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A new proof of the index formula for incomplete markets

Arkadi Predtetchinski*

Maastricht University, Faculty of Economics and Business Administration, Tongersestraat 53, 6200 MD Maastricht, Limburg, Netherlands

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Abstract

This paper gives a new proof of the index formula established by [Momi, T., 2003. The index theorem for a GEI economy when the degree of incompleteness is even. Journal of Mathematical Economics 39, 273–297] for an economy with incomplete asset markets where the difference between the number of states (S) and the number of assets (J) is an even number. The proof uses a single globally defined homotopy function on the asset pseudo-equilibrium manifold connecting the excess demand of a given economy to the individual excess demand of the unconstrained agent. We show that the asset pseudo-equilibrium manifold is orientable if the number $S - J$ is even and deduce the index formula from the homotopy invariance theorem for the degree of a map.

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1. Introduction

The index theorem was first introduced into economics by Dierker (1972). The theorem states that the indices of the individual equilibria in a regular Arrow–Debreu economy add up to $+1$. Recently Momi (2003) has proved the index formula for asset market economies where the difference between the number of states (S) and the number of available assets (J) is an even number.

* Tel.: +31 43 388 3906; fax: +31 43 388 4878.
E-mail address: A.Predtetchinski@algec.unimaas.nl (A. Predtetchinski).

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The essential difficulty involved in the proof of the index formula for incomplete markets is the discontinuity of the excess demand function, caused by changes in the rank of the asset return matrix, as prices vary. While for the Arrow–Debreu economy the index formula is implied by the fact that the aggregate excess demand function is homotopic to the individual excess demand function of a single agent via a proper homotopy map, Momi (2003) has to rely on the system of switching homotopies introduced by Brown et al. (1996) and on the system of local homotopy functions of Demarzo and Eaves (1996).

This paper presents a new proof of the result by Momi (2003). We introduce a single globally defined homotopy function between the excess demand of a given economy and the excess demand of an unconstrained agent. The domain of the homotopy function is the so-called asset pseudo-equilibrium manifold as introduced in Zhou (1997). Assuming that the number $S-J$ is even, we prove the asset pseudo-equilibrium manifold to be orientable and derive the index formula from the homotopy-invariance property of the degree of a map.

The rest of the paper is organized as follows. In Section 2 the economy with incomplete asset markets is presented. In Section 3 the index formula is stated and some motivation for the new proof of this result is provided. Section 4 discusses the mathematical concepts used in the proof of the index formula. In Section 5 we show the asset pseudo-equilibrium manifold to be an orientable manifold, provided that the number $S-J$ is even. Section 6 completes the proof of the index formula.

2. The economy

We consider two-period economies with uncertainty represented by a finite set $\{1,\ldots,S\}$ of states of nature. There are L goods in period 0 and L goods in each state of nature in period 1. The total number of time and state-contingent commodities in the economy is therefore $M=(S+1)L$. There are I agents in the economy, agent i characterized by a utility function $u^i:\mathbb{R}^M_{++}\to\mathbb{R}$ and a vector of initial endowments $e^i\in\mathbb{R}^M_{++}$. In addition, there are J assets in the economy characterized by an $(SL\times J)$-dimensional matrix A of payoffs. The entry a^j_{sl} of the matrix A specifies the amount of commodity l paid by asset j in the state of nature s.

We impose the following assumptions.

(A1). Functions u^i are twice continuously differentiable.

(A2). For each $x^i\in\mathbb{R}^M_{++}$ the vector $\mathrm{d}u^i(x^i)$ of partial derivatives of u^i at x^i belongs to $\mathbb{R}^M_{++}$.

(A3). For each $\bar{x}^i\in\mathbb{R}^M_{++}$ the closure of the set $\{x^i\in\mathbb{R}^M_{++}|u^i(x^i)\geq u^i(\bar{x}^i)\}$ is contained in $\mathbb{R}^M_{++}$.

(A4). If $x^i\in\mathbb{R}^M_{++}$ and $h\in\mathbb{R}^M\backslash\{0\}$ are such that $\mathrm{d}u^i(x^i)h=0$ then $h^\top \mathrm{d}^2u^i(x^i)h<0$.

Assumptions (A1)–(A4) are standard in the theory of incomplete markets. In particular, this set of assumptions is employed in Duffie and Shafer (1985) to demonstrate generic existence of GEI-equilibrium.

