ELSEVIER Journal of Mathematical Economics 8 (1998) 257–273

JOURNAL OF Mathematical ECONOMICS

On the existence of a continuum of constrained equilibria [1]

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Received 1 September 1993; accepted 22 July 1997

Abstract

In this paper a general equilibrium model of an economy with price rigidities and quantity rationing is considered. The equilibria in such a model are called constrained equilibria. It is well-known that there exist two so-called trivial constrained equilibria at which every consumer keeps his initial endowments. In this paper it is shown that there exists a connected set of constrained equilibria containing both these trivial constrained equilibria, hence, there exists a continuum of constrained equilibria. Moreover, it is shown that all known equilibrium existence results for the model discussed follow as easy corollaries from this result. The proof of the result combines theorems in the areas of mathematical programming with those in topology. The result is shown without using differentiability assumptions and is also extended to the case with upper semicontinuous demand correspondences.

Keywords: General equilibrium model; Price rigidities; Quantity rationing; Constrained equilibria

1. Introduction

The equilibria in a general equilibrium model of an economy with price rigidities and quantity rationing are called constrained equilibria. Many con-

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[1] This research is part of the VF-program 'Competition and Cooperation'.

PII: S0304-4068(97)00045-1

strained equilibrium existence results have been given in the past. In Drèze (1975) the existence of a constrained equilibrium without rationing on the market of an a priori chosen commodity has been shown. In van der Laan (1980b), it has been remarked that there exist two trivial constrained equilibria. In one of these trivial constrained equilibria, the prices of all commodities are set equal to their lower bound and supply of all commodities is completely rationed, while the demand of every commodity is not rationed. The second trivial constrained equilibrium is obtained by setting all prices equal to their upper bound, rationing the demand of all commodities completely and not rationing the supply of any commodity. van der Laan (1980b) and, in a somewhat different model, Kurz (1982) showed the existence of nontrivial constrained equilibria without demand rationing, while van der Laan (1982) showed the existence of a nontrivial constrained equilibrium without demand rationing and without supply rationing on at least one market. This constrained equilibrium is called a real unemployment equilibrium or supply constrained equilibrium. Similar results have been obtained in Dehez and Drèze (1984), van der Laan (1984), Weddepohl (1987), and Wu (1988) for models with a different set of admissible prices. In van der Laan and Talman (1990) and Herings (1996) more constrained equilibrium existence results have been given.

In van der Laan (1982) a theorem is given, stating that there is a connected set of constrained equilibria containing the trivial constrained equilibrium with complete supply rationing and a supply constrained equilibrium. The proof is based on properties of points generated by a simplicial algorithm applied to a model with price rigidities and quantity rationing. However, the proof is not complete since it assumes that a certain sequence of connected 1-manifold has a subsequence that converges to some connected 1-manifold. This reasoning is not valid in general. Nevertheless, the basic idea of the proof, the use of properties of the path of points generated by a simplicial algorithm in order to obtain insight into the structure of the set of constrained equilibria, will turn out to be very useful. This idea will be used to show the existence of a connected set of constrained equilibria containing both trivial constrained equilibria. Moreover, it can be shown that this connected set of constrained equilibria contains every type of constrained equilibrium mentioned above. Therefore, the result in van der Laan (1982) is generalized.

In Section 2, the model and the equilibrium concept used is briefly discussed. In Section 3, a combinatorial result is presented that is used to prove the existence of a path of approximate constrained equilibria joining the two trivial constrained equilibria. In Section 4, it is shown that the set of constrained equilibria has a component, which contains the two trivial constrained equilibria. In Section 5, it is proven that the results of Section 4 remain valid if weaker assumptions are made with respect to the economy, guaranteeing only that the total excess demand correspondence is upper semicontinuous instead of being a continuous function. In Section 6, it is shown that all the earlier mentioned constrained equilibrium existence results can be proved by a one line argument using the results of Sections 4 and 5.

2. The model

In the following, for $k \in \mathbb{N}$, define $I_k = \{1,\dots,k\}$, $Q^k = \{q \in \mathbb{R}^k | 0 \le q_j \le 1, \forall j \in I_k\}$, let $\boldsymbol{0}^k$ be a k-dimensional vector of zeros, and let $\boldsymbol{1}^k$ be a k-dimensional vector of ones. If $x, y \in \mathbb{R}^k$, then $x \ge y$ denotes that $x_j \ge y_j$, $\forall j \in I_k$, $x > y$ denotes that $x \ge y$ and there exists $j \in I_k$ such that $x_j > y_j$, and $x \gg y$ denotes that $x_j > y_j$, $\forall j \in I_k$. An exchange economy with price rigidities is denoted by $\mathscr{E} = (\{X^i, \preceq^i, w^i\}_{i=1}^m, P_{(\underline{p},\bar{p})}, (\tilde{l},\tilde{L}))$. There are m consumers indexed by $i = 1,\dots,m$ and n commodities indexed by $j = 1,\dots,n$. A consumer is defined by a consumption set X^i, a preference ordering $\preceq^i$ on X^i, and a vector of initial endowments $\mathbf{w}^i$. The set of admissible price systems is denoted by $P_{(\underline{p},\bar{p})}$, where $P_{(\underline{p},\bar{p})} = \{p \in \mathbb{R}^n | \underline{p} \le p \le \bar{p}\}$. So, $\underline{p}$ is a lower bound and $\bar{p}$ an upper bound for an admissible price system. The set of admissible price systems makes it possible to allow for a minimum price for a commodity $j \in I_n$, $\underline{p}_j$, to allow for a maximum price, $\bar{p}_j$, to describe total inflexibility of the price, $\underline{p}_j = \bar{p}_j$, or to describe a more moderate form of price rigidity, $\underline{p}_j < \bar{p}_j$. For models with other sets of admissible price systems the reader is referred to Kurz (1982), Dehez and Drèze (1984), van der Laan (1984), Weddepohl (1987), and Wu (1988). For these models similar results as in this paper can be obtained.

The equilibrium price system is required to be an element of the set $P_{(\underline{p},\bar{p})}$. Clearly, it is possible that no Walrasian equilibrium price system of the economy belongs to $P_{(\underline{p},\bar{p})}$. When a non-Walrasian price system prevails in the economy, the optimal actions of the consumers at this price system are necessarily incompatible and a description of the market mechanism should include how commodities are allocated in this case. Following Drèze (1975), the information transmitted by the market mechanism is now the price system together with a rationing scheme for each consumer, being the maximal amount a consumer is allowed to make available with regard to his initial endowment of every commodity, called the rationing scheme on supply, and the maximal amount made available to a consumer with respect to his initial endowment of every commodity, called the rationing scheme on demand. For every consumer $i \in I_m$, the constrained budget set of consumer i at a rationing scheme $(l^i, L^i) \in -\mathbb{R}^n_+ \times \mathbb{R}^n_+$ and a price system $p \in P_{(\underline{p},\bar{p})}$ is denoted by $B^i(l^i, L^i, p)$ and is defined by

$$B^i(l^i,L^i,p) = \{x^i \in X^i | p \cdot x^i \le p \cdot w^i \text{ and } l^i \le x^i - w^i \le L^i\}.$$

The restriction of l^i to $-\mathbb{R}^n_+$ and L^i to $\mathbb{R}^n_+$ implies that only cases with voluntary trading are considered. It is useful to define $l = (l^1,\dots,l^m)$, $L = (L^1,\dots,L^m)$, $l_j = (l^1_j,\dots,l^m_j)$, and $L_j = (L^1_j,\dots,L^m_j)$. The constrained demand set of a consumer $i \in I_m$ at a rationing scheme $(l^i, L^i) \in -\mathbb{R}^n_+ \times \mathbb{R}^n_+$ and a price system $p \in P_{(\underline{p},\bar{p})}$ is defined by

$$\delta^i(l^i,L^i,p) = \{\bar{x}^i \in B^i(l^i,L^i,p) | x^i \preceq^i \bar{x}^i, \forall x^i \in B^i(l^i,L^i,p)\}.$$

In general not all rationing schemes are generated by the market mechanism in the economy. Sometimes rationing schemes are required to be uniform for all consumers, sometimes they depend on the amount of initial endowment owned by the various consumers, in other cases they are determined according to some priority system, etc. The function $\tilde{l}$: $Q^n \to -\mathbb{R}^{mn}_{+}$ specifies all admissible rationing schemes on supply and the function $\tilde{L}$: $Q^n \to \mathbb{R}^{mn}_{+}$ all admissible rationing schemes on demand. An equilibrium rationing scheme on supply is required to be an element of the set $\tilde{l}(Q^n)$ and an equilibrium rationing scheme on demand of the set $\tilde{L}(Q^n)$. The pair of functions $(\tilde{l},\tilde{L})$ is called the rationing system.

