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General Existence Theorem of Zero Points[1]

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Abstract. Let X be a nonempty, compact, convex set in $\mathbb{R}^n$ and let ϕ be an upper semicontinuous mapping from X to the collection of nonempty, compact, convex subsets of $\mathbb{R}^n$. It is well known that such a mapping has a stationary point on X; i.e., there exists a point X such that its image under ϕ has a nonempty intersection with the normal cone of X at the point. In the case where, for every point in X, it holds that the intersection of the image under ϕ with the normal cone of X at the point is either empty or contains the origin 0^n, then ϕ must have a zero point on X; i.e., there exists a point in X such that 0^n lies in the image of the point. Another well-known condition for the existence of a zero point follows from the Ky Fan coincidence theorem, which says that, if for every point the intersection of the image with the tangent cone of X at the point is nonempty, the mapping must have a zero point. In this paper, we extend all these existence results by giving a general zero-point existence theorem, of which the previous two results are obtained as special cases. We discuss also what kind of solutions may exist when no further conditions are stated on the mapping ϕ. Finally, we show how our results can be used to establish several new intersection results on a compact, convex set.

Key Words. Stationary points, zero points, fixed points, normal cones, tangent cones, intersection points.

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1. Introduction

Whenever a mathematical model of some phenomenon is constructed (for instance, in engineering or in economics), the first question to ask is whether a solution to the model exists. A very powerful tool that is often used to this end, in the case where the model is a system of nonlinear functions, is the Brouwer fixed-point theorem (see Ref. 1). When the model is not a system of equations but a system of correspondences, often the Kakutani fixed-point theorem (Ref. 2) is invoked. Sometimes, models allow for a continuum of solutions and appropriate generalizations of the Brouwer and Kakutani fixed-point theorems apply, as provided by Browder (Ref. 3) and Herings, Talman, and Yang (Refs. 4–5). An alternative to fixed-point theorems consists of using intersection theorems, with the lemma of Knaster, Kuratowski, and Mazurkiewicz (Ref. 6) on the unit simplex being perhaps the most prominent example.

The existence of a solution to a nonlinear system of functions or correspondences is equivalent to the existence of the zero point of a function or a correspondence. A zero point is a point in the domain such that the origin lies in its image. In this paper, we will present a new general condition for the existence of a zero point.

Let X be a nonempty, convex, and compact set in $\mathbb{R}^n$ and let ϕ be a compact-valued, convex-valued, upper-semicontinuous mapping from X to $\mathbb{R}^n$. By Eaves (Ref. 7), it has been shown that, with respect to any such correspondence, a solution exists to the variational inequality problem; i.e., there exists a stationary point. Such a point x in X is such that its image $\phi(x)$ has a nonempty intersection with the normal cone $N(X,x)$ to X at x. From this, it follows immediately that, if at every point of X the intersection of the image and the normal cone is either empty or contains the origin, then ϕ has a zero point in X. Fan (Ref. 8) proved a coincidence result, stating a weakly separating condition under which there is a point x in X such that $\phi(x)$ has a nonempty intersection with the image at x of some other correspondence ψ on X. This condition makes use also of the normal cone at any point in X. When ψ maps every point of X to the origin and the separating condition is satisfied, a zero point of ϕ exists.

In this paper, we present a unifying theorem on the existence of zero points. The theorem puts two conditions on $\phi(x)$ at every x in X. Both conditions are related to the normal cone. More precisely, the conditions put restrictions on the set $A\phi(x)\cap\pi(v)$, where A is a nonsingular $n\times n$ matrix, v is any normalized element of the normal cone at x, and π is an upper hemicontinuous correspondence defined on the unit ball. The new theorem contains as special cases the two existence results for the zero points mentioned above. The stationary point condition is obtained when A is the identity matrix and $\pi(v)$ is equal to $\mathbb{R}^n$ for every v, while the coincidence point condition is obtained also by taking A equal to the identity matrix and $\pi(v)$ equal to the set $\{y \in \mathbb{R}^n | y^\top v \leq 0\}$. Other choices for the matrix A and the correspondence π lead to different and new zero-point existence theorems. Further, we show how the Kakutani fixed-point theorem as well as other fixed-point theorems on unbounded domains, as presented in Merrill (Ref. 9) and Eaves (Ref. 10), can be obtained as special cases of our main result. We generalize also the notion of stationary point in the case where, for at least one point in X, the two conditions are satisfied for no correspondence π. These results are exemplified in Sections 2 and 3.

Section 4 treats the special case of the zero-point problem when X is a polytope. The special structure of the polytope is exploited to obtain a sharp result on the existence of a zero point. Section 5 shows how a more general intersection theorem can be derived from our main theorem on the existence of zero points. This general intersection theorem contains as special cases several well-known intersection theorems like the ones of Knaster, Kuratowski, and Mazurkiewicz (Ref. 6), Scarf (Ref. 11), Shapley (Ref. 12), and Ichiishi (Ref. 13).

2. Zero Point Problem

Consider an arbitrary nonempty, convex, compact set X in the n-dimensional Euclidean space $\mathbb{R}^n$. For $x \in X$, the set

$$N(X, x) = \{y \in \mathbb{R}^n | (x - x')^\top y \geq 0, \text{for all } x' \in X\}$$

denotes the normal cone of the set X at the point x. Since X is compact and convex, $N(X, \cdot)$ is an upper semicontinuous, convex-valued, and closed-valued mapping.

