On the Set of (Perfect) Equilibria of a Bimatrix Game

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This article provides a new approach to the set of (perfect) equilibria. With the help of an equivalence relation on the strategy space of each player, Nash sets and Selten sets are introduced. The number of these sets is finite and each of these sets is a polytope. As a consequence the set of (perfect) equilibria is a finite union of polytopes. © 1994 John Wiley & Sons, Inc.

1. INTRODUCTION

The structure of the set of equilibria of a bimatrix game has been described in several papers such as Winkels [6], Jansen [1] and Jansen and Jurg [3]. All these authors showed that the set of equilibria of a bimatrix game is the finite union of polytopes. In [2], one of the authors, by defining an equivalence relation on the strategy space of each player, partitioned these strategy spaces in a finite number of equivalence classes. The closure of each of these classes appeared to be a polytope. By considering the so-called ϵ-proper pairs within these classes, Jansen [2] proved that the set of proper equilibria of a bimatrix game is also a finite union of polytopes.

In this article the same approach is used to (re)consider the structure of the set of (perfect) equilibria of a bimatrix game. Again we obtain a finite number of equivalence classes in the strategy space of each player. The equivalence relation in question is defined by the identification of strategies to which the other player has the same (pure) best replies. Within the closure of the product of two such equivalence classes (one for each player), we first consider the set of equilibria. We call such a set a Nash set and show that all equilibria are contained in some Nash set and that such a set is a polytope.

Next, within the product of two such equivalence classes we consider for a fixed $\epsilon > 0$ the set of ϵ-perfect equilibria. It turns out that the closure of such a set, if nonempty, is a product of polytopes. The intersection over all $\epsilon > 0$ of these closures will turn out to be a product of polytopes as well. Furthermore it is proved that the union of all the polytopes of the latter kind, called Selten sets, is identical to the set of perfect equilibria. Finally we investigate how Nash (Selten) sets intersect and how Nash sets and Selten sets are related.

NOTATION: $N := \{1, 2, \ldots\}$ is the set of positive integers. $\mathbb{R}^t$ is the vector space of t tuples of real numbers and $\Delta_t := \{p \in \mathbb{R}^t \mid p \geq 0, \sum_{i=1}^{t} p_i = 1\}$. The unit vectors in $\mathbb{R}^t$ are denoted by $e_1, e_2, \ldots, e_t$ and for $x \in \mathbb{R}^t$, $\|x\|_\infty := \max_{i=1,2,\ldots,t} |x_i|$. For a set $C \subset \mathbb{R}^t$ we denote the closure of C by $\text{cl}(C)$. For a convex set $C \subset \mathbb{R}^t$ we denote the relative interior of C by $\text{relint}(C)$. Note that $\hat{c} \in C$ is an element of $\text{relint}(C)$ if and only if for all $c \in C$ there exists an $\epsilon > 1$ such that $(1 - \epsilon)c + \epsilon\hat{c} \in C$.

2. PRELIMINARIES

Let $A := [a_{ij}]$ and $B := [b_{ij}]$ be two real $m \times n$ matrices. The $m \times n$ bimatrix game (A, B) is defined as the two-person normal form game where player 1 and 2 choose, independent of each other, a strategy $p \in \Delta_m$ and $q \in \Delta_n$, respectively. Here p_i (q_j) can be seen as the probability that player one (two) chooses his ith row (jth column). The (expected) payoffs are defined by

$$pAq = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j$$

for player 1 and

$$pBq = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i b_{ij} q_j$$

for player 2. Strategies in $\mathring{\Delta}_m := \{p \in \Delta_m | p > 0\}$ and $\mathring{\Delta}_n := \{q \in \Delta_n | q > 0\}$ are called *completely mixed*.

An *equilibrium* of the game (A, B) is a strategy pair $(p, q) \in \Delta_m \times \Delta_n$ such that $pAq \geq p'Aq$ for all $p' \in \Delta_m$ and $pBq \geq pBq'$ for all $q' \in \Delta_n$. So an equilibrium is a strategy combination with the property that no player can gain by unilaterally deviating from it. The set of all equilibria of the game (A, B) is denoted by $E(A, B)$. By a theorem of Nash [4], this set is nonempty for all bimatrix games. For a strategy $p \in \Delta_m$, we call $C(p) := \{i | p_i > 0\}$ the *carrier* of p and $PB_2(p) := \{j | pBe_j \geq pBe_l$ for all $l\}$ the set of *pure best replies* of player 2 to p. For a strategy $q \in \Delta_n$ the sets $C(q)$ and $PB_1(q)$ are defined in a similar way. The following result is well known.

