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February 2005

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"Ito's Lemma" and the Bellman equation for Poisson processes: An applied view

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- comments welcome -

Rare and randomly occurring events are important features of the economic world. In continuous time they can easily be modeled by Poisson processes. Analyzing optimal behavior in such a setup requires the appropriate version of the change of variables formula and the Hamilton-Jacobi-Bellman equation. This paper provides examples for the application of both tools in economic modeling. It accompanies the proofs in Sennewald (2005), who shows, under milder conditions than before, that the Hamilton-Jacobi-Bellman equation is both a necessary and sufficient criterion for optimality. The main example here consists of a consumption-investment problem with labor income. It is shown how the Hamilton-Jacobi-Bellman equation can be used to derive both a Keynes-Ramsey rule and a closed form solution. We also provide a new result.

Keywords: Stochastic differential equation, Poisson process, Bellman equation, Portfolio optimization, Consumption optimization

JEL classification: C61, D81, D90, G11

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1 Introduction

Poisson processes as a source of uncertainty are a standard tool for modeling rare and randomly occurring events. These processes can be found, among others, in quality-ladder models of growth (e.g., Grossman and Helpman (1991), Aghion and Howitt (1992, 1998)), in the endogenous fluctuations and growth literature with uncertainty (e.g., Wälde (2005), Steger (2005)), in the labor market matching literature (e.g., Moen (1997)), in monetary economics (e.g., Kiyotaki and Wright (1991)), and in finance (e.g., Merton (1971)). The two "major tools" required when working with Poisson processes, and with stochastic processes in general, are the change-of-variables formula (CVF) for computing stochastic differentials and, in so far as optimal control is concerned, the Hamilton-Jacobi-Bellman (HJB) equation.[3]

Despite the widespread use, applying the HJB equation as a necessary or sufficient criterion for optimality has required so far a set of restrictive or simplifying assumptions. In particular, the boundedness of the instantaneous utility (or cost) function and of the coefficients in the constraint, which is given as a stochastic differential equation (SDE), has been in most cases indispensable for the use of the HJB equation as a necessary criterion, see, e.g., Gihman and Skorohod (1972) or Dempster (1991). Other authors as, e.g., Kushner (1967) require, instead of this boundedness condition, the value function to be contained in the domain of the infinitesimal generator of the controlled process.[4] However, both conditions are not convenient for economic modeling since, on the one hand, in most cases neither utility and cost functions nor constraint coefficients are bounded and, on the other hand, to check whether the value function belongs to the mentioned domain requires in general considerably calculation. To solve this problem, Sennewald (2005) shows that the HJB equation can still be used as a necessary criterion for optimality if, instead of boundedness, only linear boundedness is assumed.[5] Apart from a terminal condition, no boundedness condition is even required for deriving the sufficiency of the HJB equation.

The present paper accompanies the rigorous proofs in Sennewald (2005) and is directed

[3] Some readers may know the CVF better under the term *Ito's lemma* and the HJB equation under the name *Bellman equation*, which are the corresponding notations for a framework with Brownian motion as noise.

[4] The domain of the infinitesimal generator of a process $X(t)$ consists of all once continously differentiable function V for that the limit $\lim_{h\searrow 0}\left[E_t V\left(X\left(t+h\right)\right)-V\left(X\left(t\right)\right)\right]/t$ exist.

[5] Notice that, if the value function is sufficiently smooth, the boundedness assumptions are sufficient for the value function to be in the domain of the extended generator. Sennewald (2005) shows implicitely that this property of the value function holds also for the more general case with linearly bounded utility and coeffcients.

at the applied model builder. It presents examples for the application of CVF and the HJB equation. These examples should allow to work with Poisson uncertainty in other setups as well. Both papers have the intention to encourage a more widespread use of Poisson processes under more general assumptions concerning the economic environment.

After presenting some versions of CVF in the subsequent section, we provide two applications for it: A derivation of a household's budget constraint and of the corresponding HJB equation for an optimum-consumption problem. In section 3 we present a typical maximization problem, consisting of determining a household's optimal consumption and investment behavior in the presence of a deterministic flow of labor income. We use the HJB equation to derive both a Keynes-Ramsey rule and a closed form solution.

The CVFs presented here are special cases of the general CVF for jump processes in Garcia and Griego (1994). The maximization problem in section 3 is a "standard" optimal consumption and portfolio problem as considered, e.g., in Merton (1969, 1971) and Aase (1984), but allows for labor income in a Poisson framework. Merton (1971) derives a solution including wages when uncertainty of the risky investment is modeled by Brownian motion. Aase (1984) extends Merton's model by introducing random jumps. But even though he gives hints how to proceed if wages as an additional source of income are taken into account, no solution for this case is presented.

Keynes-Ramsey rules have been derived before, e.g., by Cass (1965) and Koopmans (1965) in a deterministic growth model, by Steger (2003) in a Ak-type growth model with jumps, or by Wälde (1999b) for an optimum-consumption problem similar to the one presented here. But whereas Wälde (1999b) assumes that the investment into the risky asset, which is there investment into R&D, vanishes as long as R&D is not successful, we follow the "tradition" of Merton and assume that the risky asset yields at least a certain deterministic return. A Keynes Ramsey rule for this setup is a new result as well.

2 Change of Variables Formula ("Ito's Lemma")

This section first presents various versions of CVF. They are easily derived from Sennewald (2005, theorem 6.1), which in turn is a simple corollary of Garcia and Griego (1994). The CVF is a "rule" for computing the differential of functions of stochastic processes.

The second subsection provides a typical application of the CVF by showing how the budget constraint of a household can be derived via CVF. The third subsection shows how the HJB equation for a simple household's maximization problem is heuristically obtained, also by using CVF.

