr{R}_{sr}$) or the domain of strict and additive preferences ($\mathscr{R}_a$). For $\mathscr{R}_a$, we added some additive utility

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8 *Individual-rationality* means that no agent is worse off after the reallocation of the objects.

representations to the proofs in this section to demonstrate how our proofs can be adjusted. A standard argument establishes that Theorem 1 and Corollary 1 also hold on the domain of all strict preferences ($\mathscr{R}_u$).[9]

4. Solidarity

4.1. Replacement-domination

The first "solidarity" property we discuss is *welfare-domination under preference-replacement*, or *replacement-domination* for short. It incorporates a notion of solidarity among agents when a single agent changes his preference relation; that is, if an agent's preference relation is replaced by another preference relation, then, after this change, either all remaining agents are (weakly) better off or they all are (weakly) worse off.[10]

Formally, a rule φ satisfies *replacement-domination* if for all $R \in \mathscr{R}^N$, all $i \in N$, and all $R_i' \in \mathscr{R}$,

$$\text{either } [\text{for all } j \in N \backslash \{i\}, \varphi_j(R) R_j \varphi_j(R_i', R_{-i})]$$

$$\text{or } [\text{for all } j \in N \backslash \{i\}, \varphi_j(R_i', R_{-i}) R_j \varphi_j(R)].$$

It turns out that for more than two agents, *replacement-domination* and *Pareto efficiency* are not compatible.[11]

Theorem 2. *If there exist three or more agents (i.e., $n \geq 3$), then there exists no rule φ that satisfies replacement-domination and Pareto efficiency.*

Proof: Without loss of generality, assume that $N = \{1, 2, 3\}$. Choose any object $x \in K$, and let $(R_1^x, R_2^x, R_3^x) \in L(x)^N$.[12] There are exactly three *Pareto efficient* allocations for this profile: $(x, \emptyset, \emptyset)$, $(\emptyset, x, \emptyset)$, and $(\emptyset, \emptyset, x)$. Without loss of generality, assume

$$\varphi(R_1^x, R_2^x, R_3^x) = (x, \emptyset, \emptyset). \tag{5}$$

Let $y \neq x$ be another object. Change agent 1's preferences to $R_1^y \in L(y)$ and consider (R_1^y, R_2^x, R_3^x). By *Pareto efficiency*, object y goes to agent 1. Also by *Pareto efficiency*, object x goes to either agent 2 or agent 3. Without loss of generality, assume that it goes to agent 2. Then,

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[9] On the general domain of strict preferences, the formal definition of serial dictatorship must be modified slightly. The reason is that after agent i picks $B(R_i)$, the most preferred bundle for j is not necessarily $B(R_j) \backslash B(R_i)$.

[10] Moulin (1987) introduced *replacement-domination* in the context of binary choice with quasi-linear preferences. *Replacement-domination* has been studied in a variety of settings and we refer the interested reader to a recent review of the literature by Thomson (1999).

[11] Note that for assignment problems with only one object, *replacement-domination* and *Pareto efficiency* are compatible. It follows from a result by Klaus, Peters, and Storcken (1997) that the class of assignment rules for one-object assignment problems satisfying *strategy-proofness*, *replacement-domination*, and *Pareto efficiency* equals the class of serial dictatorships.

[12] If $n > 3$, choose three agents, and consider R where $R_i \in L(x)$ for these agents and $R_i \in L(\emptyset)$ for the others. We can ignore the agents with $R_i \in L(\emptyset)$ since they receive no object.

$$\varphi(R_1^y, R_2^x, R_3^x) = (y, x, \emptyset). \tag{6}$$

Now, change agent 2's preferences to $R_2^y \in L(y)$, and consider (R_1^y, R_2^y, R_3^x). By *Pareto efficiency*, agent 3 receives object x, and so he gains. Thus, agent 1 should not lose, which implies that he receives object y. Hence,

$$\varphi(R_1^y, R_2^y, R_3^x) = (y, \emptyset, x). \tag{7}$$

Now, change agent 1's preferences back to R_1^x and consider (R_1^x, R_2^y, R_3^x). By *Pareto efficiency*, agent 2 receives object y, and so he gains. Thus, agent 3 should not lose, which implies that he receives object x. Hence,

$$\varphi(R_1^x, R_2^y, R_3^x) = (\emptyset, y, x). \tag{8}$$

Finally, compare (8) and (5). As agent 2's preferences change from R_2^x to R_2^y, agent 1 loses while agent 3 gains, a contradiction. □

Remark 2: Theorem 2 also holds on $\mathscr{R}_a$, $\mathscr{R}_{sr}$, and $\mathscr{R}_u$. The theorem also applies to the assignment problem where each agent is allowed to receive at most one object (see assignment models described in Pápai (2000b) and Svensson (1999)).

