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PRICING AND INVESTMENTS IN MATCHING MARKETS

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Pricing and Investments in Matching Markets*

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Abstract

Different markets are cleared by different types of prices—seller-specific prices that are uniform across buyers in some markets, and personalized prices tailored to the buyer in others. We examine a setting in which buyers and sellers make investments before matching in a competitive market. We introduce the notion of premuneration values—the values to the transacting agents prior to any transfers—created by a buyer-seller match. Personalized price equilibrium outcomes are independent of premuneration values and exhibit inefficiencies only in the event of "coordination failures," while uniform-price equilibria depend on premuneration values and in general feature inefficient investments even without coordination failures. There is thus a trade-off between the costs of personalizing prices and the inefficient investments under uniform prices. We characterize the premuneration values under which uniform-price equilibria similarly exhibit inefficiencies only in the event of coordination failures.

Keywords: Directed search, matching, premuneration value, prematch investments, search.

JEL codes: C78, D40, D41, D50, D83

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1 Introduction

1.1 Investment and Matching Markets

We analyze a model in which agents match to generate a surplus which they then split. Prior to matching, the agents make investments that will affect the size of the surplus.

For example, suppose there is a continuum of workers and a continuum of firms, each with unit mass. Each worker and firm first makes a costly investment in an *attribute*—firms invest in technology while workers invest in human capital. In the second stage, workers and firms match and generate a surplus. In the absence of any monetary transfers, the firm owns the output produced by the worker, while the worker bears the cost of the effort exerted in the course of production and owns the value of the skills learned in the course of production. We call these costs and benefits the agents' *premuneration values* (from *pre* plus the Latin *munerare*, to give or to pay). Both the surplus and its division between buyer and seller premuneration values depend on the attributes the agents have chosen. The worker's human capital may enhance the quality of the output owned by the firm, and the firm's technology may enhance the value of on-the-job learning to the worker. The final division of the surplus between the worker and firm is determined by the premuneration values and a subsequent monetary transfer.

A large literature examines settings in which agents make investments before trading in a market. One extreme, discussed by Williamson (1975), treats the case of a single buyer and seller. The agents' post-investment market power then gives rise to a "hold-up" problem that prompts inefficient investments. At the other extreme, Cole, Mailath, and Postlewaite (2001) and Peters and Siow (2002) examine models with competitive post-investment markets, featuring a continuum of heterogenous buyers and sellers and frictionless trading, showing that equilibria with efficient investments exist.

Our analysis falls between these two. Our post-investment markets again feature continua of heterogeneous agents, but we introduce a key friction into the trading process, namely that firms (continuing with our example) cannot observe workers' attribute choices.

1.2 Personalized Pricing

The appropriate equilibrium notion in our setting is not obvious, to a large extent because we must determine the returns to attributes that nobody chooses. Continuing with our example, it is helpful to first consider the case in which firms *can* observe workers' investments. We refer to this as *personalized pricing*, since wages can be conditioned on the chosen attributes of *both* the firm and the worker. In this setting, an equilibrium would be a specification of the attribute chosen by each firm and worker, a wage function and a matching of firms and workers such that no agent can increase his utility by changing his decision and such that markets clear, i.e., the matching is one-to-one.

This equilibrium notion is similar to Walrasian equilibrium, except that the wage function attaches a value only to pairs of firm and worker attributes that are chosen in the investment stage, and not to unchosen attributes. In the language of Walrasian equilibrium, the price vector includes a price for every good present in the market, but not for nonexistent goods. We address the latter with a requirement that no firm (say) can unilaterally deviate to adopting some currently unchosen attribute and then match with a worker at her existing attribute, while splitting the surplus in such a way as to make both better off.

Environments in which people must decide which goods to bring to market or which investments to make before entering the market readily give rise to coordination failures. In the extreme, there is an autarkic equilibrium in which neither firms nor workers invest because no one expects the other side to invest. We could preclude such coordination failures by simply assuming that prices exist for *all* attributes, in and out of the market. On the one hand, we find the existence of such prices counterintuitive. More importantly, like Makowski and Ostroy (1995), we expect coordination failures to be endemic when people must decide what goods to market, and hence think it important to work with a model that does not preclude them.

Personalized-price equilibria can be shown to exist using a variant of the existence argument in Cole, Mailath, and Postlewaite (2001). There exist coordination-failure equilibria with inefficient investments, but there also exist exist efficient equilibria in which no worker-firm pair, matched or unmatched, could be made jointly better off, even if they could commit to their investments prior to matching. Premuneration values are irrelevant, in the sense that every personalized price equilibrium outcome remains an equilibrium outcome irrespective of the allocation of premuneration values.

1.3 Uniform Pricing

We are interested in the case in which firms cannot observe workers' attribute choices. Wages can then depend only on firms' attributes, and we speak of *uniform pricing* to emphasize that workers who have chosen different attributes must be offered the same wage. Our equilibrium notion is a specification of the attribute chosen by each firm and worker, a wage function, and a choice of firm on the part of each worker, such that no agent can increase his utility by changing his decision and such that markets clear. Analogous to personalized price equilibrium, the possibility of coordination failures again arises.

We show that a uniform-price equilibrium exists. However, these equilibria are in general inefficient, even if they exhibit no coordination failures. There exist efficient uniform-price equilibria if, and essentially only if, firms' premuneration values are independent of workers' attributes. Hence, premuneration values matter for uniform-price equilibria.

While it may be unrealistic to think that workers' attributes are literally unobservable, ascertaining these attributes may nonetheless be quite costly. Expanding beyond our worker-firm example, estimates from 11 highly selective liberal arts colleges indicate that they spent about $3,000 on admissions, i.e., ascertaining students' attributes, per matriculating student in 2004.[1] The cost for identifying whether a foreign high school diploma comes from a legitimate high school is $100.[2] There may thus be substantial savings from posting uniform prices and letting buyers sort themselves, if the premuneration values are such that uniform prices can do this sorting. Alternatively, if the premuneration values are such that uniform prices cannot duplicate the allocation of personalized prices, and if transactions costs or institutional considerations preclude personalized prices, then market outcomes will be inefficient.

1.4 Premuneration values

The premuneration values of the firms in our motivating example will typically depend on their employees' attributes—better skilled and more productive employees will enhance the quality and quantity of a firm's output. The business pages are filled with announcements of the good news that a

[1] Memorandum, Office of Institutional Research and Analysis, University of Pennsylvania, July 2004. We thank Barnie Lentz for his help with these data.

[2] "Vetting Those Foreign College Applications," *New York Times*, September 29, 2004, page A21.

firm has hired a particularly prized employee. Moving beyond this example, students are matched with universities after students have incurred substantial preparation costs and universities have hired faculty. Both sides care about the investments the other side has made. Universities reap benefits well beyond tuition revenues from talented students, and students clamor for spots at elite universities. Similarly, an aspiring faculty member cares about the investments a university has made in facilities and other faculty, while the university cares about the investment in knowledge and research capabilities of the potential recruit.

The central message of this paper is that there is a tradeoff between the costs of personalizing pricing and the inefficiency of uniform pricing. One might hope to ameliorate this tradeoff by reallocating the premuneration values. In particular, premuneration values are affected by the explicit and implicit property rights to the costs and benefits that flow from a match. For example, one could arrange the premuneration values in a university/student interaction so that the university owns all of the surplus. This would require a somewhat unconventional arrangement in which the university shares in the future income of students to whom it gives degrees. However, income-contingent loans in a number of countries (including Australia, Sweden and New Zealand) that effectively give the lender a share of students' future income (Johnstone, 2001) attest to the possibility of such an arrangement.[3]

There are often, however, constraints on the design of premuneration values. Moral hazard problems loom especially large. If universities owned a large share of students' enhanced future income streams, why would the students exert the effort required to realize this future income? How are we to measure and collect the increment to income attributable to the university education? Such an arrangement might also require changes in labor laws that preclude involuntary servitude. More generally, laws concerning workplace safety, the (in)ability to surrender legal rights, the division of marital

[3]In the summer of 2010, the UK debated the possibility of partially funding higher education though a "graduate tax" levied on college graduates' income (http://www.bbc.co.uk/news/education-10649459). Basketball star Yao Ming (Houston Rockets) has a contract with the China Basketball Association calling for 30% of his NBA earnings to be paid to the Chinese Basketball Association (in which he played prior to joining the Rockets), while another 20% will go to the Chinese government. Similar arrangements hold for Wang Zhizhi (Dallas Mavericks) and Menk Bateer (Denver Nuggets and San Antonio Spurs). (See the *Detroit News*, April 26, 2002, http://www.detnews.com/2002/pistons/0204/27/sports-475199.htm/.) We can view the initial match between Yao Ming and his Chinese team as producing a surplus that includes the enhanced value of his earnings as a result of developing his basketball skills, and the contract as setting premuneration values.

assets and the custody and sale of children may constrain the allocation of premuneration values. Our analysis points to the cost of such constraints or institutional arrangements, in the form of personalization costs or inefficient uniform pricing.

1.5 Related Literature

Our model is related to the literature on competitive search (see Guerrieri, Shimer, and Wright (2010) for a recent contribution and for pointers to the literature). We depart from a standard competitive search model in three respects. First, we include a first stage at which investments are made, whereas most competitive search models begin with buyers and sellers with exogenously given attributes. Second, we assume that both buyers and sellers are "totally heterogeneous," in the sense that no two buyers or sellers have the same cost of acquiring attributes. As a consequence of this heterogeneity, our equilibria (under either personalized or uniform pricing) perfectly separate investing agents—no two buyers who make nontrivial investments choose the same seller at the matching stage. Third, like Guerrieri, Shimer, and Wright (2010), we introduce a key friction into the competitive search model, asymmetric information, in the sense that sellers cannot condition prices on buyers' characteristics.

Our analysis differs from that of Guerrieri, Shimer, and Wright (2010) most notably in the nature of the prematching investment choice. In their model, only sellers make investments, and these consist of paying a fixed cost to participate in the second stage. Sellers who enter the second stage are homogenous, making it more difficult to screen buyers than in our model. Premuneration values play no role in their model and coordination failures cannot arise. The resulting equilibria are inefficient, and the inefficiencies arise not at the investment stage but out of constraints on the ability to screen workers. In contrast, in our model, the continuum of possible investments available to agents on both sides of the market is the source of inefficiencies, with the existence and nature of inefficiency depending upon the nature of the premuneration values.

Variants of competitive search models have been used to accommodate sources of friction other than asymmetric information. The most obvious such friction is to assume that buyers and sellers cannot instantly match. Instead, buyers must engage in costly search, including the prospects of being either temporarily or permanently unable to find a seller (e.g., Niederle and Yariv (2008) and Peters (2010)). We forgo including such considerations in order to focus on one friction at a time, in our case asymmetric information.

Our focus on creating incentives for efficient investments is shared by a number of other papers.[4] Acemoglu and Shimer (1999) analyze a worker-firm model in which firms (only) make ex ante investments. If wages are determined by post-match bargaining, then the resulting effective power gives rise to a standard hold-up problem inducing firms to underinvest. The hold-up problem disappears if workers have no bargaining power, but then there is excess entry on the part of firms. Acemoglu and Shimer show that efficient outcomes can be achieved if the bargaining process is replaced by wage posting on the part of firms, followed by competitive search. de Meza and Lockwood (2009) examine an investment and matching model that gives rise to *excess* investment. Their overinvestment possibility rests on a discrete set of investment choices and the presence of bargaining power in a noncompetitive post-investment stage. In contrast, the competitive post-investment markets of Cole, Mailath, and Postlewaite (2001) and Peters and Siow (2002) lead to efficient two-sided investments.

Moving from complete-information to incomplete-information matching models typically gives rise to issues of either screening, as considered here, or signaling. See Cole, Mailath, and Postlewaite (1995), Hopkins (forthcoming), Hoppe, Moldovanu, and Sela (2009), and Rege (2008) for models that incorporate signaling into matching models with investments.

2 The Model

2.1 The Market

There is a unit measure of buyers whose types are indexed by β and distributed uniformly on $[0,1]$, and a unit measure of sellers whose types are indexed by σ and distributed uniformly on $[0,1]$. For ease of reference, buyers are female and sellers male.

Buyers and sellers have an outside option (with payoff zero) that precludes participation in the matching process. If they do not take this option, they make choices in two stages. First, each buyer simultaneously chooses an attribute $b \in \mathbb{R}_+$ and each seller simultaneously chooses an attribute $s \in \mathbb{R}_+$. Second, buyers and sellers match, with each match generating a surplus to be split between the participating agents.

[4]Early indications that frictionless, competitive search might create investment incentives appear in Hosios (1990), Moen (1997) and Shi (2001). Eeckhout and Kircher (2010) provide an extension to asymmetric information, while Masters (2009) examines a model with two-sided investments.

Attributes are costly, but enhance the surplus generated in the second stage. To keep the analysis tractable, we assume that agents' types affect the first-stage cost of investment but not the second-stage surplus, which depends only on the attributes chosen by the agents. In particular, the cost of attribute $b \in \mathbb{R}_+$ to buyer β is given by $c_B(b, \beta)$ and the cost of attribute $s \in \mathbb{R}_+$ to seller σ is given by $c_S(s, \sigma)$. The total surplus from a match involving buyer attribute b and seller attribute s is given by $v(b, s)$.

Suppose that a buyer and seller match and create surplus $v(b, s)$, but (presumably counterfactually) no transfers are made. The surplus is still divided between the buyer and seller, and it may well be that both receive some of the surplus. A firm that does not pay its employee may capture much of the surplus, in the form of the value of the employee's production. The employee's surplus includes the cost of her effort, but may also include the value of her enhanced human capital stemming from her association with the firm. We refer to the portions of the surplus that accrue to the agents in the absence of transfers as their *premuneration* values. We let $h_B(b, s)$ denote the premuneration value of the buyer and $h_S(b, s)$ the premuneration value of the seller, with

$$h_B(b, s) + h_S(b, s) = v(b, s).$$

The premuneration values depend on the nature of the interaction between the two agents and the legal and institutional environment in which that interaction takes place. For example, the law may stipulate that the employer owns the output produced by an employee and owns any patents that emerge from the employees work, but that the employee owns the value of any contacts she makes while on the job.

The important point is that a match creates a surplus, independent of transfers. Some of this surplus is owned by the seller and the rest by the buyer, as specified by the premuneration values. Premuneration values are thus the counterparts of endowments in standard general equilibrium models.

Transfers alter the division of the surplus. A match between a buyer and seller with attribute choices (b, s) at a price p yields a gross (i.e., ignoring investment costs) buyer payoff of

$$h_B(b, s) - p,$$

and a gross seller payoff of

$$h_S(b, s) + p.$$

We assume that prices must be *uniform*, meaning that prices can be conditioned only on seller attributes. Any buyer who trades with a given seller does so at the same price, regardless of the buyer's attribute (though trades involving different sellers may occur at different prices).

There are several factors that would constrain prices to be uniform. First, it may be prohibitively expensive for sellers to observe buyers' characteristics. For example, firms may be unable to observe whether their potential employees have invested in effective work habits. Second, tailoring prices to buyers' attribute choices may entail prohibitive menu costs. A college may prefer to set uniform prices rather than bear the cost of an admissions department to carefully vet applicants. Similarly, it may be costless to use generic contract forms to make a standard offer to every buyer who appears, while tailoring offers to buyers' characteristics requires a costly legal process. Third, legal restrictions may prescribe uniform pricing. For example, employers may be prohibited from discriminating against potential employees whose attributes make them potentially expensive health risks, or union contracts may prohibit wage discrimination.

In each case, the constraints that give rise to uniform pricing also determine which of the two parties' attributes prices can be conditioned on. If buyer attributes are unobservable, then the only possibility is to condition prices on seller attributes. It will be convenient to consistently call the side of the market on which prices can be conditioned *sellers*. Prices may then be either positive or negative, and the agent we call a seller may in ordinary parlance be called either a buyer or seller.

2.2 Example: Basic Structure

We introduce here an example that we carry throughout the analysis. The premuneration values are such that a fixed share $\theta \in (0, 1]$ of the surplus goes to the buyer (Footnote 5 explains why $\theta = 0$ is excluded), so that

$$h_B(b, s) = \theta bs \quad \text{and} \quad h_S(b, s) = (1 - \theta)bs,$$

where the surplus function is given by $v(b, s) = bs$ and the cost functions by

$$c_B(b, \beta) = \frac{b^3}{3\beta} \quad \text{and} \quad c_S(s, \sigma) = \frac{s^3}{3\sigma}.$$

It is then a straightforward calculation (with details in Appendix A.1) that the efficient outcome entails attribute-choice functions

$$\mathbf{b}(\beta) = \beta \quad \text{and} \quad \mathbf{s}(\sigma) = \sigma,$$

and positive assortative matching, so that seller σ matches with buyer $\beta = \sigma$, and the pair produces total surplus σ^2 for a total net surplus $\frac{1}{3}\sigma^2$.

3 Equilibrium

3.1 Assumptions

Assumption 1 (Supermodularity) *The premuneration values $h_B : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ and $h_S : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ are $\mathcal{C}^2$, increasing in b and s, and satisfy*[5]

$$\frac{\partial^2 h_B}{\partial b \partial s} > 0 \qquad \textit{and} \qquad \frac{\partial^2 h_S}{\partial b \partial s} \geq 0.$$

There is a simple class of problems for which this assumption holds that includes our example: premuneration values constitute fixed shares of the surplus, or $h_B(b,s) = \theta v(b,s)$ and $h_S(b,s) = (1-\theta)v(b,s)$ for some $\theta \in (0,1]$, and the surplus function $v : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is strictly supermodular ($\partial^2 v / \partial b \partial s > 0$), as well as (twice continuously) differentiable and increasing in b and s.

Our next assumption is a "no free surplus" requirement that matches are not profitable without investments:

Assumption 2 (Essentiality) *The premuneration values $h_B(b,0)$ and $h_B(0,s)$ are constant in b and s, respectively, and*

$$h_B(0,0) + h_S(0,0) \quad = \quad 0.$$

The following single-crossing condition requires that higher-index buyers and sellers are more productive, in the sense that they have lower investment costs:

[5] The asymmetry in this assumption—it requires a strict inequality on the cross partial of h_B, but only a weak inequality on that of h_S—reflects our convention that sellers set prices. If the derivative for buyers is zero, then every buyer will attempt to purchase from the same seller, destroying all hope of sorting buyers. Peters (2010) illustrates the complications that arise if buyers' premuneration values do not depend on sellers' characteristics. However, Section 4.2 shows that there exist efficient uniform-price equilibrium outcomes if and only if seller premuneration values do not depend on buyer attribute choices, making it important to include the weak inequality for the seller. As will become clear, this zero second derivative for the seller poses no difficulty. The asymmetry that appears in the first part of Assumption 2 similarly arises out of the convention that sellers set prices, though this part of the Assumption is more technical in nature, allowing us to rule out some troublesome boundary cases.

Assumption 3 (Single-crossing) *The cost function $c_B : \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+$ is $\mathcal{C}^2$, strictly increasing and convex in b, with $c_B(0,\beta) = 0 = \partial c_B(0,\beta)/\partial b$ and*

$$\frac{\partial^2 c_B}{\partial b \partial \beta} < 0.$$

The cost function c_S satisfies analogous conditions.

Our next assumption ensures that efficient attribute choices exist and are bounded.

Assumption 4 (Boundedness) *There exists $\bar{b}$ such that for all $b > \bar{b}$, $s \in \mathbb{R}_+$, $\beta \in [0,1]$ and $\sigma \in [0,1]$,*

$$v(b,s) - c_B(b,\beta) - c_S(s,\sigma) < 0.$$

A similar statement, with an analogous $\bar{s}$, applies to sellers.

3.2 Feasible Outcomes

We next define feasible matchings between buyers and sellers. We denote by $\mathbf{b} : [0,1] \to [0,\bar{b}]$ and $\mathbf{s} : [0,1] \to [0,\bar{s}]$ the Lebesgue-measurable functions describing the attributes chosen by buyers and sellers.

The closures of the sets of attributes chosen by buyers and sellers respectively are denoted by $\mathcal{B} \equiv \mathrm{cl}(\mathbf{b}([0,1]))$ and $\mathcal{S} \equiv \mathrm{cl}(\mathbf{s}([0,1]))$. We refer to $\mathcal{B}$ and $\mathcal{S}$ as the set of *marketed attributes*. Let $\lambda_{\mathcal{B}}$ and $\lambda_{\mathcal{S}}$ be the measures induced on $\mathcal{B}$ and $\mathcal{S}$ by the agents' attribute choices: for Borel sets $\mathcal{B}' \subset \mathcal{B}$ and $\mathcal{S}' \subset \mathcal{S}$,

$$\begin{aligned} & \lambda_{\mathcal{B}}(\mathcal{B}') = \lambda\{\beta \in [0,1] : \mathbf{b}(\beta) \in \mathcal{B}'\} \\ \text{and} \qquad & \lambda_{\mathcal{S}}(\mathcal{S}') = \lambda\{\sigma \in [0,1] : \mathbf{s}(\sigma) \in \mathcal{S}'\}, \end{aligned}$$

where λ is Lebesgue measure. The measures of buyers and of sellers who choose the zero attribute are denoted by $\underline{\beta} \equiv \sup\{\beta : \mathbf{b}(\beta) = 0\}$ and $\underline{\sigma} \equiv \sup\{\sigma : \mathbf{s}(\sigma) = 0\}$.