Let ϕ be a point-to-set mapping or correspondence from X to the collection of nonempty subsets of $\mathbb{R}^n$. We assume that ϕ is an upper semicontinuous and bounded mapping and that, for every x in X, the set $\phi(x)$ is a compact and convex subset of $\mathbb{R}^n$. We are interested in conditions on the mapping ϕ, under which ϕ has a zero point, a point x^* in X satisfying $0^n \in \phi(x^*)$, where 0^n is the n-vector of zeroes. Without any conditions on ϕ, a zero point may not exist. However, as it has been shown in Eaves (Ref. 7), a stationary point of ϕ on X exists always.

Definition 2.1. A point $x^* \in X$ is a stationary point of ϕ if there exists $y^* \in \phi(x^*)$ such that $(x^* - x)^\top y^* \geq 0$ for all $x \in X$; i.e., $\phi(x^*) \cap N(X, x^*) \neq \emptyset$.

From the Eaves result, it follows immediately that, if all the stationary points of ϕ are zero points of ϕ, then ϕ has at least one zero point.

Theorem 2.1. For every $x \in X$, if it holds that $\phi(x) \cap N(X, x)$ is either empty or contains 0^n, then there exists a zero point of ϕ.

The condition in the theorem says that a zero point of ϕ exists if at any $x \in X$ no nonzero element of the image $\phi(x)$ lies in the normal cone of X at x unless the image contains 0^n. Although this condition is rather weak, it has to hold for all elements in every image set. Another existence result for zero points can be obtained from the coincidence theorem of Fan (Ref. 8).

Definition 2.2. Let ϕ and ψ be to correspondences from X to $\mathbb{R}^n$. A point $x^* \in X$ is a coincidence point of ϕ and ψ if $\phi(x^*) \cap \psi(x^*) \neq \emptyset$.

By Fan (Ref. 8), it has been proved that, if ψ is also an upper semicontinuous, bounded, convex-valued, and compact-valued correspondence from X to $\mathbb{R}^n$, and if, for every $x \in X$ and every $v \in N(X, x)$, there exists $y \in \phi(x)$ and $z \in \psi(x)$ such that $v^\top y \leq v^\top z$, then ϕ and ψ have a coincidence point. By taking $\psi(x)$ equal to $\{0^n\}$ for all x in X, we obtain the following zero point result, which is an equivalent form of the Fan coincidence theorem.

Theorem 2.2. For every $x \in X$ and every $v \in N(X, x)$, if there exists $y \in \phi(x)$ such that $v^\top y \leq 0$, then ϕ has a zero point.

The condition in this theorem says that, for every x in X, the set $\phi(x)$ should have a nonempty intersection with any halfspace that is the polar or dual cone of an element of the normal cone of X at x. The fact that two rather different conditions lead to the same existence result suggests a more general zero-point existence result. In Section 3 we give a zero-point existence theorem that contains as special cases both the theorems above and several other known existence results.

3. Existence Results

In this section, we give a unifying zero-point existence result on a compact, convex set. Both Theorems 2.1 and 2.2 as well as the Kakutani fixed-point theorem and other fixed-point and zero-point theorems are special cases of this theorem. As in Section 2, we assume that the set X is a nonempty, compact, and convex subset of $\mathbb{R}^n$ and that ϕ is an upper semicontinuous, bounded, compact-valued, and convex-valued correspondence form X to $\mathbb{R}^n$. Let B^n denote the n-dimensional unit ball.

Theorem 3.1. Suppose that there exists a nonsingular $n \times n$ matrix A and an upper semicontinuous, convex-valued, closed-valued mapping $\pi: B^n \to \mathbb{R}^n$ such that, for every $x \in X$ and every $v \in N(X, x) \cap B^n$, the following two properties hold:

(i) The set $A\phi(x) \cap \pi(v) \cap \{y | y = \mu v, \mu \geq 0\}$ is either empty or contains 0^n.

(ii) The set $A\phi(x) \cap \pi(v) \neq \emptyset$.

Then, there exists a zero point of ϕ in X.

Proof. Let the set Q be defined by

$$Q = \{q \in \mathbb{R}^n | \|q - x\|_2 \leq 1, \text{for some } x \in X\}.$$

Since X is compact, Q is a compact set. For $q \in Q$, let $p(q)$ be the orthogonal projection of q on X. Since X is a nonempty, compact, convex set, p is a continuous function from Q to X. For every $q \in Q$, it holds that

$$\|q - p(q)\|_2 \leq 1.$$

To prove the convexity of Q, take any q^1, $q^2 \in Q$ and $0 \leq \lambda \leq 1$, and let

$$q(\lambda) = \lambda q^1 + (1 - \lambda) q^2,$$
$$p(\lambda) = \lambda p(q^1) + (1 - \lambda) p(q^2).$$

Since X is convex, we have that $p(\lambda) \in X$. Moreover,

$$\|q(\lambda) - p(\lambda)\|_2 \leq \lambda \|q^1 - p(q^1)\|_2 + (1 - \lambda) \|q^2 - p(q^2)\|_2 \leq 1.$$

Therefore, $q(\lambda) \in Q$; i.e., Q is a convex set. Hence, Q is full-dimensional compact, convex set in $\mathbb{R}^n$. For $q \in Q$, let

$$v(q) = q - p(q).$$

By construction,

$$v(q) \in B^n, \qquad \text{for each } q \in Q,$$
$$\|v(q)\|_2 = 1, \qquad \text{if and only if } q \in \text{bd}(Q),$$
$$v(q) = 0^n, \qquad \text{if and only if } q \in X.$$