4.2. Population-monotonicity

So far, the set of agents has been fixed. In this section, we consider a model with a variable population. Let $P = \{1, \ldots, p\}$, $p \geq 3$, be a finite set of potential agents. We denote by $\mathscr{P}$ the set of nonempty subsets of P. A *rule* is now a function φ that associates with each set of agents $N \in \mathscr{P}$ and each preference profile $R \in \mathscr{R}^N$ an allocation $\varphi(R) = (S_i)_{i \in N}$.

Our main property in this section is *population-monotonicity*, which incorporates a notion of solidarity among agents when changes in the population occur; that is, if a group of agents leave, then, after this change, either all remaining agents are (weakly) better off or they all are (weakly) worse off.[13]

Formally, a rule φ is *population-monotonic* if for all $N \in \mathscr{P}$, all $R \in \mathscr{R}^N$, and all $M \subseteq N$,

$$\text{either [for all } i \in M, \varphi_i(R_M) R_i \varphi_i(R)]$$

$$\text{or [for all } i \in M, \varphi_i(R) R_i \varphi_i(R_M)].$$

Recall that R_M is the restriction of R to the set of agents M. This condition says that if the agents outside M leave, then the agents in M should gain together or lose together. The idea is that no one in M is responsible for the departure of $N \backslash M$, and thus it makes sense to require that the welfare of the agents M change at least in the same direction.

The following lemma states that under *Pareto efficiency*, we can ignore the

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[13] *Population-monotonicity* is introduced by Thomson (1983a,b) in the context of bargaining. An excellent survey on this axiom is Thomson (1995).

case when the agents M lose together after the agents in $N\backslash M$ leave. We omit its proof since it is standard in the literature.

Lemma 2. *Let φ be a population-monotonic and Pareto efficient rule. If $N \in \mathscr{P}$, $R \in \mathscr{R}^N$, and $M \subseteq N$, then for all $i \in M$, $\varphi_i(R_M) R_i \varphi_i(R)$.*

This result is intuitive since the departure of agents is globally beneficial for the remaining group. The objects are now allocated among fewer agents, and thus it makes sense to require a (weak) Pareto improvement for the remaining agents.

We characterize the class of rules that are *strategy-proof*, *population-monotonic*, and *Pareto efficient*. It turns out that these properties are satisfied only by serial dictatorships. By a serial dictatorship, we mean that there is a ranking of the set of potential agents, P, such that the agent with the highest ranking among those actually present chooses his best set of objects, then the agent with the next highest ranking chooses his best set of objects among the remaining objects, and so on.

Serial Dictatorship: Let π be a permutation on P.[14] Then the serial dictatorship with respect to π is defined as follows. Given a set of agents $N = \{n_1, \ldots, n_n\} \subseteq P$ such that $\pi(n_1) < \pi(n_2) < \cdots < \pi(n_n)$ and their preferences $R \in \mathscr{R}^N$,

$$\varphi_{n_1}(R) = B(R_{n_1}),$$

$$\varphi_{n_2}(R) = B(R_{n_2}) \backslash B(R_{n_1}),$$

$$\varphi_{n_3}(R) = B(R_{n_3}) \backslash [B(R_{n_1}) \cup B(R_{n_2})],$$

$$\vdots$$

$$\varphi_{n_n}(R) = B(R_{n_n}) \Big\backslash \left[\bigcup_{i=n_1}^{n_{n-1}} B(R_i) \right].$$

For example, suppose that $\pi(i) \equiv i$, $N = \{2, 4, 5\}$, $B(R_2) = \{1, 2, 3\}$, $B(R_4) = \{3, 4, 5\}$, and $B(R_5) = \{2, 3, 5, 6, 7\}$. Then agent 2 obtains $\{1, 2, 3\}$, agent 4 obtains $\{4, 5\}$, and agent 5 obtains $\{6, 7\}$.