2.1 Simple corollaries

In the following, we deal with uni- or multivariate stochastic processes $x(t)$ that, starting at time t_0 in $x(t_0)$, obey stochastic differential equations (*SDE*s) of the form

$$dx(t) = \alpha(t, x(t))\, dt + \sum_{k=1}^{d} \beta_k(t, x(t_-)) dq_k(t), \quad x(t_0) \in \mathbb{R}^n, \tag{1}$$

where α and β are non-stochastic continuous functions and $q_1, \ldots, q_d$ independent Poisson processes starting in t_0.[6] It turns out that the process $x(t)$ is a so called *cádlág* process.[7] That is, the paths of $x(t)$ are continuous from the right with left limits. The left limit is denoted by $x(t_-) \equiv \lim_{s \uparrow t} x(s)$. Thus, due to the continuity of the β_k, the left limit of $\beta_k(t, x(t))$ is given by $\beta_k(t, x(t_-))$.

At first glance, it might appear strange that one uses not $\beta_k(t, x(t))$ but its left limit $\beta_k(t, x(t_-))$ as integrand in SDE (1). But beyond analytical reasons, there is a simple heuristic explanation why this should be like this. If Poisson process $q_k(t)$ jumps, i.e., $dq_k(t) = 1$, then $x(t)$ jumps from $x(t_-)$ to $x(t)$, where the jump size is given by β_k. It would not make much sense if the jump size would depend on the post-jump state $x(t)$. It is rather convenient to assume that the jump size is determined by the state just before the jump occurs — which is formally $x(t_-)$. Thus, the jump size itself is then given by $\beta_k(t, x(t_-))$.

Corollary 2.1 *(1 Poisson process $q(t)$) Consider a univariate stochastic process $x(t)$ given as solution of the SDE*

$$dx(t) = \alpha(t, x(t))\, dt + \beta(t, x(t_-))\, dq(t).$$

Then, for a once continuously differentiable function $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$, the process $f(t, x(t))$ is cádlág, and its differential is given by

$$\begin{aligned} df(t, x(t)) &= [f_t(t, x(t)) + f_x(t, x(t))\, \alpha(t, x(t))]\, dt \\ &\quad + [f(t, x(t_-) + \beta(t, x(t_-))) - f(t, x(t_-))]\, dq(t), \end{aligned}$$

where f_t and f_x denote the partial derivatives of f with respect to the time and the state argument, t and x, respectively.

[6] A detailed analysis of SDEs with Poisson processes can be found, e.g., in Protter (1990) and Garcia and Griego (1994).

[7] The expression cádlág is an acronym from the french "continu a droite limites a gauche".

Intuitively speaking, the differential of a function is given by the "normal terms", i.e., the partial derivatives with respect to its first argument t and with respect to its second argument x times changes per unit of time (1 for the first argument and $\alpha(t, x(t))$ for the second) times dt, and by a "jump term". Whenever the process $q(t)$ increases, $x(t)$ increases by $\beta(t, x(t_-))$, and the function jumps from $f(t, x(t_-))$ to $f(t, x(t)) = f(t, x(t_-) + \beta(t, x(t_-)))$.

The cádlág property of $f(t, x(t))$ holds trivially for all continuous functions f, and we thus do not mention it anymore in the following corollaries.

Corollary 2.2 *(Many independent Poisson processes $q_k(t)$) Consider the univariate stochastic process $x(t)$ that obeys the SDE*

$$dx(t) = \alpha(t, x(t))\, dt + \sum\nolimits_{k=1}^{n} \beta_k(t, x(t_-))\, dq_k(t)\,.$$

For a once continuously differentiable function $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$, the differential of the process $f(t, x(t))$ is given by

$$\begin{aligned} df(t, x(t)) &= [f_t(t, x(t)) + f_x(t, x(t))\, \alpha(t, x(t))]\, dt \\ &\quad + \sum\nolimits_{k=1}^{n} [f(t, x(t_-) + \beta_k(t, x(t_-))) - f(t, x(t_-))]\, dq_k(t)\,. \end{aligned}$$

Again, the differential of a function is given by the "normal terms" and by a "jump term". Whenever any of the processes $q_k(t)$ increases, $x(t)$ increases by $\beta_k(t, x(t_-))$, and the function jumps from $f(t, x(t_-))$ to $f(t, x(t_-) + \beta_k(t, x(t_-)))$.

Corollary 2.3 *(Multivariate stochastic process) Consider the n-dimensional stochastic process $x(t) = (x_1(t), ..., x_n(t))$ that follows the SDE*

$$dx_i(t) = \alpha_i(t, x(t))\, dt + \beta_i(t, x(t_-))\, dq_i(t)\,, \quad i = 1, \ldots n.$$

For a once continuously differentiable function $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, the process $f(t, x(t))$ obeys

$$\begin{aligned} df(t, x(t)) &= \left[f_t(t, x(t)) + \sum\nolimits_{i=1}^{n} f_{x_i}(t, x(t))\, \alpha_i(t, x(t))\right] dt \\ &\quad + \sum\nolimits_{i=1}^{n} \left[f\left(t, x(t_-) + \overrightarrow{\beta}_i(t, x(t_-))\right) - f(t, x(t_-))\right] dq_i(t)\,, \end{aligned}$$

where $\overrightarrow{\beta}_i := (0, \cdots, 0, \beta_i, 0, \cdots, 0,)^T$ denotes a vector of functions that is $\beta_i(t, x(t_-))$ in the i-th component and 0 otherwise.

Here, the "normal terms" include partial derivatives with respect to all the x_i. Whenever any of the processes $q_i(t)$ jumps, the i-th component of $x(t)$ increases by $\beta_i(t, x(t_-))$. The "jump term" therefore makes the function jump from $f(t, x(t_-))$ to $f\left(t, x(t_-) + \overrightarrow{\beta}_i(t, x(t_-))\right)$.

2.2 Application I: The budget constraint

Most maximization problems require a constraint. For a household, this is usually the budget constraint. It is shown here how the structure of the budget constraint depends on the economic environment the household finds itself in, and how CVF is required.