LEMMA 1: A strategy pair (p, q) is an equilibrium of a bimatrix game (A, B) if and only if

$$C(p) \subset PB_1(q) \quad \text{and} \quad C(q) \subset PB_2(p).$$

Unfortunately not all equilibria of a bimatrix game are stable against small perturbations in the data of the game. In order to overcome this problem several refinements of the equilibrium concept have been formulated in the literature. For example, by requiring stability against some mistakes the players can make in choosing their strategies, Selten [5] defined perfect equilibria. Formally the definition is as follows.

DEFINITION 1: Let (A, B) be an $m \times n$ bimatrix game and let $\epsilon > 0$. A pair $(p, q) \in \Delta_m \times \Delta_n$ is called ϵ-*perfect* if p and q are completely mixed and if for all $i \in \{1, \ldots, m\}$ and all $j \in \{1, \ldots, n\}$

$$\text{if } i \notin PB_1(q), \quad \text{then } p_i \leq \epsilon,$$

and

$$\text{if } j \notin PB_2(p), \quad \text{then } q_j \leq \epsilon.$$

An equilibrium (p, q) of (A, B) is called *perfect* if there exists a sequence $(\epsilon_k)_{k \in \mathbb{N}}$ in $(0, \infty)$ converging to zero and a sequence $((p^k, q^k))_{k \in \mathbb{N}}$ in $\Delta_m \times \Delta_n$ converging to (p, q) with, for every $k \in \mathbb{N}$, (p^k, q^k) is an ϵ_k-perfect pair.

The set of perfect equilibria of the game (A, B), which is nonempty as proved by Selten, is denoted by $PE(A, B)$.

3. AN EQUIVALENCE RELATION

In this section we introduce for each player an equivalence relation on his strategy space. The closure of each equivalence class appears to be a polytope and the intersection of such polytopes is empty or a face of both polytopes.

DEFINITION 2:   For a bimatrix game two strategies p and $\bar{p}$ are called *best-reply equivalent*, denoted as $p \sim_{BR} \bar{p}$, if $PB_2(p) = PB_2(\bar{p})$. In a similar way an equivalence relation can be defined for the strategies of player 2.

Since for an $m \times n$ bimatrix game, $PB_2(p)$ is a subset of $\{1, \ldots, m\}$ for all $p \in \Delta_m$, corresponding to the equivalence relation $\sim_{BR}$ there can be only a finite number of equivalence classes, say $\mathcal{V}_1, \ldots, \mathcal{V}_M$, in Δ_m. Similarly, Δ_n is the finite union of the equivalence classes, say $\mathcal{W}_1, \ldots, \mathcal{W}_N$. For each $s \in \{1, 2, \ldots, M\}$ and each $t \in \{1, 2, \ldots, N\}$ we choose representatives p^s in $\mathcal{V}_s$ and $q^t \in \mathcal{W}_t$. Since $PB_1(q)$ and $PB_2(p)$ are given by linear inequalities, is it straightforward to prove that the closure of each equivalence class is a polytope. In order to show that the intersection of two of such polytopes, if nonempty, is a face of both polytopes, we need the following lemma.

LEMMA 2:   Let G be a face of the polytope $cl(\mathcal{V}_s)$. Then all the elements of $relint(G)$ are best-reply equivalent.

PROOF: Take $p(0), p(1) \in relint(G)$. Let $j_0 \notin PB_2(p(0))$ and suppose that $j_0 \in PB_2(p(1))$. For $\epsilon > 1$ we introduce $p(\epsilon) := (1 - \epsilon)p(0) + \epsilon p(1)$. Since $p(1)$ is an element of the relative interior of G, there is an $\epsilon > 1$ such that $p(\epsilon) \in G$ and

$$p(\epsilon)Be_{j_0} = (1 - \epsilon)p(0)Be_{j_0} + \epsilon p(1)Be_{j_0} > (1 - \epsilon)p(0)Be_{j_1} + \epsilon p(1)Be_{j_1} = p(\epsilon)Be_{j_1},$$

for some $j_1 \in PB_2(p^s) \subset PB_2(p(0))$. Therefore $j_1 \notin PB_2(p(\epsilon))$, contradicting the fact that $PB_2(p(\epsilon)) \supset PB_2(p^s)$. So $j_0 \notin PB_2(p(1))$. Hence $PB_2(p(1)) \subset PB_2(p(0))$. By interchanging the roles of $p(0)$ and $p(1)$ it follows that $p(0) \sim_{BR} p(1)$.   ∎

LEMMA 3:   If the intersection of the closure of two equivalence classes is nonempty, then this intersection is a face of both polytopes.