We introduce some notation. Given natural numbers N and K we write $M(N,K)$ to denote the set of all $(N\times K)$-dimensional matrices. The symbol $M_j(N,K)$ denotes the subset of $(N\times K)$-matrices having rank j. For $D\in M(N,K)$ we write span$D$ to denote the linear space spanned by the columns of D and D^k to denote the k^{th} column of the matrix D. Given two M-dimensional vectors $p=(p_{sl})$ and $z=(p_{sl})$ where $s=0,1,\ldots,S$ and $l=1,\ldots,L$, the symbol $p\square z$ denotes an S-dimensional vector with components $\sum_{l=1}^{L}p_{sl}z_{sl}$ for $s=1,\ldots,S$. Given a matrix $A\in M(SL,J)$

of asset payoffs and a vector $p \in \mathbb{R}^M$ of prices we write $V_A(p)$ to denote the $(S \times J)$-matrix of asset payoffs in units of account. That is, $V_A^j(p) = p\Box A^j$ for all $j = 1, \dots, J$. Given a vector $z \in \mathbb{R}^M$ we write $\dot{z}$ to denote an $[M-1]$-vector obtained from z by deleting the last component, corresponding to the commodity L in state S. Finally, we let $Y = \mathbb{R}^{M-1}$.

We parameterize the economies by the initial allocation $e = (e^1, \dots, e^I)$ and the matrix of asset returns A. The space of economies can therefore be identified with the set $\Omega = \mathbb{R}^{MI}_{++} \times M(SL, J)$.

3. The index formula

We proceed by defining the excess demand function of an economy $\omega = (e, A) \in \Omega$. We rely on the formulation of the excess demand function corresponding to the concept of no-arbitrage equilibrium and use Cass trick to ensure the properness of the excess demand functions. Let P be the set of prices $p \in \mathbb{R}^M_{++}$ such that $p_M = 1$. Given a vector $p \in P$ and a linear subspace L of $\mathbb{R}^S$ define the individual excess demand functions as

$$z^1_\omega(p) = \arg\max \left\{ u^1(e^1 + z^1) | pz^1 = 0 \right\}$$

$$z^i_\omega(p, L) = \arg\max \left\{ u^i(e^i + z^i) \,\middle|\, \begin{array}{c} pz^i = 0 \\ p\Box z^i \in L \end{array} \right\},$$

where $2 \le i \le I$. The aggregate excess demand at prices p is given by the vector

$$z_\omega(p) = z^1_\omega(p) + \sum_{i=2}^{I} z^i_\omega(p, \operatorname{span} V_A(p)).$$

The set

$$E(\omega) = \{p \in P | z_\omega(p) = 0\}$$

is the set of no-arbitrage equilibrium prices of the economy ω.

It is well-known that the excess demand function z_ω may be discontinuous at prices p where the rank of the matrix $V_A(p)$ is less than J. Whenever p is such that the matrix $V_A(p)$ has full column rank J, however, the function z_ω can be shown to be continuous, and, under the maintained assumptions, continuously differentiable at point p. Theorem 1 below is a well-known result. We therefore state the theorem without a proof.

Theorem 1. *There exists a subset Ω^* of the set Ω with a complement of Lebesgue measure zero such that for each $\omega = (e, A) \in \Omega^*$ the set $E(\omega)$ is finite and for each $p \in E(\omega)$ the matrix $V_A(p)$ has rank J. It follows that the function z_ω is continuously differentiable around each of its zeros.*

Given an economy $\omega \in \Omega^*$ we define the index of the equilibrium $p \in E(\omega)$ as

$$\operatorname{index}_\omega(p) = \operatorname{sign} \det \left[-\frac{\partial z_{\omega m}(p)}{\partial p_{m'}} \right],$$

where m and m' vary within the index set $\{1, \dots, M-1\}$. The index theorem as stated below is due to Momi (2003).

Theorem 2 (The index theorem). *Suppose that $S > J$ and that the number $S - J$ is even. Then for each $\omega \in \Omega^*$ the index formula holds:*

$$\sum_{p \in E(\omega)} \operatorname{index}_\omega(p) = 1.$$

The index formula can be restated as saying that the degree of the function $\dot{z}_\omega$ at $0 \in Y$ equals $+1$ or -1 according to whether the number M is odd or even.