Let wealth at time t, $a(t)$, be given by the number of stocks, $n(t)$, a household owns times their price, $v(t)$. That is, $a(t) = n(t)v(t)$. Let the price follow a process that is exogenous to the household (but potentially endogenous in general equilibrium),

$$dv(t) = \alpha v(t)\,dt + \beta v(t_-)\,dq(t),$$

where $\alpha, \beta \in \mathbb{R}$. Hence, the price grows with the continuous rate α and at discrete random times it jumps by β percent. The random times are modeled by the jump times of a Poisson process $q(t)$ with arrival rate λ, which is the probability that in the current period a price jump occurs. The expected (or average) growth rate is then given by $\alpha + \lambda\beta$.

Let the household earn dividend payments, $\pi(t)$ per unit of asset it owns, and labor income, $w(t)$. Assume furthermore that it spends $p(t)c(t)$ on consumption, where $c(t)$ denotes the consumption quantity and $p(t)$ the price of one unit of the consumption good. When buying stocks is the only way of saving, the number of stocks held by the household changes in a deterministic way according to

$$dn(t) = \frac{n(t)\pi(t) + w(t) - p(t)c(t)}{v(t)}dt.$$

When savings $n(t)\pi(t) + w(t) - p(t)c(t)$ are positive, the number of stocks held by the household increases by savings divided by the price of one stock. When savings are negative, the number of stocks decreases.

The change of the household's wealth, i.e., the household's budget constraint, is then given by applying CVF to $a(t) = n(t)v(t)$. Using corollary 2.3 with $f(t, x_1, x_2) = x_1x_2$, we obtain

$$\begin{aligned} da(t) &= \left\{ v(t)\frac{n(t)\pi(t) + w(t) - p(t)c(t)}{v(t)} + n(t)\alpha v(t) \right\} dt \\ &\quad + \{n(t_-)[v(t_-) + \beta v(t_-)] - n(t_-)v(t_-)\}\,dq(t) \\ &= [r(t)a(t) + w(t) - p(t)c(t)]\,dt + \beta a(t_-)\,dq(t), \end{aligned} \tag{2}$$

where the interest-rate is defined as

$$r(t) \equiv \frac{\pi(t)}{v(t)} + \alpha.$$

This is a very intuitive budget constraint: As long as the asset price does not jump, i.e., $dq(t) = 0$, the household's wealth increases by current savings, $r(t)a(t) + w(t) - p(t)c(t)$, where the interest rate, $r(t)$, consists of dividend payments in terms of the asset price plus the deterministic growth rate of the asset price. If a price jump occurs, i.e., $dq(t) = 1$, wealth jumps, as the price, by β percent, which is the stochastic part of the overall interest-rate. Altogether, the average interest rate amounts to $r(t) + \lambda\beta$.

2.3 Application II: The Hamilton-Jacobi-Bellman equation

In this subsection we show how an appropriate HJB equation can be heuristically derived if one faces a stochastic control problem. For all practical purposes, this only requires the application of CVF.

Take, for example, a simple optimum-consumption problem of a household, consisting in finding an optimal consumption process $c^*(t)$ that maximizes the expected lifetime utility,

$$E_t \int_{t_0}^{\infty} e^{-\rho(t-t_0)} u(c(t))\, dt, \tag{3}$$

subject to the budget constraint derived in the last subsection,

$$da(t) = [r(t)a(t) + w(t) - p(t)c(t)]\, dt + \beta a(t_-)\, dq(t), \quad a(t_0) > 0. \tag{4}$$

E_t denotes the expectation operator conditional on wealth in t, $a(t)$. As a starting point, one writes the HJB equation in the general form as[8]

$$\rho V(t, a(t)) = \max_{c(t)} \left\{ u(c(t)) + \frac{1}{dt} EdV(t, a(t)) \right\}, \tag{5}$$

where the maximum is achieved by the optimal consumption choice $c^*(t)$, and V denotes the value function[9]

$$V(t, a(t)) \equiv E_t \int_t^{\infty} e^{-\rho(s-t)} u(c^*(s))\, ds,$$

which is the maximized expected lifetime utility in t given wealth $a(t)$. The value function therefore presents the maximal value, in terms of utility units, an amount, $a(t)$, of wealth presents for the household at time t. The general HJB equation (5) says that the household chooses consumption in t such that it maximizes its instantaneous return from consumption, which consists of the instantaneous utility flow, $u(c(t))$, plus the change in the expected value of wealth, $\frac{1}{dt}EdV(t, a(t))$, corresponding to the consumption choice in t. That the

[8] See appendix A for a heuristical derivation.

[9] Later, in the example presented in section 3, we shall go further into detail about the considered controls.

household does not have to take into account future utility, but rather $\frac{1}{dt}EdV(t, a(t))$, is due to the derivation of the HJB equation, where optimal behavior after t is assumed, cf. appendix A. Nevertheless, the HJB equation tells furthermore that the intertemporal return from holding $a(t)$, $\rho V(t, a(t))$, is given by the return from the optimal consumption in t, $u(c^*(t)) + \frac{1}{dt}EdV(t, a^*(t))$.

Assume that V is once continuously differentiable. Obtaining the HJB equation for a specific maximization problem then requires (i) application of CVF on $V(t, a(t))$, (ii) computing expectations and (iii) "dividing" by dt.