For every $i \in I_m$, for every $j \in I_n$, component $(i-1)n+j$ of $\tilde{l}$ is denoted by $\tilde{l}^i_j$. Moreover, $\tilde{l}^i = (\tilde{l}^i_1, \dots, \tilde{l}^i_m)$ and $\tilde{l}_j = (\tilde{l}^1_j, \dots, \tilde{l}^m_j)$. Given a parameter $q \in Q^n$, $\tilde{l}^i_j(q)$ yields the rationing scheme on the supply of consumer $i \in I_m$ on market $j \in I_n$. It will be assumed that q_j determines the amount of supply rationing on market j, i.e., $\tilde{l}_j(q)$ does not depend on $q_{j'}$ for any $j' \in I_n \setminus \{j\}$. Similar remarks can be made for the function $\tilde{L}$.

The description of the rationing system is very general and includes special cases as uniform rationing (Drèze, 1975), rationing determined by initial endowments (Kurz, 1982), rationing determined by market shares (Weddepohl, 1983), rationing determined by priority (Weddepohl, 1987), or no constraints on the admissible rationing schemes (Herings, 1996). For instance, assuming that the initial endowments belong to $\mathbb{R}^n_+$ and that the consumption sets are subsets of $\mathbb{R}^n_+$, the uniform rationing system is obtained by defining, for every $i \in I_m$, for every $j \in I_n$,

$$\tilde{l}^i_j(\bar{q}) = -\bar{q}_j\left(\sum_{n \in I_m} w^n_j + \varepsilon\right), \forall \bar{q} \in Q^n,$$

$$\tilde{L}^i_j(\hat{q}) = \hat{q}_j\left(\sum_{n \in I_m} w^n_j + \varepsilon\right), \forall \hat{q} \in Q^n,$$

where ε is an arbitrarily chosen positive number.

With respect to the economy the following assumptions are made.

A1. For every $i \in I_m$, X^i is non-empty, closed, convex, $X^i \subset \mathbb{R}^n_+$, and $X^i + \mathbb{R}^n_+ \subset X^i$.

A2. For every $i \in I_m$, $\preceq^i$ is transitive, complete, continuous, weakly monotonic, and convex.

A3. For every $i \in I_m$, w^i belongs to $\text{Int}(X^i)$.

A4. $\boldsymbol{0}^n \ll \underline{p} \leq \bar{p}$.

A5. The functions $\tilde{l}$: $Q^n \to -\mathbb{R}_+^{mn}$ and $\tilde{L}$: $Q^n \to \mathbb{R}_+^{mn}$ are continuous and satisfy for every $i \in I_m$, $j \in I_n$, and $\bar{q}$, $\hat{q} \in Q^n$,

$$\tilde{l}_j^i(\bar{q}) = \tilde{l}_j^i(\bar{r}) \text{ if } \bar{r} \in Q^n \text{ and } \bar{q}_j = \bar{r}_j^1, \quad \tilde{L}_j^i(\hat{q}) = \tilde{L}_j^i(\hat{r}) \text{ if } \hat{r} \in Q^n \text{ and } \hat{q}_j = \hat{r}_j,$$
$$\tilde{l}_j^i(\bar{q}) = 0 \text{ if } \bar{q}_j = 0, \qquad \tilde{L}_j^i(\hat{q}) = 0 \text{ if } \hat{q}_j = 0,$$
$$\tilde{l}_j^i(\bar{q}) < -w_j^i \text{ if } \bar{q}_j = 1, \qquad \tilde{L}_j^i(\hat{q}) > \sum_{n \in I_m \setminus \{i\}} w_j^n \text{ if } \hat{q}_j = 1.$$

A6. For every $i \in I_m$, $\preceq^i$ is strongly convex.

A preference relation $\preceq^i$ on X^i is said to be weakly monotonic if x^i, $y^i \in X^i$ and $x^i \leq y^i$ implies $x^i \preceq^i y^i$. For x^i, $y^i \in X^i$, let $x^i \prec^i y^i$ be defined as $x^i \preceq^i y^i$ and not $y^i \preceq^i x^i$. A preference relation $\preceq^i$ on X^i is said to be convex if x^i, $y^i \in X^i$ and $x^i \prec^i y^i$ implies $x^i \prec^i \lambda x^i + (1-\lambda) y^i$, $\forall \lambda \in (0,1)$. A preference relation on $\preceq^i$ is said to be strongly convex if x^i, $y^i \in X^i$, $x^i \neq y^i$, and $x^i \preceq^i y^i$ implies $x^i \prec^i \lambda x^i + (1-\lambda) y^i$, $\forall \lambda \in (0,1)$. Assumption **A5** guarantees that the rationing scheme on the market of a commodity $j \in I_n$ is completely determined by component j of $\bar{q}$ and $\hat{q}$. If $\bar{q}_j = 0$ then there is complete supply rationing on the market of commodity j and if $\bar{q}_j = 1$ then there is no supply rationing. If $\hat{q}_j = 0$ then there is complete demand rationing on the market of commodity j and if $\hat{q}_j = 1$ then there will be no demand rationing in an equilibrium of the economy. The continuity of the rationing system guarantees that also all intermediate possibilities are feasible. Notice that the example with the uniform rationing system satisfies Assumption **A5**. Assumption **A6** will be dropped in Sections 5 and 6.

The following definition of a constrained equilibrium is closely related to the one given in Drèze (1975).

Definition 2.1 (Constrained equilibrium): A constrained equilibrium of the economy $\mathscr{E} = (\{X^i, \preceq^i, w^i\}_{i=1}^m, P_{(\underline{p},\bar{p})}, (\tilde{l}, \tilde{L}))$ is an element $(x^{*1}, \ldots, x^{*m}, l^*, L^*, p^*) \in \prod_{i=1}^m X^i \times -\mathbb{R}_+^{mn} \times \mathbb{R}_+^{mn} \times P_{(\underline{p},\bar{p})}$ such that

1. For every $i \in I_m$, $x^{*i} \in \delta^i(l^{*i}, L^{*i}, p^*)$;
2. $\sum_{i=1}^m x^{*i} - \sum_{i=1}^m w^i = \boldsymbol{0}^n$;
3. For every $j \in I_n$, $x_j^{*i'} - w_j^{i'} = L_j^{*i'}$ for some $i' \in I_m$ implies $x_j^{*i} - w_j^i > l_j^{*i}$, $\forall i \in I_m$, and $x_j^{*i'} - w_j^{i'} = l_j^{*i'}$ for some $i' \in I_m$ implies $x_j^{*i} - w_j^i < L_j^{*i}$, $\forall i \in I_m$;
4. For every $j \in I_n$, $p_j^* < \bar{p}_j$ implies $L_j^{*i} > x_j^{*i} - w_j^i$, $\forall i \in I_m$, and $p_j^* > \underline{p}_j$ implies $l_j^{*i} < x_j^{*i} - w_j^i$, $\forall i \in I_m$;
5. $l^* \in \tilde{l}(Q^n)$ and $L^* \in \tilde{L}(Q^n)$.

The first two conditions of a constrained equilibrium are standard in general equilibrium theory. Condition 3 of Definition 2.1 implies that markets are frictionless, i.e., it does not occur on any market that simultaneously a consumer is rationed on his supply while another consumer is rationed on his demand. Condition 4 of Definition 2.1 implies that there is no demand rationing on the market of a commodity $j \in I_n$ if the price of commodity j is not equal to the upper bound on the price of commodity j, and, similarly, supply rationing does not occur on the market of commodity j if the price of commodity j is greater than the lower bound on the price of commodity j.