The major difficulty involved in the proof of Theorem 2 is the discontinuity of the excess demand. The "naive" approach to the proof of Theorem 2 would be to restrict the domain of $\dot{z}_\omega$ to prices p such that the matrix $V_A(p)$ has rank J, and to connect it to the individual excess demand of agent 1 via a straight-line homotopy. The problem with this approach is that the zero set of the naive homotopy is not, in general, a compact set as it hits the set of prices where the matrix $V_A(p)$ has rank less than J. As is argued in Brown et al. (1996), the compactness property cannot be restored by perturbing the parameters of the economy.

The original proof of Momi (2003) relies on the system of switching homotopies constructed in Brown et al. (1996) and on the system of local homotopy functions introduced in Demarzo and Eaves (1996). The author shows that passing through a point of discontinuity does not affect the index of the homotopy, provided that the number $S - J$ is even.

This paper presents a different approach to the proof of Theorem 2. As a domain of functions z^i_ω, we consider the combinations of prices p and J-dimensional linear subspaces L of $\mathbb{R}^S$ such that the columns of the return matrix $V_A(p)$ span a subspace of L. This set of prices and linear spaces is known as the asset pseudo-equilibrium manifold, see Zhou (1997). We show asset pseudo-equilibrium manifold to be an orientable manifold, provided that the number $S - J$ is even. Furthermore, the excess demand function of a given economy is continuous everywhere on the asset pseudo-equilibrium manifold and it is connected to the individual excess demand function of the unconstrained agent by a proper homotopy. We are thus able to derive Theorem 2 from the homotopy-invariance theorem for the degree of a map.

4. Oriented manifolds and the degree of a map

Oriented manifolds and the degree of a map are the key tools used in the proof of Theorem 2. For a comprehensive discussion of these concepts we refer to Dold (1980) and Hirsch (1976).

An n-dimensional manifold is a Hausdorff space X which has an open cover $\{U_\alpha\}$ such that for each α there exists a homeomorphism $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$. The pair $(U_\alpha, \varphi_\alpha)$ is called a chart on X, and the collection $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ of all such charts is called an atlas on X. If for any pair of charts (U_β, φ_β) and $(U_\alpha, \varphi_\alpha)$ the transition map $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \bigcap U_\beta) \to \varphi_\beta(U_\alpha \bigcap U_\beta)$ is smooth, the atlas $\mathcal{A}$ is called smooth. If all these transition maps preserve orientation, then the atlas is called oriented. The manifold X is called smooth if it has a smooth atlas. It is called orientable if it has an oriented atlas. An orientable manifold together with an oriented atlas is called an oriented manifold.

Theorem 3 (The preimage theorem). *Let X be a smooth n-dimensional manifold and $f : X \to \mathbb{R}^m$ be a smooth function. Suppose that zero is a regular value of f. Then $f^{-1}(0)$ is an $[n - m]$-dimensional manifold. If X is orientable, so is $f^{-1}(0)$.*

Let $\mathcal{C}$ be the set of tuples (X, Y, f, y), where X and Y are oriented manifolds of the same dimension, $f : X \to Y$ is a continuous map, y is a point of Y, and the set $f^{-1}(y)$ is compact. Any tuple (X, Y, f, y) in $\mathcal{C}$ is said to be suitable for the degree theory.

Theorem 4 (The degree theorem). *There exists a map* deg *from the set $\mathcal{C}$ to the integers, called the degree theory, satisfying the axioms (P1)–(P7) below.*

(P1). If $(Y, Y, f, y) \in \mathcal{C}$ and f is the identity map, then $\deg(Y, Y, f, y) = 1$.

(P2). Let $(X, Y, f, y) \in \mathcal{C}$, let U be an open set of X containing $f^{-1}(y)$, and let V be an open set of Y containing $f(U)$. Then $\deg(X, Y, f, y) = \deg(U, V, f|U, y)$.

(P3). Let $(X, Y, f, y) \in \mathcal{C}$ and let $\mathcal{U}$ be a finite partition of X into open sets such that $(U, Y, f|U, y) \in \mathcal{C}$ for each $U \in \mathcal{U}$. Then $\deg(X, Y, f, y) = \sum_{U \in \mathcal{U}} \deg(U, Y, f|U, y)$.

(P4). Let (X, Y, f, y) and (X, Y, g, y) be elements of $\mathcal{C}$. Suppose that there exists a homotopy function $H : X \times [0, 1] \to Y$ between f and g such that $H^{-1}(y)$ is a compact set. Then $\deg(X, Y, f, y) = \deg(X, Y, g, y)$.