PROOF: We give a proof for the strategy space of player 1. Suppose that $F := cl(\mathcal{V}_s) \cap cl(\mathcal{V}_{s'}) \neq \phi$ for $s \neq s'$. Let G be the smallest face of $cl(\mathcal{V}_{s'})$ containing F. Note that for this face G, $G \cap relint(F)$ is nonempty. The proof is complete if we can show that $G \subset cl(\mathcal{V}_s)$. So let $p(0) \in relint(G)$ and consider $p(\epsilon) := \epsilon p^s + (1 - \epsilon)p(0)$ for $0 < \epsilon < 1$. We will show that $p(\epsilon) \sim_{BR} p^s$ for all $0 < \epsilon < 1$.

1. Let $j_0 \in PB_2(p^s)$. Then for an element $p(1) \in F \cap relint(G)$ we have $j_0 \in PB_2(p(1))$, and Lemma 2 implies that $j_0 \in PB_2(p(0))$. So, for all $0 < \epsilon < 1$,

    $$p(\epsilon)Be_{j_0} = \epsilon p^s Be_{j_0} + (1 - \epsilon)p(0)Be_{j_0} \geq \epsilon p^s Be_j + (1 - \epsilon)p(0)Be_j = p(\epsilon)Be_j,$$

    for all j. Consequently, $j_0 \in PB_2(p(\epsilon))$, for all $0 < \epsilon < 1$.

2. Let $j_0 \notin PB_2(p^s)$. Take $j_1 \in PB_2(p^s)$. As in part (1) one can show that $j_1 \in PB_2(p(0))$. This implies that $p(\epsilon)Be_{j_0} < p(\epsilon)Be_{j_1}$, for all $0 < \epsilon < 1$. So $j_0 \notin PB_2(p(\epsilon))$.

3. From (1) and (2) of this proof we may conclude that $p(\epsilon) \in \mathcal{V}_s$ for all $0 < \epsilon < 1$. Hence $p(0) = \lim_{\epsilon \to 0} p(\epsilon)$ is an element of $cl(\mathcal{V}_s)$. So relint(G) $\subset cl(\mathcal{V}_s)$ and obviously $G \subset cl(\mathcal{V}_s)$. ∎

4. ON THE SET OF EQUILIBRIA

In this section we consider the set of equilibria contained in the closure of the product of two of the equivalence classes (one for each player) as introduced in the foregoing section.

For $s \in \{1, 2, \ldots, M\}$ and $t \in \{1, 2, \ldots, N\}$, the set

$$\mathcal{N}_{s,t} := \{(p, q) \in cl(\mathcal{V}_s) \times cl(\mathcal{W}_t) | p_i = 0, \quad \text{if } i \notin PB_1(q^t)$$

$$\text{and} \quad q_j = 0 \quad \text{if} \quad j \notin PB_2(p^s)\},$$

if nonempty, is called a *Nash set*. Obviously a Nash set is a polytope and each equilibrium is contained in some Nash set. Further, if (p, q) is an element of some Nash set $\mathcal{N}_{s,t}$, then $p_i = 0$ if $i \notin PB_1(q) \supset PB_1(q^t)$ and $q_j = 0$ if $j \notin PB_2(p) \supset PB_2(p^s)$. In view of Lemma 1 this implies that (p, q) is an equilibrium. So we have the following theorem.

THEOREM 1: The set of equilibria of a bimatrix game is the finite union of polytopes.

In the following example it appears that a Nash subset may be included in another Nash set.