Since Q is full-dimensional, for $q \in \text{int}(Q)$ it holds that

$$N(Q, q) = \{0^n\}.$$

Now, we will show that the normal cone $N(Q, q)$ of Q at any point q on the boundary of Q is a ray. Since Q is convex and compact, $N(Q, q)$ is nonempty for every $q \in Q$. Take any point $q \in \text{bd}(Q)$ and consider the ball $B(p(q), 1)$ with radius one centered at $p(q)$. Clearly, $B(p(q), 1)$ is contained by Q and q lies also on the boundary of $B(p(q), 1)$. It follows that $N(Q, q)$ is a subset of $N(B(p(q), 1), q)$. Since the boundary of $B(p(q), 1)$ is smooth, $N(B(p(q), 1), q)$ is a ray. Consequently, $N(Q, q)$ must be a ray as well and in fact is equal to $N(B(p(q), 1), q)$. More precisely, for $q \in \mathrm{bd}(Q)$, we have

$$N(Q, q) = \{y \in \mathbb{R}^n | y = \mu v(q), \mu \geq 0\}.$$

Since p is the orthogonal projection on X, for every $q \in Q$ it holds that

$$N(Q, q) \subset N(X, p(q)).$$

Now, consider the mapping $\psi: Q \to \mathbb{R}^n$ defined by

$$\psi(q) = A\phi(p(q)) \cap \pi(q - p(q)).$$

From Condition (ii), it follows that, for every $q \in Q$, the set $\psi(q)$ is nonempty. Since A is a regular matrix, p is a continuous function, and both ϕ and π are upper semicontinuous mappings, ψ is an upper semicontinuous mapping from the full-dimensional, compact, convex set Q to $\mathbb{R}^n$. Moreover, being the intersection of a convex, compact set and a convex, closed set, $\psi(q)$ is convex and compact for any $q \in Q$. From Eaves (Ref. 7), it follows now that ψ has a stationary point on Q; i.e., there exists a point $q^* \in Q$ such that

$$\psi(q^*) \cap N(Q, q^*) \neq \emptyset.$$

Take any f^* in this intersection. Since

$$f^* \in N(Q, q^*),$$

it holds that

$$f^* = \mu^* v(q^*) \in N(X, p(q^*)), \qquad \text{for some } \mu^* \geq 0.$$

Hence,

$$f^* \in \{\mu v(q^*) | \mu \geq 0\} \cap A\phi(p(q^*)) \cap \pi(v(q^*)),$$

with

$$v(q^*) \in N(X, p(q^*)) \cap B^n.$$

Since we showed that the intersection of these three sets is nonempty, condition (i) implies that this intersection contains 0^n, from which we conclude that $p(q^*)$ is a zero point of $A\phi$ on X. Since A is a nonsingular matrix, $p(q^*)$ is a zero point of ϕ. □

The theorem says that the mapping ϕ has a zero point on X if there exists a regular matrix A and an upper semicontinuous, convex-valued, and closed-valued mapping on the unit ball B^n such that, for every element v of the normal cone of X at any x with length at most one, the image of $A\phi$ at x and the image of π at v intersect, but this intersection has no points in common with the ray determined by the vector v unless the origin is in the intersection. In case ϕ is a continuous function f from X to $\mathbb{R}^n$, conditions (i) and (ii) reduce to: for every $x \in X$ and $v \in N(X, x) \cap B^n$, it holds that

$$Af(x) \in \pi(v) \quad \text{and} \quad Af(x) \notin (\pi(v) \cap \{y | y = \mu v, \mu \geq 0\}) \backslash \{0^n\}.$$

Instead of taking a mapping on the whole unit ball, we may restrict ourselves to a mapping π on the sphere $\mathrm{bd}(B^n)$. Then, the proof is the same, by extending the mapping π to the whole unit ball as follows: $\pi(0^n)$ contains every $\pi(v)$, $v \in B^n$, and

$$\pi(v) = \pi(v/\|v\|_2), \qquad \text{for } v \in \mathrm{int}(B^n) \backslash \{0^n\}.$$

The matrix A translates the images $\phi(x)$ in a linear way, so that $A\phi(x)$ has the some properties as $\phi(x)$ has. Due to the regularity of A, a point x^* is a zero point of ϕ if and only if x^* is a zero point of $A\phi$.

The use of the matrix A expands the cases to which our result applies. For example, consider the function $f: B^n \to \mathbb{R}^n$ defined by $f(x) = x$. Then, there is no mapping π that satisfies both conditions (i) and (ii), although $f(0^n) = 0^n$. However, when we take $A = -I$, where I is the $n \times n$ identity matrix, conditions (i) and (ii) are satisfied if we take for example

$$\pi(v) = \mathbb{R}^n, \qquad \text{for all } v \in B^n.$$

In the following, we will show that several known existence results are special cases of Theorem 3.1.

Example 3.1. When $\pi(v) = \mathbb{R}^n$ for every $v \in B^n$, then condition (ii) of Theorem 3.1 is satisfied always and condition (i) reduces to the statement that $A\phi(x) \cap N(X, x)$ is empty or contains 0^n, for every $x \in X$. For $A = I$, this is precisely the condition of Theorem 2.1. However, the result holds for any regular matrix A; e.g., a zero point exists also when, for every $x \in X$, it holds that $-\phi(x) \cap N(X, x)$ is either empty or contains 0^n.

Example 3.2. When

$$\pi(v) = \{y \in \mathbb{R}^n | y^\top v \leq 0\}, \qquad \text{for every } v \in B^n,$$

then

$$\pi(v) \cap \{y \in \mathbb{R}^n | y = \mu v, \mu \geq 0\} = \{0^n\},$$

for any $v \in B^n$, and so condition (i) of Theorem 3.1 is satisfied always, while when $A = I$ condition (ii) becomes precisely the condition of Theorem 2.2. Also, now the result holds for any regular matrix A.