(P5). Let $(X, Y, f, y) \in \mathcal{C}$. Suppose K is a compact connected subset of Y containing y and $f^{-1}(K)$ is a compact set. Then $\deg(X, Y, f, y) = \deg(X, Y, f, \bar{y})$ for all $\bar{y} \in K$.

(P6). Let (X, Y, f, y) and (Y, Z, g, z) be the elements of $\mathcal{C}$. Suppose that Y is a connected space, and that f is a proper function. Then $\deg(X, Z, g \circ f, z) = \deg(Y, Z, g, z) \times \deg(X, Y, f, y)$.

(P7). If $(X, Y, f, y) \in \mathcal{C}$ and $\deg(X, Y, f, y) \neq 0$, then the set $f^{-1}(y)$ is non-empty.

Properties (P1)–(P7) imply that for X an open subset of $\mathbb{R}^n$, for $Y = \mathbb{R}^n$ and a continuously differentiable map $f : X \to Y$, the degree of f at a regular value y can be computed as

$$\deg(X, Y, f, y) = \sum_{x \in f^{-1}(y)} \operatorname{sign} \det[\mathrm{d}f(x)],$$

if the set $f^{-1}(y)$ is non-empty, and $\deg(X, Y, f, y) = 0$ otherwise.

5. The domain of the homotopy

Let $G_J(\mathbb{R}^S)$ be the set of all J-dimensional linear subspaces of $\mathbb{R}^S$. One introduces an identification topology on $G_J(\mathbb{R}^S)$ by considering it a quotient space of $M_J(S \times J)$ where matrices B and B' are equivalent if $\operatorname{span} B = \operatorname{span} B'$. Then $G_J(\mathbb{R}^S)$ is a compact topological space. Following Zhou (1997) we define an asset pseudo-equilibrium manifold of an economy $\omega = (e, A) \in \Omega$ as

$$X_\omega = \left\{(p, L) \in P \times G_J(\mathbb{R}^S) | \operatorname{span} V_A(p) \subset L\right\}.$$

The space X_ω is the domain of our homotopy function connecting the aggregate excess demand of the economy ω to the individual excess demand of the unconstrained agent. The main result of this section is Lemma 3 that shows X_ω to be an orientable manifold in the case $S - J$ is an even number. As a preliminary step, we consider the space Z defined as

$$Z = \left\{(L, K) \in G_J(\mathbb{R}^S) \times M(S, J) | \operatorname{span} K \subset L\right\}.$$

Lemma 1. *The space Z is a smooth $[SJ]$-dimensional manifold. If $S - J$ is an even number then the manifold Z is orientable.*

Proof. We construct a smooth atlas $\mathcal{A}$ on Z and verify that $\mathcal{A}$ is oriented when $S - J$ is even. Let (L_α, K_α) be a point in Z. Let an $[S \times J]$-dimensional matrix B_α represent a basis of the linear space L_α, and let an $[S \times (S - J)]$-dimensional matrix $B_\alpha^\perp$ represent a basis of the orthogonal complement to the linear space L_α. We can choose these matrices in such a way that the $[S \times S]$-

matrix $[B_\alpha, B_\alpha^\perp]$ is orthogonal and its determinant is equal to $+1$. Let U_α be an open neighborhood of the point (L_α, K_α) in Z defined as

$$U_\alpha = \left\{ (L, K) \in Z \,\middle|\, \begin{array}{c} \det(B_\alpha^\top B) \neq 0 \text{ for each matrix} \\ B \in M(S, J) \text{ representing a basis of } L \end{array} \right\}.$$

Define the map $\phi_\alpha : M(S-J, J) \times M(J, J) \to U_\alpha$ as follows. Given $(E, Q) \in M(S-J, J) \times M(J, J)$ let an $[S \times J]$-dimensional matrix $B_{E\alpha}$ be given by $B_{E\alpha} = B_\alpha^\perp E + B_\alpha$. We let $\phi_\alpha(E, Q) = (L, K)$ where

$$L = \mathrm{span}[B_{E\alpha}]$$
$$K = [B_{E\alpha}]Q.$$

Observe that the matrix $B_{E\alpha}$ has rank J, so its columns span a J-dimensional linear subspace of $\mathbb{R}^S$. Moreover, $B_\alpha^\top B_{E\alpha}$ is the identity matrix. Hence, the point $\phi_\alpha(E, Q)$ is indeed an element of the set U_α.