Taking the budget constraint (4), CVF from corollary 2.1 yields

$$\begin{aligned} dV(t, a(t)) &= \{V_t(t, a(t)) + V_a(t, a(t))[r(t)a(t) + w(t) - p(t)c^*(t)]\} dt \\ &\quad + [V(t, (1+\beta)a(t_-)) - V(t, a(t_-))] dq(t). \end{aligned}$$

With $Edq_t = \lambda dt$, we get

$$\begin{aligned} EdV(t, a(t)) &= \{V_t(t, a(t)) + V_a(t, a(t))[r(t)a(t) + w(t) - p(t)c^*(t)]\} dt \\ &\quad + \lambda [V(t, (1+\beta)a(t)) - V(t, a(t))] dt. \end{aligned}$$

Dividing by dt gives finally the HJB equation for the maximization problem consisting of (3) and (4):

$$\rho V(t, a(t)) = \max_{c(t)\geq 0} \left\{ \begin{array}{c} u(c(t)) + V_t(t, a(t)) \\ +V_a(t, a(t))[r(t)a(t) + w(t) - p(t)c^*(t)] \\ +\lambda [V(t, (1+\beta)a(t)) - V(t, a(t))] \end{array} \right\}. \qquad (6)$$

This approach is very practical, a rigorous background with the necessary assumptions can be found in Sennewald (2005). Note that with this derivation we have implicitly shown that the HJB equation is a necessary criterion for optimality. Hence, the value function must satisfy the HJB equation (6), and the maximum must be attained by the optimal consumption. In the following section, we show how this fact can be used to do further analysis (e.g., to derive a Keynes-Ramsey) if one does not explicitly know neither the value function nor the optimal control.

3 A typical maximization problem

We now present a typical maximization problem, which consists in determining a household's optimal consumption and investment behavior. Finding closed form expressions for the optimal controls is usually restricted to special cases. Nevertheless, for optimum-consumption problems it is usually possible to derive a Keynes-Ramsey rule. We show how this can be achieved, making use of the HJB equation as a necessary criterion for optimality. Then the closed form solution is presented, and its optimality is verified by the fact that the HJB equation together with a certain terminal condition yields a sufficient criterion for optimality.

3.1 The problem

3.1.1 The Setup

Consider a household that is endowed with some initial wealth $a(t_0) > 0$. At each instant, the household can invest its wealth $a(t)$ in both a risky and a safe asset. The share of wealth the household holds in the risky asset is denoted by $\theta(t)$. The price $v_1(t)$ of one unit of the risky asset obeys the SDE

$$dv_1(t) = r_1 v_1(t)\, dt + \beta v_1(t_-)\, dq(t), \tag{7}$$

where $r_1 \in \mathbb{R}$ and $\beta > 0$. That is, the price of the risky asset grows at each instant with a fixed rate r_1 and at random points in time it jumps by β percent. The randomness comes from the well-known Poisson process $q(t)$ with arrival rate λ. The price $v_2(t)$ of one unit of the safe asset is assumed to follow

$$dv_2(t) = r_2 v_2(t)\, dt, \tag{8}$$

where $r_2 \geq 0$. Let the household receive a fixed wage income w and spend $c(t) \geq 0$ on consumption.[10] Then, in analogy to (2) or derived as in appendix B by the “self-financing” concept, the household's budget constraint reads[11]

$$da(t) = \{[\theta(t) r_1 + (1-\theta(t)) r_2] a(t) + w - c(t)\} dt + \beta\theta(t_-) a(t_-)\, dq(t). \tag{9}$$

We allow wealth to become negative, but assume the debts always be covered by the household's lifetime labor income discounted with the safe interest rate, r_2. That is, $a(t) > -w/r_2$ for all t. Let the household's time preference rate be given by the constant $\rho > 0$ and assume that the planning horizon is infinite. Forming expectations about future consumption

[10] Unlike in subsections 2.2 and 2.3, we consider here real variables expressed in terms of the consumption good.

[11] Another approach to derive the budget constraint is to start with the assumption of a "self-financing portfolio". The derivation is presented in appendix B.

streams and given the CRRA[12] instantaneous utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}, \quad \sigma > 0, \quad \sigma \neq 1,^{13} \tag{10}$$

the household's objective is given by maximizing the expected lifetime utility,

$$E \int_{t_0}^{\infty} e^{-\rho(t-t_0)} u(c(t))\, dt, \tag{11}$$

subject to its budget constraint (9). The control variables of the household are the nonnegative consumption stream, $c(t)$, and the share, $\theta(t)$, held in the risky asset. To avoid a trivial investment problem we assume

$$r_1 < r_2 < r_1 + \lambda\beta. \tag{12}$$

That is, the guaranteed return of the risky asset, r_1, is lower than the return of the riskless asset, r_2, whereas, on the other hand, the expected return of the risky asset, $r_1 + \lambda\beta$, shall be greater than r_2.

3.1.2 Classes of controls

There exist various types of controls that may be considered, for example, feedback controls, which depend on the whole history of $a(t)$, Markov controls, which depend on current time and wealth, or generalized controls, which do not depend on "anything" and are merely assumed to be adapted. Here, and usually in other applications as well, only Markov controls are considered. Reasons therefor are:

- Markov controls are easy to handle. That is, since they depend only on current time and wealth, one exactly knows what do to if at time t wealth $a(t)$ is observed.
- The HJB equation provides a very powerful tool to characterize and verify optimal Markov controls.
- If the controlled process (which is here the household's wealth) is a Markov process, the performance of optimal Markov controls is in general as that good as for generalized controls, see, e.g., Sennewald (2004, theorem 5.5).

[12] CRRA: constant relative risk aversion.

[13] The special case $\sigma = 1$, i.e. $u(c) = \log c$, is considered in appendix C, where Keynes-Ramsey rule and closed form solution are presented.