Thus, both Theorem 2.1 and Theorem 2.2 are special cases of Theorem 3.1. For $x \in X$, let the tangent cone of X at x be defined by

$$T(X,x) = \{z \in \mathbb{R}^n | z^\top y \leq 0, \text{for all } y \in N(X,x)\}.$$

The next result says that ϕ has a zero point if, for every x in X, the set $\phi(x)$ has a nonempty intersection with $T(X,x)$.

Theorem 3.2. For every $x \in X$, if it holds that $\phi(x) \cap T(X,x) \neq \emptyset$, then ϕ has a zero point.

Proof. We show that the conditions of Theorem 3.1 are satisfied for

$$\pi(v) = \{y \in \mathbb{R}^n | y^\top v \leq 0\} \quad \text{and} \quad A = I,$$

and so ϕ has a zero point. Condition (i) of Theorem 3.1 is satisfied because

$$\pi(v) \cap \{y | y = \mu v, \mu \geq 0\} = \{0^n\},$$

so

$$\phi(x) \cap \pi(v) \cap \{y | y = \mu v, \mu \geq 0\}$$

is either empty or contains $\{0^n\}$. When

$$v \in N(X,x),$$

it follows that

$$T(X,x) \subset \pi(v),$$

so

$$\phi(x) \cap \pi(v) \neq \emptyset, \qquad \text{if } \phi(x) \cap T(X,x) \neq \emptyset,$$

and condition (ii) follows. □

Obviously, ϕ has also a zero point on X if there exists a regular matrix A such that

$$A\phi(x) \cap T(X,x) \neq \emptyset, \qquad \text{for every } x \in X.$$

The condition in Theorem 3.2 is very simple and in general easy to check. From Theorem 3.2, we get immediately the Kakutani fixed-point theorem.

Example 3.3. The Kakutani fixed-point theorem states that, if ϕ is a correspondence from X into itself, it has at least one fixed point; i.e., there exists $x^* \in X$ satisfying $x^* \in \phi(x^*)$. Define the mapping ψ from X to $\mathbb{R}^n$ by

$$\psi(x) = \phi(x) - \{x\}, \qquad \text{for all } x \in X.$$

Since

$$X-\{x\}\subset T(X,x) \quad \text{and} \quad \phi(x)\subset X, \qquad \text{for all } x\in X,$$

we have that

$$\psi(x)\subset T(X,x)$$

and so

$$\psi(x)\cap T(X,x)\neq\emptyset, \qquad \text{for all } x\in X.$$

From Theorem 3.2, it follows that there exists $x^*\in X$ such that $0^n\in\psi(x^*)$. Clearly, x^* is a fixed point of ϕ.

The set $\pi(v)$ is not necessarily a half-space or the whole space as it is illustrated in the next example.

Example 3.4. Fix some strictly positive vector $m\in\mathbb{R}^n$. Let $\pi(v)$ be given by

$$\pi(0^n)=\mathbb{R}^n$$

and for $v\in B^n\setminus\{0^n\}$ by

$$\begin{aligned}\pi(v)=\{y\in\mathbb{R}^n|y_i\leq m_i(1-v_i/\max_j|v_j|), &\text{if } v_i>0;\\ y_i\geq m_i(-1-v_i/\max_j|v_j|), &\text{if} v_i<0\}.\end{aligned}$$

Clearly, π is an upper semicontinuous, convex-valued, and closed-valued correspondence on B^n. Moreover, for every $v\in B^n$, it holds that

$$\pi(v)\cap\{y\in\mathbb{R}^n|y=\mu v,\mu\geq 0\}=\{0^n\},$$

and so condition (i) of Theorem 3.1 is satisfied always. If condition (ii) holds for this π, then there exists a zero point of ϕ on X.

The result in Example 3.4 was introduced in Herings, van der Laan, and Talman (Ref. 14) to prove the existence of a continuum of quantity-constrained equilibria in an exchange economy with prices restricted to an arbitrary convex, compact set. The next fixed-point theorem is due to Eaves (Ref. 10) and is used to guarantee the convergence of simplicial homotopy algorithms.

Example 3.5. Let X be full-dimensional and suppose that there exists $c\in\text{int}(X)$ such that, for all $x\in\text{bd}(X)$, it holds that $c\in\phi(x)$. Then, there exists a fixed point of ϕ in X. Define

$$\psi(x) = \phi(x) - \{x\}, \qquad \text{for all } x \in X.$$

For $x \in \mathrm{bd}(X)$, it holds that

$$c - x \in T(X, x) \cap \psi(x),$$

and for $x \in \mathrm{int}(X)$, it holds that

$$T(X, x) = \mathbb{R}^n.$$

Hence,

$$T(X, x) \cap \psi(x) \neq \phi, \qquad \text{for every } x \in X;$$

i.e., the mapping ψ satisfies the condition of Theorem 3.2. Therefore, ψ has a zero point on X, which is a fixed point of ϕ.

The following fixed-point theorem is due to Merrill (Ref. 9) and has also applications in constrained and unconstrained optimization.