Theorem 3. *A rule φ is strategy-proof, population-monotonic, and Pareto efficient if and only if it is a serial dictatorship.*

Proof: It is easy to verify that any serial dictatorship satisfies the properties named in the theorem; the proof is left to the reader. Serial dictatorships do not satisfy *population-monotonicity* when we allow for non-separable preferences; see Example 2 below.

To prove the uniqueness part, let φ be a rule satisfying the properties named in the theorem. We know from Theorem 1 that in two-agent economies, φ is a

[14] That is, π is a one-to-one function from P to P.

serial dictatorship. We write $i \succ j$ if agent i is the first dictator in two-agent economies with agents i and j.

We first show that $\succ$ produces no cycle. To see this, suppose that $1 \succ 2$, $2 \succ 3$, and $3 \succ 1$. Let $(R_1^x, R_2^x, R_3^x) \in L(x)^{\{1,2,3\}}$. Without loss of generality, assume that $\varphi(R_1^x, R_2^x, R_3^x) = (x, \varnothing, \varnothing)$. Since $3 \succ 1$, we have $\varphi(R_1^x, R_3^x) = (\varnothing, x)$. This means that when agent 2 leaves, agent 1 loses while agent 3 gains, a contradiction.

This shows that $\succ$ produces a strict ranking over the set of potential agents P. Let π be the permutation defined by $\pi(i) < \pi(j) \Leftrightarrow i \succ j$. We show that φ is the serial dictatorship with respect to π.

Take any $N \in \mathscr{P}$ and $R \in \mathscr{R}^N$. We first claim that for all $i, j \in N$,

$$i \succ j \Rightarrow \varphi_j(R) \cap B(R_i) = \varnothing. \tag{9}$$

That is, if $i \succ j$, then agent j does not receive any object that is "good" for agent i. To prove this, let $R_j' \in \mathscr{R}$ be such that $B(R_j') = \varphi_j(R)$. By *strategy-proofness*, $\varphi_j(R_j', R_{-j}) R_j' \varphi_j(R)$, which implies $\varphi_j(R_j', R_{-j}) = \varphi_j(R)$. Suppose now that agents $N \backslash \{i, j\}$ leave. Since agent j should not lose, $\varphi_j(R_i, R_j') = \varphi_j(R)$. Since $i \succ j$, we have $\varphi_i(R_i, R_j') = B(R_i)$, which establishes (9).

Now, denote $N = \{i_1, i_2, \ldots, i_q\}$ where $i_1 \succ i_2 \succ \cdots \succ i_q$. Then (9) implies that no one in $\{i_2, \ldots, i_q\}$ receives objects in $B(R_{i_1})$. This, together with *Pareto efficiency*, implies that agent i_1 receives $B(R_{i_1})$. Similarly, no one in $\{i_3, \ldots, i_q\}$ receives objects in $B(R_{i_1}) \cup B(R_{i_2})$. Since agent i_1 receives $B(R_{i_1})$ and φ is *Pareto efficient*, it follows that agent i_2 receives $B(R_{i_2}) \backslash B(R_{i_1})$. It is easy to see how this argument applies for the remaining agents. □

Theorem 3 also holds on domains $\mathscr{R}_{sr}$ and $\mathscr{R}_a$. However, it does not hold on $\mathscr{R}_u$, as the following example shows.

Example 2. Let $K = \{x, y\}$, $P = \{1, 2, 3\}$, and $\pi(i) \equiv i$. Let φ denote the serial dictatorship that corresponds to π; that is, agent 1 is the first dictator, agent 2 the second dictator and agent 3 chooses last.[15] Let $R = (R_1, R_2, R_3)$ be such that $R_1 \in L(x)$, $R_3 \in L(y)$, and $\{x, y\} P_2 x P_2 \varnothing P_2 y$. Then $\varphi(R) = (x, \varnothing, y)$ and $\varphi(R_2, R_3) = \{\{x, y\}, \varnothing\}$. Hence, when agent 1 leaves, agent 2 gains and agent 3 loses.