We simplify the analysis by restricting attention to equilibrium attribute-choice functions that are strictly increasing when positive (i.e., $\mathbf{b}(\beta) > 0$ and $\beta' > \beta$ imply $\mathbf{b}(\beta') > \mathbf{b}(\beta)$, and similarly for $\mathbf{s}$) and that assign equal masses of buyers and sellers to zero attribute choices. We show that equilibria exist with attribute choice functions satisfying these restrictions. More general feasible matchings could be defined, but at the cost of considerable technical complication.

Definition 1 *Suppose* $\mathbf{b}$ *and* $\mathbf{s}$ *are strictly increasing when positive and that* $\underline{\sigma} = \underline{\beta}$*. A* feasible matching *is a pair of measure-preserving functions* $\tilde{b} : (\mathcal{S}, \lambda_S) \to (\mathcal{B}, \lambda_B)$ *and* $\tilde{s} : (\mathcal{B}, \lambda_B) \to (\mathcal{S}, \lambda_S)$ *satisfying*

$$\tilde{s}(\tilde{b}(s)) = s \text{ for all } s \in \mathbf{s}((\underline{\sigma}, 1]), \tag{1}$$

$$\text{and} \qquad \tilde{b}(\tilde{s}(b)) = b \text{ for all } b \in \mathbf{b}((\underline{\beta}, 1]). \tag{2}$$

Given a feasible matching $(\tilde{b}, \tilde{s})$, $\tilde{b}(s)$ specifies the buyer attribute matched to a seller with attribute s, and $\tilde{s}(b)$ specifies the seller attribute matched to a buyer with attribute b. Observe that equations (1) and (2) imply that $\tilde{s}$ is one-to-one on $\mathbf{b}((\underline{\beta}, 1])$ and $\tilde{b}$ is one-to-one on $\mathbf{s}((\underline{\sigma}, 1])$. The measure-preserving requirement on $\tilde{b}$ ensures that the measure of any set of sellers is equal to the measure of the set of buyers with whom they are matched, i.e., $\lambda_{\mathcal{B}}(\tilde{b}(\mathcal{S}')) = \lambda_{\mathcal{S}}(\mathcal{S}')$ for all Borel $\mathcal{S}' \subset \mathcal{S}$ (and similarly for $\tilde{s}$).

We have simplified the analysis by defining the matching functions $\tilde{b}$ and $\tilde{s}$ on the closures $\mathcal{S}$ and $\mathcal{B}$ of the sets of chosen attributes. In many cases of interest, efficient attribute-choice functions are discontinuous (see Cole, Mailath, and Postlewaite (2001, Section 2) for an example of discontinuous attribute-choice functions with personalized pricing (cf. Section 6.1)). Since the sets $\mathcal{B}$ and $\mathcal{S}$ are the *closures* of the sets of attribute choices, a seller σ (with attribute choice $s(\sigma)$) may be matched with a buyer attribute choice b that is not chosen by any buyer. We interpret such a seller as matching with a buyer whose attribute choice is arbitrarily close to b, while retaining the convenience of saying that $s(\sigma)$ matches with b. Defining feasible matchings on either the agents directly or on the sets of attributes (rather than their closures) would avoid this interpretation, at the cost of requiring the equivalent but more complicated formulation used in Cole, Mailath, and Postlewaite (2001).

Definition 2 *A* feasible outcome $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s})$ *is a pair of attribute-choice functions* $\mathbf{b}$ *and* $\mathbf{s}$ *that are strictly increasing when positive and satisfy* $\underline{\sigma} = \underline{\beta}$*, along with a feasible matching* $(\tilde{b}, \tilde{s})$*.*

3.3 Uniform Pricing

Sellers post prices that depend on their own attribute choices, but not the attributes of buyers. We describe these prices by a *uniform-price function* $p_U : \mathcal{S} \to \mathbb{R}$.

Given a feasible outcome $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s})$ and a uniform-price function p_U the payoffs to a buyer β choosing $b \in \mathcal{B}$ and a seller σ choosing $s \in \mathcal{S}$ are

$$\Pi_B(b, \beta) \equiv h_B(b, \tilde{s}(b)) - p_U(\tilde{s}(b)) - c_B(b, \beta)$$

and

$$\Pi_S(s, \sigma) \equiv h_S(\tilde{b}(s), s) + p_U(s) - c_S(s, \sigma).$$

Under uniform pricing, sellers cannot condition on buyer attributes. Consequently, sellers choose only their own attributes. Buyers, on the other hand, choose attributes and can choose any marketed seller attribute regardless of their own attribute choice. These choices should maximize payoffs. A buyer β *optimizes (at* $\mathbf{b}$*) given* p_U if

$$\Pi_B(\mathbf{b}(\beta), \beta) = \max_{(b,s) \in \mathbb{R}_+ \times \mathcal{S}} h_B(b, s) - p_U(s) - c_B(b, \beta). \tag{3}$$

Similarly, a seller σ *optimizes (at* $\mathbf{s}$*) given* p_U if

$$\Pi_S(\mathbf{s}(\sigma), \sigma) = \max_{s \in \mathcal{S}} \ h_S(\tilde{b}(s), s) + p_U(s) - c_S(s, \sigma). \tag{4}$$

3.4 Equilibrium

The uniform-price function p_U determines the payoff to a buyer for *any* attribute he chooses and any seller he matches with, since prices do not depend on the buyers' attribute choices. It also determines the payoff to any seller who chooses a marketed attribute (i.e., $s \in \mathcal{S}$), but not for nonmarketed attributes, since such attributes are not priced by the function p_U. We think of a seller who chooses a nonmarketed attribute as naming the price at which he is willing to trade, and then trading with one of the buyers willing to trade at this price, if there are any. However, this attribute and price combination potentially attracts many buyer attributes, all of which are indistinguishable to the seller. The following definition requires that the seller's deviation to (s', p) with $s' \notin \mathcal{S}$ be profitable *irrespective* of the buyer attracted.[6]

Definition 3 *Given* $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s}, p_U)$*, there is a* profitable seller deviation *if there exists* σ *such that either (i)* $\Pi_S(\mathbf{s}(\sigma), \sigma) < 0$ *or (ii) there exists an unmarketed attribute choice* $s' \notin \mathcal{S}$*, a price* $p \in \mathbb{R}$*, and at least one buyer* $b' \in \mathcal{B}$ *such that*

[6] We could extend Definition 3 to cover deviations to any seller attribute (rather than simply unmarketed seller attributes), as well as deviations to other prices at the seller's current attribute. Appendix B shows that if buyers optimize given p_U and sellers have no profitable deviations in this extended sense, then sellers must also be optimizing given p_U.

$$h_B(b', s') - p \;>\; h_B(b', \tilde{s}(b')) - p_U(\tilde{s}(b')), \tag{5}$$

and for any such b'*,*

$$h_S(b', s') + p - c_S(s', \sigma) \;>\; \Pi_S(\mathbf{s}(\sigma), \sigma).$$

If $\Pi_S(\mathbf{s}(\sigma), \sigma) < 0$, the outside option is better for the seller than the prescribed choice. This part of the definition plays only a technical role in the analysis, ensuring that we are not inappropriately forcing our agents to participate in the market. We will make greater use of the second requirement, that a profitable seller deviation arises if there is some seller who can choose an unmarketed attribute and set a price that attracts some buyers, and then earn a higher payoff from any attracted buyer than in the putative equilibrium.

Remark 1 (Profitable Deviations) A seller is defined to have a profitable deviation under uniform pricing only if he is better off when matched with *any* buyer who is attracted to the deviation. Why make sellers so pessimistic? One could alternatively think of requiring only that the seller be better off given a random draw from the set of attracted buyers. Though the details of the calculations (and the existence proof) would differ considerably, the qualitative forces behind our results would remain. In particular, the essence of uniform pricing is that the seller cannot stipulate which buyers he is willing to trade with and which he is not. This inability affects the seller most starkly when we assume the seller draws the worst buyer from the set of willing buyers, but the effects remain as long as the seller cannot select the *best* buyer.

Adopting the pessimistic formulation that seller deviations must be profitable when matched with the worst willing buyer makes seller deviations less attractive and hence enlarges the set of uniform-price equilibria. Our key results (Propositions 1 and 2), establishing conditions under which there exist efficient uniform price equilibria, are rendered more powerful by such a permissive definition of equilibrium. ♦

Definition 4 *A feasible outcome* $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s})$ *and a uniform-price function* $p_U : \mathcal{S} \to \mathbb{R}$ *constitute a* uniform-price equilibrium *if all agents optimize given* p_U *and the seller has no profitable deviations.*

Remark 2 The definition of a uniform-price equilibrium is reminiscent of that of a subgame-perfect equilibrium of a game, but with many of the details of the game left unspecified. In particular, given a candidate equilibrium, the deviations in the agents' choices (attribute choices and matching) that would preclude this outcome and price from being an equilibrium are identified without specifying the precise result of the deviations. For example, suppose that given an outcome $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s})$, buyer β could get a higher payoff by deviating and choosing seller attribute s' rather than the prescribed seller attribute $\tilde{s}(b(\beta))$. This would result in there being two buyers matched with seller s', and if we were to model this as a well-defined game we would have to specify which buyer ends up matched with the seller. One could provide such specificity, but doing so gives rise to a number of arbitrary choices and technical issues that obscure the underlying economics. Analogous to the definition of Walrasian equilibrium, we simply say that an outcome and price is an equilibrium when no such deviations exist. ♦

Remark 3 (Complete Pricing) By altering Definition 4 to require p_U to have domain $[0, \bar{s}]$, thereby setting a price for *every* seller attribute (whether marketed or not), and expanding to $[0, \bar{s}]$ the set of seller attribute choices over which the buyer optimizes, we obtain a *complete* uniform-price equilibrium. Notice, however, that the matching function is still restricted to marketed attributes, and hence the seller's payoff when choosing an unmarketed attribute is still separately defined as in Definition 3. ♦

Remark 4 (Hedonic Pricing) In a uniform-price equilibrium, each buyer faces prices over seller attributes, and so it is tempting to interpret the prices as hedonic prices. However, since sellers care about buyer attributes and the prices are not a function of these attributes, all payoff-relevant characteristics are not priced.[7] Accordingly, a uniform-price equilibrium is not an equilibrium in hedonic prices. ♦

3.5 Example: A Uniform-Price Equilibrium

Under uniform pricing, buyer β faces a uniform-price schedule p_U and chooses a buyer attribute b and a seller attribute $s \in \mathcal{S}$ to solve

$$\max_{b,s} \ \theta b s - p_U(s) - \frac{b^3}{3\beta}.$$

[7]Of course, in equilibrium, each seller can infer the buyer attribute that is matched with each marketed attribute at the equilibrium price.

When choosing an attribute s, the seller is selected by a buyer with attribute $b = \tilde{b}(s)$ and receives prices p_U. The seller σ thus solves

$$\max_s \; (1-\theta)\tilde{b}(s)s + p_U(s) - \frac{s^3}{3\sigma}.$$

The uniform-price equilibrium is given by the following collection (the derivation appears in Appendix A.2):

$$\mathbf{b}(\beta) = \theta^{\frac{2}{3}}(2-\theta)^{\frac{1}{3}}\beta, \tag{6}$$

$$\mathbf{s}(\sigma) = \theta^{\frac{1}{3}}(2-\theta)^{\frac{2}{3}}\sigma, \tag{7}$$

$$p_U(s) = \frac{\theta}{2}\left(\frac{\theta}{2-\theta}\right)^{1/3} s^2, \tag{8}$$

$$\text{and} \qquad \tilde{b}(s) = \left(\frac{\theta}{2-\theta}\right)^{1/3} s. \tag{9}$$

When $\theta = 1$, this uniform-price equilibrium gives the efficient outcome calculated in Section 2.2. In this case, the restriction to uniform pricing imposes no efficiency costs, and giving sellers the ability to condition prices on buyer attributes would have no effect on behavior or payoffs. Conversely, when $\theta < 1$, the uniform-price equilibrium is inefficient, in that the generated surplus of almost all matched pairs is not maximized. We discuss this inefficiency further in Section 4.3.

Note that the equilibrium is not unique. In particular, all buyers and sellers choosing the zero attribute is also an equilibrium outcome.

4 Efficiency

When are uniform-price equilibrium outcomes efficient? Efficiency fails (i.e., total surplus is not maximized) when either the wrong agents are matched or the wrong attributes agents are chosen by matched.

4.1 Efficient Matching

Efficiency requires that the second-stage matching be positively assortative in attributes. The supermodularity assumptions on premuneration values guarantee this positive assortativity in equilibrium.

Lemma 1 *In any uniform-price equilibrium* $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s}, p_U)$, $\tilde{b}$ *and* $\tilde{s}$ *are strictly increasing for strictly positive attributes, and so the matching is positively assortative in attributes.*

Proof. Suppose $\tilde{b}$ is not strictly increasing. Since $\tilde{b}$ is one-to-one on $\mathbf{s}((\underline{\sigma}, 1])$ (see Definition 1 and its following comment), there exists $0 < s_1 < s_2$ with $b_1 \equiv \tilde{b}(s_1) > \tilde{b}(s_2) \equiv b_2$. Adding

$$h_B(b_1, s_1) - p_U(s_1) \geq h_B(b_1, s_2) - p_U(s_2)$$

and

$$h_B(b_2, s_2) - p_U(s_2) \geq h_B(b_2, s_1) - p_U(s_1)$$

gives

$$h_B(b_1, s_1) + h_B(b_2, s_2) \geq h_B(b_1, s_2) + h_B(b_2, s_1),$$

contradicting the strict supermodularity of h_B.

Equation (2) then implies that $\tilde{s}$ is strictly increasing. ∎

4.2 Efficient Investments

Efficiency at the investment stage requires that the attribute choice functions $(\mathbf{b}, \mathbf{s})$ satisfy

$$(\mathbf{b}(\phi), \mathbf{s}(\phi)) \in \arg\max_{b,s \in \mathbb{R}_+} W(b, s, \phi),$$

where

$$W(b, s, \phi) \equiv v(b, s) - c_B(b, \phi) - c_S(s, \phi).$$

This efficiency is not guaranteed. We begin with some intuition, appropriate when equilibrium is characterized by first-order conditions. Fix a uniform-price equilibrium. By standard incentive compatibility arguments, the uniform-price function is differentiable. The first-order conditions implied for the buyer's choice of attribute b and matching attribute choice s in a uniform-price equilibrium are

$$0 = \frac{dh_B(b, s)}{db} - \frac{dc_B(b, \beta)}{db} \tag{10}$$

$$\text{and} \qquad 0 = \frac{dh_B(b, s)}{ds} - \frac{dp_U(s)}{ds}, \tag{11}$$

while the seller's first-order condition for choosing s is (assuming $\tilde{b}$ is differentiable)

$$0 = \frac{dh_S(\tilde{b}(s), s)}{db} \frac{d\tilde{b}(s)}{ds} + \frac{dh_S(\tilde{b}(s), s)}{ds} + \frac{dp_U(s)}{ds} - \frac{dc_S(s, \sigma)}{ds}. \tag{12}$$

Using (11) to eliminate $dp_U(s)/ds$ in (12) and then using the identity $v(b,s) = h_B(b,s) + h_S(b,s)$ in (10) and (12), these three first-order conditions can be reduced to

$$0 = \frac{dv(b,s)}{db} - \frac{dh_S(b,s)}{db} - \frac{dc_B(b,\beta)}{db}$$

and
$$0 = \frac{dh_S(b,s)}{db}\frac{d\tilde{b}(s)}{ds} + \frac{dv(b,s)}{ds} - \frac{dc_S(s,\sigma)}{ds}.$$

Efficiency requires than any matched buyer and seller maximize the difference between the surplus they generate and their investment costs, giving rise to the first-order conditions:

$$\begin{aligned} 0 &= \frac{dv(b,s)}{db} - \frac{dc_B(b,\beta)}{db} \\ 0 &= \frac{dv(b,s)}{ds} - \frac{dc_S(s,\sigma)}{ds}. \end{aligned} \tag{13}$$

Comparing these, it is immediate that the solution to the first-order conditions for an efficient allocation will be a solution for the first-order conditions for the uniform-price equilibrium if $dh_S(b,s)/db = 0$, that is, if each seller's premuneration value is independent of the attribute choice of the buyer with whom the seller is matched. Moreover, the same argument shows that when seller premuneration values are independent of buyer attributes, every uniform-price equilibrium is constrained efficient, in that no efficiency gains can be achieved without a simultaneous deviation to unmarketed buyer and seller attributes. In other words, inefficiency arises only out of coordination failure.

These arguments are summarized in the following proposition. The proof follows the preceding intuition (though it requires no differentiability assumptions), and so is relegated to Appendix C.

Proposition 1 *Suppose the sellers' premuneration values do not depend on the buyer's attribute. There exist efficient uniform-price equilibria. In addition, every uniform-price equilibrium outcome* $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s})$ *is constrained efficient:*

$$\begin{aligned} W(\mathbf{b}(\phi), \mathbf{s}(\phi), \phi) &= \max_{\substack{b \in \mathbf{b}([0,1]), \\ s \in \mathbb{R}_+}} W(b,s,\phi) \\ &= \max_{\substack{b \in \mathbb{R}_+, \\ s \in \mathbf{s}([0,1])}} W(b,s,\phi). \end{aligned}$$

The constancy of $h_S(b, s)$ in b is also essentially necessary for personalized-price equilibria to be achieved via uniform pricing. The "essentially" here is that this constancy need not hold for pairs (b, s) that are not matched in equilibrium.[8]

Proposition 2 *Suppose the efficient outcome* $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s})$ *can be supported as a uniform-price equilibrium outcome. Then for all* $s \in \mathcal{S}$*,*

$$\frac{dh_S(\tilde{b}(s), s)}{db} = 0.$$

Proof. It follows from (10) and (13) (again, without any differentiability assumptions beyond those placed on the primitives of the model in Assumptions 1 and 3), that if $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s}, p_P)$ is and efficient outcome that can be supported by uniform prices, then

$$\frac{dh_B(\tilde{b}(s), s)}{db} = \frac{dv(\tilde{b}(s), s)}{db},$$

implying $dh_S(\tilde{b}(s), s)/db = 0$. ■

4.3 Example: Efficiency

Suppose first that sellers own none of the surplus (i.e., $\theta = 1$, and hence $h_S(b, s) = 0$ and $dh_S(b, s)/db = 0$). In this case, the uniform-price equilibrium of Section 3.5 results in an efficient outcome. Consequently, no seller would gain by personalizing his price even if he could and the ability to personalize prices is irrelevant.

In the efficient outcome, the buyer's equilibrium attribute choice is $\mathbf{b}(\beta) = \beta$. Buyer attributes in the uniform-price equilibria are again a linear function of the buyer's index, with slope $\theta^{2/3}(2-\theta)^{1/3}$. This slope is below 1 for all $\theta < 1$, that is, buyers' investments are inefficiently low. The inability to personalize prices prevents sellers from offering buyers lower prices in return for higher buyer attributes. As a result, the return on buyers' investments under uniform pricing is less than the social return, and buyers choose lower attributes than would be efficient.

The magnitude of the inefficiency decreases as θ increases. The smaller the buyers' premuneration values, the larger the extent to which their attribute choices fall short of efficient levels.

[8] Analogously, the single-crossing condition is essentially necessary for a separating equilibrium in a signaling model.