EXAMPLE 1: Let (A, B) be the 2 × 3-bimatrix game defined by

$$(A, B) = \begin{bmatrix} (2, 2) & (1, 0) & (0, 0) \\ (2, 2) & (0, 0) & (1, 0) \end{bmatrix}.$$

Let
$$\mathcal{V} := \{p \in \Delta_2 | PB_2(p) = \{1\}\} = \Delta_2,$$

$$\mathcal{W}_1 := \{q \in \Delta_3 | PB_1(q) = \{1, 2\}\} = \{q \in \Delta_3 | q_2 = q_3\},$$

and

$$\mathcal{W}_2 := \{q \in \Delta_3 | PB_1(q) = \{2\}\} = \{q \in \Delta_3 | q_2 < q_3\}.$$

Then we have the Nash sets

$$\mathcal{N}_1 := \{(p, q) \in cl(\mathcal{V}) \times cl(\mathcal{W}_1) | q_2 = q_3 = 0\} = \Delta_2 \times \{e_1\}$$

and

$$\mathcal{N}_2 := \{(p, q) \in cl(\mathcal{V}) \times cl(\mathcal{W}_2) | p_1 = 0 \quad \text{and} \quad q_2 = q_3 = 0\} = \{(e_2, e_1)\}.$$

A Nash subset not properly contained in another Nash set is called a *maximal* Nash set. Since the number of Nash sets is finite, each Nash set is contained in a maximal one.

Note that a Nash set $\mathcal{N}_{s,t}$ is the Cartesian product of the two polytopes

$$\mathcal{N}^1_{s,t} := \{p \in \mathrm{cl}(\mathcal{V}_s) | p_i = 0, \quad \text{if} \quad i \notin PB_1(q^t)\}$$

and

$$\mathcal{N}^2_{s,t} := \{q \in \mathrm{cl}(\mathcal{W}_t) | q_j = 0, \quad \text{if} \quad j \notin PB_2(p^s)\}.$$

Since $\mathcal{N}^1_{s,t}$ is a face of $\mathrm{cl}(\mathcal{V}_s)$ and $\mathcal{N}^2_{s,t}$ is a face of $\mathrm{cl}(\mathcal{W}_t)$, Lemma 3 implies the following.

LEMMA 4:   The intersection of two (maximal) Nash sets is empty or a face of both Nash sets.

More particularly, we have the following.

COROLLARY 1:   If $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$ is a Nash set contained in the (maximal) Nash set $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$, then $\mathcal{N}_i$ is a face of $\mathcal{M}_i$ for $i = 1, 2$.

5. ON THE SET OF PERFECT EQUILIBRIA

In this section the equivalence relations introduced in Section 3 again play an important role. We show that when we take the intersection over all $\epsilon > 0$ of the closure of the collections of ϵ-perfect pairs which are contained in a particular product of two equivalence classes (one for each player), we get a subset of $PE(A, B)$. Such a subset turns out to be the product of two polytopes and the union over all these subsets is $PE(A, B)$.

Because completely mixed strategies are crucial in the definition of perfect equilibria, we need a small adjustment of the equivalence classes introduced before: for $s \in \{1, \ldots, M\}$ and $t \in \{1, \ldots, N\}$, $\mathcal{V}^*_s$ is the set of completely mixed strategies in $\mathcal{V}_s$ and $\mathcal{W}^*_t$ is the set of completely mixed strategies in $\mathcal{W}_t$. Note that some of these sets might be empty. However, if $\mathcal{V}^*_s$ is nonempty, then $\mathrm{cl}(\mathcal{V}^*_s) = \mathrm{cl}(\mathcal{V}_s)$ and $\mathrm{cl}(\mathcal{V}^*_s)$ is a polytope. Naturally, the same result holds for $\mathrm{cl}(\mathcal{W}^*_t)$ if $\mathcal{W}^*_t$ is nonempty.

For $s \in \{1, \ldots, M\}, t \in \{1, \ldots, N\}$, and $\epsilon > 0$ we introduce the set

$$\mathcal{S}_{s,t}(\epsilon) := \{(p, q) \in \mathcal{V}^*_s \times \mathcal{W}^*_t | p_i \le \epsilon \quad \text{if} \quad i \notin PB_1(q) \text{ and } q_j \le \epsilon \text{ if } j \notin PB_2(p)\}$$

of ϵ-perfect pairs in $\mathcal{V}_s \times \mathcal{W}_t$. Note that $\mathrm{cl}(\mathcal{S}_{s,t}(\epsilon))$ is a polytope for each $\epsilon > 0$. Finally,

$$\mathcal{S}_{s,t} := \bigcap_{\epsilon > 0} \mathrm{cl}(\mathcal{S}_{s,t}(\epsilon)),$$

if nonempty, is called a *Selten set*.

After these preparations we prove that the set of perfect equilibria is the union of the Selten sets.