Corollary 2. *Suppose that there exist at least three potential agents (i.e., $|P| \geq 3$). Then on the general domain of strict preferences $\mathscr{R}_u$, there exists no rule that is strategy-proof, population-monotonic, and Pareto efficient.*

5. Consistency

Our last property of rules is a condition of stability. To understand this condition, suppose that after objects are allocated according to a rule, some agents leave the economy with their allotments, and the remaining agents "renegotiate" among themselves the assignment of the remaining objects. What if

[15] As in the two-person case, the formal definition of serial dictatorship has to be adjusted on the general domain of strict preferences.

the *same* rule is applied to their "renegotiation problem"? The rule might be considered "unstable" or "inconsistent" if its assignment to the renegotiating agents differ from its original assignment to them.

The condition of consistency has been formulated in a variety of economic and game-theoretic contexts.[16] Thus it is interesting to see what solution concepts are characterized by consistency in our context.

To formulate *consistency*, we have to consider a model where both the set of agents and the set of objects are variable. Let $O = \{1, \dots, o\}$ be a finite set of potential objects. We denote by $\mathcal{O}$ the set of all non-empty subsets of O. Preferences R_i are defined over 2^O. A *rule* is now a function φ that associates with each $N \in \mathcal{P}$, each $R \in \mathcal{R}^N$, and each $K \in \mathcal{O}$ an allocation $\varphi(R; K)$ where $\bigcup_{i \in N} \varphi_i(R; K) \subseteq K$. Here $\varphi_i(R; K)$ denotes the set of objects that agent i receives when the set of available objects is K.

A rule φ is *consistent* if for all $K \in \mathcal{O}$, all $N \in \mathcal{P}$, all $R \in \mathcal{R}^N$, all $M \subseteq N$, and all $i \in M$,

$$\varphi_i(R; K) = \varphi_i\left(R_M; \bigcup_{j \in M} \varphi_j(R; K)\right).$$

The right-hand side is the outcome when φ is reapplied to reallocating the objects received by the agents M to themselves. This condition says that such a "renegotiation" does not change the outcome for the agents in M.

We use the notation $B(R_i; K) = B(R_i) \cap K$. This is the set of "good" objects in K.

It turns out, that *consistency* together with *strategy-proofness* and *Pareto efficiency* implies serial dictatorships.

Serial Dictatorship: Let π be a permutation on P. Then the serial dictatorship with respect to π is defined as follows. When an economy consists of agents $N = \{n_1, \dots, n_n\} \subseteq P$ such that $\pi(n_1) < \pi(n_2) < \cdots < \pi(n_n)$, their preferences are $R \in \mathcal{R}^N$, and the set of available objects is $K \in \mathcal{O}$, then

$$\varphi_{n_1}(R; K) = B(R_{n_1}; K),$$

$$\varphi_{n_2}(R; K) = B(R_{n_2}; K) \backslash B(R_{n_1}; K),$$

$$\varphi_{n_3}(R; K) = B(R_{n_3}; K) \backslash [B(R_{n_1}; K) \cup B(R_{n_2}; K)],$$

$$\vdots$$

$$\varphi_{n_n}(R; K) = B(R_{n_n}; K) \Bigg\backslash \left[\bigcup_{i=n_1}^{n_{n-1}} B(R_i; K)\right].$$

Theorem 4. *A rule φ is strategy-proof, consistent, and Pareto efficient if and only if it is a serial dictatorship.*

[16] *Consistency* is a key property of a number of major solution concepts in economics and game theory. Examples include Walrasian equilibrium, Nash bargaining solution, core, nucleolus, Shapley value, and Nash equilibrium. An exhaustive survey on *consistency* is Thomson (1996).

Proof: It is easy to check that any serial dictatorship satisfies the properties named in the theorem; the proof is left to the reader.

To prove the uniqueness part, let φ be a rule satisfying the properties named in the theorem. Consider the two-person economies with agents i and j where the set of available objects is $K \in O$. We know that within these economies, φ is a serial dictatorship (Theorem 1). We write $i \succ^K j$ if agent i is the first dictator.

We first show that $i \succ^O j$ implies that $i \succ^K j$ for all $K \in \mathcal{O}$. To see this, suppose that $B(R_i) = B(R_j) = K$. Since $i \succ^O j$, $\varphi(R_i, R_j; O) = (K, \emptyset)$. By *consistency*, $\varphi(R_i, R_j; K) = (K, \emptyset)$. This implies that when the set of available objects is K, agent j is not the first dictator, and thus the first dictator is agent i. From now on, we write $i \succ j$ whenever $i \succ^O j$.