Let some commodity $j \in I_n$ be given. In principle there are three possible regimes on the market of this commodity in a constrained equilibrium $(x^{*1}, \dots, x^{*m}, l^*, L^*, p^*)$ of the economy $\mathscr{E}$. Either there is a consumer $i' \in I_m$ such that $x_j^{*i'} - w_j^{i'} = L_j^{*i}$, so by Condition 3 of Definition 2.1, $x_j^{*i} - w_j^i < L_j^{*i}$, $\forall i \in I_m$, and by Condition 4 of Definition 2.1, $p_j^* = \underline{p}_j$. Or there is no rationing on the market of commodity j, i.e., $l_j^{*i} < x_j^{*i} - w_j^i < L_j^{*i}$, $\forall i \in I_m$, and, obviously, the price of commodity j is between $\underline{p}_j$ and $\bar{p}_j$. Or there is a consumer $i' \in I_m$ such that $x_j^{*i'} - w_j^{i'} = L_j^{*i'}$, so by Condition 3 of Definition 2.1, $x_j^{*i} - w_j^i < L_j^{*i}$, $\forall i \in I_m$, and by Condition 4 of Definition 2.1, $p_j^* = \bar{p}_j$. Therefore, there is no loss of generality to consider only price systems and rationing schemes which are in the image set of the function $(\hat{l}, \hat{L}, \hat{p})$: $Q^n \to -\mathbb{R}_+^{mn} \times \mathbb{R}_+^{mn} \times P_{(\underline{p}, \bar{p})}$ defined by

$$\hat{p}_j(q) = \max\left\{\underline{p}_j, \min\left\{(2 - 3q_j)\underline{p}_j + (3q_j - 1)\bar{p}_j, \bar{p}_j\right\}\right\}, \forall q \in Q^n, \forall j \in I_n,$$

$$\hat{l}(q) = \tilde{l}(\min\{1^n, 3q\}), \forall q \in Q^n,$$

$$\hat{L}(q) = \tilde{L}(\min\{1^n, 3\,1^n - 3q\}), \forall q \in Q^n,$$

where the minimum of two vectors is taken componentwise. For $\hat{l}$ and $\hat{L}$ the same notational conventions are used as for $\tilde{l}$ and $\tilde{L}$. Notice that, for every $j \in I_n$, if $0 \le q_j \le \frac{1}{3}$ then $\hat{L}_j^i(q) > \sum_{n \in I_m \setminus \{i\}} w_j^n$, $\forall i \in I_m$ and $\hat{p}_j(q) = \underline{p}_j$, if $\frac{1}{3} \le q_j \le \frac{2}{3}$ then $\hat{l}_j^i(q) < -w_j^i$, $\forall i \in I_m$, $\hat{L}_j^i(q) > \sum_{n \in I_m \setminus \{i\}} w_j^n$, $\forall i \in I_m$, and $\underline{p}_j \le \hat{p}_j(q) \le \bar{p}_j$, and if $\frac{2}{3} \le q_j \le 1$ then $\hat{l}_j^i(q) < -w_j^i$, $\forall i \in I_m$ and $\hat{p}_j(q) = \bar{p}_j$.

Since the preference relations are only assumed to be weakly monotonic, it is not guaranteed that a consumer spends his entire budget at every consumption bundle in his demand set. Clearly, this holds in a constrained equilibrium. Therefore, for every $q \in Q^n$, define the set $W(q)$ of vectors in $\mathbb{R}^n$ orthogonal to $\hat{p}(q)$, so $W(q) = \{z \in \mathbb{R}^n | \hat{p}(q) \cdot z = 0\}$, and define the total excess demand correspondence $\hat{\zeta}: Q^n \to \mathbb{R}^n$ by

$$\hat{\zeta}(q) = \left(\sum_{i=1}^{m} \delta^i\left(\hat{l}^i(q), \hat{L}^i(q), \hat{p}(q)\right) - \sum_{i=1}^{m} \{w^i\}\right) \cap W(q), \forall q \in Q^n.$$

Using the results in Herings (1996) the following theorem can be shown.

Theorem 2.2. *Let the economy* $\mathscr{E} = (\{X^i, \preceq^i, w^i\}_{i=1}^m, P_{(\underline{p}, \bar{p})}, (\tilde{l}, \tilde{L}))$ *satisfy the Assumptions A1–A5. Then the correspondence* $\hat{\zeta}: Q^n \to \mathbb{R}^n$ *satisfies the following conditions:*

1. $\hat{\zeta}$ *is a non-empty valued, compact valued, convex valued, upper semicontinuous correspondence;*

2. $\forall q \in Q^n, \forall z \in \hat{\zeta}(q), \forall j \in I_n$, $z_j \geq 0$ if $q_j = 0$;
3. $\forall q \in Q^n, \forall z \in \hat{\zeta}(q), \forall j \in I_n$, $z_j \leq 0$ if $q_j = 1$;
4. $\forall q \in Q^n, \forall z \in \hat{\zeta}(q)$, $\hat{p}(q) \cdot z = 0$.

If in addition Assumption A6 is satisfied, then $\hat{\zeta}$ *is a continuous function.*

A result similar to Theorem 2.2 can be given for δ^i, $\forall i \in I_m$. Whenever $\hat{\zeta}$ is considered as a function instead of a correspondence it will be denoted by $\hat{z}$.

If for some $q \in Q^n$ it holds that $\mathbf{0}^n \in \hat{\zeta}(q)$, then it is easily verified that q induces the constrained equilibrium $(x^{*1}, \ldots, x^{*m}, \hat{l}(q), \hat{L}(q), \hat{p}(q))$, for some $x^{*i} \in \delta^i(\hat{l}^i(q), \hat{L}^i(q), \hat{p}(q))$, $\forall i \in I_m$. Condition 1 of Definition 2.1 is guaranteed by the definition of the correspondences δ^i and $\hat{\zeta}$. Condition 2 is satisfied since $\mathbf{0}^n \in \hat{\zeta}(q)$. Finally, Conditions 3, 4, and 5 are satisfied by the definition of the function $(\hat{l}, \hat{L}, \hat{p})$.

Let $q = \mathbf{0}^n$. By Properties 2 and 4 of Theorem 2.2 it follows immediately that $\hat{\zeta}(\mathbf{0}^n) = \{\mathbf{0}^n\}$. Moreover, $\mathbf{0}^n$ induces the trivial constrained equilibrium with complete supply rationing, $(w^1, \ldots, w^m, \mathbf{0}^{mn}, \hat{L}(\mathbf{0}^n), \underline{p})$. Let $q = \mathbf{1}^n$. By Properties 3 and 4 of Theorem 2.2 it follows that $\hat{\zeta}(\mathbf{1}^n) = \{\mathbf{0}^n\}$. Moreover, $\mathbf{1}^n$ induces the trivial constrained equilibrium with complete demand rationing, $(w^1, \ldots, w^m, \hat{l}(\mathbf{1}^n), \mathbf{0}^{mn}, \bar{p})$. The two equilibria corresponding to $\mathbf{0}^n$ and $\mathbf{1}^n$ are referred to as the trivial constrained equilibria in the sequel.

3. The existence of a path of approximate constrained equilibria

In this section a combinatorial result about the existence of a sequence of adjacent complete simplices is presented. This result will be used to show the existence of a path of approximate zero points of the total excess demand correspondence of the economy $\mathscr{E}$. The path will be shown to join the elements $\mathbf{0}^n$ and $\mathbf{1}^n$, inducing the two trivial constrained equilibria. In this section, it is always assumed that the economy $\mathscr{E}$ satisfies the Assumptions A1–A6. First some preliminaries are given.