Define the map $\varphi_\alpha : U_\alpha \to M(S-J, J) \times M(J, J)$ as follows. Given $(L, K) \in U_\alpha$ let $\varphi_\alpha(L, K) = (E, Q)$ where

$$E = B_\alpha^{\perp\top} B [B_\alpha^\top B]^{-1}$$
$$Q = B_\alpha^\top B [B^\top B]^{-1} B^\top K,$$

and B is an $[S \times J]$-dimensional matrix representing a basis of the linear space L. Observe that φ_α is well-defined: the value for $\varphi_\alpha(L, K)$ depends only on the linear space L and matrix K and not on the particular choice of the basis for L.

It is straightforward to check that ϕ_α is the inverse of φ_α. Since both φ_α and ϕ_α are continuous, one concludes that φ_α is a homeomorphism, and that the pair $(U_\alpha, \varphi_\alpha)$ is a chart on Z. Let the atlas $\mathcal{A}$ on Z consist of all such charts.

Choose a pair of charts, say $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β), from the atlas $\mathcal{A}$ and consider the transition map $t = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \bigcap U_\beta) \to \varphi_\beta(U_\alpha \bigcap U_\beta)$. Given (E, Q) in the domain of t, we can write $t(E, Q) = (t_1(E), t_2(E, Q))$ where

$$t_1(E) = B_\beta^{\perp\top} B_{E\alpha} [B_\beta^\top B_{E\alpha}]^{-1}$$
$$t_2(E, Q) = [B_\beta^\top B_{E\alpha}]Q.$$

The transition map t is clearly a smooth map. This proves that Z is a smooth $[SJ]$-dimensional manifold.

Next we demonstrate that, when $S - J$ is an even number, the transition map t is an orientation-preserving diffeomorphism. We compute the Jacobian matrix of t and prove its determinant to be positive. For the purpose of differentiation we identify all the involved matrices with the vectors of the Euclidean space by means of a vectorization operator. With this convention, the Jacobian matrix of t at point (E, Q) can be written as

$$\mathrm{d}t(E, Q) = \begin{bmatrix} \mathrm{d}_E t_1(E) & 0 \\ \mathrm{d}_E t_2(E, Q) & \mathrm{d}_Q t_2(E, Q) \end{bmatrix}.$$

To compute the upper-left block of $\mathrm{d}t(E, Q)$, we notice that the map t_1 is implicitly defined by the equation $\left[t_1 B_\beta^\top - B_\beta^{\perp\top} \right] B_{E\alpha} = 0$. The Implicit Function Theorem applies to show that

$$\mathrm{d}_E t_1(E) = -[(B_{E\alpha}^\top B_\beta) \otimes I_{S-J}]^{-1} [I_J \otimes (t_1(E) B_\beta^\top - B_\beta^{\perp\top}) B_\alpha^\perp],$$

where the symbol $\otimes$ denotes the Kronecker product and I_J is the J-dimensional identity matrix. Using the assumption that the matrix $[B_\beta, B_\beta^\perp]$ is orthogonal and that its determinant is +1, one shows the equality

$$\det \mathrm{d}_E t_1(E) = \det[B_\beta^\top B_{E\alpha}]^{-S}.$$

The computation of the lower-right block of $\mathrm{d}t(E, Q)$ is trivial, for t_2 depends linearly on Q. We have

$$\det \mathrm{d}_Q t_2(E, Q) = \det[B_\beta^\top B_{E\alpha}]^J.$$

Finally, the determinant of $\mathrm{d}t(E, Q)$ is given by the product of the determinants of its upper-left and lower-right blocks. It is clearly positive, if the number $S - J$ is even. □

Lemma 2. *The set X_ω is closed in $P \times G_J(\mathbb{R}^S)$.*

Proof. Let $q : M_J(S, J) \to G_J(\mathbb{R}^S)$ be a quotient map, and let 1_P be the identity map on P. Since P is a locally compact Hausdorff space, the Cartesian product $1_P \times q : P \times M_J(S, J) \to P \times G_J(\mathbb{R}^S)$ is a quotient map, see Munkres (1984), p. 113, Theorem 20.1. The full preimage of X_ω under the map $1_P \times q$ is given by

$$\{(p, B) \in P \times M_J(S, J) | \mathrm{span} V_A(p) \subset \mathrm{span} B\},$$

a closed subset of $P \times M_J(S, J)$. The result follows. □

In the sequel we shall abbreviate the phrase "there exists a subset Ω^* of Ω with a complement of Lebesgue measure zero such that for each $\omega \in \Omega^*$ property P holds" to a phrase "for generic ω, property P holds". With this convention, our next result is as follows.