We next show that $\succ$ produces no cycle. To see this, suppose that $1 \succ 2$, $2 \succ 3$, and $3 \succ 1$. Let $(R_1^x, R_2^x, R_3^x) \in L(x)^{\{1,2,3\}}$. Without loss of generality, assume $\varphi(R_1^x, R_2^x, R_3^x; O) = (x, \emptyset, \emptyset)$. By *consistency*,

$$\varphi(R_1^x, R_3^x; x) = (x, \emptyset).$$

This contradicts $3 \succ^x 1$.

This shows that $\succ$ produces a strict ranking over the set of agents P. Let π be the permutation on P defined by $\pi(i) < \pi(j) \Leftrightarrow i \succ j$. We show that φ is the serial dictatorship with respect to π.

Take any $N \in \mathcal{P}$, $R \in \mathcal{R}^N$, and $K \in \mathcal{O}$. Take any $i, j \in N$ such that $i \succ j$. Then by *consistency*,

$$\begin{aligned}\varphi_i(R; K) &= \varphi_i(R_i, R_j; \varphi_i(R; K) \cup \varphi_j(R; K)) \\ &= B(R_i) \cap [\varphi_i(R; K) \cup \varphi_j(R; K)]\end{aligned}$$

where the second equality follows from $i \succ j$. It follows that $\varphi_j(R; K) \cap B(R_i) = \emptyset$ if $i \succ j$.

Now, denote $N = \{i_1, i_2, \ldots, i_q\}$ where $i_1 \succ i_2 \succ \cdots \succ i_q$. Since no one in $\{i_2, \ldots, i_q\}$ receives objects in $B(R_{i_1})$, *Pareto efficiency* implies that agent i_1 receives $B(R_{i_1}; K)$. Similarly, since no one in $\{i_3, \ldots, i_q\}$ receives objects in $B(R_{i_1}) \cup B(R_{i_2})$ and agent i_1 receives $B(R_{i_1}; K)$, *Pareto efficiency* implies that agent i_2 receives $B(R_{i_2}; K) \backslash B(R_{i_1}; K)$. It is easy to see how this argument applies for the remaining agents in N. □

Remark 3: Theorem 4 also holds on $\mathcal{R}_{sr}$, $\mathcal{R}_a$, and $\mathcal{R}_u$.[17]

6. Independence of the axioms

We conclude with the independence of the axioms in Theorems 1, 3, and 4.

Example 3. Consider the rule that does not allocate the objects at all: $\varphi_i(R, K) \equiv \emptyset$. This rule is obviously *strategy-proof*, *population-monotonic*, *consistent*, and not *Pareto efficient*.

[17] As before, the formal definition of serial dictatorship must be modified on the domain $\mathcal{R}_u$.

Example 4. The following rule is *strategy-proof* and *Pareto efficient*, while it is neither *population-monotonic* nor *consistent*. For $N = \{i_1, \ldots, i_n\}$, φ is the "sequential" dictatorship where the first dictator is agent i_1 and, for all $k \in \{2, \ldots, n\}$, the k-th dictator is given by

$$\pi^{-1}(k) = \begin{cases} i_k & \text{if } B(R_{i_1}) = K, \\ i_{n+2-k} & \text{otherwise.} \end{cases}$$

That is, the order in which agents can choose their bundles after agent i_1 depends on the preferences of agent i_1.

The class of sequential dictatorships is a superset of the class of serial dictatorships that allows for certain changes in the sequence of dictators depending on the choices that previous dictators make; see Ehlers and Klaus (2000).

Example 5. The following rule is *population-monotonic*, *consistent*, and *Pareto efficient*, but it is not *strategy-proof*. Let $N = \{i_1, \ldots, i_n\}$ be such that $i_1 < i_2 < \cdots < i_n$. If $K = O$ and $B(R_i; O) = O$ for some $i \in N$, then $\varphi_j(R; O) = O$ where $j = \arg\min\{i : B(R_i; O) = O\}$. Otherwise, φ is the serial dictatorship where the k-th dictator is i_k.

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