A t-simplex in $\mathbb{R}^k$, denoted by σ, is defined as the convex hull of $t+1$ finely independent points of $x^1, \ldots, x^{t+1}$ of $\mathbb{R}^k$, so $\sigma = \mathrm{co}(\{x^1, \ldots, x^{t+1}\})$, and is also denoted by $\sigma(x^1, \ldots, x^{t+1})$. The points $x^1, \ldots, x^{t+1}$ are called the vertices of $\sigma(x^1, \ldots, x^{t+1})$. The barycentre of $\sigma(x^1, \ldots, x^{t+1})$ is given by $\sum_{s \in I_{t+1}} [1/(t+1)] x^s$ and is an element of $\sigma(x^1, \ldots, x^{t+1})$. A $(t-1)$-simplex τ being the convex hull of t vertices of the simplex $\sigma(x^1, \ldots, x^{t+1})$ is called a facet of σ. An s-simplex τ, for some $s \leq t$, is called a face of σ if all the vertices of τ are vertices of σ.

A triangulation of a compact, convex, k-dimensional set S is defined as a finite collection Σ of k-simplices such that

1. $\bigcup_{\sigma \in \Sigma} \sigma = S$;

2. The intersection of two simplices in Σ is either empty or the convex hull of $t \leq k+1$ common vertices.

Let Σ be a triangulation of a non-empty compact, convex set S. It can be shown that if a facet τ of a simplex $\sigma^1 \in \Sigma$ lies in the boundary of S then there is no $\sigma^2 \in \Sigma$ such that $\sigma^2 \neq \sigma^1$ and τ is a facet of σ^2, and if τ does not lie in the boundary of S then there is exactly one $\sigma^2 \in \Sigma$ such that $\sigma^2 \neq \sigma^1$ and τ is a facet of σ^2. The mesh size of the triangulation Σ of S, denoted by mesh (Σ), is given by mesh $(\Sigma) = \max\{\|x-y\|_\infty | \exists \sigma \in \Sigma$ such that $x, y \in \sigma\}$. Let Σ be a triangulation of Q^n. For $J \subset I_n$, define the sets

$$A(J) = \left\{q \in Q^n | q_j = 0, \forall j \in I_n \setminus J\right\},$$

$$\Sigma(J) = \left\{\sigma \cap A(J) | \sigma \in \Sigma \text{ and } \dim(\sigma \cap A(J)) = |J|\right\}.$$

From Theorem 2.3 in Todd (1976), it follows immediately that $\Sigma(J)$ is a triangulation of $A(J)$.

Let a labelling function ϕ: $Q^n \to I_{n+1}$ and a triangulation Σ of Q^n be given. Let J be a non-empty subset of I_{n+1} with $|J| = t$. A $(t-1)$-simplex $\tau(q^1,\dots,q^t)$ being a face of a simplex σ of Σ is J-complete if $\phi(\{q^1,\dots,q^t\}) = J$. A $(t-1)$-simplex is said to be complete if it is J-complete for some non-empty subset J of I_{n+1} with $|J| = t$. The two simplices $\bar{\tau}$ and $\hat{\tau}, \bar{\tau} \neq \hat{\tau}$, are said to be adjacent complete simplices if $\bar{\tau}$ and $\hat{\tau}$ are both J-complete facets of the same t-simplex σ of $\Sigma(J)$ for some non-empty subset J of I_n with $|J| = t$, or if $\bar{\tau}$ is a J-complete facet of the complete t-simplex $\hat{\tau}$ of $\Sigma(J)$ for some non-empty subset J of I_n with $|J| = t$, or if $\hat{\tau}$ is a J-complete facet of the complete t-simplex $\bar{\tau}$ of $\Sigma(J)$ for some non-empty subset J of I_n with $|J| = t$. The labelling function ϕ: $Q^n \to I_{n+1}$ is said to be proper if, for every $q \in Q^n$, for every $j \in I_n$, $q_j = 1$ implies $\phi(q) \neq j$, and $q_j = 0$ implies $\phi(q) \neq n+1$.

Let a proper labelling function ϕ: $Q^n \to I_{n+1}$ and a triangulation Σ of Q^n be given. Then it can be shown by means of a simplicial algorithm, along the lines of van der Laan (1980a, 1982), that there exists a unique finite sequence of adjacent complete simplices of varying dimension starting with the $\{\phi(\mathbf{0}^n)\}$-complete simplex $\{\mathbf{0}^n\}$ and terminating with an I_{n+1}-complete simplex.

Theorem 3.1. *Let a proper labelling function ϕ: $Q^n \to I_{n+1}$ and a triangulation Σ of Q^n be given. Then there exists a unique finite sequence of complete simplices $\tau^1,\dots,\tau^K$ such that $\tau^1 = \{\mathbf{0}^n\}$, τ^K is an I_{n+1}-complete simplex, and any two successive simplices in the finite sequence are adjacent complete simplices.*

For the case $n = 2$, Theorem 3.1 is illustrated in Fig. 1. The finite sequence of simplices starts with the $\{2\}$-complete simplex $\tau^1 = \{\mathbf{0}^n\}$ being a facet of a uniquely determined 1-simplex τ^2 of $\Sigma(\{2\})$. The finite sequence of simplices terminates with the $\{1,2,3\}$-complete simplex $\tau^{11} = \mathrm{co}(\{(\frac{1}{3},\frac{2}{3}),(\frac{2}{3},\frac{2}{3}),(\frac{2}{3},1)\})$ of

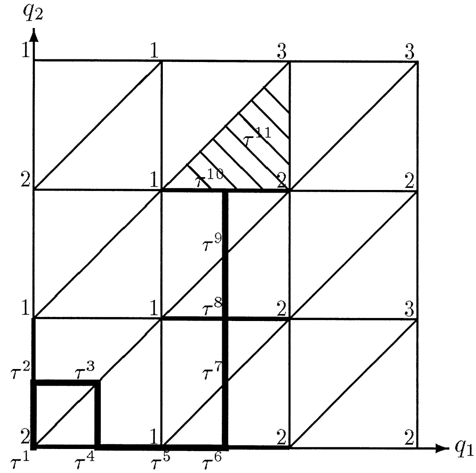

Fig. 1. Illustration of Theorem 3.1, $n=2$.

$\Sigma(\{1,2\})=\Sigma$. After the starting simplex τ^1 three {1,2}-complete simplices being facets of simplices of $\Sigma(\{1,2\})$ are generated. Then the {1}-complete simplex τ^5 and five {1,2}-complete simplices are generated. Notice that the 0-simplex τ^1 and the 1-simplex τ^4 are not adjacent complete simplices. The barycentres of any two adjacent complete simplices in the sequence have been joined by a straight line.

Now a specific labelling function is constructed, based on the total excess demand function $\hat{z}$ of the economy $\mathscr{E}$. For every $q \in Q^n$, define the set $I(q)$ by

$$I(q)=\left\{j^* \in I_n | q_{j^*} \neq 1 \text{ and } \hat{z}_{j^*}(q)=\max_{j \in I_n} \hat{z}_j(q)\right\} \cup \{n+1\},$$

and define the labelling function $\hat{\phi}\colon Q^n \to I_{n+1}$ by $\hat{\phi}(q)=\min I(q),$, $\forall q \in Q^n$. Whenever there will be made a reference to the finite sequence of adjacent complete simplices given in Theorem 3.1 in the sequel, it is assumed that the labelling function $\hat{\phi}$ is used.

Theorem 3.2. *Let the economy $\mathscr{E}$ satisfy Assumptions A1–A6. Then the labelling function $\hat{\phi}\colon Q^n \to I_{n+1}$ is proper.*

Proof. Let some $q \in Q^n$ with $q_j=0$ for some $j \in I_n$ be given. For every $j' \in I_n$ with $q'_j=1$ it holds that $\hat{z}_{j'}(q) \leq 0$ by Property 3 of Theorem 2.2. Hence, $I(q) \neq \{n+1\}$ and therefore, $\hat{\phi}(q) \neq n+1$. Let some $q \in Q^n$ with $q_j=1$ for some $j \in I_n$ be given. Then $j \notin I(q)$ and therefore, $\hat{\phi}(q) \neq j$. Q.E.D.

Now it is proven that $\mathbf{1}^n$ is a vertex of the last adjacent complete simplex of the finite sequence given in Theorem 3.1. This is done by showing that $\mathbf{1}^n$ is the only point of Q^n having label $n+1$.