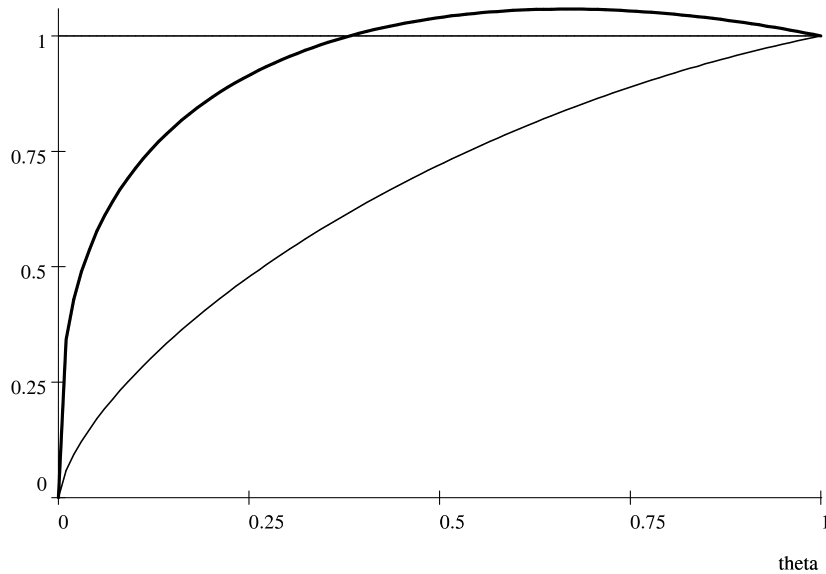

Figure 1: Uniform-price equilibrium attribute choices as a function of θ, the buyers' premuneration-value share of the surplus. The lower curved line is the coefficient of the (linear) buyer attribute-choice function, while the upper curved line is that of the seller attribute-choice function. Both coefficients are 1 in the efficient outcome.

Sellers' attribute choices in the uniform-price equilibrium are similarly a linear function of index, with slope $\theta^{1/3}(2-\theta)^{2/3}$. Since this exceeds the buyer coefficient, buyers choose smaller attributes than sellers, with buyers of attribute choice level b matching with values $s > b$.

Perhaps surprisingly, the sellers' investment behavior is not monotonic in θ, as illustrated in Figure 1. For low levels of θ—when the sellers' share of the surplus is near 1—sellers invest very little. This is to be expected since the value of their investment depends on buyers' investment, which is low in this case. The slope of the seller attribute-choice function initially increases in θ, a consequence of the increase in buyers' attribute choices and the increase in the price a seller attribute fetches. When $\theta \approx .38$, sellers make precisely the attribute choices under uniform pricing that they would in the efficient outcome. The equilibrium is still inefficient, however, as buyers invest too little. For larger values of θ, uniform pricing leads sellers to invest *more* than they do in the efficient outcome.

To understand this seller behavior, notice that a seller would like to screen the buyers to whom he sells, but the inability to personalize prices precludes doing so directly. The key to screening buyers is that high-attribute buyers have a higher willingness to pay for high-attribute sellers than do low-attribute buyers. Sellers then have an incentive to choose higher attributes (than the efficient level) and charge higher prices. As θ increases, buyer attribute choices increase, making screening all the more valuable to sellers. As a result, seller attribute choices continue to increase above their efficient levels as θ increases above .38.

Once θ reaches 2/3, sellers' attribute choices no longer increase (though seller attribute choices remain above efficient levels). Buyers' attribute choices continue to increase as θ increases, but the decreasing share that sellers receive makes screening less valuable, and hence investment less attractive.

Sellers' incentives to screen buyers lead not only to attribute choices that exceed the efficient investments, but also to attribute choices that are inefficiently high *given* the buyers' (inefficiently low) attribute choices, for all $\theta < 1$. In equilibrium seller σ is matched with buyer $\beta = \sigma$, who makes attribute choice $\theta^{2/3}(2-\theta)^{1/3}\sigma$. The net surplus (ignoring the cost of b) from a match of seller σ with such a buyer is

$$s\theta^{2/3}(2-\theta)^{1/3}\sigma - \frac{s^3}{3\sigma}.$$

The seller attribute maximizing this surplus is

$$s(\sigma) = \sigma\theta^{1/3}(2-\theta)^{1/6},$$

which is smaller than the seller's equilibrium attribute choice of $\sigma\theta^{1/3}(2-\theta)^{1/3}$.

5 Existence of Equilibrium

Appendix D establishes the existence of uniform-price equilibria, by showing the existence of complete uniform-price equilibria (see Remark 3).

Proposition 3 *If there exists* $(b, s) \in (0, \bar{b}] \times (0, \bar{s}]$ *with*

$$h_B(b,s) + h_S(0,s) - c_B(b,1) - c_S(s,1) > 0, \tag{14}$$

then there exists a complete uniform-price equilibrium in which some buyers and some sellers make strictly positive attribute choices.

Moreover, if for all $\phi \in (0,1]$, there exists $(b,s) \in (0,\bar{b}] \times (0,\bar{s}]$

$$h_B(b,s) + h_S(0,s) - c_B(b,\phi) - c_S(s,\phi) > 0, \tag{15}$$

then there exists a complete uniform-price equilibrium with $\mathbf{b}(\beta), \mathbf{s}(\sigma) > 0$ for $\beta, \sigma \in (0,1]$.

In general, condition (14) is stronger than the requirement that there be a positive surplus for the most efficient match (though (14) is implied by that requirement if $h_S(b,s)$ is independent of b, the condition of Proposition 1). Uniform-pricing equilibria are inefficient when $h_S(b,s)$ depends on b, and if this dependence is too extreme, (14) may fail and there may be no investment on either side.

Two significant complications must be confronted in the proof of existence of uniform-price equilibria: Equilibrium attribute-choice functions may be discontinuous, and we must preclude profitable deviations to attributes not in the market. These complications preclude the direct application of a fixed point theorem. We proceed indirectly, constructing a simultaneous-move three-player game whose equilibria capture the relevant behavior of uniform-price equilibria. The players include a buyer, whose payoff corresponds to the total buyer payoff in our model, a seller whose payoff is analogous but who does not set prices, and a price-setter who is penalized for market imbalance. In constructing this game, we define seller payoffs in a manner incorporating the pessimism inherent in our definition of uniform-price equilibrium. Glicksberg's fixed point theorem establishes the existence of Nash equilibria in the three-player game when strategies are constrained to be Lipschitz continuous. We then examine the limit as this constraint is removed, showing that the result corresponds to a uniform-price equilibrium of the underlying economy.

6 Discussion

6.1 Comparison with Personalized Pricing

6.1.1 Personalized Price Equilibrium

The obvious point of comparison for a uniform price equilibrium is with a scenario in which prices can be conditioned on both buyer and seller characteristics. In such a scenario, there is a *personalized-price function* $p_P : \mathcal{B} \times \mathcal{S} \to \mathbb{R}$, where $p_P(b,s)$ is the (possibly negative) price that a seller with attribute choice $s \in \mathcal{S}$ receives when selling to a buyer with attribute choice $b \in \mathcal{B}$. This gives rise to a *personalized price equilibrium*, analogous to that of a uniform price equilibrium except that sellers can charge different prices to different buyers, and the possibility of a profitable deviation to an unmarketed attribute is now open to buyers as well as sellers. Appendix E develops the details, establishing the following results.

- Personalized price equilibria exist, and are, modulo some technical differences in the specification, equivalent to the *ex post contracting equilibria* of Cole, Mailath, and Postlewaite (2001).

- Personalized price equilibria are constrained efficient, in the sense that there is no alternative, Pareto superior allocation that restricts buyers and sellers to choosing attributes marketed in the equilibrium. Personalized price equilibria may exhibit "coordination failure" inefficiencies, in which mutual gains could be realized if buyers and sellers both bring currently unmarketed attributes to the market. There exists an efficient personalized-price equilibrium.

- Premuneration values are irrelevant for personalized-price equilibria. For a given specification of premuneration values and attendant personalized price equilibrium, any other specification of premuneration values admits a personalized-price equilibrium whose outcome, including investments, matching function, and payoffs, duplicate that of the original equilibrium.

- Under the conditions of Proposition 1, uniform and personalized price equilibria coincide. In this case, the ability to personalize prices is irrelevant. Personalization brings sellers no advantage, and even the slightest cost of personalization would suffice to ensure that we see uniform pricing.

The essence of our results, culminating in Propositions 1 and 2, is to establish conditions under which personalization is redundant. If these conditions hold, uniform prices also lead to efficient equilibrium outcomes. If not, uniform prices are inextricably linked to inefficient investments. Under uniform pricing, premuneration values matter.

6.1.2 Example: Personalized Pricing

Returning to our example, suppose that sellers observe buyers' attribute choices and so can personalize their prices. If buyers and sellers optimize given the personalized-price function

$$p_P(b,s) = \frac{s^2}{2} - (1-\theta)bs, \tag{16}$$

the result is a feasible and efficient outcome.[9] In particular, given the pricing function (16), buyer β chooses the attribute $b = \mathbf{b}(\beta) = \beta$ and chooses to match with seller attribute $s = \mathbf{b}(\beta)$. The seller chooses attribute $s = \mathbf{s}(\sigma) = \sigma$. The resulting matching of buyers and sellers clears the seller attribute market (in that the distributions of demanded and supplied seller attributes agree) and the resulting outcome is efficient. Appendix A.3 contains the details and confirms that this is a personalized-price equilibrium.

If $\theta < 1$ in our example, then all buyers receive lower payoffs under uniform than under personalized prices.[10] A natural conjecture is that sellers are necessarily disadvantaged by the inability to personalize prices. The seller's equilibrium payoff in the uniform-price equilibrium is given by

$$(1-\theta)\tilde{b}(s(\sigma))s(\sigma) + p_U(s(\sigma)) - \frac{(s(\sigma))^3}{3\sigma} = \frac{1}{6}\theta(2-\theta)^2\sigma^2.$$

When $\theta = 1$, this duplicates the payoff from the personalized-price equilibrium. For θ for which sellers' attributes exceed the personalized-price equilibrium level, every seller actually earns a *higher* payoff under the uniform-price equilibrium. This higher payoff results from the higher prices that buyers are willing to pay for the higher attributes chosen by sellers when they cannot personalize prices.

Why don't we see such higher prices under personalized pricing? Suppose that given a uniform-price equilibrium, a single seller had the ability to personalize prices. Such a seller could profitably reduce his attribute choice and the price at which he trades, using personalization to exclude the undesirable buyers that render such a deviation unprofitable under uniform pricing.

6.1.3 Which Prices are Personalized?

Personalizing prices requires a seller to set a price for every buyer attribute in the market. However, personalized-price outcomes can be achieved with

[9] Note that for any seller attribute s, the price that a seller would receive in a match with a buyer with attribute b is decreasing in b—higher values of b are more valuable, and hence sellers are willing to charge less for them.

[10] The buyer's payoff under uniform pricing, $\theta\tilde{s}(b(\beta)) - p_U(\tilde{s}(b(\beta))) - \frac{(b(\beta))^3}{3\beta} = \frac{1}{6}\theta^2(2-\theta)\beta^2$, falls short of the buyer's payoff in the personalized-price equilibrium.

much simpler pricing schemes. The apparent absence of complicated pricing schemes thus need not signal the absence of personalized pricing.

The critical feature of personalized pricing is the seller's ability to exclude buyers with attribute choices lower than the seller's equilibrium match. In particular, by charging a sufficiently high price to specific buyer attribute choices, a seller can ensure that buyers with those attributes will chose not to buy. We denote this sufficiently high price by P. A personalized-price function p_P is a *uniform-rationing price* if it has the form

$$p_P(b, s) = \begin{cases} p_{UR}(s), & \forall b \geq \tilde{b}(s), \\ P, & \text{otherwise}, \end{cases}$$

for some $p_{UR} : \mathcal{S} \to \mathbb{R}_+$ and $\tilde{b} : \mathcal{S} \to \mathcal{B}$. Under uniform-rationing pricing, a seller with attribute choice s sets a uniform price $p(s) = p_{UR}(s)$, but then excludes any buyers with $b < b^{\dagger}(s)$.

Appendix E.4 provides the straightforward argument that any personalized-price equilibrium outcome can be supported by a uniform-rationing price. Hence, personalized pricing may be ubiquitous without one observing complete menus of prices. Whenever we observe sellers rejecting some buyers—colleges denying some applicants, or firms rejecting some workers as unqualified—we are observing forms of personalized pricing.[11]

6.2 Information

Suppose sellers are constrained to set uniform prices because buyers' attributes are not observable, but that buyers can certify these attributes, perhaps by taking exams or completing internships that demonstrate their skills. One might suspect that if the cost to buyers of certifying their attribute is not too high, the uncertainty might "unravel": high-attribute buyers would reveal themselves, making it optimal for the highest-attribute buyers in the remaining pool to reveal themselves, and so on until all buyers' attributes are known.[12] In addition, it seems that this cascading information revelation must make at least lower-ranked buyers worse off, if not all buyers. Indeed, to avoid such unraveling, Harvard Business School students have successfully lobbied for policies that prohibit students' divulging

[11] Peters (2010) examines a model in which personalized prices are achieved via uniform rationing.

[12] See Grossman (1981), Milgrom (1981), or Okuno-Fujiwara, Postlewaite, and Suzumura (1990) for analyses of such unraveling.

their grades to potential employers, while the Wharton student government adopted a policy banning the release of grades.[13]

In contrast, in our example, *all* buyers may be worse off when information about their attributes is suppressed than when it is known. This result holds no matter what (nonzero) share the buyers own of the surplus, and holds for all buyers. It is the distorted incentives to invest that ensure even the lowest attribute buyers would be made worse off if buyer-attribute information were suppressed.

6.3 Who Should Set Prices?

Suppose we could design the informational or legal context so that one side of the market can set prices, but cannot observe the characteristics on the other side of the market. Which should we choose? We return to our example. When $\theta = 0$, so the seller owns all of the surplus, the equilibrium collapses into the trivial equilibrium in which no surplus is generated. In this case, a buyer's payoff is solely the price p_U, which will have to be negative in order to bring buyers into the market, and buyers will choose the seller posting the smallest ("largest negative") price. Because sellers cannot condition prices on buyer attribute choice, every buyer will choose $b = 0$ in equilibrium. Similarly, when θ is positive but small, the equilibrium is markedly inefficient, featuring tiny attribute choices. This is an indication that the "wrong" side of the market is setting prices, that is, the side setting prices owns little of the surplus. Suppose personalization by a price setter is precluded for some reason other than informational asymmetries (such as legal restrictions or transaction costs), but that an alternative market design would allow buyers to post uniform prices (i.e., prices that only depend on buyer attributes). While it is more efficient for sellers to be the price setters for $\theta > \frac{1}{2}$, it would be more efficient to have buyers post prices when $\theta < \frac{1}{2}$.

6.4 Overinvestment or Underinvestment?

The inefficiencies arising in hold-up problems are well understood. The inefficiencies that arise under uniform pricing are qualitatively different, as is easily seen from the overinvestment by sellers in our example for some values of θ. The inefficiencies in uniform price equilibria in our model stem from sellers' use of attribute choice as an instrument to screen buyers, in addition to price, and the response of buyers.

[13] Ostrovsky and Schwarz (2010) investigate the optimal amount of information to disclose from the students' perspective.

It is unclear whether these inefficiencies lead in general to over- or underinvestment. To provide insight into the nature of the forces involved, it is useful to analyze why the outcome of a personalized-price equilibrium $(\mathbf{b}^P, \mathbf{s}^P, \tilde{b}^P, \tilde{s}^P, p_P)$ is not a uniform-price equilibrium outcome. Consider the outcome $(\mathbf{b}^P \mathbf{s}^P, \tilde{b}^P, \tilde{s}^P)$ with uniform price $p_U(s) = p_P(\tilde{b}(s), s)$. With this uniform price, buyers who previously matched with low attribute sellers find sellers with higher attributes more attractive, since the uniform price does not penalize low attribute buyers. This suggests that in a uniform-price equilibrium, a seller could discourage low attribute buyers by raising his price, and to avoid losing the high attribute buyers, also raising his attribute. The supermodularity in premuneration values ensures that it is possible to screen buyers in this way.

However, simply altering the seller attributes and p_U is not sufficient in general to obtain a uniform price equilibrium. There are two distinct issues. First, in a personalized-price equilibrium, from the envelope theorem, the impact on a buyer β of a marginal deviation from $\mathbf{b}^P(\beta)$ is given by

$$\frac{\partial h_B(b,s)}{\partial b} - \frac{\partial p^P(b,s)}{\partial b} - \frac{\partial c_B(b,\beta)}{\partial b},$$

evaluated at $s = \tilde{s}(b)$. In contrast, in a uniform-price equilibrium, the second term $(\partial p^P/\partial b)$ is absent. Second, the seller attributes (and prices) are different than in the personalized-price equilibrium. It is a priori unclear which effect will dominate. In our running example, the buyers underinvest, and for θ not too small, the sellers overinvest.

Characterizing the nature of the investment inefficiencies in uniform-price equilibria will necessarily depend on the specifics of the premuneration values and the attribute cost functions. Similarly, little can be said about whether it would be more efficient for one side or the other to set prices at a general level; in particular, it will not be a function only of the premuneration values of the two sides.

6.5 Premuneration Values

Our main result is that under uniform pricing, the decomposition of the total surplus of a match into the buyer and seller premuneration values affects the efficiency of prematch investments. Appropriately specified premuneration values can allow us to avoid either the cost (or impossibility) of personalizing prices or the inefficiencies of uniform pricing.

Premuneration values can sometimes be rearranged by appropriate legal and institutional innovations. The match of researchers and universities generates a surplus that includes the value of marketable patents from faculty research. Historically, universities have owned these patents, but another institutional arrangement could grant them to the faculty. The feasibility of such ownership is reflected in the decisions of many universities to unilaterally grant professors shares in the revenues from patents stemming from their research. Why aren't all premuneration values specified so as to allow efficient uniform-pricing outcomes?

Section 1.4 highlighted moral hazard problems. Monitoring considerations may also play a role. Consider a collection of heterogeneous and risk averse agents who are to be matched with risk neutral principals. One could ensure that the principal's premuneration values are independent of agent characteristics by assigning ownership of the technology to the agents. Uniform pricing per se would then impose no costs, but the agents would inefficiently bear all of the risk associated with the match, leading to inefficient actions and less valuable matches. We could instead let the principal own some or all of the technology, but now the principal's premuneration value will no longer be independent of the characteristics of the agent with whom he is matched.[14] Finally, legal restrictions may be at work.[15]

Putting these considerations together, new monitoring and contracting technologies may be valuable, not only because they can create better incentives within a match, but also because they can create more leeway for designing premuneration values and hence better matching.

[14]Even before the incentive-design stage, simply measuring and contracting on the relevant variables may pose difficulties. The University of New Mexico sued a former researcher for rights to patents that he applied for four years after he had left the university, arguing that the patents stemmed from research that he had done before leaving. ("Universities Try to Keep Inventions From Going 'Out the Back Door,' " *Chronicle of Higher Education*, May 17, 2002.) In principle, the owner of the rights to a song is entitled to a payment each time the song is played on the radio in a bar or health club, but collection is impractical.

[15]For example, Bulow and Levin (2006) note that the National Residency Matching Program matching medical residents and hospitals constrains hospitals to make the same offers to all residents. They argue that the primary effect is not inefficient matching but a transfer of surplus to the hospitals (with Niederle and Roth (2003, 2005) offering an alternative view). However, Nicholson (2003) argues that the result is an inefficient allocation of residents to specialties. Medical students who do their residency acquire training that dramatically increases their future earnings. Nicholson argues that this part of the surplus from the match (which is owned by the student) is so large in some specialties (such as dermatology, general surgery, orthopedic surgery and radiology) that if personalized prices were employed, medical students would pay hospitals handsomely for the opportunity to do their residency in these specialities. This is as compared to their stipend, which was $44,700 in 2007/8 (Association of American Medical College Survey of Household Stipends, Benefits and Funding, Autumn 2007 Report).