Example 3.6. Let ψ be an upper-semicontinuous mapping from $\mathbb{R}^n$ to the collection of compact, convex subsets of $\mathbb{R}^n$. Suppose that there exists $w \in \mathbb{R}^n$ and $\mu > 0$ such that, for all $x \notin B(w, \mu)$ and $f \in \psi(x)$,

$$(f - x)^\top (w - x) > 0.$$

Then, ψ has a fixed point in $B(w, \mu)$. Take

$$X = B(w, \mu) \quad \text{and} \quad \pi(v) = \{y \in \mathbb{R}^n | y^\top v \leq 0\}, \qquad \text{for } v \in B^n.$$

For x on the boundary of $B(w, \mu)$, it holds that $x - w \in N(X, x)$ and there is $f \in \psi(x)$ such that

$$(f - x)^\top (w - x) \geq 0.$$

For those x, condition (ii) of Theorem 3.1 holds for the mapping ϕ on X defined by

$$\phi(x) = \psi(x) - \{x\}.$$

For v in the interior of $B(w, \mu)$, condition (ii) is trivially satisfied. Also, condition (i) is satisfied for ϕ, since for every $x \in \mathrm{bd}(X)$ it holds that

$$N(X, x) = \{y \in \mathbb{R}^n | y = \mu(x - w), \mu \geq 0\} \quad \text{and} \quad (x - w)^\top (x - w) > 0.$$

Hence, there exists x^* in X satisfying $0^n \in \phi(x^*)$, and therefore $x^* \in \psi(x^*)$.

In Theorem 3.1, we have provided a sufficient condition for the existence of a zero point of a mapping on an arbitrary compact, convex set. In case the conditions of Theorem 3.1 are not satisfied, a zero point may not exist. In this case, it is possible to obtain a generalization of the notion of a stationary point, without losing existence of a stationary point under standard assumptions.

Definition 3.1. Let $\pi: B^n \rightarrow \mathbb{R}^n$ be a convex-valued, closed-valued, upper-semicontinuous mapping, and let A be any nonsingular $n\times n$ matrix. A point $x^* \in X$ is a stationary point with respect to π and A of the mapping ϕ from X to $\mathbb{R}^n$ if $0^n \in \phi(x^*)$ or $A\phi(x^*)\cap\pi(v)\cap\{y|y=\mu v, \mu\geq 0\}\neq\emptyset$ for some $v\in N(X,x^*)$ or $A\phi(x^*)\cap\pi(v)=\emptyset$ for some $v\in N(X,x^*)$.

Notice that, when

$$\pi(v)=\mathbb{R}^n, \qquad \text{for all } v\in B^n,$$

and when A is the identity matrix, then the above definition is reduced to the usual definition of a stationary point; see Definition 2.1. As a consequence of Theorem 3.1, we have the following theorem.

Theorem 3.3. Let X be nonempty, compact, and convex, and let ϕ be upper semicontinuous, bounded, convex-valued, and compact-valued. Then, for every convex-valued, closed-valued, upper semicontinuous mapping $\pi: B^n \rightarrow \mathbb{R}^n$, and for every nonsingular $n\times n$ matrix A, ϕ has a stationary point with respect to π and A.

4. Zero Points on Polytopes

In this section, we consider the case that the compact, convex set X is a polytope. Let a polytope P be described in polyhedral form by

$$P=\{x\in\mathbb{R}^n | a^{i\top}x\leq b_i, i\in I_m\},$$

where for every $i\in I_m=\{1,\ldots,m\}$, the vector a^i is a nonzero vector in $\mathbb{R}^n$ and $b_i\in\mathbb{R}$. Without loss of generality, we assume that P is full-dimensional, simple and that there are no redundant constraints. For $I\subset I_m$, define

$$F(I)=\{x\in P | a^{i\top}x=b_i, i\in I\},$$

$$C(I)=\left\{y\in\mathbb{R}^n | y=\sum_{i\in I}\mu_i a^i, \mu_i\geq 0, i\in I\right\}.$$

Notice that

$$F(\emptyset)=P \quad \text{and} \quad C(\emptyset)=\{0^n\}.$$

When $F(I)\neq\emptyset$, we call $F(I)$ a face of P. Let $\mathcal{I}$ be the collection of subsets I of I_m such that $F(I)$ is a face of P. For $x\in P$, define

$$I^x=\{i\in I_m | a^{i\top}x=b_i\};$$

i.e., $F(I^x)$ is the unique face of P of which x is an interior point. Clearly, $C(I^x)$ is the normal cone of P at $x \in P$; i.e.,

$$C(I^x) = N(P, x).$$

The next theorem gives a sufficient condition for the existence of a zero point of a mapping on P.

Theorem 4.1. Let P be a polytope and let ϕ be a mapping from P to $\mathbb{R}^n$ satisfying the conditions stated before. Suppose that there exists a nonsingular $n \times n$ matrix A and a collection of closed convex cones $Y(I)$, $I \in \mathscr{I}$, such that $Y(I) \subset Y(J)$ whenever $J \subset I$ and such that the following two properties hold for every $x \in P$:

(i) The set $A\phi(x) \cap Y(I^x) \cap C(I^x)$ is either empty or contains 0^n.
(ii) The set $A\phi(x) \cap Y(I^x) \neq \emptyset$.

Then, there exists a zero point of ϕ in P.

Proof. Since P is assumed to be simple and there are no redundant constraints, for every vector $v \in B^n$ there is a unique index set $I \in \mathscr{I}$ for which it holds that $v \in \text{int}(C(I))$. For $v \in B^n$, define $\pi(v) = Y(I)$ for the unique $I \in \mathscr{I}$ for which $v \in \text{int}(C(I))$. Clearly, $\pi(v)$ is a convex and closed set for every $v \in B^n$. To prove upper semicontinuity, let $(v^k, k \in \mathbb{N})$ be a convergent sequence of points in B^n and let v be its limit point. For $k \in \mathbb{N}$, let I_k be such that $v^k \in \text{int}(C(I_k))$ and let I be such that $v \in \text{int}(C(I))$. Since I and all the I_k, $k \in \mathbb{N}$, are uniquely determined and v^k converges to v, it holds that $I \subset I_k$ for sufficiently large $k \in \mathbb{N}$. Hence, $Y(I_k) \subset Y(I)$ for sufficiently large $k \in \mathbb{N}$ and therefore π is an upper semicontinuous mapping. Moreover, because of conditions (i) and (ii), π satisfies conditions (i) and (ii) of Theorem 3.1. Consequently, there exists a zero point of ϕ on P. □

The conditions in the theorem for a point x in P are completely determined by the set of indices that determines the face of P in which x lies.