Theorem 3.3. *Let the economy* $\mathscr{E}$ *satisfy Assumptions A1–A6. Then* $q \in Q^n$ *satisfies* $\hat{\phi}(q) = n+1$ *if and only if* $q = \mathbf{1}^n$.

Proof. Clearly, $I(\mathbf{1}^n) = \{n+1\}$ and hence, $\hat{\phi}(\mathbf{1}^n) = n+1$. Let some $q \in Q^n \setminus \{\mathbf{1}^n\}$ be given. If $q_j = 1$ for some $j \in I_n$, then $\hat{z}_j(q) \leq 0$ by Property 3 of Theorem 2.2. Hence, by Property 4 of Theorem 2.2, $\hat{z}_j(q) \geq 0$ for some $j \in I_n$ for which $q_j < 1$. Therefore, $I(q) \neq \{n+1\}$ and $\hat{\phi}(q) \neq n+1$. Q.E.D.

Let a triangulation Σ of Q^n be given. Then it can be shown that any point of any simplex of Σ containing one of the adjacent complete simplices in the finite sequence given in Theorem 3.1 induces a state of the markets of the economy at which the total excess demand is arbitrarily close to zero if the mesh size of Σ is small enough.

Theorem 3.4. *Let the economy* $\mathscr{E}$ *satisfy Assumptions A1–A6. Let* Σ *be a triangulation of* Q^n. *Then, for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if* mesh$(\Sigma) < \delta$ *and* $\sigma \in \Sigma$ *contains one of the adjacent complete simplices in the finite sequence given in Theorem 3.1, then* $\|\hat{z}(q)\|_\infty < \varepsilon$, $\forall q \in \sigma$.

Proof. Let some $\varepsilon > 0$ be given. Let $\delta > 0$ be such that q^1, $q^2 \in Q^n$ and $\|q^1 - q^2\|_\infty < \delta$ implies $\|\hat{z}(q^1) - \hat{z}(q^2)\|_\infty < [(\min_{j \in I_n} \underline{p}_j)/(\Sigma_{j \in I_n} \bar{p}_j)]\varepsilon$, and let mesh$(\Sigma) < \delta$. Since $\hat{z}$ is a continuous function by Theorem 2.2 and the set Q^n is compact, such a δ exists. Let σ be an n-simplex of Σ containing one of the adjacent complete simplices of the finite sequence given in Theorem 3.1. Then there exists a non-empty subset $\bar{J}$ of I_n and a $\bar{J}$-complete simplex $\bar{\tau} \subset A(\bar{J})$ being a face of σ. For every $q \in \bar{\tau}$, for every $j \in I_n \setminus \bar{J}$, $q_j = 0$ and therefore, $\hat{z}_j(q) \geq 0$ by Property 2 of Theorem 2.2. For every $j' \in \bar{J}$, there exists a vertex q of $\bar{\tau}$ such that $\hat{\phi}(q) = j'$ and it follows that $\hat{z}_{j'}(q) = \max_{j \in I_n} \hat{z}_j(q) \geq 0$. So, for every $j \in I_n$, there exists $q \in \bar{\tau}$ such that $\hat{z}_j(q) \geq 0$.

Let some $q \in \sigma$ be given. Since mesh$(\Sigma) < \delta$, it follows from the previous paragraph that $\hat{z}_{j'}(q) > -[(\min_{j \in I_n} \underline{p}_j)/(\Sigma_{j \in I_n} \bar{p}_j)]\varepsilon \geq -\varepsilon$, $\forall j' \in I_n$. Moreover, it holds that

$$\hat{p}_{j'}(q)\hat{z}_{j'}(q) = -\sum_{j \in I_n \setminus \{j'\}} \hat{p}_j(q)\hat{z}_j(q) < \frac{\min_{j \in I_n} \underline{p}_j}{\sum_{j \in I_n} \bar{p}_j} \varepsilon \sum_{j \in I_n \setminus \{j'\}} \hat{p}_j(q), \forall j' \in I_n,$$

and therefore $\hat{z}_{j'}(q) < [(\min_{j \in I_n} \underline{p}_j)/(\hat{p}_{j'}(q))]\varepsilon \leq \varepsilon$, $\forall j' \in I_n$. Hence, $\|\hat{z}(q)\|_\infty < \varepsilon$. Q.E.D.

Summarizing the results of Theorems 3.1, 3.2, and 3.3, it follows that there exists a unique finite sequence of adjacent complete simplices such that $\{\mathbf{0}^n\}$ is the first simplex and $\mathbf{1}^n$ is a vertex of the last simplex of the sequence. This result combined with Theorem 3.4 is now used to prove the existence of a path of approximate zero points of the total excess demand function, where the path is shown to join the elements $\mathbf{0}^n$ and $\mathbf{1}^n$, i.e., the elements inducing the two trivial constrained equilibria.

Theorem 3.5. *Let the economy $\mathscr{E}$ satisfy Assumptions A1–A6. Then, for every $r \in \mathbb{N}$, there exists a continuous function f^r: $[0,1] \to Q^n$ such that $f^r(0) = \mathbf{0}^n$, $f^r(1) = \mathbf{1}^n$, and $\|\hat{z}(f^r(t))\|_\infty < \frac{1}{r}$, $\forall t \in [0,1]$.*

Proof. Let some $r \in \mathbb{N}$ be given. Let Σ be a triangulation of Q^n with mesh$(\Sigma) < \delta$ where δ is chosen such that Theorem 3.4 holds for $\varepsilon = \frac{1}{r}$. Consider the finite sequence of adjacent complete simplices $\tau^1, \ldots, \tau^K$ given in Theorem 3.1. For every $k \in I_K$, let $q^k \in Q^n$ be defined as the barycentre of τ^k and let q^{K+1} be defined by $q^{K+1} = \mathbf{1}^n$. Clearly, $q^1 = \mathbf{0}^n$. By Theorem 3.3 it holds that $\mathbf{1}^n \in \tau^K$. Moreover, by the definition of adjacent complete simplices it holds that for every $k \in I_K$ there exists $\sigma^k \in \Sigma$ containing q^k and q^{k+1}. For $t \in \mathbb{R}$, define $\lfloor t \rfloor$ as the greatest integer which is less than or equal to t. By the convexity of simplices and by Theorem 3.4, it is easily verified that the function f^r: $[0,1] \to Q^n$, defined by

$$f^r(t) = (1 - Kt + \lfloor Kt \rfloor) q^{\lfloor 1+Kt \rfloor} + (Kt - \lfloor Kt \rfloor) q^{1+\lfloor 1+Kt \rfloor}, \forall t \in [0,1),$$
$$f^r(1) = q^{K+1},$$

satisfies all conditions of the theorem. Q.E.D.

The set E is defined as the set of all elements of Q^n inducing a constrained equilibrium of the economy, so $E = \{q^* \in Q^n | \hat{z}(q^*) = \mathbf{0}^n\}$. Clearly, $\mathbf{0}^n \in E$ and $\mathbf{1}^n \in E$. Moreover, E is a closed set by the continuity of the function $\hat{z}$.

For a non-empty, compact subset S of $\mathbb{R}^n$, define the continuous function d_S: $\mathbb{R}^n \to \mathbb{R}$ by $d_S(x) = \min\{\|x - y\|_\infty | y \in S\}$. The following theorem shows that the elements of Q^n inducing the approximate constrained equilibria given by the image of the function f^r of Theorem 3.5 are uniformly close to the set E if r is large enough.