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Appendix

A Example, Detailed Calculations

A.1 Efficiency

Efficiency requires that for each matched pair β and σ, attribute choices b and s solve

$$\max_{b,s} bs - \frac{b^3}{3\beta} - \frac{s^3}{3\sigma},$$

giving first-order conditions

$$s - \frac{b^2}{\beta} = 0 \quad \text{and} \quad b - \frac{s^2}{\sigma} = 0.$$

Efficiency also requires positive assortative matching in attribute (and so in index, since the cost functions guarantee that attribute choices will be increasing in index). We can accordingly solve by setting $\sigma = \beta$, which in turn implies $s = b$, giving the efficient attribute-choice functions

$$\mathbf{b}(\beta) = \beta \quad \text{and} \quad \mathbf{s}(\sigma) = \sigma.$$

A.2 Derivation of (6)–(9)

The buyer chooses an attribute b and a seller attribute s with whom to match in order to solve

$$\max_{b,s} \ \theta bs + p_U(s) - \frac{b^3}{3\beta}.$$

Assuming p_U is differentiable, the first-order conditions for the buyer are

$$\begin{aligned} \theta s - \frac{b^2}{\beta} &= 0 \\ \text{and} \qquad \theta b - p_U'(s) &= 0. \end{aligned}$$

When choosing an attribute s, the seller is selected by a buyer with attribute $b = \tilde{b}(s)$. The seller σ thus solves

$$\max_{s} \ (1-\theta)\tilde{b}(s)s + p_U(s) - \frac{s^3}{3\sigma},$$

implying (assuming $\tilde{b}$ is differentiable) the first-order condition

$$(1-\theta)[\tilde{b}'(s)s + \tilde{b}(s)] + p_U'(s) - \frac{s^2}{\sigma} = 0.$$

Begin by conjecturing that the equilibrium attribute-choice functions are given by the linear functions

$$\mathbf{b}(\beta) = A\beta \tag{A.1}$$

and

$$\mathbf{s}(\sigma) = B\sigma. \tag{A.2}$$

Then, assuming that in equilibrium, a buyer of type β matches with seller of type $\sigma = \beta$, we have $\tilde{b}(s) = As/B$. Using this, rewrite the buyer's second first-order condition as $\theta As/B - p_U'(s) = 0$ and solve for the price function

$$p_U(s) = \frac{\theta A}{2B}s^2.$$

The requirement that low index traders be willing to participate in the market implies that the constant of integration equals 0. Similarly, rewrite the buyer's first first-order condition as $\theta Bb/A - b^2/\beta = 0$ and solve for b, yielding

$$b = \frac{\theta B}{A}\beta. \tag{A.3}$$

Turning to the seller, write the first-order condition as $2(1-\theta)As/B + \theta As/B - s^2/\sigma = 0$ and solve for s,

$$s = \frac{(2-\theta)A}{B}\sigma. \tag{A.4}$$

Combining (A.1) with (A.3) and (A.2) with (A.4), yields $A = \theta^{\frac{2}{3}}(2-\theta)^{\frac{1}{3}}$ and $B = \theta^{\frac{1}{3}}(2-\theta)^{\frac{2}{3}}$. It is straightforward to verify that the second order conditions are satisfied, and so the conjecture is verified.

A.3 Personalized Prices

Suppose that sellers observe buyers' attribute choices and so can personalize their prices. Consider the candidate personalized-price function

$$p_P(b,s) = \frac{s^2}{2} - (1-\theta)bs. \tag{A.5}$$

Given the pricing function (A.5), buyer β chooses a buyer attribute b and a seller attribute s (i.e., chooses to match with a seller with that attribute) to solve

$$\max_{b,s}\ \theta bs - \frac{s^2}{2} + (1-\theta)bs - \frac{b^3}{3\beta} = \max_{(b,s)}\ bs - \frac{s^2}{2} - \frac{b^3}{3\beta}.$$

Hence, buyer β chooses the attribute $b = \mathbf{b}(\beta) = \beta$ and chooses to match with seller attribute $s = \mathbf{b}(\beta)$. The implied distribution of demanded seller attributes is uniform on $[0,1]$.

When choosing an attribute s, the seller is selected by a buyer with attribute $b = \tilde{b}(s) = s$. The seller σ thus solves

$$\max_{s}\ (1-\theta)\tilde{b}(s)s + \frac{s^2}{2} - (1-\theta)\tilde{b}(s)s - \frac{s^3}{3\sigma} = \max_{s}\ \frac{s^2}{2} - \frac{s^3}{3\sigma},$$

yielding the attribute choice $s = \mathbf{s}(\sigma) = \sigma$. The implied distribution of supplied seller attributes is uniform on $[0,1]$.

The resulting matching of buyers and sellers clears the seller attribute market (in that the distributions of demanded and supplied seller attributes agree). We thus have a personalized-price equilibrium.

Equilibrium payoffs to the seller and buyer are

$$\begin{aligned} & \frac{(\mathbf{s}(\sigma))^2}{2} - \frac{(\mathbf{s}(\sigma))^3}{3\sigma} &=& \frac{\sigma^2}{2} - \frac{\sigma^3}{3\sigma} &=& \frac{1}{6}\sigma^2 \\ \text{and}\quad & \frac{(\mathbf{b}(\beta))^2}{2} - \frac{(\mathbf{b}(\beta))^3}{3\beta} &=& \frac{\beta^2}{2} - \frac{\beta^3}{3\beta} &=& \frac{1}{6}\beta^2. \end{aligned}$$

B The Absence of Profitable Deviations and Optimization given p_U

Say that a seller has an extended profitable deviation if either he has a profitable seller deviation in the sense of Definition 3, or there exists an attribute $s \in \mathcal{S}$ for which property (ii) of Definition 3 holds. Note that this includes the possibility of charging a different price for $\mathbf{s}(\sigma)$.

Lemma B.1 *Fix a feasible outcome* $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s})$ *and a uniform price* p_U*. Suppose all buyers optimize at* $\mathbf{b}$ *given* p_U *(i.e., (3) holds). If seller* σ *has no extended profitable deviation, then he is optimizing at* $\mathbf{s}$ *given* p_U*.*

Proof. Suppose there exists a seller σ and attribute choice $s' \in \mathcal{S}$ such that

$$\Pi_S(\mathbf{s}(\sigma), \sigma) < \Pi_S(s', \sigma) = h_S(\tilde{b}(s'), s') + p_U(s') - c_S(s', \sigma).$$

Let $\varepsilon = [\Pi_S(s', \sigma) - \Pi_S(\mathbf{s}(\sigma), \sigma)]/4 > 0$. Then, there exists $\delta > 0$ such that for all $b \geq \tilde{b}(s') - \delta$,

$$h_S(b, s') + p_U(s') - c_S(s', \sigma) > \Pi_S(\mathbf{s}(\sigma), \sigma) + 3\varepsilon. \tag{B.1}$$

Denote by p'' the price for an attribute s'' that makes the buyer with attribute $\tilde{b}(s')$ indifferent between s' (her equilibrium match) and s'', i.e.,

$$h_B(\tilde{b}(s'), s'') - p'' = h_B(\tilde{b}(s'), s') - p_U(s').$$

Choose $s'' > s'$ sufficiently close to s' so that

$$\left| h_S(b, s'') - c_S(s'', \sigma) - h_S(b, s') + c_S(s', \sigma) \right| < \varepsilon, \quad \forall b \in \mathcal{B}, \tag{B.2}$$

holds and $|p'' - p_U(s')| < \varepsilon/2$. From Assumption 1 (Supermodularity),

$$h_B(\tilde{b}(s') - \delta, s'') - p'' < h_B(\tilde{b}(s') - \delta, s') - p_U(s').$$

For $\hat{p} < p''$ sufficiently close to p'', we have $p'' - \hat{p} > \varepsilon/2$ and

$$h_B(\tilde{b}(s') - \delta, s'') - \hat{p} < h_B(\tilde{b}(s') - \delta, s') - p_U(s').$$

Moreover the buyer with attribute $\tilde{b}(s')$ receives strictly higher payoff from $(s'', \hat{p})$ than from $(s', p_U(s')))$. Another application of Assumption 1 shows that for all $b \leq \tilde{b}(s') - \delta$,

$$h_B(b, s'') - \hat{p} < h_B(b, s') - p_U(s').$$

From (3), for all $b \in \mathcal{B}$,

$$h_B(b, s') - p_U(s') \leq h_B(b, \tilde{s}(b)) - p_U(\tilde{s}(b)),$$

and so no buyer with attribute $b \leq \tilde{b}(s') - \delta$ finds $(s'', \hat{p})$ attractive. Thus, the pair $(s'', \hat{p})$ is a profitable deviation for seller σ, since

$$\begin{aligned} h_S(\tilde{b}(s') - \delta, s'') + \hat{p} - c_S(s'', \sigma) >& h_S(\tilde{b}(s') - \delta, s') + \hat{p} - c_S(s', \sigma) - \varepsilon \\ \geq& \Pi_S(\mathbf{s}(\sigma), \sigma) + 3\varepsilon + (\hat{p} - p'') + (p'' - p_U(s')) - \varepsilon \\ =& \Pi_S(\mathbf{s}(\sigma), \sigma) + \varepsilon, \end{aligned}$$

where the first inequality follows from (B.2) and the second from (B.1). ∎

C Proof of Proposition 1: Efficient Uniform Pricing

Let $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s}, p_P)$ be an efficient personalized-price equilibrium (see Section E; existence is established in Proposition E.1). We first show that the price function can be altered so that the seller is indifferent over buyer attributes. In particular, $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s}, \hat{p}_P)$ is a personalized-price equilibrium, where $\hat{p}_P$ is the personalized-price function given by

$$\hat{p}_P(b,s) = p_P(\tilde{b}(s),s) + h_S(\tilde{b}(s),s) - h_S(b,s), \quad \forall (b,s) \in \mathcal{B} \times \mathcal{S}. \tag{C.1}$$

Moreover, under $\hat{p}_P$, the seller is indifferent over all marketed buyer attributes.

To verify this, note that seller indifference is immediate, and it is then immediate that the seller is optimizing given $\hat{p}_P$. We then need show only that the buyer is optimizing given p_U. Suppose (E.1) fails at some β. Then, for some $(b,s) \in \mathcal{B} \times \mathcal{S}$ and for sufficiently small $\varepsilon > 0$,

$$h_B(b,s) - (\hat{p}_P(b,s) + \varepsilon) - c_B(s,\beta) > \Pi_B(\mathbf{b}(\beta),\beta).$$

Since no buyer has a profitable out-of-market deviation,

$$h_S(\tilde{b}(s),s) + \hat{p}_P(\tilde{b}(s),s) \geq h_S(b,s) + \hat{p}_P(b,s) + \varepsilon.$$

But this, with (C.1), yields a contradiction.

We now notice that if $h_S(b,s)$ does not depend on b, then neither does $\hat{p}_P$, implying that $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s}, p_U)$ for $p_U(s) = \hat{p}_P(\cdot, s)$ is a uniform-price equilibrium.

The constrained efficiency of uniform-price equilibria when the seller premuneration values do not depend on buyer attributes follows from the observation that such equilibria are also personalized price equilibria and Lemma E.2.

D Proof of Proposition 3: Existence of Equilibrium.

The existence proof is involved and indirect. We would like to construct a game Γ whose equilibria induce uniform-price equilibria. However, the obvious such game Γ is itself difficult to handle, so we work with an approximating sequence of games Γ^n. We verify that each Γ^n has an equilibrium, take limits, and show that the limiting strategy profile induces a uniform-price equilibrium. Loosely, the n index allows us to accommodate (in the limit) the possibility of jumps in the attribute-choice functions (precluded in game Γ^n).

D.1 Preliminaries

Let $P = \max\{h_B(\bar{b}, \bar{s}), h_S(\bar{b}, \bar{s})\}$. Then P is sufficiently large that no buyer would be willing to purchase any seller attribute choice $s \in [0, \bar{s}]$ at a price exceeding P, nor would any seller be willing to sell to a buyer $b \in [0, \bar{b}]$ at price less than $-P$. We can thus limit prices to the interval $[-P, P]$.

Since buyer premuneration values are $\mathcal{C}^2$, there is a Lipschitz constant Δ such that for all $\varepsilon > 0$, $s \in [0, \bar{s} - \varepsilon]$, and $b \in [0, \bar{b}]$, we have $h_B(b, s + \varepsilon) - h_B(b, s) < \Delta\varepsilon$. As a result, given a choice between seller s and seller $s + \varepsilon$ at a price higher by $\Delta\varepsilon$, buyers would always choose the former. Equilibrium prices will thus never need to increase at a rate faster than Δ.

D.2 The game Γ^n

Each game Γ^n has three players, consisting of a buyer, a seller, and a price-setter.

D.2.1 Strategy spaces

We begin by defining the strategy spaces for Γ^n.

The buyer chooses a pair of functions, $(\mathbf{b}, \mathbf{s}_B)$, where $\mathbf{b} : [0, 1] \to [0, \bar{b}]$ specifies a buyer attribute choice and $\mathbf{s}_B : [0, 1] \to [0, \bar{s}]$ a seller attribute with which to match, each as a function of the buyer's type. We denote the set of pairs of increasing functions $(\mathbf{b}, \mathbf{s}_B)$ normed by the sum of the L^1 norms on the component functions by Υ_B. In Γ^n, the buyer is restricted to the subset of Υ_B, denoted by Υ_B^n, of functions satisfying (D.1) and (D.2):

$$(\beta' - \beta)/n \leq \mathbf{b}(\beta') - \mathbf{b}(\beta) \leq n(\beta' - \beta), \quad \forall \beta < \beta' \in [0, 1], \tag{D.1}$$

and

$$(\beta' - \beta)/n \leq \mathbf{s}_B(\beta') - \mathbf{s}_B(\beta) \leq n(\beta' - \beta), \quad \forall \beta < \beta' \in [0, 1]. \tag{D.2}$$

The seller chooses an increasing function $\mathbf{s}$, where $\mathbf{s} : [0, 1] \to [0, \bar{s}]$ specifies a seller attribute choice as a function of seller's type. We denote the set of increasing functions $\mathbf{s}$ endowed with the L^1 norm by Υ_S. In Γ^n, the seller is restricted to the subset of Υ_S, denoted by Υ_S^n, of functions satisfying (D.3),

$$(\sigma' - \sigma)/n \leq \mathbf{s}(\sigma') - \mathbf{s}(\sigma) \leq n(\sigma' - \sigma), \quad \forall \sigma < \sigma' \in [0,1]. \tag{D.3}$$

The price-setter chooses an increasing function $p_U : [0,\bar{s}] \rightarrow [-P, P]$ satisfying

$$p_U(s') - p_U(s) < 2\Delta(s' - s) \tag{D.4}$$

for all $s < s' \in [0,\bar{s}]$. Denote the set of increasing functions p_U satisfying (D.4), endowed with the sup norm, by Υ_P (note that Υ_P is not indexed by n). Every function in Υ_P is continuous; indeed the collection Υ_P is equicontinuous.

The set $\Upsilon \equiv \Upsilon_B \times \Upsilon_S \times \Upsilon_P$, when normed by the sum of the three constituent norms, is a compact metric space.* It is immediate that $\Upsilon^n \equiv \Upsilon_B^n \times \Upsilon_S^n \times \Upsilon_P$ is a closed subset of Υ, and so also compact.

D.2.2 Buyer and Price-Setter Payoffs

The buyer. The buyer's payoff from $(\mathbf{b}, \mathbf{s}_B) \in \Upsilon_B^n$, when the price-setter has chosen $p_U \in \Upsilon_P$ is

$$\int \left(h_B(\mathbf{b}(\beta), \mathbf{s}_B(\beta)) - p_U(\mathbf{s}_B(\beta)) - c_B(\mathbf{b}(\beta), \beta)\right) d\beta. \tag{D.5}$$

Note that the buyer's payoff is independent of seller behavior.

For any $\mathbf{s}_B$ and $\mathbf{s}$, define

$$F_B(s) \equiv \lambda\{\beta : \mathbf{s}_B(\beta) \leq s\}$$
$$\text{and} \qquad F_S(s) \equiv \lambda\{\sigma : \mathbf{s}(\sigma) \leq s\}.$$

---

*It suffices for this conclusion to show that Υ is sequentially compact, since sequential compactness is equivalent to compactness for metric spaces (Dunford and Schwartz, 1988, p. 20). An argument analogous to that of Helly's theorem (Billingsley, 1986, Theorem 25.9) shows Υ is sequentially compact. In particular, given a sequence $\{(\mathbf{b}^m, \mathbf{s}_B^m, \mathbf{s}^m, p_U^m)\}$, we can choose a subsequence along which each function converges at every rational value in its domain to a limit $\{(\mathbf{b}^\infty, \mathbf{s}_B^\infty, \mathbf{s}^\infty, p_U^\infty)\}$. Because each function in the sequence $\{(\mathbf{b}^m, \mathbf{s}_B^m, \mathbf{s}^m, p_U^m)\}$ is increasing, so must be each limiting function $\{(\mathbf{b}^\infty, \mathbf{s}_B^\infty, \mathbf{s}^\infty, p_U^\infty)\}$. This ensures convergence at every continuity point of the limit functions, and hence almost everywhere for the functions $\mathbf{b}^m$, $\mathbf{s}_B^m$ and $\mathbf{s}^m$ and everywhere for the functions $p_U^m)\}$, sufficing (for bounded functions) for L^1 convergence in the former three cases and convergence in the sup norm in the latter.

The price-setter. The price-setter's payoff from $p_U \in \Upsilon_P$, when the buyer and seller have chosen $(\mathbf{b}, \mathbf{s}_B, \mathbf{s}) \in \Upsilon_B^n \times \Upsilon_S^n$ is given by

$$\int_0^{\bar{s}} p_U(s) \left(F_B(s) - F_S(s)\right) ds. \tag{D.6}$$

Hence, the price-setter has an incentive to raise the price of seller attribute choices in excess demand and lower the price of seller attribute choices in excess supply.

D.2.3 Seller Payoffs

The specification of the seller's payoff is complicated by the need to incorporate incentives arising from the possibility of profitable seller deviations to attribute choices outside $\mathcal{S}$. Given an attribute choice s, price p, and price function p_U, set

$$B(s, p, p_U) \equiv \left\{ b \in [0, \bar{b}] : h_B(b, s) - p \geq \max_{s' \in [0, \bar{s}]} \{h_B(b, s') - p_U(s')\} \right\}. \tag{D.7}$$

Hence, $B(s, p, p_U)$ is the set of buyer attributes that find attribute s at price p (weakly) more attractive than any attribute $s' \in [0, \bar{s}]$ at price $p_U(s')$. Note that since the buyer is constrained in Γ^n to choose seller attributes so that (D.2) is satisfied, a maximizing buyer's payoff from an attribute b (ignoring costs) need not be given by $\max_{s' \in [0, \bar{s}]}\{h_B(b, s') - p_U(s')\}$. Note also that for all s and $p_U \in \Upsilon_P$, since there is no a priori restriction on p, $B(s, p, p_U)$ is nonempty for low p (possibly requiring $p < -P$, e.g., if $p_U \equiv -P$), and it is empty if $p > p_U(s)$. Indeed, for sufficiently low p, $B(s, p, p_U) = [0, \bar{b}]$.

Lemma D.1 *(1) If $B(s, p, p_U) \neq \varnothing$, then $B(s, p, p_U) = [b_1, b_2]$ with $b_1 \leq b_2$.*

(2) For fixed s and p_U, let $\bar{p}(s, p_U) \equiv \sup\{p : B(s, p, p_U) \neq \varnothing\}$ and write $[b_1(p), b_2(p)]$ for $B(s, p, p_U)$ when $p \leq \bar{p}(s, p_U)$. Denote the set of discontinuity points in the domain of $b_j(p)$ by $\mathcal{D}_j(s, p_U)$ ($j = 1, 2$). The set $\{s : \mathcal{D}_j(s, p_U) \neq \varnothing\}$ has zero Lebesgue measure.

(3) Suppose $\{(s^\ell, p^\ell, p_U^\ell)\}_\ell$ is a sequence converging to (s, p, p_U) with $\varnothing \neq B(s^\ell, p^\ell, p_U^\ell) \equiv [b_1^\ell, b_2^\ell]$. Then $B(s, p, p_U) \neq \varnothing$, and so $B(s, p, p_U) = [b_1, b_2]$, where

$$b_1 \leq \liminf\nolimits_\ell b_1^\ell \leq \limsup\nolimits_\ell b_2^\ell \leq b_2. \tag{D.8}$$

(4) Moreover, if $p \notin \mathcal{D}_j(s, p_U) \cup \{\bar{p}(s, p_U)\}$, then $b_j = \lim_\ell b_j^\ell$.

Proof. (1) Suppose b_1, $b_2 \in B(s, p, p_U)$ with $b_1 < b_2$, and $\hat{b} \notin B(s, p, p_U)$ for some $\hat{b} \in (b_1, b_2)$. Then there exists $\hat{s} \in [0, \bar{s}]$ such that

$$h_B(\hat{b}, s) - p < h_B(\hat{b}, \hat{s}) - p_U(\hat{s}).$$

If $\hat{s} > s$, then Assumption 1 implies

$$\begin{aligned} h_B(b_2, \hat{s}) - h_B(b_2, s) &\geq h_B(\hat{b}, \hat{s}) - h_B(\hat{b}, s) \\ &> p_U(\hat{s}) - p, \end{aligned}$$

contradicting $b_2 \in B(s, p, p_U)$. Similarly, $\hat{s} < s$ contradicts $b_1 \in B(s, p, p_U)$, and so $\hat{s} = s$. But $b_2 \in B(s, p, p_U)$ then implies $p_U(s) \geq p$ while $\hat{b} \notin B(s, p, p_U)$ implies $p_U(s) < p$, the final contradiction, and so $\hat{b} \in B(s, p, p_U)$. It is immediate that $B(s, p, p_U)$ is closed.

(2) Since $B(s, p', p_U) \supset B(s, p, p_U)$ for $p' < p$, $b_1(p)$ and $b_2(p)$ are monotonic functions of p, and so are continuous except at a countable number of points. Moreover, we can apply the maximum theorem (since each of the functions in the maximum are continuous) to conclude that the right side of the inequality in (D.7) is continuous in b, and so b_1 and b_2 are left-continuous functions of p (as (D.7) features a weak inequality bounding p from above).