5. Intersection Theorems

In this section, we give a general intersection theorem on compact, convex sets. Let X be again a nonempty, compact, convex set in $\mathbb{R}^n$. For some finite set of indices $\mathscr{J}$, let $\{D^j | j \in \mathscr{J}\}$ be a finite closed covering of X; i.e., for every $j \in \mathscr{J}$, the set D^j is a closed, possibly empty, subset of X and the union of all these sets is X. Let $\{c^j | j \in \mathscr{J}\}$ be some collection of vectors in $\mathbb{R}^n$. For a subset J of $\mathscr{J}$, let $C(J)$ be defined by

$$C(J) = \text{con}\{c^j | j \in J\}.$$

A collection $\{c^j | j \in J\}$ or the set J itself is called balanced if J is a nonempty subset of $\mathscr{J}$ and $0^n \in C(J)$. A point $x^* \in X$ is called an intersection point if $x^* \in \cap_{j \in J} D^j$ for some balanced set J. For $x \in X$, define the index set J^x as

$$J^x = \{j \in J | x \in D^j|\}.$$

Because $\{D^j | j \in J\}$ is a covering of X, we have that J^x is nonempty for every $x \in X$. By definition, x^* is an intersection point if and only if the index set J^{x^*} is balanced. The next theorem gives a sufficient condition for the existence of an intersection point.

Theorem 5.1. Let $\{D^j | j \in \mathscr{J}\}$ be a finite, closed covering of a nonempty, compact, convex set X in $\mathbb{R}^n$ and let $\{c^j | j \in \mathscr{J}\}$ be a collection of vectors in $\mathbb{R}^n$. Suppose that there exists a closed-valued, convex-valued, upper semicontinuous mapping $\pi: B^n \to \mathbb{R}^n$ such that, for every $x \in X$ and $v \in N(X, x) \cap B^n$, the following two properties hold:

(i) The set $C(J^x) \cap \pi(v) \cap \{y | y = \mu v, \mu \geq 0\}$ is either empty or contains 0^n.
(ii) The set $C(J^x) \cap \pi(v) \neq \emptyset$.

Then, there exists an intersection point.

Proof. Define $\phi: X \to \mathbb{R}^n$ by

$$\phi(x) = C(J^x), \qquad x \in X.$$

Since $\{D^j | j \in \mathscr{J}\}$ is a closed covering of X, we have that ϕ is an upper semicontinuous mapping. Moreover, for every $x \in X$, since J^x is nonempty and $C(J^x)$ is the convex hull of a finite number of points, $\phi(x)$ is nonempty, convex, and compact. Because of conditions (i) and (ii), the mapping ϕ satisfies all the conditions of Theorem 3.1 and therefore there exists $x^* \in X$ satisfying $0^n \in \phi(x^*)$; i.e., x^* is an intersection point. □

In the remaining part of this section, we will show that several known intersection theorems, like the ones of KKM (Ref. 6), Scarf (Ref. 11), Shapley (Ref. 12), and Ichiishi (Ref. 13), follow as special cases of Theorem 5.1.

The set

$$S^n = \left\{ x \in \mathbb{R}^n_+ | \sum_{i=1}^{n} x_i = 1 \right\}$$

is called the unit simplex, which is a simple example of polytope. For $h \in I_n$, S^n_h denotes the facet

$$S^n_h = \{x \in S^n | x_h = 0\},$$

and for $T \subset I_n$,

$$S^n(T) = \bigcap_{h \in T} S^n_h.$$

We define the jth unit vector in $\mathbb{R}^n$ by e^j. The first result is the classical KKM lemma.

Example 5.1. Let $\{D^j | j \in I_n\}$ be a collection of closed sets covering the unit simplex S^n such that, for every $T \subset I_n$, the face $S^n(T)$ is contained in $\cup_{j \notin T} D^j$. Then, $\cap_{j \in I_n} D^j \neq \emptyset$.

Proof. From the definition of the simplex,

$$S^n = \{x \in \mathbb{R}^n | -x_i \leq 0, i \in I_n, \text{ and } 1^n \cdot x = 1\},$$

it follows immediately that

$$N(X, x) = \left\{ v \in \mathbb{R}^n | v = \alpha 1^n - \sum_{\{i | x_i = 0\}} \beta_i e^i, \alpha \in \mathbb{R}, \beta_i \geq 0 \right\}.$$

We define

$$\pi(v) = \{y \in \mathbb{R}^n | y^\top v \leq 0\} \quad \text{and} \quad c^j = (1/n)1^n - e^j, \quad j \in I_n.$$

Notice that the collection $\{c^j | j \in J\}$ is balanced if and only if $J = I_n$. To show the KKM-lemma, it remains to verify the two conditions of Theorem 5.1.

Since

$$\pi(v) \cap \{y | y = \mu v, \mu \geq 0\}$$

equals 0^n, condition (i) is clearly satisfied.