Assume that there exists optimal Markov controls, $c^*(t)$ and $\theta^*(t)$, maximizing the expected lifetime utility (11) subject to budget constraint (9). Then we define the value function V as[14]

$$V(a(t_0)) \equiv E\int_{t_0}^{\infty} e^{-\rho(t-t_0)} u(c^*(t))\,dt.$$

3.2 Solution strategies

Finding the optimal Markov controls can be undertaken by the HJB equation, which derived as in subsection 2.3 or taken from Sennewald (2005), reads for all $a > -w/r_2$

$$\rho V(a) = \max_{\{c \geq 0, \theta \in \mathbb{R}\}} \left\{ \begin{array}{c} u(c) + [(\theta r_1 + (1-\theta) r_2) a + w - c] V'(a) \\ +\lambda [V(\tilde{a}) - V(a)] \end{array} \right\}, \quad (13)$$

where $\tilde{a} \equiv (1 + \theta\beta) a$ denotes the post-jump wealth if at wealth a a jump in the risky asset price occurs. The maximum is achieved by the optimal Markov control values, c^* and θ^*, corresponding to state a.[15] Since, together with a terminal condition, the HJB equation presents a sufficient criterion for optimality (cf. Sennewald (2005, theorem 5.3)), it can be used to verify whether a candidate for the optimal solution is indeed optimal. How to derive such candidates and how to undertake the verification is shown in subsection 3.4.

Unfortunately, finding explicit expressions for the optimal controls is rather the exception. Nonetheless, starting with the HJB equation one can at least derive some characteristics of the optimal behavior for the case where a closed form solution cannot be achieved. For this purpose one uses that the HJB equation presents also a necessary condition for optimality, see Sennewald (2005, theorem 5.1). In the considered optimum consumption and portfolio problem this leads to a stochastic form of the Keynes-Ramsey rule for the optimal consumption path, as is shown in the subsection 3.3. Notice that, while a closed form solution yields the absolute level of optimal consumption, this rule describes "only" the optimal change in consumption over time.

3.3 The Keynes-Ramsey rule

In the present section we show how, starting from the HJB equation as a necessary criterion for optimality, one can derive a Keynes-Ramsey rule if a candidate for a closed form solution

[14] One can show that the value function in this example does not depend on initial time but on initial wealth only. An "ex-post proof" is given by subsection 3.4, where we derive an explicit expression for the value function, see equation (32).

[15] Since the value function does not depend on current time, one can show that the optimal policy does not depend on time neither.

is not deducible. This rule tells us how the optimal consumption must evolve over time. For the HJB equation to become necessary, certain conditions must be satisfied, see Sennewald (2005, theorem 5.1). In particular, the utility function (10) as well as the coefficients in the budget constraint (9) are required to be linearly bounded. That is, there must exist real numbers $\kappa, \mu_i, \nu_i \in \mathbb{R}_+$, $i = 1, 2$, such that for all $a > -w/r_2, c \geq 0$ and $\theta \in \mathbb{R}$,

$$|u(c)| \leq \kappa (1 + c), \tag{14}$$

$$|[\theta r_1 + (1-\theta) r_2] a + w - c| \leq \mu_1 + \nu_1 |a|, \tag{15}$$

and

$$|\beta \theta a| \leq \mu_2 + \nu_2 |a|. \tag{16}$$

Condition (14) is trivially satisfied if the risk aversion parameter σ in utility function (10) lies between 0 and 1.[16] In the case of log-utility or for $\sigma > 1$, utility is bounded as long as consumption does not tend to 0. We assume therefore that there exists a threshold $\varepsilon > 0$ the consumption expenditure never falls below. This assumption is justified if one considers that the marginal utility becomes ∞ as consumption tends to 0. Thus, the household will smooth its consumption stream such that consumption never becomes 0. Then, if we choose ε small enough, we can conclude that utility is bounded by $\left(\varepsilon^{-(\sigma-1)} - 1\right) / (\sigma - 1)$.

For inequalities (15) and (16) to be satisfied, we introduce the following control space constraint. Assume that consumption shall not exceed current wealth plus lifetime labor income, w/r_2. That is,

$$0 \leq c(t) \leq a(t) + \frac{w}{r_2}. \tag{17}$$

Furthermore, we do not allow short-selling of the risky asset, whereas, on the other hand, the household can finance risky investment by short-selling the safe asset. The limit for this kind of borrowing is again given by lifetime labor income. That is, $[1 - \theta(t)] a(t) \geq -w/r_2$. The constraint for the share held in the risky asset thus reads:

$$0 \leq \theta(t) a(t) \leq a(t) + \frac{w}{r_2}. \tag{18}$$

Then the set of admissible controls contains all cádlág processes $c(t)$ and $\theta(t)$ satisfying conditions (17) and (18) such that the associated wealth process always remains above the level $-w/r_2$. Now it is easy to show that for all admissible $c(t)$ and $\theta(t)$ the linear boundedness conditions (15) and (16) are satisfied. Assume that the optimal Markov controls, $c^*(t)$ and $\theta^*(t)$, are in the set of admissible controls.

[16] Choose, e.g., $\kappa = \frac{1}{1-\sigma}$.

Beside the boundedness conditions, a certain regularity condition must hold, and the expected present values of the optimal controls must be finite, see assumption (H3) and (H4) in Sennewald (2005). But for these technical conditions to be satisfied, we have merely to assume the time preference rate ρ to be high enough, cf. remark 3.1(ii) in Sennewald (2004). Then, assuming that the value function is sufficiently smooth, the HJB equation is a necessary criterion for optimality.

Since c^* and θ^* maximize the righ-hand side in the HJB equation (13), the following first-order conditions must be satisfied, if c^* and θ^* are not corner solutions with respect to the constraints (17) and (18):

$$u'(c^*) = V'(a) \tag{19}$$

and

$$V'(a)(r_1 - r_2)a + \lambda V'(\tilde{a}^*)\beta a = 0, \tag{20}$$

where $\tilde{a}^* \equiv (1 + \theta^* \beta) a$ denotes the post-jump wealth for the optimal investment behavior. Replacing in equation (20) V' with u' according to (19) yields for $a \neq 0$

$$\frac{u'(\tilde{c}^*)}{u'(c^*)} = \frac{r_2 - r_1}{\lambda\beta}, \tag{21}$$

where $\tilde{c}^*$ denotes the optimal consumption choice for $\tilde{a}^*$. Hence, the ratio for optimal consumption after and before a jump is constant:

$$\frac{\tilde{c}^*}{c^*} = \left(\frac{\lambda\beta}{r_2 - r_1}\right)^{1/\sigma}. \tag{22}$$

Since by assumption (12) the term on the right-hand side is greater than 1, this equation shows that consumption jumps upwards if a jump in the risky asset price occurs. This result is not surprising since, if the risky asset price jumps upwards, so does the household's wealth.