Suppose $p \in \mathcal{D}_1(s, p_U)$, and let $b_1^+ \equiv \lim_{p' \searrow p} b_1(p')$. Since b_1 is left-continuous, $b_1(p) < b_1^+$. Then for all $b \in [b_1(p), b_1^+]$,

$$h_B(b, s) - p = \max_{s' \in [0, \bar{s}]} h_B(b, s') - p_U(s'). \tag{D.9}$$

From the envelope theorem (Milgrom and Segal, 2002, Theorem 2), this implies for all $b \in (b_1(p), b_1^+)$,

$$\frac{\partial h_B(b, s)}{\partial b} = \frac{\partial h_B(b, s'(b))}{\partial b},$$

where $s'(b) \in \arg\max_{s' \in [0, \bar{s}]} h_B(b, s') - p_U(s')$. Assumption 1 then implies $s = s'(b)$ for all $b \in (b_1(p), b_1^+)$, and so $p = p_U(s)$.

Since $b_1^+ \in B(s, p_U(s), p_U)$, for all $s'' > s$,

$$h_B(b_1^+, s'') - h_B(b_1^+, s) \leq p_U(s'') - p_U(s)$$

so that

$$\frac{\partial h_B(b_1^+, s)}{\partial s} \leq \liminf_{s'' > s} \frac{p_U(s'') - p_U(s)}{s'' - s}.$$

On the other hand, for all $s' < s$,

$$p_U(s) - p_U(s') \leq h_B(b_1(p), s) - h_B(b_1(p), s'),$$

so that
$$\limsup_{s'<s} \frac{p_U(s) - p_U(s')}{s - s'} \leq \frac{\partial h_B(b_1(p), s)}{\partial s}.$$
Consequently, since
$$\frac{\partial h_B(b_1(p), s)}{\partial s} < \frac{\partial h_B(b_1^+, s)}{\partial s},$$
the price function p_U cannot be differentiable at s. Finally, since p_U is a monotonic function, it is differentiable almost everywhere (Billingsley, 1986, Theorem 31.2), and hence $\{s : \mathcal{D}_1(s, p_U) \neq \varnothing\}$ has zero Lebesgue measure. A similar argument shows that $\{s : \mathcal{D}_2(s, p_U) \neq \varnothing\}$ has zero Lebesgue measure.

(3) Suppose $\{(s^\ell, p^\ell, p_U^\ell)\}_\ell$ is a sequence converging to (s, p, p_U), and let $\{b^\ell\}$ be a sequence of attributes with $b^\ell \in B(s^\ell, p^\ell, p_U^\ell)$ for all ℓ. By taking a subsequence if necessary, we can assume $\{b^\ell\}$ is a convergent sequence with limit b. Since
$$h_B(b^\ell, s^\ell) - p^\ell \geq \max_{s' \in [0, \bar{s}]} \{h_B(b^\ell, s') - p_U^\ell(s')\}, \qquad \forall \ell,$$
taking limits gives
$$h_B(b, s) - p \geq \max_{s' \in [0, \bar{s}]} \{h_B(b, s') - p_U(s')\},$$
and so $b \in B(s, p, p_U)$. Hence, from part 2 of the lemma, $B(s, p, p_U) = [b_1, b_2]$, and (D.8) the follows from taking sequences $\{b^\ell\}$ with limits $\liminf_\ell b_1^\ell$ and $\leq \limsup_\ell b_2^\ell$.

(4) Consider b_2 and suppose $p \notin \mathcal{D}_2(s, p_U) \cup \{\bar{p}(s, p_U)\}$ (and so $p < \bar{p}(s, p_U)$). Hence, $b_2 = b_2^+ \equiv \lim_{p' \searrow p} b_2(p')$. Consider $b \in (b_1^+, b_2)$. For $p' > p$ sufficiently close to p, we have $b \in B(s, p', p_U)$, and so
$$h_B(b, s) - p > \max_{s' \in [0, \bar{s}]} \{h_B(b, s') - p_U(s')\}.$$
Consequently, for ℓ sufficiently large,
$$h_B(b, s^\ell) - p^\ell > \max_{s' \in [0, \bar{s}]} \{h_B(b, s') - p_U(s')\},$$
i.e., $b \in B(s^\ell, p^\ell, p_U^\ell)$. This implies that $b_2^\ell(p^\ell) \geq b$, and hence $\liminf b_2^\ell(p^\ell) \geq b$. Since this holds for all $b \in (b_1^+, b_2)$ and $\limsup_\ell b_2^\ell \leq b_2$, we have $\lim_\ell b_2^\ell = b_2$. The argument for b_1 is an obvious modification of this argument. ■

Fix (s, p, p_U) and suppose $\lambda(\{\beta : \mathbf{b}(\beta) \in B(s, p, p_U)\}) > 0$. Since $\mathbf{b}$ is strictly increasing and continuous, it then follows from Lemma D.1 that $\mathbf{b}([0,1]) \cap B(s, p, p_U) = [b'_1, b'_2]$ for some $0 \leq b'_1 < b'_2 \leq \bar{b}$. The payoff to the seller from $(s, p, \mathbf{b}, p_U)$ is given by

$$H(s, p, \mathbf{b}, p_U) \equiv h_S(b'_1, s) + p. \tag{D.10}$$

This function depends upon p_U and $\mathbf{b}$ through the dependence of b'_1 on $B(s, p, p_U)$ and $\mathbf{b}$. For later reference, note that for fixed s, $\mathbf{b}$, and p_U, the function $H(s, p, \mathbf{b}, p_U)$ is continuous from the left in p (since $\mathbf{b}$ satisfies (D.1) and both $b_1(p)$ and $b_2(p)$, defined just before Lemma D.1, are left-continuous).

We set

$$\tilde{P}(s, \mathbf{b}, p_U) \equiv \{p : \lambda(\{\beta : \mathbf{b}(\beta) \in B(s, p, p_U)\}) > 0\},$$

and noting that this set is nonempty, define

$$\bar{H}(s, \mathbf{b}, p_U) \equiv \max\left\{\sup\nolimits_{p \in \tilde{P}(s, \mathbf{b}, p_U)} H(s, p, \mathbf{b}, p_U),\ h_S(0, s) + p_U(s)\right\}. \tag{D.11}$$

Notice that if $p \in \tilde{P}(s, \mathbf{b}, p_U)$ for all $p < p_U(s)$, then the first term in (D.11) will be the maximum.[†]

The seller's payoff from $\mathbf{s} \in \Upsilon_S^n$ when the buyer and price-setter have chosen $(\mathbf{b}, \mathbf{s}_B, p_U) \in \Upsilon_B^n \times \Upsilon_P$ is then

$$\int \left(\bar{H}(\mathbf{s}(\sigma), \mathbf{b}, p_U) - c_S(\mathbf{s}(\sigma), \sigma)\right) d\sigma. \tag{D.12}$$

Taking the maximum over $\sup_{p \in \tilde{P}(s, \mathbf{b}, p_U)} H(s, p, \mathbf{b}, p_U)$ and $h_S(0, s) + p_U(s)$ effectively assumes that the seller can always sell attribute choice s at the posted price $p_U(s)$, though perhaps only attracting buyer attribute choice 0.

Note that the seller, when considering the payoff implications of altering the attribute-choice function over an interval of seller types, can ignore the seller types outside the interval, since feasibility of buyer responses is irrelevant (the comparison in B for buyer attributes is always to her payoffs, which is independent of seller behavior).

[†]It need not be true that for $s \in \mathbf{s}_B([0,1])$, $p \in \tilde{P}(s, \mathbf{b}, p_U)$ for all $p < p_U(s)$. Moreover, we may have $\bar{H}(s, \mathbf{b}, p_U) \neq h_S(\mathbf{b}(\mathbf{s}_B^{-1}(s)), s) + p_U(s)$ (see the discussion just before Lemma D.1).

D.3 Equilibrium in game Γ^n

Our next task is to show that each game Γ^n has a Nash equilibrium, and that the price-setter plays a pure strategy in any such equilibrium. To do this, we first note that the price-setter's payoff is concave in p_U (note that the buyer's and sellers's payoffs need not be even quasiconcave). If the payoff functions in game Γ^n are continuous, then Glicksberg's fixed point theorem, applied to the game where we allow the buyer and seller to randomize, yields a Nash equilibrium in which the buyer and seller may randomize, but the price-setter does not.

Lemma D.2 *The buyer, price-setter and seller payoff functions given by (D.5),(D.6) and (D.12), are continuous functions of* $(\mathbf{b}, \mathbf{s}_B, \mathbf{s}, p_U)$ *on* Υ^n*.*

Proof. We first note that for increasing, bounded functions on a compact set, L^1 convergence implies convergence almost everywhere.[‡]

Consider first the buyer. The functions $\mathbf{b}$, $\mathbf{s}_B$, and p_U are bounded functions on compact sets, and hence the absolute value of each of these functions is dominated by an integrable function. The continuity of the buyer's payoff then follows immediately from Lebesgue's dominated convergence theorem, if we can show that the convergence of $\mathbf{b}$, p_U, and $\mathbf{s}_B$ in the L^1 norm (and hence almost everywhere) implies the convergence almost everywhere of $h_B(\mathbf{b}, \mathbf{s}_B)$, $p_U(\mathbf{s}_B)$, and $c_B(\mathbf{b}(\cdot), \cdot)$ (note that we are talking about sequences of functions within a given game Γ^n). The first and the third of these follows from the continuity of h_B and c_B (from Assumptions 1 and 3), while for the remaining case it suffices to note that the collection Υ_P is equicontinuous.

Consider now the price-setter. Suppose $\mathbf{s}^\ell$ converges in L^1, and so almost everywhere, to $\mathbf{s}$. Then F_S^ℓ converges weakly to F_S (and so a.e.).[§] Similarly,

[‡]Suppose $\{f_n\}_n$, with each f_n increasing, converges in L^1 norm to an increasing function f without converging almost everywhere. Then since f is discontinuous on a set of measure zero, there exists (for example) a continuity point x of f with $\limsup f_n(x) > f(x)$ (with the case $\liminf f_n(x) < f(x)$ analogous). The continuity of f at x then ensures that for some point $y > x$, some $\varepsilon > 0$, all $z \in [x, y]$ and for infinitely many n, we have $f_n(z) \geq f_n(x) \geq f(y) + \varepsilon \geq f(z) + \varepsilon$. This in turn ensures that $\int |f_n(z) - f(z)| dz > (y - x)\varepsilon$ infinitely often, precluding the L^1 convergence of $\{f_n\}_{n=1}^\infty$ to f.

[§]Fix $\varepsilon > 0$. By Egoroff's theorem (Royden, 1988, p.73), $\mathbf{s}^\ell$ converges uniformly to $\mathbf{s}$ on a set E of measure at least $1 - \varepsilon$. Suppose s is a continuity point of F_S. There then exists $\delta > 0$ such that $|F_S(s) - F_S(s')| < \varepsilon$ for all $|s - s'| \leq \delta$. There exists ℓ' such that, for all $\sigma \in E$, for all $\ell > \ell'$, $|\mathbf{s}^\ell(\sigma) - \mathbf{s}(\sigma)| < \delta$. Consequently, $F_S^\ell(s) = \lambda\{\sigma : \mathbf{s}^\ell(\sigma) \leq s\} \leq \lambda\{\sigma : \mathbf{s}(\sigma) - \delta \leq s\} + \varepsilon = F_S(s + \delta) + \varepsilon$ and $F_S(s - \delta) - \varepsilon \leq F_S^\ell(s)$, and so $|F_S^\ell(s) - F_S(s)| < 2\varepsilon$. Hence, F_S^ℓ converges weakly to F_S.

if $\mathbf{s}_B^\ell$ converges in L^1 to $\mathbf{s}_B$, then F_B^ℓ converges a.e. to F_B. Continuity for the price-setter's payoff then follows from arguments analogous to those applied to the buyer, since we have convergence almost everywhere of $p_U[F_B - F_S]$.

Finally, we turn to the seller, where the proof of continuity is more involved. It suffices to argue that $\bar{H}(s, \mathbf{b}, p_U)$ is continuous in $(s, \mathbf{b}, p_U)$ for almost all s (since $\mathbf{s}_B$ is irrelevant in the determination of the seller's payoff and the continuity with respect to $\mathbf{s}$ is then obvious, at which point another appeal to Lebesgue's dominated convergence theorem completes the argument).

Fix a sequence $(s^\ell, \mathbf{b}^\ell, p_U^\ell)$ converging to some point $(\hat{s}, \hat{\mathbf{b}}, \hat{p}_U)$. Since we need continuity for only almost all $s \in [0, \bar{s}]$, we can assume $\mathcal{D}_1(\hat{s}, \hat{p}_U) \cup \mathcal{D}_2(\hat{s}, \hat{p}_U) = \varnothing$ (or, equivalently, that $\hat{p}_U$ is differentiable at $\hat{s}$, see the proof of Lemma D.1.2). We thus need only prove the following claim.

Claim 1 $\lim_{\ell\to\infty} \bar{H}(s^\ell, \mathbf{b}^\ell, p_U^\ell) = \bar{H}(\hat{s}, \hat{\mathbf{b}}, \hat{p}_U)$.

Proof. Since $\bar{H}^k(s, \mathbf{b}, p_U)$ is the maximum of two terms, it suffices to show that

$$\begin{aligned} \lim_{\ell\to\infty} \sup\nolimits_{p\in\tilde{P}(s^\ell,\mathbf{b}^\ell,p_U^\ell)} H(s^\ell, p, \mathbf{b}^\ell, p_U^\ell) &= \sup_{p\in\tilde{P}(\hat{s},\hat{\mathbf{b}},\hat{p}_U)} H(\hat{s}, p, \hat{\mathbf{b}}, \hat{p}_U) \\ \text{and} \qquad \lim_{\ell\to\infty} h_S(0, s^\ell) + p_U^\ell(s^\ell) &= h_S(0, \hat{s}) + \hat{p}_U(\hat{s}). \end{aligned}$$

The second is immediate from the continuity of h_S and $\hat{p}_U$ at $\hat{s}$.

We accordingly turn to the first. To conserve on notation, we define $\sup_{p\in\tilde{P}(s,\mathbf{b},p_U)} H(s, p, \mathbf{b}, p_U) \equiv \bar{\bar{H}}(s, \mathbf{b}, p_U)$.

We first show that

$$\liminf_{\ell\to\infty} \bar{\bar{H}}(s^\ell, \mathbf{b}^\ell, p_U^\ell) \geq \bar{\bar{H}}(\hat{s}, \hat{\mathbf{b}}, \hat{p}_U). \tag{D.13}$$

For all $\varepsilon > 0$ there exists $\hat{p} \in \tilde{P}(\hat{s}, \hat{\mathbf{b}}, \hat{p}_U)$ such that

$$H(\hat{s}, \hat{p}, \hat{\mathbf{b}}, \hat{p}_U) + \varepsilon/2 \geq \bar{\bar{H}}(\hat{s}, \hat{\mathbf{b}}, \hat{p}_U).$$

Since $H(\hat{s}, p, \hat{\mathbf{b}}, \hat{p}_U)$ is continuous from the left in p, there exists $\hat{p}' \notin \mathcal{D}_1(\hat{s}, \hat{p}_U) \cup \mathcal{D}_2(\hat{s}, \hat{p}_U) \cup \{\bar{p}(\hat{s}, \hat{p}_U)\}$ with $\hat{p}' \leq \hat{p}$ satisfying

$$|H(\hat{s}, \hat{p}, \hat{\mathbf{b}}, \hat{p}_U) - H(\hat{s}, \hat{p}', \hat{\mathbf{b}}, \hat{p}_U)| < \varepsilon/2,$$

and so

$$H(\hat{s}, \hat{p}', \hat{\mathbf{b}}, \hat{p}_U) + \varepsilon \geq \bar{\bar{H}}(\hat{s}, \hat{\mathbf{b}}, \hat{p}_U).$$

Since $\hat{\mathbf{b}}$ satisfies (D.1), for sufficiently large ℓ, $\hat{p}' \in \tilde{P}(s^\ell, \mathbf{b}^\ell, p_U^\ell)$, and so (applying Lemma D.1.3)

$$\lim_{\ell\to\infty} H(s^\ell, \hat{p}', \mathbf{b}^\ell, p_U^\ell) = H(\hat{s}, \hat{p}', \hat{\mathbf{b}}, \hat{p}_U).$$

Hence,

$$\liminf_{\ell\to\infty} \bar{\bar{H}}(s^\ell, \mathbf{b}^\ell, p_U^\ell) + \varepsilon \geq \bar{\bar{H}}(\hat{s}, \hat{\mathbf{b}}, \hat{p}_U), \qquad \forall \varepsilon > 0,$$

yielding (D.13).

We now argue that

$$\bar{\bar{H}}(\hat{s}, \hat{\mathbf{b}}, \hat{p}_U) \geq \limsup_{\ell\to\infty} \bar{\bar{H}}(s^\ell, \mathbf{b}^\ell, p_U^\ell), \tag{D.14}$$

which with (D.13) gives continuity.

Fix $\varepsilon > 0$. For each ℓ, there exists $p^\ell \in \tilde{P}(s^\ell, \mathbf{b}^\ell, p_U^\ell)$ such that

$$H(s^\ell, p^\ell, \mathbf{b}^\ell, p_U^\ell) + \varepsilon \geq \bar{\bar{H}}(s^\ell, \mathbf{b}^\ell, p_U^\ell). \tag{D.15}$$

Without loss of generality, we can assume $\{p^\ell\}_\ell$ is a convergent sequence, with limit $\hat{p}$. Suppose first that $\hat{p} \in \tilde{P}(\hat{s}, \hat{\mathbf{b}}, \hat{p}_U)$. If $\hat{p} \neq \{\bar{p}(\hat{s}, \hat{p}_U)\}$, it is immediate that

$$H(\hat{s}, \hat{p}, \hat{\mathbf{b}}, \hat{p}_U) + \varepsilon \geq \limsup_{\ell\to\infty} \bar{\bar{H}}(s^\ell, \mathbf{b}^\ell, p_U^\ell), \tag{D.16}$$

which (since it holds for all ε) implies (D.14).

Suppose now that $\hat{p} \notin \tilde{P}(\hat{s}, \hat{\mathbf{b}}, \hat{p}_U)$ or $p = \bar{p}(\hat{s}, \hat{p}_U)$. Since $\hat{p}_U$ is differentiable at $\hat{s}$, there cannot be a nondegenerate interval of buyer attributes indifferent between $(\hat{s}, \hat{p})$ and the unconstrained optimal seller attribute under $\hat{p}_U$. This implies $\hat{\mathbf{b}}([0,1]) \cap B(\hat{s}, \hat{p}, \hat{p}_U) = \{\hat{b}\}$ for some $\hat{b}$, and so

$$\bar{\bar{H}}(\hat{s}, \hat{\mathbf{b}}, \hat{p}_U) \geq h_S(\hat{b}, \hat{s}) + \hat{p}.$$

From Lemma D.1.3,

$$\lim_{\ell\to\infty} H(s^\ell, p^\ell, \mathbf{b}^\ell, p_U^\ell) + \varepsilon = h_S(\hat{b}, \hat{s}) + \hat{p} + \varepsilon,$$

and so (taking the lim sup of both sides of (D.15))

$$\bar{\bar{H}}(\hat{s}, \hat{\mathbf{b}}, \hat{p}_U) + \varepsilon \geq \limsup_{\ell\to\infty} \bar{\bar{H}}(s^\ell, \mathbf{b}^\ell, p_U^\ell),$$

which (since it holds for all $\varepsilon > 0$) implies (D.14). ■ ■

Allowing the buyer and seller to choose mixed strategies then gives us a game whose best responses consist of closed, convex sets. As a result, we can apply Glicksberg (1952) to conclude that we have a Nash equilibrium in which the price-setter plays a pure strategy, while the buyer and seller may mix:

Lemma D.3 *The game Γ^n has a Nash Equilibrium, $(\xi_B^n, \xi_S^n, p_U^n) \in \Delta(\Upsilon_B) \times \Delta(\Upsilon_S) \times \Upsilon_P$.*

D.4 The limit $n \to \infty$

We now examine the limit as $n \to \infty$. In particular, let $\{(\xi_B^n, \xi_S^n, p_U^n)\}_n \subset \Delta(\Upsilon_B) \times \Delta(\Upsilon_S) \times \Upsilon_P$ be a sequence of Nash equilibria of the games Γ^n. Without loss of generality (since the relevant spaces are sequentially compact), we may assume that both the sequence of equilibria converges to some limit $(\xi_B^*, \xi_S^*, p_U^*)$, and that players' payoffs also converge.