Theorem 3.6. *Let the economy $\mathscr{E}$ satisfy Assumptions A1–A6. For every $r \in \mathbb{N}$, let f^r: $[0,1] \to Q^n$ be a function such that $\|\hat{z}(f^r(t))\|_\infty < \frac{1}{r}$, $\forall t \in [0,1]$. Then, for every $\varepsilon > 0$, there exists $R \in \mathbb{N}$ such that, for every $r \geq R$, $d_E(f^r(t)) < \varepsilon$, $\forall t \in [0,1]$.*

Proof. Suppose there exists $\varepsilon > 0$ such that for every $r \in \mathbb{N}$ there exists $s^r \geq r$ and $t^r \in [0,1]$ satisfying $d_E(f^{s^r}(t^r)) \geq \varepsilon$. Consider the sequence $(f^{s^r}(t^r))_{r \in \mathbb{N}}$ in

Q^n. Without loss of generality, this sequence can be assumed to converge to some $\bar{q} \in Q^n$. By the continuity of the function d_E it holds that

$$d_E(\bar{q}) = d_E\Big(\lim_{r\to\infty} f^{s^r}(t^r)\Big) = \lim_{r\to\infty} d_E\big(f^{s^r}(t^r)\big) \geq \varepsilon.$$

However, by the continuity of the function $\hat{z}$ it holds that

$$0 \leq \|\hat{z}(\bar{q})\|_\infty = \Big\|\hat{z}\Big(\lim_{r\to\infty} f^{s^r}(t^r)\Big)\Big\|_\infty = \Big\|\lim_{r\to\infty} \hat{z}\big(f^{s^r}(t^r)\big)\Big\|_\infty \leq \lim_{r\to\infty} \frac{1}{s^r} = 0.$$

So, $\bar{q} \in E$ and therefore $d_E(\bar{q}) = 0$, a contradiction. Q.E.D.

4. The existence of a continuum of constrained equilibria

In this section it is always assumed that the economy $\mathscr{E}$ satisfies Assumptions A1–A6. Using the existence of the path of points inducing approximate constrained equilibria shown in the previous section, it will be shown that the set E has a connected subset C such that $\mathbf{0}^n \in C$ and $\mathbf{1}^n \in C$. Before proving the main theorem, a few definitions and properties are given first.

A topological space X is said to be connected if it is not the union of two non-empty, disjoint, closed sets. A subset of a topological space is connected if it becomes a connected space when given the induced topology. The component of a point x in a topological space X is the union of all connected subsets of X containing x. It is easily seen that each component is connected and therefore, the component of an element x is the largest connected subset containing x. The quasi-component of an element x in a topological space X is the intersection of all subsets of X which are both open and closed and contain x. Following Dugundji (1965) it is not difficult to show that the collection of all components partitions X. Similarly, the collection of all quasi-components partitions X. Moreover, it is easily seen that the component of a point is a subset of the quasi-component of this point.

The main argument used in the proof of the result that the set E has a component containing both $\mathbf{0}^n$ and $\mathbf{1}^n$ is that the quasi-component of $\mathbf{0}^n$ in E contains $\mathbf{1}^n$. However, the following example makes clear that the quasi-component of a point in a subset of Q^n is not necessarily connected. It is a modification of Example 115 in Steen and Seebach (1970).

Example 4.1. Let the sets R^0, R^1, and S^r, $\forall r \in \mathbb{N}$, be defined by $R^0 = \{(q_1, q_2) \in Q^2 | q_1 = 0\}$, $R^1 = \{(q_1, q_2) \in Q^2 | q_1 = 1\}$, and $S^r = \{(q^1, q^2) \in Q^2 | \|(q_1, q_2) - (\frac{1}{2}, \frac{1}{2})\|_\infty = r/[2(r+1)], \forall r \in \mathbb{N}$. For every $r \in \mathbb{N}$, S^r is a square with center $(\frac{1}{2}, \frac{1}{2})$ and diameter $r/(r+1)$. Consider the set $T = R^0 \cup R^1 \cup (\cup_{r\in\mathbb{N}} S^r)$. Give T the topology induced by the topology of the Euclidean space $\mathbb{R}^2$. Since, for every $k \in \mathbb{N}$, the set $\bigcup_{r=1}^k S^r$ is open and closed in T, it holds that the quasi-component of (0,0) is a subset of $R^0 \cup R^1$. Clearly, R^0 is the component of (0,0). Hence, R^0 is a subset of the quasi-component of (0,0). Since every open and closed set in T containing R^0 has to contain R^1, it has to hold that the quasi-component of (0,0) is equal to $R^0 \cup R^1$.

Since $\hat{z}$ is a continuous function, the set E is closed. Therefore, it cannot be equal to a set like T considered in Example 4.1. The following theorem, stated in page 235 of Munkres (1975), gives sufficient conditions guaranteeing that the component and the quasi-component of a point coincide.

Theorem 4.2. *Let X be a compact subset of the Euclidean space $\mathbb{R}^n$. Then the component and the quasi-component of each point of the set X coincide.*

It will be useful to define, for non-empty, compact subsets S, T of $\mathbb{R}^n$, the number $e(S,T)$ by $e(S,T) = \min\{\|x-y\|_\infty | x \in S,\ y \in T\}$. It follows immediately that $e(S,T)$ is well-defined. Clearly, if S and T are disjoint, then $e(S,T) > 0$. The next theorem finally gives the desired result.

Theorem 4.3. *Let the economy $\mathscr{E}$ satisfy Assumptions A1–A6. Then the set E contains a component C such that $\mathbf{0}^n \in C$ and $\mathbf{1}^n \in C$.*

Proof. Suppose that the component of $\mathbf{0}^n$ in E does not contain $\mathbf{1}^n$. Then, by Theorem 4.2, the quasi-component of $\mathbf{0}^n$ in E does not contain $\mathbf{1}^n$. Hence, there exists a set E^0 being both open and closed in E, containing $\mathbf{0}^n$, but not containing $\mathbf{1}^n$. Let the set E^1 be defined by $E^1 = E \setminus E^0$. Clearly, E^1 is both open and closed in E and contains $\mathbf{1}^n$. Since E is a closed subset of $\mathbb{R}^n$, it follows that E^0 and E^1 are compact. Moreover, there exists $\varepsilon > 0$ such that $e(E^0, E^1) > \varepsilon$. By Theorems 3.5 and 3.6 there exists a continuous function f: $[0,1] \to Q^n$ such that $d_E(f(t)) < \frac{1}{2}\varepsilon$, $\forall t \in [0,1]$, while $f(0) = \mathbf{0}^n$ and $f(1) = \mathbf{1}^n$. Let the function g: $[0,1] \to \mathbb{R}$ be defined by $g(t) = d_{E^0}(f(t)) - d_{E^1}(f(t))$, $\forall t \in [0,1]$. From the continuity of the functions f, d_{E^0}, and d_{E^1} it follows that the function g is continuous. Moreover, $g(0) < -\varepsilon$ and $g(1) > \varepsilon$. Hence, there exists $\bar{t} \in [0,1]$ such that $g(\bar{t}) = 0$, and therefore, $d_{E^0}(f(\bar{t})) = d_{E^1}(f(\bar{t})) = d_E(f(\bar{t})) < \frac{1}{2}\varepsilon$. Consequently, there exists $q^0 \in E^0$ and $q^1 \in E^1$ such that $\|f(\bar{t}) - q^0\|_\infty < \frac{1}{2}\varepsilon$ and $\|f(\bar{t}) - q^1\|_\infty < \frac{1}{2}\varepsilon$. Hence,

$$\varepsilon < e(E^0, E^1) \le \|q^0 - q^1\|_\infty \le \|f(\bar{t}) - q^0\|_\infty + \|f(\bar{t}) - q^1\|_\infty < \varepsilon,$$

a contradiction. Q.E.D.

Define the function ψ: $Q^n \to \mathbb{R}^{3mn+n}$ by

$$\psi(q) = \Big(\delta^1\big(\hat{l}^1(q), \hat{L}^1(q), \hat{p}(q)\big), \ldots, \delta^m\big(\hat{l}^m(q), \hat{L}^m(q), \hat{p}(q)\big), \hat{l}(q),$$
$$\hat{L}(q), \hat{p}(q)\Big), \forall q \in Q^n.$$

It can easily be shown that the function ψ is continuous if the economy $\mathscr{E}$ satisfies the Assumptions A1–A6. Moreover, $\psi(E)$ is the set of constrained equilibria of the economy $\mathscr{E}$. Since the image of a connected set under a continuous function is connected, the following corollary follows immediately.