Lemma 3. *For generic ω, the space X_ω is an $[M - 1]$-dimensional smooth manifold. If $S - J$ is an even number then the manifold X_ω is orientable.*

Proof. Consider a smooth function

$$\begin{aligned} &\Omega \times P \times Z \xrightarrow{f} M(S, J) \\ &(e, A, p, L, K) \mapsto V_A(p) - K. \end{aligned}$$

To see that the function f is a submersion, write $V_A(p) = QA$ where Q is an $[S \times SL]$-dimensional matrix given by

$$Q = \begin{bmatrix} p_{11} & \cdots & p_{1L} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & p_{21} & \cdots & p_{2L} & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & p_{S1} & \cdots & p_{SL} \end{bmatrix}.$$

Identifying all involved matrices with the vectors of the Euclidean space by means of a vectorization operator, we can write the derivative of f with respect to A as $I_J \otimes Q$. Here the symbol $\otimes$ denotes the Kronecker product. Since the matrix Q has full row rank S, the matrix $I_J \otimes Q$ has full row rank SJ.

The Transversality theorem applies to show that for generic $\omega = (e, A)$, zero is a regular value of the function f_ω, where f_ω is the restriction of f to the set $\{\omega\} \times P \times Z$. By the preimage theorem, $f_\omega^{-1}(0)$ is a smooth submanifold of $P \times Z$ with codimension SJ. If the manifold Z is orientable, so is the manifold $f_\omega^{-1}(0)$. Now observe that the space $f_\omega^{-1}(0)$ is homeomorphic to

X_ω. Given a smooth (oriented) atlas on $f_\omega^{-1}(0)$, this homeomorphism induces a smooth (oriented) atlas on the space X_ω, in a natural way. □

Given an economy $\omega = (e, A) \in \Omega$ define

$$P^*_\omega = \{p \in P | \text{rank}\, V_A(p) = J\} \text{ and } X_\omega{}^* = X_\omega \cap (P^*_\omega \times G_J(\mathbb{R}^S)).$$

In the rest of this section we prove that P^*_ω is a path-connected space. To do so, we first establish an auxiliary Lemma 4.

Lemma 4. *Let $C_1, \ldots, C_m$ be smooth submanifolds of $\mathbb{R}^n$ each having a codimension greater than one. Then $\mathbb{R}^n \setminus \bigcup_{i=1}^m C_i$ is a path-connected set.*

Proof. Let x and x' be any points in $\mathbb{R}^n \setminus \bigcup_{i=1}^m C_i$. Consider a family of curves c_y in $\mathbb{R}^n$ parameterized by $y \in \mathbb{R}^n$ connecting points x and x', $c_y(t) = (1-t)x + tx' + (1-t)ty$. The function $(t, y) \mapsto c_y(t)$ is transversal to each manifold C_i. By Transversality theorem there exists some $\bar{y} \in \mathbb{R}^n$ such that $c_{\bar{y}}$ is transversal to C_i for all $i = 1, \ldots, m$. Since the codimension of C_i is greater than one, the intersection of $c_{\bar{y}}([0, 1])$ with C_i must be empty. Thus, the curve $c_{\bar{y}}$ is entirely contained in the set $\mathbb{R}^n \setminus \bigcup_{i=1}^m C_i$. □

Lemma 5. *Suppose that $S > J$. Then for generic ω, P^*_ω is a path-connected space.*

Proof. Notice that $P^*_\omega = P \setminus \bigcup_{j=0}^{J-1} P^j_\omega$, where $P^j_\omega = \{p \in P \,|\, \text{rank} V_A(p) = j\}$. By the preceding lemma, it is sufficient to demonstrate that each P^j_ω is a smooth submanifold of P with codimension greater than one.