We examine the limit $(\xi_B^*, \xi_S^*, p_U^*)$. Intuitively, we would like to think of this profile as the equilibrium of a "limit game." However, the definition of this limit game is not straightforward, since the definition of the seller's payoffs in the game Γ^n relies on the strategies $\mathbf{b}$, $\mathbf{s}_B$, and $\mathbf{s}$ having properties (such as strict monotonicity and continuity) that need not carry over to their limits. In establishing properties of $(\xi_B^*, \xi_S^*, p_U^*)$, we accordingly typically begin our argument in the limit, and then pass back to the approximating equilibrium profile $(\xi_B^n, \xi_S^n, p_U^n)$ to obtain a contradiction. The latter step of the argument is notationally cumbersome, and we do not always make the approximation explicit.

Note that while the seller is best responding to ξ_B^n in choosing $\mathbf{s}$, the choice of p implicit in (D.11) is made after $(\mathbf{b}, \mathbf{s}_B)$ is realized.

While the L^1 topology does not distinguish between functions that agree almost everywhere, it will be important for some of the later developments that we make the selection indicated in the next lemma from the equivalence classes of functions that agree almost everywhere.

Lemma D.4 *The limit profile $(\xi_B^*, \xi_S^*, p_U^*)$ is pure, which we denote by $(\mathbf{b}^*, \mathbf{s}_B^*, \mathbf{s}^*, p_U^*)$. The limit functions can be (and subsequently are) taken to be increasing, and the functions $\mathbf{b}^*$, $\mathbf{s}_B^*$, and $\mathbf{s}^*$ can be (and subsequently are) taken to be continuous from the left.*

Proof. Consider the buyer (the case of the seller is analogous). Toward a contradiction, suppose the buyer's strategy (ξ_B^*, ξ_S^*) is not pure. Let $\xi_{B,b}^*$ and $\xi_{B,s}^*$ denote the marginal distributions induced on buyer and seller attributes chosen by the buyer. Then define a pair of increasing functions $\mathbf{b}' : [0,1] \to [0,\bar{b}]$ and $\mathbf{s}'_B : [0,1] \to [0,\bar{s}]$ by

$$\begin{aligned} & \mathbf{b}'(\beta) = \inf\{b : \xi^*_{B,b}(b) \geq \beta\} \\ \text{and} \qquad & \mathbf{s}'_B(\beta) = \inf\{s : \xi^*_{B,s}(s) \geq \beta\}. \end{aligned}$$

These functions constitute pure strategies for the buyer giving the same distribution of buyer and seller attributes chosen by the buyer. (For example, for any set $[\underline{b}, \bar{b}]$ of buyer attributes with $\underline{b} > 0$, $\xi^*_{B,b}[\underline{b}, \bar{b}] = \beta'' - \beta'$, where $\mathbf{b}'(\beta'') = \bar{b}$ and $\mathbf{b}'(\beta') = \underline{b}$.) However, the $\mathbf{b}'$ and $\mathbf{s}'_B$ feature positive assortativity between the buyer's types and attribute choice, and between the buyer's and the seller's attribute with which the buyer matches, while $\xi^*_{B,b}$ and $\xi^*_{B,s}$, being mixed, do not. From Assumptions 1 and 3, this positive assortativity increases the buyer's payoff, and so the constructed pure strategy strictly increases the buyer's payoff. It then follows from straightforward continuity arguments that for sufficiently large n, i.e., for a game in which the slope requirements on the buyer's strategy are sufficiently weak and the equilibrium profile $(\xi^n_B, \xi^n_S, p^n_U)$ is sufficiently close to $(\xi^*_B, \xi^*_S, p^*_U)$, there is a pure strategy sufficiently close to $\mathbf{b}'$ and $\mathbf{s}'_B$ giving the buyer a payoff higher than his supposed equilibrium payoff in Γ^n a contradiction. Hence, the buyer cannot mix.

The conclusion that each function is increasing is an implication of the observation that if a sequence of increasing functions $\{f_n\}$ converges in L^1 to a function f, then that function is increasing. ∎

It is helpful to keep in mind the nature of convergence in $\Delta(\Upsilon_B) \times \Delta(\Upsilon_S) \times \Upsilon_P$. Recalling that Υ_B, Υ_S, are each endowed with the L^1 norm and Υ_P with the sup norm, and the definition of the Prohorov metric (which metrizes weak convergence), $(\xi^n_B, \xi^n_S, p^n_U)$ converges to the pure profile $(\mathbf{b}^*, \mathbf{s}^*_B, \mathbf{s}^*, p^*_U)$ if, and only if, the following holds: For all $\varepsilon > 0$ there exists n' such that for all $n \geq n'$,

$$\xi^n_B \left(\left\{(\mathbf{b}, \mathbf{s}_B) \in \Upsilon^n_B : \int |\mathbf{b}(\beta) - \mathbf{b}^*(\beta)| d\beta < \varepsilon, \ \int |\mathbf{s}_B(\beta) - \mathbf{s}^*_B(\beta)| d\beta < \varepsilon\right\}\right) \geq 1 - \varepsilon,$$

$$\xi^n_S \left(\left\{\mathbf{s} \in \Upsilon^n_S : \int |\mathbf{s}(\sigma) - \mathbf{s}^*(\sigma)| d\sigma < \varepsilon\right\}\right) \geq 1 - \varepsilon,$$

and

$$\sup |p^n_U(s) - p^*_U(s)| < \varepsilon.$$

We next restate the nature of convergence in a more useful form:

Lemma D.5 *For all* $\varepsilon > 0$*, there exists a set* $E^\varepsilon \subset [0,1]$ *with* $\lambda(E^\varepsilon) \geq 1-\varepsilon$ *and* n_ε *such that for all* $n \geq n_\varepsilon$*,*

$$\xi_B^n\left(\{(\mathbf{b}, \mathbf{s}_B) \in \Upsilon_B^n : |\mathbf{b}(\beta) - \mathbf{b}^*(\beta)| < \varepsilon,\ |\mathbf{s}_B(\beta) - \mathbf{s}_B^*(\beta)| < \varepsilon,\ \forall \beta \in E^\varepsilon\}\right) \geq 1-\varepsilon,$$

$$\xi_S^n\left(\{\mathbf{s} \in \Upsilon_S^n : |\mathbf{s}(\sigma) - \mathbf{s}^*(\sigma)| < \varepsilon,\ \forall \sigma \in E^\varepsilon\}\right) \geq 1-\varepsilon,$$

and

$$|p_U^n(s) - p_U^*(s)| < \varepsilon,\ \forall s.$$

Moreover, the sets E^ε *are nested:* $E^{\varepsilon'} \subset E^\varepsilon$ *if* $\varepsilon < \varepsilon'$*.*

Proof. Fix $\varepsilon > 0$. We prove that there is a set E_S^ε with $\lambda(E_S^\varepsilon) > 1 - \varepsilon/3$ and an integer n'_S such that

$$\xi_S^n\left(\{\mathbf{s} \in \Upsilon_S^n : |\mathbf{s}(\sigma) - \mathbf{s}^*(\sigma)| < \varepsilon,\ \forall \sigma \in E_S^\varepsilon\}\right) \geq 1-\varepsilon \tag{D.17}$$

for all $n > n'_S$. The same argument implies a set E_B^ε and integer n'_B for the function $\mathbf{b}^*$, and a $\hat{E}_B^\varepsilon$ and n''_B for the function $\mathbf{s}_B^*$.[¶] The desired set is $E^\varepsilon = E_S^\varepsilon \cap E_B^\varepsilon \cap \hat{E}_B^\varepsilon$ and integer is $n_\varepsilon = \max\{n'_S, n'_B, n''_B\}$.

Let $\{\sigma^k\}$ be an enumeration of the discontinuities of $\mathbf{s}^*$. Since $\mathbf{s}^*$ is bounded, there exists K such that the total size of the discontinuities over $\{\sigma^k\}_{k>K}$ is less than $\varepsilon/6$.

Fix $L > 2$ such that $\{(\sigma^k - 2^{-kL}, \sigma^k + 2^{-kL})\}_{k=1}^K$ is pairwise disjoint and $2^{1-L} < \varepsilon/6$. Defining

$$E_S^\varepsilon = [0,1] \setminus \bigcup_k (\sigma^k - 2^{-kL}, \sigma^k + 2^{-kL})$$

yields a set of measure at least $1 - \varepsilon/3$.

Let E_S^K be the set given by $[0,1] \setminus \cup_{k=1}^K (\sigma^k - 2^{-kL}, \sigma^k + 2^{-kL})$; clearly $E_S^\varepsilon \subset E_S^K$. The set E_S^K can be written as the disjoint union of closed intervals

[¶] More precisely, the sets can be chosen so that, for $n > n'_B$,

$$\xi_B^n(\{(\mathbf{b}, \mathbf{s}_B) \in \Upsilon_B^n : |\mathbf{b}(\beta) - \mathbf{b}^*(\beta)| < \varepsilon,\ \forall \beta \in E_B^\varepsilon\}) \geq 1 - \varepsilon/2,$$

and, for $n > n''_B$,

$$\xi_B^n(\{(\mathbf{b}, \mathbf{s}_B) \in \Upsilon_B^n : |\mathbf{s}_B(\beta) - \mathbf{s}_B^*(\beta)| < \varepsilon,\ \forall \beta \in \hat{E}_B^\varepsilon\}) \geq 1 - \varepsilon/2,$$

so that, for $n > \max\{n'_B, n''_B\}$,

$$\xi_B^n(\{(\mathbf{b}, \mathbf{s}_B) \in \Upsilon_B^n : |\mathbf{b}(\beta) - \mathbf{b}^*(\beta)| < \varepsilon, |\mathbf{s}_B(\beta) - \mathbf{s}_B^*(\beta)| < \varepsilon,\ \forall \beta \in E_B^\varepsilon \cap \hat{E}_B^\varepsilon\}) \geq 1 - \varepsilon.$$

I_k, $k = 0, 1, \ldots, K$. There exists an $\eta > 0$ such that for all k and for all $\sigma, \sigma' \in I_k$, if $|\sigma - \sigma'| < \eta$ then $|\mathbf{s}^*(\sigma) - \mathbf{s}^*(\sigma')| < \varepsilon/3$.

Let $\{x_\ell\} \subset I_k$ be an η-grid of I_k, i.e., $x_{\ell+1} - \eta < x_\ell < x_{\ell+1}$ for all ℓ.

Consider an increasing function $\mathbf{s}$ satisfying $\int |\mathbf{s} - \mathbf{s}^*| < \varepsilon\eta/3$. We claim that for all $\sigma \in E_S^K$ (and so for all $\sigma \in E_S^\varepsilon$), $|\mathbf{s} - \mathbf{s}^*| < \varepsilon$. Observe that (D.17) then follows, since n'_S can be chosen so that $\xi_S^n\left(\left\{\mathbf{s} \in \Upsilon_S^n : \int |\mathbf{s} - \mathbf{s}^*| < \varepsilon\eta/3\right\}\right) \geq 1 - \varepsilon$ holds for all $n > n'_S$.

The claim follows from two observations:

1. $|\mathbf{s}(x_\ell) - \mathbf{s}^*(x_\ell)| < 2\varepsilon/3$: Suppose $\mathbf{s}(x_\ell) \geq \mathbf{s}^*(x_\ell) + 2\varepsilon/3$ (the other possibility is handled mutatis mutandis). Then, for all $\sigma \in (x_\ell, x_{\ell+1})$,

$$\mathbf{s}(\sigma) \geq \mathbf{s}(x_\ell) \geq \mathbf{s}^*(x_\ell) + 2\varepsilon/3 > \mathbf{s}^*(\sigma) + \varepsilon/3.$$

But this is impossible, since it would imply $\int |\mathbf{s} - \mathbf{s}^*| > \varepsilon\eta/3$.

2. For all ℓ and all $\sigma \in (x_\ell, x_{\ell+1})$, $|\mathbf{s}(\sigma) - \mathbf{s}^*(\sigma)| < \varepsilon$: Suppose $\mathbf{s}(\sigma) \geq \mathbf{s}^*(\sigma) + \varepsilon$ (the other possibility is handled mutatis mutandis). Then,

$$\mathbf{s}(x_{\ell+1}) \geq \mathbf{s}(\sigma) \geq \mathbf{s}^*(\sigma) + \varepsilon \geq \mathbf{s}^*(x_{\ell+1}) + 2\varepsilon/3,$$

contradicting the previous observation.

The last assertion of Lemma D.5 is immediate from the definition of E_S^ε.

■

Lemma D.6 *The profile* $(\mathbf{b}^*, \mathbf{s}_B^*, \mathbf{s}^*, p_U^*)$ *balances the market, i.e.,* $F_B^*(s) = F_S^*(s)$ *for all* s*. Hence,* $\mathbf{s}_B^*(x) = \mathbf{s}^*(x)$ *for almost all* $x \in [0,1]$.

Proof. Since F_B^* and F_S^* are continuous from the right, it suffices to show that they agree almost everywhere. We first argue that $F_B^*(s) - F_S^*(s) \leq 0$ almost everywhere. Suppose this is not the case, so there exists $\hat{s} < \bar{s}$ with $F_B^*(\hat{s}) - F_S^*(\hat{s}) = \varepsilon > 0$ and with $\hat{s}$ a continuity point of $F_B^* - F_S^*$. Then there exists s_1 and s_2 with $\hat{s} \in (s_1, s_2)$, $F_B^*(s) - F_S^*(s) \geq \varepsilon/2$ on $[s_1, s_2]$, and either $s_1 = 0$ or, for every $\eta > 0$, there is a value $s_\eta \in [s_1 - \eta, s_1)$ with $F_B^*(s_\eta) - F_S^*(s_\eta) < \varepsilon/2$ (note that $F_B^*(s_\eta) - F_S^*(s_\eta)$ may be negative, and so is bounded below by -1). We consider the case in which $s_1 > 0$ and $p_U^*(s_1) < p_U^*(s_2)$, with the remaining cases a straightforward simplification.

Since $F_B^*(s) - F_S^*(s) > 0$ on $[s_1, s_2]$, for fixed $p_U^*(s_1)$ and $p_U^*(s_2)$, the price-setter must be setting prices as large as possible on this interval. If not, there is a price function $\hat{p}_U \in \Upsilon_P$ with $\hat{p}_U(s) \geq p_U^*(s)$ for all s and

$\hat{p}_U(s) > p_U^*(s)$ for some s yielding strictly higher payoffs to the price-setter than p_U^* in Γ^n for sufficiently large n, when the buyer and seller choose (ξ_B^n, ξ_S^n). But this contradicts the equilibrium property of $(\xi_B^n, \xi_S^n, p_U^n)$.

Hence, there exists $s' \in [s_1, s_2]$ such that $dp_U^*(s)/ds = 2\Delta$ on (s_1, s') and $p_U^*(s) = p_U^*(s_2)$ for $s \in [s', s_2]$. That is, prices increase at the maximum rate possible until hitting $p_U^*(s_2)$ (with $s' = s_2$ possible, but since $p_U^*(s_1) < p_U^*(s_2)$, we have $s_1 < s'$). Consequently, $\mathbf{s}_B([0,1]) \cap [s_1, s_2] \subset \{s_1, s_2\}$, i.e., buyers demand only seller attribute choices s_1 and s_2 from this interval. (Since all seller attribute choices in $[s', s_2]$ command the same price, buyers demand only attribute choice s_2 from this set, while the price of a seller attribute choice increases sufficiently quickly on $[s_1, s']$ that from this set buyers demand only s_1.)

Since for every $\eta > 0$, there exists $s_\eta \in [s_1 - \eta, s_1)$ with $F_B^*(s_\eta) - F_S^*(s_\eta) < \varepsilon/2$ and yet $F_B^*(s_1) - F_S^*(s_1) \geq \varepsilon$, the buyer must choose attributes arbitrarily close to s_1 for some buyer types. This implies that there is a range of seller attributes just below s_1 with prices that are not too low, that is, there exists $\eta' > 0$ such that

$$p_U^*(s) > p_U^*(s_1) - \Delta(s_1 - s)$$

for all $s \in [s_1 - \eta', s_1)$. Consider now the price function $p_U^\eta \in \Upsilon_P$ given by

$$p_U^\eta(s) \equiv \begin{cases} p_U^*(s), & \text{if } s \geq s', \\ \min\{p_U^*(s_1 - \eta) + 2\Delta(s - s_1 + \eta),\ p_U^*(s')\}, & \text{if } s \in (s_1 - \eta, s'), \\ p_U^*(s), & \text{if } s \leq s_1 - \eta, \end{cases}$$

and note that $p_U^0 = p_U^*$. Since $p_U^\eta \geq p_U^*$, the price-setter's payoff from choosing $p_U^\eta \in \Upsilon_P$ less the payoff from p_U^* is bounded below by

$$-\int_{s_1-\eta}^{s_1} \left(p_U^\eta(s) - p_U^*(s)\right)\, ds + \int_{s_1}^{s'} \left(p_U^\eta(s) - p_U^*(s)\right) \varepsilon/2\, ds. \tag{D.18}$$

For $\eta < \eta'$ and $s \in (s_1 - \eta, s_1)$,

$$\begin{aligned} p_U^\eta(s) - p_U^*(s) &\leq p_U^*(s_1 - \eta) + 2\Delta(s - s_1 + \eta) - p_U^*(s_1) + \Delta(s_1 - s) \\ &= p_U^*(s_1 - \eta) - p_U^*(s_1) - \Delta(s_1 - s) + 2\Delta\eta \\ &\leq 2\Delta\eta. \end{aligned}$$

Moreover, for $s \in (s_1,\ s_1 + (s' - s_1)/2)$, if η is sufficiently close to 0, we have

$p_U^\eta(s) < p_U^*(s')$ and so

$$\begin{aligned} p_U^\eta(s) - p_U^*(s) &= p_U^*(s_1 - \eta) + 2\Delta(s - s_1 + \eta) - p_U^*(s) \\ &\geq p_U^*(s_1) - \Delta\eta + 2\Delta(s - s_1 + \eta) - p_U^*(s) \\ &= p_U^*(s_1) - \Delta\eta + 2\Delta(s - s_1 + \eta) - p_U^*(s_1) - 2\Delta(s - s_1) \\ &= \Delta\eta. \end{aligned}$$

Since $p_U^\eta(s) \geq p_U^*(s)$ for all s, the expression in (D.18) is bounded below by

$$-\int_{s_1-\eta}^{s_1} 2\Delta\eta \, ds + \int_{s_1}^{s_1+(s'-s_1)/2} \Delta\eta\varepsilon/2 \, ds,$$

which is clearly positive for sufficiently small η. Since the lower bound is strictly positive, the price-setter has a profitable deviation (in Γ^n for large n), a contradiction.

We conclude that $F_B^*(s) - F_S^*(s) \leq 0$ for almost all s. It remains to argue that it is not negative on a set of positive measure. Suppose it is. Then there must exist a seller characteristic $\hat{s} > 0$ such that $p_U(s) = -P$ for $s < \hat{s}$, $F_B^*(s) - F_S^*(s) < 0$ for a positive-measure subset of $[0, \hat{s}]$, and $F_B^*(s) - F_S^*(s) = 0$ for almost all $s > \hat{s}$. But then no seller would choose attributes in $[0, \hat{s})$, a contradiction. ∎

We now seek a characterization of the seller's payoffs. Intuitively, we would like to use Lemma D.6 and the monotonicity of $\mathbf{b}^*$ and $\mathbf{s}_B^*$ to conclude that there is positive assortative matching, and indeed that a seller of type σ matches with a buyer of type $\beta = \sigma$. However, these properties may not hold if $\mathbf{b}^*$ and $\mathbf{s}_B^*$ are not strictly increasing. Moreover, even if we had such a matching, the specification of the seller's payoffs given by (D.12) leaves open the possibility that the (gross) payoff to a seller of type σ choosing attribute s may not be given by $h_S(\tilde{b}(s), s) + p_U(s)$. Hence, the buyers that sellers are implicitly choosing in their payoff calculations may not duplicate those whose seller choices balance the market.

Our first step in addressing these issues is to show that the buyer's limiting attribute-choice function is indeed strictly increasing. Intuitively, if a positive measure of buyer types choose the same attribute, by having some higher types in the pool choose a slightly higher attribute, and some lower types choose a slightly lower attribute, we can keep the average attribute unchanged, while saving costs (from Assumption 3).

Lemma D.7 *The function* $\mathbf{b}^*$ *is strictly increasing when nonzero.*

Proof. By construction, $\mathbf{b}^*$ is weakly increasing. We show that $\beta'' > \beta'$ and $\mathbf{b}^*(\beta') > 0$ imply $\mathbf{b}^*(\beta'') > \mathbf{b}^*(\beta')$. Suppose to the contrary that $b = \mathbf{b}^*(\beta) > 0$ for two distinct values of β.