In the next step, we compute the evolution of $V'(a^*(t))$, where $a^*(t)$ denotes the wealth process associated to the optimal consumption and investment behavior. Assume that V is twice continuously differentiable. Then, due to budget constraint (9), CVF from corollary 2.1 yields

$$\begin{aligned} dV'(a^*(t)) &= \{[\theta^*(t) r_1 + (1 - \theta^*(t)) r_2] a^*(t) + w - c^*(t)\} V''(a^*(t)) dt \\ &\quad + [V'(\tilde{a}^*(t_-)) - V'(a^*(t_-))] dq(t). \end{aligned} \tag{23}$$

On the other hand, differentiating the maximized HJB equation (13) evaluated at $a^*(t)$

yields under application of the envelope theorem

$$\begin{aligned}
\rho V'(a^*(t)) &= \{[\theta^*(t) r_1 + (1-\theta^*(t)) r_2] a^*(t) + w - c^*(t)\} V''(a^*(t)) \\
&\quad + \{\theta^*(t) r_1 + [1-\theta^*(t)] r_2\} V'(a^*(t)) \\
&\quad + \lambda \{V'(\tilde{a}^*(t)) [1+\theta^*(t)\beta] - V'(a^*(t))\}.
\end{aligned}$$

Rearranging gives

$$\begin{aligned}
& \{[\theta^*(t) r_1 + (1-\theta^*(t)) r_2] a^*(t) + w - c^*(t)\} V''(a^*(t)) \\
= & \{\rho - [\theta^*(t) r_1 + (1-\theta^*(t)) r_2]\} V'(a^*(t)) \\
& - \lambda \{V'(\tilde{a}^*(t)) [1+\theta^*(t)\beta] - V'(a^*(t))\}.
\end{aligned}$$

Inserting this into (23) yields

$$\begin{aligned}
dV'(a^*(t)) &= \left\{ \begin{array}{c} \{\rho - [\theta^*(t) r_1 + (1-\theta^*(t)) r_2]\} V'(a^*(t)) \\ -\lambda \{[1+\theta^*(t)\beta] V'(\tilde{a}^*(t)) - V'(a^*(t))\} \end{array} \right\} dt \\
&\quad + [V'(\tilde{a}^*(t_-)) - V'(a^*(t_-))] \, dq(t).
\end{aligned}$$

By replacing V' with u' according to the first-order condition for optimal consumption, equation (19), we obtain

$$\begin{aligned}
du'(c^*(t)) &= \left\{ \begin{array}{c} \{\rho - [\theta^*(t) r_1 + (1-\theta^*(t)) r_2]\} u'(c^*(t)) \\ -\lambda \{[1+\theta^*(t)\beta] u'(\tilde{c}^*(t)) - u'(c^*(t))\} \end{array} \right\} dt \\
&\quad + [u'(\tilde{c}^*(t_-)) - u'(c^*(t_-))] \, dq(t).
\end{aligned}$$

Now applying the CVF from corollary 2.1 to $f(x) = (u')^{-1}(x)$ leads to the Keynes-Ramsey rule for general utility functions u,

$$\begin{aligned}
-\frac{u''(c^*(t_-))}{u'(c^*(t_-))} dc^*(t) &= \left\{ \begin{array}{c} \theta^*(t) r_1 + [1-\theta^*(t)] r_2 - \rho \\ -\lambda \left\{ 1 - [1+\theta^*(t)\beta] \frac{u'(\tilde{c}^*(t))}{u'(c^*(t))} \right\} \end{array} \right\} dt \\
&\quad - [\tilde{c}^*(t_-) - c^*(t_-)] \frac{u''(c^*(t_-))}{u'(c^*(t_-))} dq(t).
\end{aligned}$$

For the CRRA utility function as given as in (10) we get by eliminating $u'(\tilde{c}_t^*)$ according to (21) and $\tilde{c}_t^*$ according to (22)

$$\frac{dc^*(t)}{c^*(t_-)} = \frac{1}{\sigma} \left[r_2 - \lambda \left(1 - \frac{r_2 - r_1}{\lambda\beta} \right) - \rho \right] dt + \left[\left(\frac{\lambda\beta}{r_2 - r_1} \right)^{1/\sigma} - 1 \right] dq(t).$$

The optimal change in consumption can thus be expressed in terms of well-known parameters. As long as the price of the risky asset does not jump, optimal consumption grows constantly by the rate $\left[r_2 - \lambda\left(1 - \frac{r_2 - r_1}{\lambda\beta}\right) - \rho\right]/\sigma$. The higher the risk-free interest rate, r_2, and the lower the guaranteed interest rate of the risky asset, r_1, the discrete growth rate, β, the probability of a price jump, λ, the time preference rate, ρ, and the risk aversion parameter, σ, the higher becomes the consumption growth rate. If the risky asset price jumps, consumption jumps as well to its new higher level $c^*(t) = [(\lambda\beta)/(r_2 - r_1)]^{1/\sigma} c^*(t_-)$. Here the growth rate depends positively on λ, β, and r_1, whereas r_2 and σ have negative influence.