The set $M_j(S \times J)$ of $(S \times J)$-dimensional matrices having rank j is a smooth manifold with codimension $(S - j)(J - j)$ in $M(S, J)$, see Jongen et al. (2000), p. 310. By a similar argument as that in the proof of Lemma 3, the function

$$\Omega \times P \xrightarrow{g} M(S \times J)$$
$$(e, A, p) \mapsto V_A(p)$$

is a submersion, and is therefore transversal to each $M_j(S \times J)$. The Transversality theorem applies to show that for generic $\omega = (e, A)$ the function g_ω intersects the manifolds $M_j(S \times J)$ transversally for each $j = 0, \ldots, J - 1$, where g_ω denotes the restriction of g to the set $\{\omega\} \times P$. Therefore P^j_ω, being the preimage of $M_j(S \times J)$ under the function g_ω, is a smooth submanifold of P with codimension $(S - j)(J - j) > 1$ for all $j = 0, \ldots, J - 1$. □

6. The homotopy

Let the economy $\omega = (e, A) \in \Omega$ be given. Lemma 6 below summarizes the relevant properties of the individual excess demand functions z^i_ω as defined in Section 3. As all five properties are well-known, the proof is omitted. For the rest of this section we use the symbol z^0_ω rather than z_ω to denote the aggregate excess demand of the economy ω.

Lemma 6.

1. *The function z^1_ω is continuously differentiable throughout P. The function z^0_ω is continuously differentiable on P^*_ω. For each $2 \le i \le I$, the function z^i_ω is continuous on $P \times G_J(\mathbb{R}^S)$.*
2. *For each $1 \le i \le I$ the function z^i_ω is bounded from below by $-e^i$.*
3. *For each $1 \le i \le I$ the function z^i_ω satisfies Walras' law.*

4. *The function* z^1_ω *is proper.*
5. *There exists a unique price vector* $p \in P$ *satisfying the equation* $z^1_\omega(p) = 0$. *Moreover,*

$$\operatorname{sign}\det\left[\frac{\partial z^1_{\omega m}(p)}{\partial p_{m'}}\right] = (-1)^{M-1},$$

where m *and* m' *vary within the index set* $\{1, \ldots, M-1\}$.

Define the maps $f^0_\omega, f^1_\omega, \ldots, f^I_\omega : X_\omega \to Y$ by letting $f^1_\omega(p, L) = \dot{z}^1(p)$ for agent 1, $f^i_\omega(p, L) = \dot{z}^i(p, L)$ for each $2 \leq i \leq I$ and $f^0_\omega = f^1_\omega + \cdots + f^I_\omega$. We remark that each zero of the function f^0_ω is a pseudo-equilibrium of the economy ω as defined in Duffie and Shafer (1985).

Lemma 7. *Let* $S - J$ *be even. For generic* $\omega \in \Omega$ *the tuples* $(X_\omega, Y, f^t_\omega, 0)$ *for* $t = 0, 1$ *are suitable for the degree theory, i.e.* $(X_\omega, Y, f^t_\omega, 0) \in \mathcal{C}$. *Furthermore,* $\deg(X_\omega, Y, f^0_\omega, 0) = \deg(X_\omega, Y, f^1_\omega, 0)$.

To prove Lemma 7 we show that for each $\omega \in \Omega$ the zero set of a straight-line homotopy map between the functions f^0_ω and f^1_ω is a compact set (Lemma 8 below). It then follows immediately that the zero set of each function f^t_ω is a compact set. This shows that for each ω as in Lemma 3 both tuples $(X_\omega, Y, f^t_\omega, 0)$ for $t = 0, 1$ are suitable for the degree theory. The equality of the respective degrees follows from the homotopy-invariance (P4). Recall that a straight-line homotopy between the functions f^0_ω and f^1_ω is defined as $H^t_\omega = f^1_\omega + t[f^2_\omega + \cdots + f^I_\omega]$.

Lemma 8. *Let* $H_\omega : X_\omega \times [0, 1] \to Y$ *be a straight-line homotopy map between the functions* f^0_ω *and* f^1_ω. *Then* $H^{-1}_\omega(0)$ *is a compact set.*

Proof. Given an economy $\omega = (e, A) \in \Omega$ let K be a compact subset of $\mathbb{R}^M$ defined as $K = \{z \in \mathbb{R}^M | -e^1 \leq z \leq e^2 + \cdots + e^I\}$, and let $C \subset P$ denote the preimage of K under the function z^1_ω. Since K is a compact set, and z^1_ω is a proper function, the set C is compact.