Define $\beta_1 \equiv \inf\{\beta : \mathbf{b}^*(\beta) = b\}$, $\beta_2 \equiv \sup\{\beta : \mathbf{b}^*(\beta) = b\}$, and $\bar{\beta} = (\beta_1 + \beta_2)/2$. We assume $0 < \beta_1$ and $\beta_2 < 1$ (if equality holds in either case, then the argument is modified in the obvious manner). We now define a new attribute-choice function (as a function of a parameter $\eta > 0$) that is strictly increasing on a neighborhood of $[\beta_1, \beta_2]$ and agrees with $\mathbf{b}^*$ outside that neighborhood. First, define

$$\beta_1^\eta = \inf\{\beta \le \beta_1 : \mathbf{b}^*(\beta) \ge b + \eta(\beta - \bar{\beta})\}$$

and

$$\beta_2^\eta = \sup\{\beta \ge \beta_2 : \mathbf{b}^*(\beta) \le b + \eta(\beta - \bar{\beta})\}.$$

Note that as $\eta \to 0$, $\beta_j^\eta \to \beta_j$ for $j = 1, 2$. Finally, define

$$\mathbf{b}^\eta(\beta) \equiv \begin{cases} \mathbf{b}^*(\beta), & \text{if } \beta > \beta_2^\eta, \\ b + \eta(\beta - \bar{\beta}), & \text{if } \beta \in [\beta_1^\eta, \beta_2^\eta], \\ \mathbf{b}^*(\beta), & \text{if } \beta < \beta_1^\eta. \end{cases}$$

The difference in payoffs to the buyer under $\mathbf{b}^\eta$ and under $\mathbf{b}^*$ is given by

$$\int_{\beta_1^\eta}^{\beta_2^\eta} h_B(\mathbf{b}^\eta(\beta), \mathbf{s}_B^*(\beta)) - h_B(\mathbf{b}^*(\beta), \mathbf{s}_B^*(\beta)) - [c_B(\mathbf{b}^\eta(\beta), \beta) - c_B(\mathbf{b}^*(\beta), \beta)]\, d\beta. \tag{D.19}$$

Now,

$$\begin{aligned} &\int_{\beta_1}^{\beta_2} [c_B(\mathbf{b}^\eta(\beta), \beta) - c_B(\mathbf{b}^*(\beta), \beta)]\, d\beta \\ &\qquad = \int_{\beta_1}^{\beta_2} \left[\frac{\partial c_B(b, \beta)}{\partial b} \eta(\beta - \bar{\beta}) + o(\eta) \right] d\beta \\ &\qquad = \eta \int_0^{(\beta_2 - \beta_1)/2} \left[\frac{\partial c_B(b, \bar{\beta} + x)}{\partial b} - \frac{\partial c_B(b, \bar{\beta} - x)}{\partial b} \right] x\, dx + o(\eta). \end{aligned}$$

From Assumption 3, the integrand is strictly negative, and so the integral is strictly negative and independent of η. Since $\mathbf{s}_B^*$ is increasing, a similar argument applied to the difference in the premuneration values shows that

$$\begin{aligned} &\int_{\beta_1}^{\beta_2} h_B(\mathbf{b}^\eta(\beta), \mathbf{s}_B^*(\beta)) - h_B(\mathbf{b}^*(\beta), \mathbf{s}_B^*(\beta)) - [c_B(\mathbf{b}^\eta(\beta), \beta) - c_B(\mathbf{b}^*(\beta), \beta)]\, d\beta \\ &\qquad \ge \eta \int_0^{(\beta_2 - \beta_1)/2} \left[\frac{\partial c_B(b, \bar{\beta} - x)}{\partial b} - \frac{\partial c_B(b, \bar{\beta} + x)}{\partial b} \right] x\, d\beta + o(\eta). \end{aligned}$$

It remains to argue that the contribution to (D.19) from the intervals $[\beta_1^\eta, \beta_1)$ and $(\beta_2, \beta_2^\eta]$ is of order $o(\eta)$. But this is immediate, since $|\mathbf{b}^\eta(\beta) - \mathbf{b}^*(\beta)| \leq \eta$ and $\beta_j^\eta \to \beta_j$ as $\eta \to 0$ (for $j = 1, 2$). Hence, for $\eta > 0$ sufficiently small, $\mathbf{b}^\eta$ gives the buyer a strictly higher payoff under (D.5) than $\mathbf{b}^*$. But, then by a now familiar argument, the buyer has a profitable deviation in Γ^n for sufficiently large n, a contradiction. So $\mathbf{b}^*$ is strictly increasing when nonzero. ∎

We next show that the seller's payoffs converge to the payoff one would expect the seller to receive by matching with his corresponding buyer type.

Lemma D.8 *For almost all σ satisfying $\mathbf{b}^*(\sigma) > 0$,*

$$\lim_n \int \bar{H}(\mathbf{s}(\sigma), \mathbf{b}, p_U^n) - c_S(\mathbf{s}(\sigma), \sigma) d\xi^n$$
$$= h_S(\mathbf{b}^*(\sigma)), \mathbf{s}^*(\sigma)) + p_U^*(\mathbf{s}^*(\sigma)) - c_S(\mathbf{s}^*(\sigma), \sigma).$$

The functions $\mathbf{s}$ and $\mathbf{b}$ on the left side of this expression are strategies in the game Γ^n, and are the objects over which the equilibrium ξ^n mixes.

Proof. Suppose the claim is false. Then, since the limit exists, there exists n'' and $\eta > 0$ such that for all σ in a set G of sellers of measure at least η whose "matched" buyers choose positive attributes (i.e., $\mathbf{b}^*(\sigma) > 0$), for all $n > n''$,

$$\int \bar{H}(\mathbf{s}(\sigma), \mathbf{b}, p_U^n) - c_S(\mathbf{s}(\sigma), \sigma) d\xi^n$$

is at least η distant from

$$h_S(\mathbf{b}^*(\sigma), \mathbf{s}^*(\sigma)) + p_U^*(\mathbf{s}^*(\sigma)) - c_S(\mathbf{s}^*(\sigma), \sigma).$$

Since G has positive measure, we may assume that every index in G is a continuity point of the limit functions $(\mathbf{b}^*, \mathbf{s}_B^*, \mathbf{s}^*)$.

For any $\varepsilon > 0$, let $E^\varepsilon \subset [0, 1]$ be the set from Lemma D.5 satisfying $\lambda(E^\varepsilon) \geq 1 - \varepsilon$.

Fix an index $\sigma' \in G \cap E^{\varepsilon'}$ for some $\varepsilon' > 0$ (since E^ε is monotonic in ε, $\sigma' \in G \cap E^\varepsilon$ for all smaller ε). Since $\mathbf{b}^*$ is strictly increasing, without loss of generality, we may assume that, for all $\zeta > 0$, there is a positive measure set of buyers with $\mathbf{b}^*(\beta) \in (\mathbf{b}^*(\sigma') - \zeta, \mathbf{b}^*(\sigma'))$. Indeed, a positive measure set of buyers in E^ε does so for all ε sufficiently small. Formally,

$$\forall \zeta > 0 \; \exists \varepsilon'' \; \forall \varepsilon < \varepsilon'', \quad \lambda\{\beta \in E^\varepsilon : \mathbf{b}^*(\beta) \in (\mathbf{b}^*(\sigma') - \zeta, \mathbf{b}^*(\sigma'))\} > 0. \quad \text{(D.20)}$$

Consider some $\varepsilon < \varepsilon'$ and suppose $n > \max\{n_\varepsilon, n''\}$, where n_ε is from Lemma D.5. Let $(\mathbf{b}, \mathbf{s}_B, \mathbf{s}) \in \Upsilon_B^n \times \Upsilon_S^n$ be a triple of functions with the property that $|\mathbf{b}(\beta) - \mathbf{b}^*(\beta)| < \varepsilon$ and $|\mathbf{s}_B(\beta) - \mathbf{s}_B^*(\beta)| < \varepsilon$ for all $\beta \in E^\varepsilon$, and $|\mathbf{s}(\sigma) - \mathbf{s}^*(\sigma)| < \varepsilon$ for all $\sigma \in E^\varepsilon$. (Recall that, from Lemma D.5, ξ^n assigns high probability to such functions for large n.)

By Lemma D.6, $\mathbf{s}^*$ and $\mathbf{s}_B^*$ are equal almost surely, so without loss of generality, we may assume that $\mathbf{s}^*(x) = \mathbf{s}_B^*(x)$ for all $x \in E$.

Observe first that if the max in (D.11) is achieved by $h_S(0, \mathbf{s}(\sigma)) + p_U^n(\mathbf{s}(\sigma))$, then

$$\begin{aligned}\bar{H}(\mathbf{s}(\sigma), \mathbf{b}, p_U^n) - c_S(\mathbf{s}(\sigma), \sigma) &= h_S(0, \mathbf{s}(\sigma)) + p_U^n(\mathbf{s}(\sigma)) - c_S(\mathbf{s}(\sigma), \sigma) \\ &\leq h_S(\mathbf{b}(\sigma), \mathbf{s}(\sigma)) + p_U^n(\mathbf{s}(\sigma)) - c_S(\mathbf{s}(\sigma), \sigma).\end{aligned}$$

We claim that, for sufficiently small $\varepsilon > 0$, the set $\tilde{P}(\mathbf{s}(\sigma), \mathbf{b}, p_U^n)$ contains all $p < p_U^n(\mathbf{s}(\sigma))$. This follows from (D.20) and the observation that buyers in E^ε receive a payoff (ignoring costs) arbitrarily close to $h_B(\mathbf{b}^*(\beta), \mathbf{s}^*(\beta)) - p_U^*(\mathbf{s}^*(\beta))$.

Consequently, for p sufficiently close to $p_U^n(\mathbf{s}(\sigma))$, single crossing (Assumption 1) implies that a buyer β with attribute satisfying $\mathbf{b}^*(\beta) < \mathbf{b}^*(\sigma)$ will not be attracted (for sufficiently large n). This implies that

$$\sup_{p \in \tilde{P}(\mathbf{s}(\sigma), \mathbf{b}, p_U^n)} H(\mathbf{s}(\sigma), p, \mathbf{b}, p_U^n) = h_S(\mathbf{b}(\sigma), \mathbf{s}(\sigma)) + p_U^n(\mathbf{s}(\sigma)).$$

By choosing ε small (or, equivalently, n large), the right side can be made arbitrarily close to

$$h_S(b_1(s'), s') + p_U^*(s') = h_S(b_1(\mathbf{s}^*(\sigma)), \mathbf{s}^*(\sigma)) + p_U^*(\mathbf{s}^*(\sigma)).$$

Hence, the max in (D.11) is achieved by the first term, and we have a contradiction. ∎

With this payoff characterization in hand, we can show that seller attribute choices are strictly increasing in types (when positive), as are the types of sellers with whom buyers attempt to match.

Lemma D.9 *The functions $\mathbf{s}_B^*$ and $\mathbf{s}^*$ are strictly increasing on $\{\beta : \mathbf{b}^*(\beta) > 0\}$.*

Proof. From Lemma D.6, $\mathbf{s}_B^*(x) = \mathbf{s}^*(x)$ for almost all $x \in [0, 1]$, and so it suffices to prove the result for $\mathbf{s}^*$. Suppose to the contrary there is a strictly positive constant $\hat{s}$ and associated maximal nondegenerate interval (σ_1, σ_2)

with $\mathbf{s}^*(\sigma) = \hat{s}$ and $\mathbf{b}^*(\sigma) > 0$ for all $\sigma \in (\sigma_1, \sigma_2)$. From Lemma D.6, we also have $\mathbf{s}^*_B(\beta) = \hat{s}$ for all $\beta \in (\sigma_1, \sigma_2)$.

Define $b_1 \equiv \lim_{\beta \downarrow \sigma_1} \mathbf{b}^*(\beta)$ and $b_2 \equiv \lim_{\beta \uparrow \sigma_2} \mathbf{b}^*(\beta)$.

Define $\sigma(\eta) \equiv \inf\{\sigma : \mathbf{s}^*(\sigma) \geq \hat{s} + \eta\}$, and notice that $\lim_{\eta \to 0} \sigma(\eta) = \sigma_2$. The seller attribute-choice function $\mathbf{s}'$ given by

$$\mathbf{s}'(s) = \begin{cases} \mathbf{s}^*(\sigma), & \text{if } \sigma \notin (\sigma_1, \sigma(\eta)), \\ \hat{s} + \eta, & \text{if } \sigma \in (\sigma_1, \sigma(\eta)), \end{cases}$$

is weakly increasing. Consider the price $\hat{p} > p^*_U(\hat{s})$ for attribute $\hat{s} + \eta$ satisfying

$$\hat{p} = \sup\{p : B(\hat{s} + \eta, p, p^*_U) \neq \varnothing\}.$$

(This is $\bar{p}(\hat{s} + \eta, p, p^*_U)$ from Lemma D.1(1).) The price $\hat{p}$ is at least as high as the value p' satisfying $h_B(b_2, \hat{s}) - p_U(\hat{s}) = h_B(b_2, \hat{s} + \eta) - p'$. At the price $\hat{p}$ for attribute choice $\hat{s} + \eta$, the seller ensures that attribute choice $\hat{s} + \eta$ is chosen by a buyer at least as high as b_2 (the single-crossing condition on buyer premuneration values ensures that no lower attribute buyers will choose $\hat{s} + \eta$). From Lemma D.8, we have then have that the switch to attribute-choice function $\mathbf{s}'$ increases the seller's payoff by at least

$$\begin{aligned} \int_{\sigma_1}^{\sigma_2} (h_S(b_2, \hat{s} + \eta) + \hat{p})\, d\sigma - \int_{\sigma_1}^{\sigma_2} (h_S(b_1, \hat{s}) + p^*_U(\hat{s}))\, d\sigma \\ - \int_{\sigma_1}^{\sigma(\eta)} (c_s(\hat{s} + \eta, \sigma) - c_S(\mathbf{s}^*(\sigma), \sigma) d\sigma \\ > (\sigma_2 - \sigma_1)[h_S(b_2, \hat{s} + \eta) - h_S(b_1, \hat{s}] \\ - (\sigma(\eta) - \sigma_1)[c_S(\hat{s} + \eta, \sigma_1) - c_S(\hat{s}, \sigma_1)]. \end{aligned}$$

The first term after the inequality is bounded away from zero as η approaches zero, while the second approaches zero as does η, ensuring that there is some $\eta > 0$ for which the payoff difference is positive. Intuitively, each seller in the interval (σ_1, σ_2) experiences a discontinuous increase in expected buyer (at a higher price) when increasing her attribute choice, while sellers in the interval $(\sigma_2, \sigma(\eta))$ experience a continuous increase in cost. The attribute-choice function $\mathbf{s}'$ increases the seller's payoff for sufficiently small η, yielding the result. ∎

The limiting mass of buyers and seller choosing zero attributes are equal:

Lemma D.10

$$\lambda(\{\sigma : \mathbf{s}^*(\sigma) = 0\}) = \lambda(\{\beta : \mathbf{b}^*(\beta) = 0\}).$$

Proof. First, suppose $\lambda(\{\beta : \mathbf{b}^*(\beta) = 0\}) > \lambda(\{\sigma : \mathbf{s}^*(\sigma) = 0\})$. Then because $\mathbf{s}^*_B = \mathbf{s}^*$ almost everywhere, there exists a positive mass of buyers for whom $\mathbf{b}^*(\beta) = 0$ and $\mathbf{s}^*_B(\beta) > 0$. By Assumption 2, $h_B(0, s)$ is independent of s, and so, since p^*_U is strictly increasing, the buyers choosing $b = 0$ can increase their payoff by choosing $s = 0$. The buyer's equilibrium strategy must then be suboptimal in the game Γ^n for sufficiently large n, a contradiction.

Now, suppose $\lambda(\{\beta : \mathbf{b}^*(\beta) = 0\}) < \lambda(\{\sigma : \mathbf{s}^*(\sigma) = 0\})$. Then there exists a positive mass of buyers for whom $\mathbf{b}^*(\beta) > 0$ and $\mathbf{s}^*_B(\beta) = 0$. By Lemma D.6, there is then a positive mass of buyers choosing a zero seller attribute and positive buyer attribute. Since $h_B(b, 0)$ is independent of b (Assumption 2) and $c_B(b, \beta)$ is strictly increasing in b, such buyers can increase their payoff by choosing $b = 0$. The buyer's equilibrium strategy must then be suboptimal in the game Γ^n for sufficiently large n, a contradiction.
■

We now turn to feasible matchings. For $b \in [0, \mathbf{b}^*(1)]$ and $s \in [0, \mathbf{s}^*(1)]$, we define

$$\tilde{b}^*(s) \equiv \begin{cases} \mathbf{b}^*((\mathbf{s}^*)^{-1}(s)), & s \in \mathbf{s}^*([0,1]), s > 0, \\ \max\{0, \sup_{b \in \mathcal{B}}\{b < \mathbf{b}^*(\inf\{\sigma : \mathbf{s}^*(\sigma) > s\})\}\}, & \text{otherwise,} \end{cases}$$

and

$$\tilde{s}^*(b) \equiv \begin{cases} \mathbf{s}^*((\mathbf{b}^*)^{-1}(b)), & b \in \mathbf{b}^*([0,1]), b > 0, \\ \max\{0, \sup_{s \in \mathcal{S}}\{s < \mathbf{s}^*(\inf\{\beta : \mathbf{b}^*(\beta) > b\})\}\}, & \text{otherwise.} \end{cases}$$

The maximum in the specification of $\tilde{b}^*$ (with $\tilde{s}^*$ similar) ensures that $\tilde{b}^*$ is well defined when $\mathbf{s}^*$ is continuous at $\inf\{\sigma | \mathbf{s}^*(\sigma) > 0\}$ (in which case, the supremum is taken over the empty set and so has value $-\infty$).

Lemma D.11 *The pair $(\tilde{b}^*, \tilde{s}^*)$ is a feasible matching. In addition, for all values $b > 0$ and $s > 0$, we have*

$$\tilde{s}^*(b) = \mathbf{s}^*((\mathbf{b}^*)^{-1}(b))$$

where

$$(\mathbf{b}^*)^{-1}(b) = \begin{cases} \inf\{\beta : \mathbf{b}^*(\beta) > b\}, & \text{for } b \le \mathbf{b}^*(1), \\ 1, & \text{for } b > \mathbf{b}^*(1), \end{cases}$$

and

$$\tilde{b}^*(s) = \mathbf{b}^*((\mathbf{s}^*)^{-1}(s)),$$

where

$$(\mathbf{s}^*)^{-1}(s) = \begin{cases} \inf\{\sigma : \mathbf{s}^*(\sigma) > s\}, & \text{for } s \leq \mathbf{s}^*(1), \\ 1, & \text{for } s > \mathbf{s}^*(1). \end{cases}$$

Proof. From Lemma D.10, we can assume that $\mathbf{b}^*$ and $\mathbf{s}^*$ share a common set $[0, x]$ on which they are zero. It is then immediate that $(\tilde{b}^*, \tilde{s}^*)$ is a feasible matching.

The final two statements follow immediately from the left continuity of the attribute-choice functions (see Lemma D.4) and the definitions of $\tilde{s}^*$ and $\tilde{b}^*$. ■

Finally, we show that the seller's payoff satisfies an optimality condition.

Lemma D.12 *For almost all σ,*

$$\begin{aligned} \lim_n \int \bar{H}(\mathbf{s}(\sigma), \mathbf{b}, p_U^n) - c_S(\mathbf{s}(\sigma), \sigma) d\xi^n & \\ &= h_S(\mathbf{b}^*(\sigma), \mathbf{s}^*(\sigma)) + p_U^*(\mathbf{s}^*(\sigma)) - c_S(\mathbf{s}^*(\sigma), \sigma) \\ &= \max_{s \in \mathcal{S}} h_S(\tilde{b}^*(s), s) + p_U^*(s) - c_S(s, \sigma). \end{aligned}$$

Proof. The first inequality duplicates Lemma D.8.

Single-crossing (Assumption 3) implies that the attribute choices maximizing $h_S(\tilde{b}^*(s), s) + p_U^*(s) - c_S(s, \sigma)$ are increasing in σ, and so if the second equality fails, in games Γ^n for sufficiently large n, the seller has a profitable deviation. ■

D.5 Uniform-Price Equilibria

We finally argue that the profile $(\mathbf{b}^*, \mathbf{s}^*, \tilde{b}^*, \tilde{s}^*, p_U^*)$ induces a uniform-price equilibrium of the matching market with identical attribute choices and matching function (but perhaps a vertical shift in the price function).