3.4 A closed form solution

3.4.1 General approach: Guessing the value function

Obtaining a closed form solution for the optimal controls is not obvious. Looking for such a solution has a long tradition in finance (see, e.g., Merton (1969, 1971) or Framstad et al. (2001)) and also in macroeconomics (see, e.g., Wälde (1999a)). Finding a closed form solution is in general the result of an "educated guess". That means, we consider already solved optimization problems that are similar to ours and try to deduce a solution from them. After having found a candidate for a solution, it has to be verified. To this end, one can use a so called verification theorem. Such a theorem tells us that, if the candidate for the optimal solution solves the HJB equation and if furthermore certain limiting conditions are satisfied, the candidate for the optimal solution is indeed optimal, see, e.g., theorem 5.2 in Sennewald (2005).

From similar consumption and investment problems in Merton (1969, 1971) and elsewhere we can guess that the value function is of the form

$$J(a) = \frac{K[a+L]^{1-\sigma} - M}{1-\sigma} \tag{24}$$

with unknown constants K, L, and M. In the following steps, this rather vague expression for the candidate of the value function is used to derive the optimal consumption and investment behavior and explicit expression for K, L and M.

3.4.2 Deriving and verifying optimal consumption and investment

Let us for the moment abandon the control space contraints (17) and (18), introduced in subsection 3.3. Starting from the candidate for the value function in (24) and using the verification theorem 5.2 in Sennewald (2005), we show how the optimal consumption and investment behavior can be both derived and verified at the same time. The verification consists of two steps:

1.) Does the candidate for the value function solve the HJB equation

$$\rho J(a) = \max_{\{c\geq 0,\theta\in\mathbb{R}\}} \left\{ \begin{array}{c} u(c) + [(\theta r_1 + (1-\theta) r_2) a + w - c] J'(a) \\ +\lambda [J(\tilde{a}) - J(a)] \end{array} \right\}? \tag{25}$$

And is the maximum in (25) attained by the candidates for the optimal controls, c^* and θ^*?

2.) Are the limiting conditions

$$\lim_{t\to\infty} E\left[e^{-\rho t} J(a^*(t))\right] = 0 \tag{26}$$

and

$$\lim_{t\to\infty} E\left[e^{-\rho t} J(a(t))\right] \geq 0 \tag{27}$$

satisfied, where $a(t)$ denotes the wealth process associated to an arbitrary admissible Markov control?

At first, we derive in step 1.) the constants K, L, M and the candidates for the optimal controls such that the HJB equation (25) holds. Then we show in step 2.) that these candidates satisfy limiting conditions (26) and (27).

Step 1.) Since the right-hand side of the HJB equation (25) is strictly concave in c and θ, the HJB equation holds if the following two points are satisfied:

a) Do the candidates for the optimal controls solve the first-order conditions for the maximum on the right-hand side in (25)?

b) Do the candidates for the optimal controls yield equality in (25)?

Point a) makes sure that c^* and θ^* maximize the right-hand side in (25). If furthermore point b) is satisfied, we can conclude that the HJB equation holds.

ad a) The first-order conditions read (cf. also (19) and (20))

$$u'(c^*) = J'(a) \tag{28}$$

and

$$J'(a)(r_1 - r_2) a + \lambda J'(\tilde{a}^*) \beta a = 0. \tag{29}$$

Rearranging the last equation yields for $a \neq 0$

$$(a+L)^{-\sigma}(r_2 - r_1) = \lambda [(1+\beta\theta^*) a + L]^{-\sigma} \beta.$$

Therefore, the optimal consumption must be

$$c^* = K^{-1/\sigma}[a+L], \tag{30}$$

and the optimal share invested in the risky asset

$$\theta^* = \left[\left(\frac{\lambda\beta}{r_2 - r_1}\right)^{1/\sigma} - 1\right]\frac{a+L}{\beta a}, \quad a \neq 0. \tag{31}$$

ad b) Inserting (30) and (31) into the maximized HJB equation (25) gives unique expressions for K, L, and M, such that finally the candidate for the value function reads[17]

$$J(a) = \frac{\frac{1}{\psi^\sigma}\left(a + \frac{w}{r_2}\right)^{1-\sigma} - \frac{1}{\rho}}{1-\sigma}, \tag{32}$$

with the constant

$$\psi = \frac{1}{\sigma}(\rho + \lambda) - \frac{1-\sigma}{\sigma}\left(r_2 + \frac{r_2 - r_1}{\beta}\right) - \lambda\left(\frac{\lambda\beta}{r_2 - r_1}\right)^{\frac{1-\sigma}{\sigma}}. \tag{33}$$

Thus, according to (30), optimal consumption must obey

$$c^* = \psi\left[a + \frac{w}{r_2}\right], \tag{34}$$

whereas, by (31), the optimal share of wealth held in the risky asset can only be

$$\theta^* = \left[\left(\frac{\lambda\beta}{r_2 - r_1}\right)^{\frac{1}{\sigma}} - 1\right]\frac{a + \frac{w}{r_2}}{\beta a}, \quad a \neq 0. \tag{35}$$

In order to derive economically meaningful solutions, we require ψ to be positive. That is, the time preference must be enough, namely

$$\rho > (1-\sigma)\left(r_2 + \frac{r_2 - r_1}{\beta}\right) + \sigma\lambda\left(\frac{\lambda\beta}{r_2 - r_1}\right)^{\frac{1-\sigma}{\sigma}} - \lambda. \tag{36}$$

Notice that with (34) and (35), we have derived the (only) controls corresponding to the guessed value function (24) that maximize the HJB equation. Thus, if now the terminal conditions in step 2.) are satisfied, we know that these controls are optimal.

Step 2.) This step requires some calculation. At first, we check limiting condition (26). Due to the shape of J as given as in (32), it suffice to show

$$\lim_{t\to\infty} e^{-\rho t} E\left(a^*(t) + \frac{w}{r_2}\right)^{1-\sigma} = 0. \tag{37}$$

To this end, we derive an explicit expression for $(a^*(t) + w/r_2)^{1-\sigma}$. According to CVF, the total wealth process $a^*(t) + w/r_2$ obeys the budget constraint (9) with starting point

[17] See appendix D.