Consider $x \in S^n$. If $x \in \mathrm{int}(S^n)$, then

$$N(X, x) = \{v \in \mathbb{R}^n | v = \alpha 1^n, \alpha \in \mathbb{R}\},$$

so

$$v \in N(X, x) \setminus \{0^n\} \quad \text{implies } \pi(v) = \{x \in \mathbb{R}^n | 1^n \cdot x = 0\},$$

so

$$C(J^x) \cap \pi(v) = C(J^x) \neq \emptyset.$$

If $x \in \mathrm{bd}(S^n)$, say $x \in S^n(T)$ with $T = \{j | x_j = 0\}$, then

$$x \in D^j, \qquad \text{for some } j \in I_n \backslash T.$$

The corresponding $c^j \in C(J^x)$ satisfies

$$c^j_j = 1/n - 1 \quad \text{and} \quad c^j_k = 1/n, \quad \text{if } k \neq j.$$

Whenever $v \in N(X, x)$, it holds that

$$v = \alpha 1^n - \sum_{i \in T} \beta_i e^i,$$

so

$$v^\top c^j = -\sum_{i \in T} \beta_i (1/n) \leq 0.$$

It follows that $c^j \in \pi(v)$. □

The next example is due to Scarf (Ref. 11) and can be viewed as a dual version of the KKM lemma.

Example 5.2. Let $\{D^j | j \in I_n\}$ be a collection of closed sets covering the unit simplex S^n such that, for every $j \in I_n$, the facet S^n_j is contained in D^j. Then, $\cap_{j \in I_n} D^j \neq \emptyset$.

Proof. We define

$$\pi(v) = \{y \in \mathbb{R}^n | y^\top v \leq 0\}$$

and

$$c^j = e^j - (1/n) 1^n, \qquad j \in I_n.$$

Notice that the collection $\{c^j | j \in J\}$ is balanced if and only if $J = I_n$. To show the Scarf lemma, it remains to verify the two conditions of Theorem 5.1.

Condition (i) is satisfied for the same reason as in Example 5.1.

Consider $x \in S^n$. If $x \in \mathrm{int}(S^n)$, then

$$v \in N(X, x) \backslash \{0^n\} \quad \text{implies } \pi(v) = \{x \in \mathbb{R}^n | 1^n \cdot x = 0\},$$

so

$$C(J^x) \cap \pi(v) = C(J^x) \neq \emptyset.$$

If $x \in \mathrm{bd}(S^n)$, say $x \in S^n(T)$ with $T = \{j | x_j = 0\}$, then

$$x \in D^j, \qquad \text{for all } j \in T.$$

The vector $\bar{c} \in C(J^x)$ defined by $\bar{c} = \sum_{j \in T} (1/|T|)c^j$ satisfies

$$\bar{c}_j = 1/|T| - 1/n, \qquad \text{if } j \in T,$$

$$\bar{c}_j = -1/n, \qquad \text{if } j \in I_n \backslash T.$$

Whenever $v \in N(X, x)$, it holds that

$$v = \alpha 1^n - \sum_{j \in T} \beta_j e^j,$$

so

$$v^\top \bar{c} = -\sum_{j \in T} (1/|T| - 1/n)\beta_j \leq 0.$$

It follows that

$$\bar{c} \in \pi(v).$$ □

We continue with the Shapley lemma. We define the collection of nonempty subsets of I_n by $\mathscr{I}_n$. For $S \subset \mathscr{I}_n$, we define $e^S \in \mathbb{R}^n$ as the vector satisfying

$$e_i^S = 1, \qquad \text{if } i \in S,$$

$$e_i^S = 0, \qquad \text{otherwise.}$$

We say that a collection $\mathscr{B} = \{B_1, \ldots, B_k\}$ of members of $\mathscr{I}$ is set-balanced if there exist nonnegative numbers λ_j, $j = 1, \ldots, k$, such that

$$\sum_{j=1}^{k} \lambda_j e^{B_j} = 1^n.$$

Example 5.3. Let $\{D^S | S \in \mathscr{I}_n\}$ be a collection of closed sets covering the unit simplex S^n such that, for every $T \subset I_n$, the face $S^n(T)$ is contained in $\cup_{S \subset I_n \backslash T} D^S$. Then, there is a set-balanced family $\mathscr{B} = \{B_1, \ldots, B_k\}$ of elements of $\mathscr{I}_n$ for which $\cap_{j=1}^{k} D^{B_j} \neq \emptyset$.

Proof. We define

$$\pi(v) = \{y \in \mathbb{R}^n | y^\top v \leq 0\}$$

and

$$c^S = (1/n)1^n - e^S, \qquad S \in \mathscr{I}_n.$$

Notice that the collection $\{c^{S_1}, \ldots, c^{S_k}\}$ is balanced if an only if $\{S_1, \ldots, S_k\}$ is set-balanced. To show the KKMS-lemma, it remains to verify the two conditions of Theorem 5.1.

Since

$$\pi(v) \cap \{y | y = \mu v,\ \mu \geq 0\}$$

equals 0^n, condition (i) is clearly satisfied.