THEOREM 2:   Let (A, B) be an $m \times n$ bimatrix game. An equilibrium (p, q) is perfect if and only if there is a pair (s, t) such that $(p, q) \in \mathcal{S}_{s,t}$.

PROOF: (1) Let (p, q) be a perfect equilibrium. Then we can find a decreasing sequence $(\epsilon_k)_{k \in \mathbb{N}}$ in $(0, \infty)$ converging to zero and a sequence $((p^k, q^k))_{k \in \mathbb{N}}$ in $\Delta_m \times \Delta_n$ converging to (p, q) such that (p^k, q^k) is ϵ_k-perfect for every $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ there is a pair $(s(k), t(k))$ such that $(p^k, q^k) \in \mathscr{S}_{s(k),t(k)}(\epsilon_k)$. Because $\{1, \ldots, M\} \times \{1, \ldots, N\}$ is a finite collection, we may assume that for some s and t, $s = s(k)$ and $t = t(k)$, for every $k \in \mathbb{N}$. Because $\epsilon_1, \epsilon_2, \ldots$ is a decreasing sequence, $\mathscr{S}_{s,t}(\epsilon_1) \supset \mathscr{S}_{s,t}(\epsilon_2) \supset \cdots$. This implies in particular that $\mathscr{S}_{s,t}(\epsilon_k)$ contains, for all k, a tail of the sequence $(p^1, q^1), (p^2, q^2), \ldots$ and therefore

$$(p, q) \in \mathrm{cl}(\mathscr{S}_{s,t}(\epsilon_k)), \quad \text{for every } k \in \mathbb{N}.$$

Now we can conclude that

$$(p, q) \in \bigcap_{k=1}^{\infty} \mathrm{cl}(\mathscr{S}_{s,t}(\epsilon_k)) = \mathscr{S}_{s,t}.$$

(2) If $(p, q) \in \mathscr{S}_{s,t}$ for some pair (s, t), then $(p, q) \in \mathrm{cl}(\mathscr{S}_{s,t}(1/k))$ for all $k \in \mathbb{N}$. This means that we can find a sequence $(\delta_k)_{k \in \mathbb{N}}$ converging to zero and for all k a pair (p^k, q^k) such that $(p^k, q^k) \in \mathscr{S}_{s,t}(1/k)$ and $\|(p^k, q^k) - (p, q)\|_\infty < \delta_k$. But $(p^k, q^k) \in \mathscr{S}_{s,t}(1/k)$ means that (p^k, q^k) is a $1/k$-perfect pair. Since the sequence $((p^k, q^k))_{k \in \mathbb{N}}$ converges to (p, q) as $k \to \infty$, (p, q) is a perfect equilibrium. ■

In order to show that a Selten set is a polytope, for $\epsilon \geq 0$ we introduce the sets

$$P_t(\epsilon) := \bigcap_{i \notin PB_1(q^t)} \{p \in \Delta_m | p_i \leq \epsilon\},$$

and

$$Q_s(\epsilon) := \bigcap_{j \notin PB_2(p^s)} \{q \in \Delta_n | q_j \leq \epsilon\}.$$

Note that these sets are polytopes. Furthermore, for $\epsilon > 0$,

$$\mathscr{S}_{s,t}(\epsilon) = (\mathscr{V}_s^* \cap P_t(\epsilon)) \times (\mathscr{W}_t^* \cap Q_s(\epsilon)).$$

LEMMA 5: If for some pair (s, t), $\mathscr{S}_{s,t}$ is nonempty, then for $\epsilon > 0$

$$\mathrm{cl}(\mathscr{S}_{s,t}(\epsilon)) = (\mathrm{cl}(\mathscr{V}_s^*) \cap P_t(\epsilon)) \times (\mathrm{cl}(\mathscr{W}_t^*) \cap Q_s(\epsilon)).$$

PROOF: Let $\epsilon > 0$. In view of the observation preceding this lemma,

$$\mathrm{cl}(\mathscr{S}_{s,t}(\epsilon)) = \mathrm{cl}(\mathscr{V}_s^* \cap P_t(\epsilon)) \times \mathrm{cl}(\mathscr{W}_t^* \cap Q_s(\epsilon)).$$