The first task is to show that equilibrium payoffs are nonnegative, so that agents would not prefer to be out of the market. Suppose $\{\xi_B^n, \xi_S^n, p_U^n\}_n$ is the sequence whose limit induces $(\mathbf{b}^*, \mathbf{s}_B^*, \tilde{b}^*, \tilde{s}^*, p_U^*)$. We have

$$\begin{aligned} h_B(0,0) - p_U^*(0) &= h_B(0,0) - p_U^*(0) - c_B(0,\beta) \\ &\leq h_B(\mathbf{b}(\beta), \tilde{s}^*(\mathbf{b}^*(\beta))) - p_U^*(\tilde{s}^*(\mathbf{b}^*(\beta))) - c_B(\mathbf{b}(\beta), \beta) \quad \text{(D.21)} \end{aligned}$$

and

$$\begin{aligned} h_S(0,0) + p_U^*(0) &= h_S(0,0) + p_U^*(0) - c_S(0,\sigma) \\ &\le h_S(\tilde{b}^*(\mathbf{s}(\sigma)), \mathbf{s}(\sigma)) + p_U^*(\mathbf{s}(\sigma)) - c_S(\mathbf{s}(\sigma),\sigma). \end{aligned} \tag{D.22}$$

Let

$$\kappa^* \equiv h_B(0,0) - p_U^*(0) \ge -h_S(0,0) - p_U^*(0)$$

(where the inequality follows from Assumption 2) and replace the price function p_U^* with $p_U^* + \kappa^*$. Both ξ_B^n and ξ_S^n remain best responses given price $p_U^n + \kappa^*$ and markets still clear in the limit of $n \to \infty$. Moreover, replacing p_U^* with $p_U^* + \kappa^*$ in (D.21)–(D.22) gives nonnegative payoffs.

It is immediate from the formulation of the buyer's payoffs in the game and from Lemma D.12 that almost all buyers and sellers are optimizing given p_U^*.

It remains to consider deviations by a seller of type σ to a value s not chosen by any seller under $\mathbf{s}^*$. If there is a profitable such deviation for seller σ, then there is a price p such that $B(s, p, p_U^*)$ is nonempty and for all $b \in B(s, p, p_U^*)$,

$$\Pi_S(\mathbf{s}^*(\sigma), \sigma) < h_S(b, s) + p - c_S(s, \sigma).$$

But then for all sufficiently large n, $B(s, p', p_U^n)$ is again nonempty for p' less than but close to p, contradicting the fact that $\Pi_S(\mathbf{s}^*(\sigma), \sigma)$ is close to $\int \bar{H}(\mathbf{s}(\sigma), \mathbf{b}, p_U^n) - c_S(\mathbf{s}(\sigma), \sigma)\, d\xi^n$.

D.6 Nontriviality

Partial nontriviality. We now show that under (14), the profile $(\mathbf{b}^*, \mathbf{s}^*, \tilde{b}^*, \tilde{s}^*, p_U^*)$ is nontrivial. If the equilibrium is trivial, $\mathbf{b}^*$ and $\mathbf{s}^*$ are identically zero, so that there is no agent for whom it is profitable to trade at price p_U, and hence for all $(b, s) \in (0, \bar{b}] \times (0, \bar{s}]$,

$$\begin{aligned} & h_S(0,s) + p_U(s) - c_S(s,1) \le 0 \\ \text{and} \qquad & h_B(b,s) - p_U(s) - c_B(b,1) \le 0, \end{aligned}$$

where we focus on agents $\beta = 1 = \sigma$ since they are the most likely to want to trade. Notice that we are using here the maximum that appears in the building block (D.11) for the specification of the seller's payoff, and which effectively allows the seller to sell any attribute choice $s \in [0, \bar{s}]$ at price

$p_U(s)$, assuming in the process that he can attract at least a zero-attribute buyer. For these two inequalities to hold, it must be that

$$h_B(b,s) + h_S(0,s) \leq c_B(b,1) + c_S(s,1),$$

contradicting (14).

Full nontriviality. We now assume (15) holds. Suppose that there is an interval of seller types $[0,\sigma']$ with $\sigma' > 0$ who choose zero attributes. By Lemma D.9, we then have $\mathbf{b}^*(\beta) = 0$ for all $\beta \in [0,\sigma']$. If neither agent of type $\phi \in (0,\beta')$ chooses a strictly positive attribute, it must be that

$$\begin{aligned} & h_S(0,s) + p_U(s) - c_S(s,\phi) \leq 0 \\ \text{and} \qquad & h_B(b,s) - p_U(s) - c_B(b,\phi) \leq 0, \end{aligned}$$

where (b,s) are a pair of attributes satisfying (15). But summing these two inequalities yields an inequality contradicting (15).

E Personalized Pricing

E.1 Prices

A *personalized-price function* is a function $p_P : \mathcal{B} \times \mathcal{S} \to \mathbb{R}$; where $p_P(b,s)$ is the (possibly negative) price that seller with attribute choice $s \in \mathcal{S}$ receives when selling to a buyer with attribute choice $b \in \mathcal{B}$. We emphasize that a personalized-price function prices only matches between *marketed* attributes.

E.2 Equilibrium

Given a feasible outcome $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s})$ and a personalized price p_P, the payoffs to a buyer β who chooses $b \in \mathcal{B}$ and to a seller σ who chooses $s \in \mathcal{S}$ are given by

$$\begin{aligned} & \Pi_B(b,\beta) \equiv h_B(b,\tilde{s}(b)) - p_P(b,\tilde{s}(b)) - c_B(b,\beta) \\ \text{and} \qquad & \Pi_S(s,\sigma) \equiv h_S(\tilde{b}(s),s) + p_P(\tilde{b}(s),s) - c_S(s,\sigma). \end{aligned}$$

The specification of equilibrium begins with appropriate modifications of the notions of buyer and seller optimization given p_U (i.e., (3) and (4)):

Definition E.1 *Given a feasible outcome* $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s})$*, buyer* β *is* optimizing at $\mathbf{b}$ given p_P *if*

$$(\mathbf{b}(\beta), \tilde{s}(\mathbf{b}(\beta))) \in \underset{(b,s)\in\mathcal{B}\times\mathcal{S}}{\arg\max}\, h_B(b,s) - p_P(b,s) - c_B(b,\beta). \tag{E.1}$$

Seller σ *is* optimizing at $\mathbf{s}$ given p_P *if*

$$(\tilde{b}(\mathbf{s}(\sigma)), \mathbf{s}(\sigma)) \in \underset{(b,s)\in\mathcal{B}\times\mathcal{S}}{\arg\max}\, h_S(b,s) + p_P(b,s) - c_S(s,\sigma). \tag{E.2}$$

Since the personalized-price function p_P prices only pairs of marketed attributes, the stipulation that a seller optimize given p_P says nothing about what might happen if this seller chooses an attribute $s \notin \mathcal{S}$. We require that no seller can choose an attribute $s \notin \mathcal{S}$, find a target buyer attribute $b \in \mathcal{B}$ with whom to match, and find a way to split the resulting surplus so that the seller and target buyer are both better off than in equilibrium:[‖]

Definition E.2 *Given* $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s}, p_P)$*, there is a* profitable seller deviation *if there exists a seller* σ *such that either (*i*)* $\Pi_S(\mathbf{s}(\sigma),\sigma) < 0$ *or (*ii*) there exist an unmarketed seller attribute choice* $s \notin \mathcal{S}$*, a marketed buyer attribute* $b \in \mathcal{B}$*, and a price* $p \in \mathbb{R}$ *such that*

$$h_B(b, \tilde{s}(b)) - p_P(b, \tilde{s}(b)) \;<\; h_B(b,s) - p \tag{E.3}$$

and

$$\Pi_S(\mathbf{s}(\sigma),\sigma) \;<\; h_S(b,s) + p - c_S(s,\sigma). \tag{E.4}$$

The definition of a buyer's profitable deviation is similar:

Definition E.3 *Given* $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s}, p_P)$*, there is a* profitable buyer deviation *if there exists a buyer* β *such that either (*i*)* $\Pi_B(\mathbf{b}(\beta),\beta) < 0$ *or (*ii*) there exist an unmarketed buyer attribute choice* $b \notin \mathcal{B}$*, a marketed seller attribute* $s \in \mathcal{S}$*, and a price* $p \in \mathbb{R}$ *such that*

$$\Pi_B(\mathbf{b}(\beta),\beta) < h_B(b,s) - p - c_B(b,\beta)$$

and

$$h_S(\tilde{b}(s), s) + p_P(\tilde{b}(s), s) < h_S(b,s) + p.$$

Definition E.4 *A feasible outcome* $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s})$ *and a personalized-price function* p_P *constitute a* personalized-price equilibrium *if all buyers and sellers are optimizing given* p_P *and no buyer or seller has a profitable deviation.*

[‖] Similar to footnote 6, Definitions E.2 and E.3 can be extended to cover deviations to any seller attribute, as well as to prices that differ from the personalized-price function. See Mailath, Postlewaite, and Samuelson (2010) for details.

Remark E.1 (Premuneration Values) Since personalized prices can compensate for any alterations of the division of $v(b,s)$, the decomposition of the surplus $v(b,s)$ between the buyer's and seller's premuneration values plays no role in the efficiency of a personalized-price equilibrium outcome. In particular, it is a straightforward calculation that if $(\mathbf{b},\mathbf{s},\tilde{b},\tilde{s},p_P)$ is a personalized-price equilibrium with premuneration values $h_B(b,s)$ and $h_S(b,s)$, then $(\mathbf{b},\mathbf{s},\tilde{b},\tilde{s},p'_P)$ is a personalized-price equilibrium with premuneration values $h'_B(b,s)$ and $h'_S(b,s)$, where

$$p'_P(b,s) = p_P(b,s) + h'_B(b,s) - h_B(b,s) = p_P(b,s) + h_S(b,s) - h'_S(b,s).$$

♦

Remark E.2 (Ex Post Contracting Equilibrium) Cole, Mailath, and Postlewaite (2001) study continua of buyers and sellers who first simultaneously choose attributes (as here), and then match and bargain to divide the resulting surplus $v(b,s)$, with the matching/bargaining stage being modeled as a cooperative game (more specifically, an assignment game). An *ex post contracting equilibrium* in Cole, Mailath, and Postlewaite (2001) is a Nash equilibrium of the noncooperative attribute-choice game, where the payoffs from the attribute choices are determined by stable (equivalently, core) allocations in the induced assignment game.[**] The set of outcomes and implied payoffs are essentially the same under the two notions (modulo some technical differences). In particular, if all buyers and sellers are optimizing in the market given p_P, then no buyer-seller pair with attributes $(b,s) \in \mathcal{B} \times \mathcal{S}$ can block the equilibrium. Moreover, a seller σ has a profitable out-of-market deviation if and only if there is a blocking pair consisting of that seller (with some attribute s) and some buyer with an attribute $b \in \mathcal{B}$. An analogous comment applies to buyers. ♦

E.3 Efficiency

Lemma E.1 *In any personalized-price equilibrium* $(\mathbf{b},\mathbf{s},\tilde{b},\tilde{s},p_P)$*,* $\tilde{b}$ *and* $\tilde{s}$ *are strictly increasing for strictly positive attributes.*

Proof. We consider only $\tilde{b}$ (since $\tilde{s}$ is almost identical). Suppose $\tilde{b}$ is not strictly increasing. Since $\tilde{b}$ is one-to-one on $\mathbf{s}((\underline{\sigma},1])$ (see Definition 1 and its

[**] Consequently, matching is over buyers and sellers, not attributes as here. This difference results in some technical complications.

following comment), there exists $0 < s_1 < s_2$ with $b_1 \equiv \tilde{b}(s_1) > \tilde{b}(s_2) \equiv b_2$. From (E.1) for the buyer choosing b_1 and from (E.2) for the seller choosing s_2, we have

$$\begin{aligned} & h_B(b_1, s_1) - p_P(b_1, s_1) \geq h_B(b_1, s_2) - p_P(b_1, s_2) \\ \text{and} \qquad & h_S(b_2, s_2) + p_P(b_2, s_2) \geq h_S(b_1, s_2) + p_P(b_1, s_2), \end{aligned}$$

and so

$$h_B(b_1, s_1) + h_S(b_2, s_2) - p_P(b_1, s_1) + p_P(b_2, s_2) \geq v(b_1, s_2).$$

Adding this to the analogous inequality obtained from (E.1) for the buyer choosing b_2 and from (E.2) for the seller choosing s_1, we obtain

$$v(b_1, s_1) + v(b_2, s_2) \geq v(b_1, s_2) + v(b_2, s_1).$$

But Assumption 1 requires the reverse (strict) inequality, a contradiction. ■

From Lemma E.1, matching in a personalized-price equilibrium is positively assortative in attributes. Since the attribute-choice functions are strictly increasing in index when positive, we can accordingly define the ex ante surplus for buyer and seller types $\beta = \sigma = \phi \in [0, 1]$ as

$$\begin{aligned} W(b, s, \phi) &\equiv h_B(b, s) + h_S(b, s) - c_B(b, \phi) - c_S(s, \phi) \\ &= v(b, s) - c_B(b, \phi) - c_S(s, \phi). \end{aligned}$$

An efficient choice of attributes maximizes $W(b, s, \phi)$ for (almost) all ϕ.

Personalized-price equilibrium outcomes are constrained efficient in the sense that no matched pair of agents can increase its net surplus without both agents deviating to attribute choices outside the sets of marketed attributes $\mathcal{B}$ and $\mathcal{S}$:[††]

Lemma E.2 *Suppose* $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s}, p_P)$ *is a personalized-price equilibrium. Then, for all* $\phi \in [0, 1]$, $b \in \mathcal{B}$, $s \in \mathcal{S}$ *and all* b' *and* s',

$$\begin{aligned} & W(b, s', \phi) \leq W(\mathbf{b}(\phi), \mathbf{s}(\phi), \phi) \\ \textit{and} \qquad & W(b', s, \phi) \leq W(\mathbf{b}(\phi), \mathbf{s}(\phi), \phi). \end{aligned}$$

[††]This is essentially Cole, Mailath, and Postlewaite (2001, Lemma 2), which describes a constrained efficiency property of ex post contracting equilibria (see Remark E.2). The current formulation allows a more transparent statement and proof.

Proof. En route to a contradiction, suppose there exists $\phi \in [0,1]$, $b \in \mathcal{B}$ and $s' \in [0,\bar{s}]$ such that $W(b,s',\phi) > W(\mathbf{b}(\phi),\mathbf{s}(\phi),\phi)$. The other possibility is handled analogously.

Let $\varepsilon = [W(b,s',\phi) - W(\mathbf{b}(\phi),\mathbf{s}(\phi),\phi)]/3 > 0$ and set $p = h_B(b,s') - h_B(b,\tilde{s}(b)) + p_P(b,\tilde{s}(b)) - \varepsilon$. The seller of type $\sigma = \phi$ can induce a buyer with attribute choice b to buy from him by choosing s' and offering a price p. Moreover, this deviation is strictly preferred by the seller ϕ:

$$\begin{aligned}
h_S(b,s') &+ p - c_S(s',\phi) \\
&= h_S(b,s') + h_B(b,s') - h_B(b,\tilde{s}(b)) + p_P(b,\tilde{s}(b)) - \varepsilon - c_S(s',\phi) \\
&> \Pi_S(\mathbf{s}(\phi),\phi) + [h_B(\mathbf{b}(\phi),\mathbf{s}(\phi)) - p_P(\mathbf{b}(\phi),\mathbf{s}(\phi)) - c_B(\mathbf{b}(\phi),\phi)] \\
&\qquad\qquad - [h_B(b,\tilde{s}(b)) - p_P(b,\tilde{s}(b)) - c_B(b,\phi)] + \varepsilon \\
&\geq \Pi_S(\mathbf{s}(\phi),\phi) + \varepsilon,
\end{aligned}$$

where the equality uses the definition of p, the strict inequality follows from $W(b,s',\phi) > W(\mathbf{b}(\phi),\mathbf{s}(\phi),\phi) + 2\varepsilon$, and the last inequality is an implication of (E.1). ■

Lemma E.2 does not ensure that a personalized-price equilibrium outcome is efficient. The possibility remains that $W(b,s,\phi)$ may be maximized by a pair of values $b \notin \mathcal{B}$ and $s \notin \mathcal{S}$. In this sense, the inefficiency is the result of a coordination failure. For example, for the premuneration values $h_B(b,s) = \theta bs$ and $h_S(b,s) = (1-\theta)bs$, it is an equilibrium for all agents to choose attribute 0, giving a constrained-efficient outcome that is in fact quite inefficient. The possible inefficiency of a uniform-price equilibrium can be viewed as reflecting incomplete markets.

We could ensure efficiency by ensuring that a price exists for every attribute combination, whether marketed or not:

Definition E.5 *The feasible outcome* $(\mathbf{b},\mathbf{s},\tilde{b},\tilde{s})$ *and personalized price* p_P *is a* complete personalized-price equilibrium *if there is an extension of* p_P *to* $[0,\bar{b}] \times [0,\bar{s}]$ *(also denoted by* p_P*) such that for all* β *and all* σ*,*

$$0 \leq \Pi_B(\mathbf{b}(\beta),\beta) = \sup_{(b,s)\in[0,\bar{b}]\times[0,\bar{s}]} h_B(b,s) - p_P(b,s) - c_B(b,\beta)$$

and

$$0 \leq \Pi_S(\mathbf{s}(\sigma),\sigma) = \sup_{(b,s)\in[0,\bar{b}]\times[0,\bar{s}]} h_S(b,s) + p_P(b,s) - c_S(s,\sigma).$$

Though the names suggest that every complete personalized-price equilibrium outcome is indeed a personalized-price equilibrium outcome, this is not immediate, as we have replaced the prohibition on profitable deviations with the requirement that agents be optimizing given p_P with respect to all attribute choices. However, we have:

Lemma E.3

(E.3.1) Every complete personalized-price equilibrium outcome is a personalized-price equilibrium outcome.

(E.3.2) A complete personalized-price equilibrium outcome is efficient.

Proof. Fix a complete personalized-price equilibrium $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s}, p_P)$. To show that this is a personalized-price equilibrium, we must show there are no profitable deviations. We discuss seller deviations; the buyer case is analogous. Suppose the seller has a profitable deviation, so there exists a type σ and an attribute choice $s' \notin \mathcal{S}$, a price $p \in \mathbb{R}$, and $b' \in \mathcal{B}$ with

$$\Pi_S(\mathbf{s}(\sigma), \sigma) < h_S(b', s') + p - c_S(s', \sigma) \tag{E.5}$$

and

$$h_B(b', \tilde{s}(b')) - p_P(b', \tilde{s}(b')) < h_B(b', s') - p. \tag{E.6}$$

Since $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s}, p_P)$ is a complete personalized-price equilibrium, (E.5) implies $p > p_P(b', s')$.

There exists some $\beta \in [0, 1]$ for which $b' = \mathbf{b}(\beta)$, and so subtracting $c_B(b', \beta)$ from both sides of (E.6) and again using the assumption that $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s}, p_P)$ is a complete personalized-price equilibrium gives $p < p_P(b', s')$, a contradiction.

Since every pair of attributes is priced, the efficiency of complete personalized-price equilibria is a straightforward calculation. ■

One route to existence is to note that a personalized-price equilibrium is essentially equivalent to Cole, Mailath, and Postlewaite (2001) ex post contracting equilibrium, and then to refer to that paper for conditions for the existence of an ex post contracting equilibria. We take an alternative route here, building on the relationship between personalized-price and uniform-price equilibria.

Proposition E.1 *There exists an efficient personalized-price equilibrium.*

Proof. Suppose first that $h_S(b, s) = 0$ and hence $h_B(b, s) = v(b, s)$ for all pairs (b, s). Proposition 3 ensures that there exists a complete uniform-price equilibrium. Since $h_S(b,s) = 0$, this is also a complete personalized-price equilibrium. Then, if $h_S(b,s) \neq 0$, by setting

$$p'_P(b,s) = p_P(b,s) - h_S(b,s) = p_P(b,s) + h_B(b,s) - v(b,s),$$

we again have a complete (and hence efficient) personalized-price equilibrium. ■

E.4 Uniform Rationing Equilibria

Lemma E.4 *Any personalized-price equilibrium outcome is a uniform-rationing equilibrium outcome.*

Proof. Let $(\mathbf{b}, \mathbf{s}, \tilde{b}, \tilde{s})$ be a personalized-price equilibrium outcome and consider its associated uniform-rationing price. The conditions for the latter to be a personalized-price equilibrium are implied by the former, with the exception that there may now be profitable deviations by a buyer β with attribute choice $\mathbf{b}(\beta)$ to match with a seller with $s < \tilde{s}(\mathbf{b}(\beta))$ (and hence $\tilde{b}(s) < \mathbf{b}(\beta)$). But since $h_S(b,s)$ is increasing in b, the seller in question would welcome such a match. Hence, if this match is a profitable deviation in the uniform-rationing equilibrium, it is a profitable deviation in the personalized-price equilibrium, a contradiction. ■

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