Consider $x \in S^n$. If $x \in \mathrm{int}(S^n)$, then

$$v \in N(X, x) \backslash \{0^n\} \quad \text{implies } \pi(v) = \{x \in \mathbb{R}^n | 1^n \cdot x = 0\},$$

so

$$C(J^x) \cap \pi(v) = C(J^x) \neq \emptyset.$$

If $x \in \mathrm{bd}(S^n)$, say $x \in S^n(T)$ with $T = \{j | x_j = 0\}$, then

$$x \in D^S, \qquad \text{for some } S \subset I_n \backslash T.$$

The corresponding $c^S \in C(J^x)$ satisfies

$$c_j^S = 1/n - 1/|S|, \qquad \text{if } j \in S$$

$$c_j^S = 1/n, \qquad \text{if } j \notin S.$$

Whenever $v \in N(X, x)$, it holds that

$$v = \alpha 1^n - \sum_{i \in T} \beta_i e^i,$$

so

$$v^\top c^S = -\sum_{i \in T} \beta_i c_i^S = -\sum_{i \in T} \beta_i (1/n) \leq 0.$$

It follows that $c^S \in \pi(v)$. □

The next result is due to Ichiishi (Ref. 13), which can be seen as a dual version of the Shapley intersection lemma.

Example 5.4. Let $\{D^S | S \in \mathscr{I}_n\}$ be a collection of closed sets covering the unit simplex S^n such that, for every $T \in \mathscr{I}_n$, the face $S^n(T)$ is contained in $\cup_{T \subset S} D^S$. Then, there is a set-balanced family $\mathscr{B} = \{B_1, \ldots, B_k\}$ of elements of $\mathscr{I}_n$ for which $\cap_{j=1}^k D^{B_j} \neq \emptyset$.

Proof. We define

$$\pi(v) = \{y \in \mathbb{R}^n | y^\top v \leq 0\}$$

and

$$c^S = e^S - (1/n) 1^n, \qquad S \in \mathscr{I}_n.$$

Notice that the collection $\{c^{S_1}, \ldots, c^{S_k}\}$ is balanced if and only if $\{S_1, \ldots, S_k\}$ is set-balanced. To show the Ichiishi lemma, it remains to verify the two conditions of Theorem 5.1.

Condition (i) is satisfied for the same reason as in Example 5.1.

Consider $x \in S^n$. If $x \in \text{int}(S^n)$, then

$$v \in N(X, x) \setminus \{0^n\} \quad \text{implies } \pi(v) = \{x \in \mathbb{R}^n | 1^n \cdot x = 0\},$$

so

$$C(J^x) \cap \pi(v) = C(J^x) \neq \emptyset.$$

If $x \in \text{bd}(S^n)$, say $x \in S^n(T)$ with $T = \{j | x_j = 0\}$, then

$$x \in D^S, \qquad \text{for some } S \supset T.$$

The vector $c^S \in C(J^x)$ satisfies

$$c_j^S = 1/|S| - 1/n, \qquad \text{if } j \in S$$

$$c_j^S = -1/n, \qquad \text{if } j \in I_n \setminus S.$$

Whenever $v \in N(X, x)$, it holds that

$$v = \alpha 1^n - \sum_{j \in T} \beta_j e^j,$$

so

$$v^\top c^S = -\sum_{j \in T} (1/|S| - 1/n)\beta_j \leq 0.$$

It follows that $c^S \in \pi(v)$. □

Finally, we will show that a quite general intersection theorem of van der Laan, Talman, and Yang (Ref. 15) follows also from Theorem 5.1 as a particular case. To state their result, we define first, for $I \subset I_m$, the set $A^*(I)$ by

$$A^*(I) = \{y \in \mathbb{R}^n | y^\top x \leq 0 \text{ for all } x \in A(I)\}.$$

Their theorem reads as follows.

Theorem 5.2. Let $\{D^j | j \in \mathcal{J}\}$ be a finite closed covering of a full-dimensional polytope $P = \{x \in \mathbb{R}^n | a^{i^\top} x \leq \alpha_i, i \in I\}$ and set $\{c^j | j \in \mathcal{J}\}$ be a collection of vectors in $\mathbb{R}^n$. Suppose that, for every $x \in \text{bd}(P)$, it holds that $C(J^x) \cap A^*(I^x) \neq \emptyset$. Then, there exists a balanced set $J \subset \mathcal{J}$ for which $\cap_{j \in J} D^j \neq \emptyset$.

Proof. Define

$$\pi(v) = \{y \in \mathbb{R}^n | y^\top v \leq 0\}.$$

Since P is a full-dimensional polytope, it follows that, for $x \in \mathrm{int}(P)$, it holds that $N(X, x) = \{0^n\}$, so conditions (i) and (ii) of Theorem 5.1 are obviously satisfied.

Consider $x \in \mathrm{bd}(P)$. Condition (i) of Theorem 5.1 is satisfied for the same reason as in Example 5.1. Let y be an element of $C(J^x) \cap A^*(I^x)$. Then,

$$y^\top v \leq 0, \qquad \text{for all } v \in N(X, x),$$

so

$$y \in C(J^x) \cap \pi(v), \qquad \text{for all } v \in N(X, x),$$

and condition (ii) of Theorem 5.1 is satisfied as well. It follows that there is an intersection point; i.e., there exists a balanced set $J \subset \mathscr{J}$ for which $\cap_{j \in J} D^j \neq \emptyset$. □

Theorem 5.1 generalizes Theorem 5.2 in two respects. First, it treats the case of an arbitrary nonempty, compact, and convex set X, thereby generalizing the assumption that X be a polytope. Secondly, it weakens the boundary condition.

Theorem 5.2 contains generalizations of the lemmas of KKM, Scarf, Shapley, and Ichiishi to the polytope as special cases, as well as lemmas on the cube by Freund (Ref. 16) and lemmas on the polytope by Ichiishi and Idzik (Ref. 17). Since Theorem 5.2 is a special case of Theorem 5.1, these results follow as special cases of Theorem 5.1 as well. We also refer to Gale (Ref. 18), Herings and Talman (Ref. 19), and Yang (Ref. 20–21) for other types of intersection results.

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