Next we prove that $\mathrm{cl}(\mathscr{V}_s^* \cap P_t(\epsilon)) = \mathrm{cl}(\mathscr{V}_s^*) \cap P_t(\epsilon)$. It is obvious that $\mathrm{cl}(\mathscr{V}_s^* \cap P_t(\epsilon)) \subset \mathrm{cl}(\mathscr{V}_s^*) \cap P_t(\epsilon)$. In order to prove the other inclusion, we take a $p(1) \in \mathrm{cl}(\mathscr{V}_s^*) \cap P_t(\epsilon)$. Because $\mathscr{S}_{s,t} \neq \phi$, we can find a point $p(0) \in \mathscr{V}_s^* \cap P_t(\epsilon)$. Now for $\lambda \in [0, 1]$, we introduce $p(\lambda) := \lambda p(1) + (1 - \lambda)p(0)$. Then $p(\lambda)$ is a completely mixed element of $P_t(\epsilon)$, for all $\lambda \in [0, 1)$. Furthermore it is easy to check that for every $\lambda \in [0, 1)$, $p(\lambda) \sim_{\mathrm{BR}} p^s$. So, for such λ, $p(\lambda) \in \mathscr{V}_s^* \cap P_t(\epsilon)$. Because $\lim_{\lambda \to 1} p(\lambda) = p(1)$, we find that $p(1) \in \mathrm{cl}(\mathscr{V}_s^* \cap P_t(\epsilon))$. ■

By the description in the foregoing lemma, a Selten set $\mathcal{S}_{s,t}$ can be written as

$$\mathcal{S}_{s,t} = \bigcap_{\epsilon>0} (\mathrm{cl}(\mathcal{S}_{s,t}(\epsilon))) = \bigcap_{\epsilon>0} (\mathrm{cl}(\mathcal{V}_s^*) \cap P_t(\epsilon)) \times \bigcap_{\epsilon>0} (\mathrm{cl}(\mathcal{W}_t^*) \cap Q_s(\epsilon))$$

$$= \left(\mathrm{cl}(\mathcal{V}_s^*) \cap \bigcap_{\epsilon>0} P_t(\epsilon) \right) \times \left(\mathrm{cl}(\mathcal{W}_t^*) \cap \bigcap_{\epsilon>0} Q_s(\epsilon) \right)$$

$$= (\mathrm{cl}(\mathcal{V}_s^*) \cap P_t(0)) \times (\mathrm{cl}(\mathcal{W}_t^*) \cap Q_s(0)). \tag{1}$$

Since the four sets occurring in this last expression are polytopes, $\mathcal{S}_{s,t}$ is the product of two polytopes, and therefore is a polytope itself. In combination with Theorem 2 we find the following.

THEOREM 3: For a bimatrix game the set of perfect equilibria is the union of a finite number of polytopes.

Furthermore, the description given in (1) implies that a Selten set is also a Nash set. Hence a Selten set $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ is contained in a (unique) maximal Nash set, say $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$, and Corollary 1 implies that $\mathcal{S}_i$ is a face of $\mathcal{N}_i$ for $i = 1, 2$.

The bimatrix game introduced in Example 1 has two Selten sets. For this game and an $\epsilon > 0$ all ϵ-perfect pairs are contained in

$$\mathcal{S}_1(\epsilon) = \{(p, q) \in \mathcal{V}^* \times \mathcal{W}_1^* | q_2 \leq \epsilon \text{ and } q_3 \leq \epsilon\}$$

$$= \mathring{\Delta}_2 \times \{q \in \mathring{\Delta}_3 | q_2 = q_3 \leq \epsilon\}$$

or

$$\mathcal{S}_2(\epsilon) = \{(p, q) \in \mathcal{V}^* \times \mathcal{W}_2^* | q_2 \leq \epsilon, q_3 \leq \epsilon, \text{ and } p_1 \leq \epsilon\}$$

$$= \{p \in \mathring{\Delta}_2 | p_1 \leq \epsilon\} \times \{q \in \mathring{\Delta}_3 | q_2 < q_3 \leq \epsilon\}.$$

This leads to the Selten sets

$$\mathcal{S}_1 := \bigcap_{\epsilon>0} \mathrm{cl}(\mathcal{S}_1(\epsilon)) = \Delta_2 \times \{e_1\}$$

and

$$\mathcal{S}_2 := \bigcap_{\epsilon>0} \mathrm{cl}(\mathcal{S}_2(\epsilon)) = \{(e_2, e_1)\}.$$

Observing that the second Selten set is properly contained in the first one, we introduce *maximal* Selten sets in a similar way as we did for Nash sets.

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