Corollary 4.4. *Let the economy $\mathscr{E}$ satisfy Assumptions A1–A6. Then there exists a connected set of constrained equilibria of the economy $\mathscr{E}$, containing the trivial constrained equilibria $(w^1,\ldots,w^m, \mathbf{0}^{mn}, \hat{L}(\mathbf{0}^n),\underline{p})$ and $(w^1,\ldots,w^m, \hat{l}(\mathbf{1}^n), \mathbf{0}^{mn}, \bar{p})$.*

5. The case of upper semicontinuous correspondences

In this section it is always assumed that the economy $\mathscr{E}$ satisfies Assumptions A1–A5. Theorem 4.3 and Corollary 4.4 will be shown to hold also under these assumptions. It holds by Theorem 2.2 that $\hat{\zeta}$ is an upper semicontinuous correspondence, satisfying that for every $q \in Q^n$ the set $\hat{\zeta}(q)$ is non-empty, convex, and compact. The desired results will be shown by approximating this upper semicontinuous correspondence by a sequence of continuous functions satisfying the properties of the previous section. To do so, the notion of a piecewise linear approximation of a correspondence is useful. Let a triangulation Σ of Q^n and a correspondence φ: $Q^n \to \mathbb{R}^n$ be given. A function F: $Q^n \to \mathbb{R}^n$ is called a piecewise linear approximation of φ with respect to Σ if for every vertex x of any $\sigma \in \Sigma$ it holds that $F(x) \in \varphi(x)$ and for every element x of S it holds that $F(x) = \Sigma_{k \in I_{n+1}} \lambda^k F(x^k)$, when $x \in \sigma(x^1,\ldots,x^{n+1})$ for some n-simplex $\sigma \in \Sigma$ and $x = \Sigma_{k \in I_{n+1}} \lambda^k x^k$ for some $\lambda^k \in \mathbb{R}_+$, $\forall k \in I_{n+1}$, with $\Sigma_{k \in I_{n+1}} \lambda^k = 1$. A piecewise linear approximation F of a correspondence with respect to a triangulation Σ is uniquely determined when the images by F of all the vertices in the triangulation are specified.

Let the set E be defined by $E = \{q \in Q^n | \mathbf{0}^n \in \hat{\zeta}(q)\}$. Again, E is the set of all elements of Q^n inducing a constrained equilibrium. Clearly, $\mathbf{0}^n \in E$ and $\mathbf{1}^n \in E$.

Theorem 5.1. *Let the economy $\mathscr{E}$ satisfy Assumptions A1–A5. Then the set E contains a component C such that $\mathbf{0}^n \in C$ and $\mathbf{1}^n \in C$.*

Proof. Since $\hat{\zeta}$ is an upper semicontinuous correspondence, it follows that E is closed in Q^n and hence compact. Suppose that the component of $\mathbf{0}^n$ in E does not contain $\mathbf{1}^n$. Then, by Theorem 4.2, the quasi-component of $\mathbf{0}^n$ in E does not contain $\mathbf{1}^n$. Hence, there exist sets E^0 and E^1 both being closed in E and satisfying $E^0 \cap E^1 = \emptyset$, $E^0 \cup E^1 = E$, $\mathbf{0}^n \in E^0$, and $\mathbf{1}^n \in E^1$. Hence, $e(E^0, E^1) > \varepsilon$ for some $\varepsilon > 0$. Let the correspondence $\bar{\zeta}$: $Q^n \to \mathbb{R}^n$ be defined by

$$\bar{\zeta}(q) = \left\{\bar{z} \in \mathbb{R}^n | \exists \hat{z} \in \hat{\zeta}(q), \bar{z}_j = \hat{p}_j(q)\hat{z}_j, \forall j \in I_n\right\}, \forall q \in Q^n.$$

For every $r \in \mathbb{N}$, let Σ^r be a triangulation of Q^n with mesh $(\Sigma^r) \le \frac{1}{r}$ and let the function Z^r: $Q^n \to \mathbb{R}^n$ be a piecewise linear approximation of $\bar{\zeta}$ with respect to

Σ^r. For every $r \in \mathbb{N}$, it follows that Z^r is a continuous function and, for every $q \in Q^n$, $Z_j^r(q) \geq 0$ if $q_j = 0$ for some $j \in I_n$, $Z_j^r(q) \leq 0$ if $q_j = 1$ for some $j \in I_n$, and $\mathbf{1}^n \cdot Z^r(q) = 0$. So, by Theorem 4.3, for every $r \in \mathbb{N}$ there exists a connected set C^r containing 0^n and 1^n and satisfying that $Z^r(q) = 0^n$. For every $r \in \mathbb{N}$, let the function g^r: $C^r \to \mathbb{R}$ be defined by $g^r(q) = d_{E^0}(q) - d_{E^1}(q)$, $\forall q \in C^r$. Clearly, for every $r \in \mathbb{N}$, the function g^r is continuous. Moreover, $g^r(\mathbf{0}^n) < -\varepsilon$ and $g^r(\mathbf{1}^n) > \varepsilon$, so there exists $\bar{q}^r \in C^r$ such that $g^r(\bar{q}^r) = 0$. Obviously, $d_{E^0}(\bar{q}^r) = d_{E^1}(\bar{q}^r) = d_E(\bar{q}^r) > \frac{1}{2}\varepsilon$. For every $r \in \mathbb{N}$, let $(\lambda^k)^r \geqslant 0$, $\forall k \in I_{n+1}$, and let $\sigma^r \in \Sigma^r$ with vertices $(q^k)^r$, $\forall k \in I_{n+1}$, be such that $\bar{q}^r = \Sigma_{k \in I_{n+1}}(\lambda^k)^r(q^k)^r$ and $\Sigma_{k \in I_{n+1}}(\lambda^k)^r = 1$. Moreover, let $(\hat{z}^k)^r \in \hat{\zeta}((q^k)^r)$ be such that $Z_j^r((q^k)^r) = \hat{p}_j((q^k)^r)(\hat{z}_j^k)^r$, $\forall j \in I_n$. Since $\hat{\zeta}$ is a compact valued, upper semicontinuous correspondence and Q^n is compact, the sequence $((\lambda^1)^r, \ldots, (\lambda^{n+1})^r, (\hat{z}^1)^r, \ldots, (\hat{z}^{n+1})^r, \bar{q}^r)_{r \in \mathbb{N}}$ in $\prod_{k \in I_{n+1}}[0,1] \times \prod_{k \in I_{n+1}}\mathbb{R}^n \times Q^n$ can, without loss of generality, be assumed to converge to an element $(\bar{\lambda}^1, \ldots, \bar{\lambda}^{n+1}, \bar{z}^1, \ldots, \bar{z}^{n+1}, \bar{q}) \in \prod_{k \in I_{n+1}}[0,1] \times \prod_{k \in I_{n+1}}\mathbb{R}^n \times Q^n$. Since mesh $(\Sigma^r) \leq \frac{1}{r}$, it holds that $(q^k)^r \to \bar{q}$, $\forall k \in I_{n+1}$. Since $\hat{\zeta}$ is a compact valued, upper semicontinuous correspondence it holds that $\bar{z}^k \in \hat{\zeta}(\bar{q}), \forall k \in I_{n+1}$. It also holds that $\hat{\zeta}$ is a convex valued correspondence, so $\Sigma_{k \in I_{n+1}}\bar{\lambda}^k \bar{z}^k \in \hat{\zeta}(\bar{q})$. Moreover, since $Z^r(\bar{q}^r) = \mathbf{0}^n$, it follows that, for every $j \in I_n$,

$$0 = \lim_{r \to +\infty} Z_j^r(\bar{q}^r) = \lim_{r \to +\infty} \sum_{k \in I_{n+1}} (\lambda^k)^r \hat{p}_j\left((q^k)^r\right)\left(\hat{z}_j^k\right)^r = \sum_{k \in I_{n+1}} \bar{\lambda}^k \hat{p}_j(\bar{q})\bar{z}_j^k$$

$$= \hat{p}_j(\bar{q}) \sum_{k \in I_{n+1}} \bar{\lambda}^k \bar{z}_j^k.$$

Therefore, $\Sigma_{k \in I_{n+1}}\bar{\lambda}^k \bar{z}^k = \mathbf{0}^n$, so $\mathbf{0}^n \in \hat{\zeta}(\bar{q})$ and $d_E(\bar{q}) = 0$. Consequently, it follows that $0 = d_E(\bar{q}) = d_E(\lim_{r \to +\infty}\bar{q}^r) = \lim_{r \to +\infty} d_E(\bar{q}^r) \geq \frac{1}{2}\varepsilon$ a contradiction. Q.E.D.