If $H_\omega(p, L, t) = 0$, then the Walras' law implies that $z^1_\omega(p) + t[z^2_\omega(p, L) + \cdots + z^I_\omega(p, L)]$ is a zero vector of $\mathbb{R}^M$. Since each z^i_ω is bounded below by $-e^i$, the vector $z^1_\omega(p)$ is an element of K, and so the price vector p is an element of C. We have thus showed that $H^{-1}_\omega(0) \subseteq C \times G_J(\mathbb{R}^S) \times [0, 1]$. Now the set $H^{-1}_\omega(0)$ is a closed subset of $X_\omega \times [0, 1]$, by continuity of the function H_ω. The set $X_\omega \times [0, 1]$ is closed in $P \times G_J(\mathbb{R}^S) \times [0, 1]$ by Lemma 2. Thus $H^{-1}_\omega(0)$ is closed in $P \times G_J(\mathbb{R}^S) \times [0, 1]$. In particular, it is closed in $C \times G_J(\mathbb{R}^S) \times [0, 1]$. Since $C \times G_J(\mathbb{R}^S) \times [0, 1]$ is a compact set, $H^{-1}_\omega(0)$ is a compact set as well. □

Lemma 9. *For generic* ω, *the following conditions hold: For each* $t = 0, 1$ *(a) zero is a regular value of the function* $\dot{z}^t_\omega | P^*_\omega$, *(b) the zero set of the function* $\dot{z}^t_\omega$ *is entirely contained in* P^*_ω, *and (c) the zero set of the function* f^t_ω *is entirely contained in* $X_\omega{}^*$.

Each part of Lemma 9 is obtained by an easy application of the Transversality theorem. In particular, the fact that the zero set of the function $\dot{z}^0_\omega$ is entirely contained in P^*_ω is a restatement of Theorem 1. That the zero set of the function f^0_ω is contained in $X_\omega{}^*$ follows from another well-known result: every pseudo-equilibrium (p, L) of a generic economy $\omega = (e, A)$ is such that the matrix $V_A(p)$ has rank J. For a proof of this result see, for example, Demarzo and Eaves (1996).

Lemma 10. *Let* $S > J$ *and suppose that* $S - J$ *is even. Then for generic* $\omega \in \Omega$ *the tuples* $(P^*_\omega, Y, \dot{z}^t_\omega | P^*_\omega, 0)$ *for* $t = 0, 1$ *are suitable for the degree theory and* $\deg(P^*_\omega, Y, \dot{z}^0_\omega | P^*_\omega, 0) = \deg(P^*_\omega, Y, \dot{z}^1_\omega | P^*_\omega, 0)$.

Proof. Let $t = 0, 1$. The tuple $(P^*_\omega, Y, \dot{z}^t_\omega|P^*_\omega, 0)$ is suitable for the degree theory because P^*_ω can be seen as an open subset of Y, the function $\dot{z}^t_\omega$ is continuously differentiable throughout P^*_ω and because zero is a regular value of $\dot{z}^t_\omega|P^*_\omega$.

To show an equality of the respective degrees, observe that a natural projection $X_\omega{}^* \to P^*_\omega$ is a homeomorphism. We let $\iota : P^*_\omega \to X_\omega{}^*$ denote its inverse. Since P^*_ω (and therefore $X_\omega{}^*$) is a connected space by Lemma 4, Property (P5) of the degree theory shows that $\deg(P^*_\omega, X_\omega{}^*, \iota, x)$ is the same for all $x \in X_\omega$. We let $\deg(\iota)$ denote this common degree.

As $X_\omega{}^*$ is an open subset of X_ω that contains all zeros of the function f^t_ω, Property (P2) of the degree theory implies that $\deg(X_\omega, Y, f^t_\omega, 0) = \deg(X_\omega{}^*, Y, f^t_\omega|X_\omega{}^*, 0)$. Furthermore, the map $\dot{z}^t_\omega|P^*_\omega$ is a composite map that equals ι followed by $f^t_\omega|X_\omega{}^*$. Thus Property (P6) implies that $\deg(P^*_\omega, Y, \dot{z}^t_\omega|P^*_\omega, 0) = \deg(X_\omega{}^*, Y, f^t_\omega|X_\omega{}^*, 0) \times \deg(\iota)$. The result now follows from Lemma 7. □

To complete the proof of the index theorem we observe that for generic $\omega \in \Omega$,

$$\deg(P^*_\omega, Y, \dot{z}^0_\omega|P^*_\omega, 0) = (-1)^{M-1} \sum_{p \in E(p)} \mathrm{index}_\omega(p)$$

$$\deg(P^*_\omega, Y, \dot{z}^1_\omega|P^*_\omega, 0) = (-1)^{M-1}.$$

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