Since the image of a connected set by a non-empty valued, compact valued, convex valued, upper semicontinuous correspondence is connected, Corollary 5.2 is obtained in a similar way as Corollary 4.4.

Corollary 5.2. *Let the economy $\mathscr{E}$ satisfy Assumptions A1–A5. Then there exists a connected set of constrained equilibria of the economy $\mathscr{E}$, containing the trivial constrained equilibria $(w^1, \ldots, w^m, \mathbf{0}^{mn}, \hat{L}(\mathbf{0}^n), \underline{p})$ and $(w^1, \ldots, w^m, \hat{l}(\mathbf{1}^n), \mathbf{0}^{mn}, \bar{p})$.*

6. Constrained equilibrium existence results

In this section it is assumed that the economy $\mathscr{E}$ satisfies Assumptions A1–A5.

A Drèze equilibrium with respect to a commodity $j \in I_n$ of the economy is defined as a constrained equilibrium $(x^{*1}, \ldots, x^{*m}, l^*, L^*, p^*)$ of $\mathscr{E}$ satisfying $l_j^{*i} < x_j^{*i} - w_j^i < L_j^{*i}$, $\forall i \in I_m$. The existence of a Drèze equilibrium with respect to commodity $j \in I_n$ is shown in Drèze (1975). It is easily seen that the existence of a Drèze equilibrium with respect to commodity $j \in I_n$ is equivalent to the set $E \cap \{q \in Q^n | \frac{1}{3} \le q_j \le \frac{2}{3}\}$ being non-empty. In van der Laan and Talman (1990) it is shown that for an economy with uniform rationing schemes it holds that, for every $j \in I_n$, for every $\alpha \in [0, 1]$,

$$E \cap \left\{q \in Q^n | q_j = \alpha\right\} \neq \emptyset \tag{1}$$

A supply constrained equilibrium of the economy $\mathscr{E}$ is defined as a constrained equilibrium $(x^{*1},\dots,x^{*m}, l^*, L^*, p^*)$ of $\mathscr{E}$ satisfying $x_j^{*i} - w_j^i < L_j^{*i}$, $\forall i \in I_n$, and there exists $j' \in I_n$ such that $l_{j'}^{*i} < x_{j'}^{*i} - w_{j'}^i < L_{j'}^{*i}$, $\forall i \in I_m$. For the motivation of this definition, see van der Laan (1980b, 1982) and Kurz (1982). It is easily seen that the existence of a supply constrained equilibrium is equivalent to the set $E \cap \{q \in Q^n | \frac{1}{3} \le \max_{j \in I_n} q_j \le \frac{2}{3}$ being non-empty. In Herings (1996) it is shown that, for every $\overline{\alpha} \in Q^n$,

$$E \cap \left\{q \in Q^n | q \le \overline{\alpha}, \exists j \in I_n, q_j = \overline{\alpha}_j\right\} \neq \emptyset. \tag{2}$$

Moreover, it is shown in Herings (1996) that, for every $\underline{\alpha} \in Q^n$,

$$E \cap \left\{q \in Q^n | q \ge \underline{\alpha}, \exists j \in I_n, q_j = \underline{\alpha}_j\right\} \neq \emptyset. \tag{3}$$

Using Theorem 5.1 the existence of each of the constrained equilibria mentioned above is very easily shown. The existence results follow as easy corollaries to Theorem 6.1.

Theorem 6.1. *Let the economy $\mathscr{E}$ satisfy Assumptions A1–A5. Let f: $Q^n \to \mathbb{R}$ be a continuous function satisfying $f(\boldsymbol{0}^n) \le 0$ and $f(\boldsymbol{1}^n) \ge 0$. Then $E \cap f^{-1}(\{0\}) \neq \emptyset$.*

Proof. By Theorem 5.1, E has a component C such that $\boldsymbol{0}^n \in C$ and $\boldsymbol{1}^n \in C$. Since C is connected and f is a continuous function, it follows that $f(C)$ is connected. Since $\boldsymbol{0}^n \in C$, $\boldsymbol{1}^n \in C$, $f(\boldsymbol{0}^n) \le 0$, and $f(\boldsymbol{1}^n) \ge 0$, it holds that $0 \in f(C) \subset f(E)$.

Let some $j \in I_n$ and $\alpha \in [0,1]$ be given. Then Eq. (1) is obtained by defining f: $Q^n \to \mathbb{R}$ by $f(q) = q_j - \alpha$, $\forall q \in Q^n$, and applying Theorem 6.1. Let some $\overline{\alpha} \in Q^n$ be given. Then Eq. (2) is obtained by defining f: $Q^n \to \mathbb{R}$ by $f(q) = \max\{q_j - \overline{\alpha}_j | j \in I_n\}$, $\forall q \in Q^n$, and applying Theorem 6.1. Finally, let some $\underline{\alpha} \in Q^n$ be given. Then Eq. (3) is obtained by defining f: $Q^n \to \mathbb{R}$ by $f(q) = \min\{q_j - \underline{\alpha}_j | j \in I_n\}$, $\forall q \in Q^n$, and applying Theorem 6.1. The proof of Theorem 6.1 does not only show the existence of each one of the above-mentioned constrained equilibria, but also shows that the component of the set of constrained equilibria of the economy containing the two trivial constrained equilibria contains every constrained equilibrium with properties as given in Eqs. (1)–(3).

Acknowledgements

The author would like to thank Dolf Talman, Jan van Geldrop, and an anonymous referee for their valuable comments on previous drafts of this paper. The author is financially supported by the Cooperation Centre Tilburg and Eindhoven Universities, The Netherlands.

References

Dehez, P., Drèze, J.H., 1984. On supply-constrained equilibria. J. Econ. Theory 33, 172–182.
Drèze, J.H., 1975. Existence of an exchange equilibrium under price rigidities. Int. Econ. Rev. 16, 301–320.
Dugundji, J., 1965. Topology. Allyn Bacon, Boston.
Herings, P.J.J., 1996. Equilibrium existence results for economics with price rigidities. Econ. Theory 7, 63–80.
Kurz, M., 1982. Unemployment equilibrium in an economy with linked prices. J. Econ. Theory 26, 100–123.
Munkres, J.R., 1975. Topology. A First Course. Prentice-Hall, Englewood Cliffs, NJ.
Steen, L.A., Seebach Jr., J.A., 1970. Counter Examples in Topology. Holt, Rinehart, Winston, New York.
Todd, M.J., 1976. The computation of fixed points and applications. Lecture Notes in Economics and Mathematical Systems 124. Springer-Verlag, Berlin.
van der Laan, G., 1980a. Simplicial fixed point algorithms. Mathematical Centre Tracts 129. Mathematisch Centrum, Amsterdam.
van der Laan, G., 1980b. Equilibrium under rigid prices with compensation for the consumers. Int. Econ. Rev. 21, 63–73.
van der Laan, G., 1982. Simplicial approximation of unemployment equilibria. J. Math. Econ. 9, 83–97.
van der Laan, G., 1984. Supply-constrained fixed price equilibria in monetary economies. J. Math. Econ. 13, 171–187.
van der Laan, G., Talman, A.J.J., 1990. Price rigidities and rationing. In: Van der Ploeg, F. (Ed.), Advanced Lectures in Quantitative Economics. Academic Press, London, pp. 123–168.
Weddepohl, C., 1983. Fixed price equilibria in a multifirm model. J. Econ. Theory 29, 95–108.
Weddepohl, C., 1987. Supply-constrained equilibria in economies with indexed prices. J. Econ. Theory 43, 203–222.
Wu, H.-M., 1988. Unemployment equilibrium in a random economy. J. Math. Econ. 17, 385–400.