$a(t_0) + w/r_2$. Inserting the candidates for optimal consumption and investment from (34) and (35) into the budget constraint yields

$$d\left[a^*(t) + \frac{w}{r_2}\right] = \eta_1 \left[a^*(t) + \frac{w}{r_2}\right] dt + \eta_2 \left[a^*(t_-) + \frac{w}{r_2}\right] dq(t),$$

where $\eta_1 = \frac{1}{\sigma}\left[r_2 - \lambda\left(1 - \frac{r_2 - r_1}{\lambda\beta}\right) - \rho\right]$ and $\eta_2 = \left(\frac{\lambda\beta}{r_2 - r_1}\right)^{\frac{1}{\sigma}} - 1$. The solution of this linear stochastic differential equation reads (see, e.g., Garcia and Griego (1994)):

$$a^*(t) + \frac{w}{r_2} = \left(a(t_0) + \frac{w}{r_2}\right) \exp^{\eta_1(t-t_0)+\ln(1+\eta_2)q(t)}.$$

Hence,[18]

$$E\left(a^*(t) + \frac{w}{r_2}\right)^{1-\sigma} = \left(a(t_0) + \frac{w}{r_2}\right)^{1-\sigma} \exp^{\left\{(1-\sigma)\eta_1 + \lambda\left[(1+\eta_2)^{1-\sigma} - 1\right]\right\}(t-t_0)}.$$

Therefore, (37), and thus (26) as well, are satisfied if and only if

$$\rho > (1-\sigma)\eta_1 + \lambda\left[(1+\eta_2)^{1-\sigma} - 1\right],$$

which after some rearranging is equivalent to

$$\rho > (1-\sigma)\left(r_2 + \frac{r_2 - r_1}{\beta}\right) + \sigma\lambda\left(\frac{\lambda\beta}{r_2 - r_1}\right)^{\frac{1-\sigma}{\sigma}} - \lambda.$$

But this parameter constellation is already met by (36). Thus, limiting condition (26) is satisfied. This well-known result, the connection between positive consumption and limiting condition (26), was also found by Merton (1990) in a revised version of its paper from 1969 for the case with Brownian motion as noise.

It remains to be shown that limiting inequality (27) holds for any arbitrary admissible Markov control. For the case $0 < \sigma < 1$, we use that the candidate for the value function, (32), is always greater than $-[\rho(1-\sigma)]^{-1}$. Therefore,

$$\lim_{t\to\infty} E\left[e^{-\rho t} J(a(t))\right] \geq -\lim_{t\to\infty} \frac{e^{-\rho t}}{\rho(1-\sigma)} = 0$$

is trivially satisfied.

For $\sigma \geq 1$, finding a lower bound for $J(a(t))$ is less simple since we can not rule out that $J(a(t))$ approaches $-\infty$, which happens if $a(t)$ approaches the boundary of the state space, $-w/r_2$. Thus, for (27) to be satisfied, we have to show that $J(a(t))$ tends to $-\infty$

[18] Here we use $E \exp^{aX+b} = \exp^{\lambda\left(\exp^b - 1\right)+a}$, where X is a Poisson distributed random variable with parameter λ.

with a rate less than ρ. For this purpose, we derive at first the lowest $a(t)$ the household can achieve. Assume without loss of generality that the household is in debt. That is, $a(t) < 0$. Now, introducing again control space constraints (17) and (18), one can show easily that the infinitesimal change of $a(t)$ is always greater than $-(1-r_1)[a(t) + w/r_2]$. Thus, using a comparison principle as, e.g., corollary 3.5 in Bassan et al. (1993), we can conclude that $a(t) \geq \tilde{a}(t)$, where $\tilde{a}(t)$ is the solution of

$$d\tilde{a}(t) = -(1-r_1)\left[\tilde{a}(t) + \frac{w}{r_2}\right]dt, \quad \tilde{a}(t_0) = a(t_0).$$

Solving this linear differential equation yields

$$\tilde{a}(t) = \left(1 - \frac{w}{r_2}\right)\exp^{-(1-r_1)[t-t_0]} a(t_0) - \frac{w}{r_2}.$$

Hence,

$$\lim_{t\to\infty} E\left[e^{-\rho t} J(a(t))\right] \geq \lim_{t\to\infty} e^{-\rho t} J(\tilde{a}(t))$$
$$= -\frac{\left[\left(1-\frac{w}{r_2}\right)a(t_0)\right]^{-(\sigma-1)}\exp^{-\rho t_0}}{(\sigma-1)\psi^\sigma}\lim_{t\to\infty}\exp^{-[\rho-(\sigma-1)(1-r_1)][t-t_0]}.$$

Thus, for limiting condition (27) to be satisfied, we must again require the time preference parameter ρ to be high enough, namely $\rho > (\sigma-1)(1-r_1)$.

Finally, we have verified that the derived candidates for the optimal controls, (34) and (35), are indeed the optimal Markov controls. For this purpose, we required only the time preference rate to be high enough, and for the case $\sigma > 1$ we introduced again control space constraints (17) and (18). Therefore, we now have to make sure that for $\sigma > 1$ the optimal controls indeed satisfy these constraints. Inserting the expression for optimal consumption and investment, (34) and (35), into (17) and (18), respectively, shows that a sufficient parameter constellation is given if $\psi \leq 1$, i.e.,

$$\rho \leq \sigma + (1-\sigma)\left(r_2 + \frac{r_2 - r_1}{\beta}\right) + \sigma\lambda\left(\frac{\lambda\beta}{r_2 - r_1}\right)^{\frac{1-\sigma}{\sigma}} - \lambda$$

and

$$\frac{\lambda\beta}{r_2 - r_1} \leq (1+\beta)^\sigma.$$

The latter inequality means that the expected return from a jump in the risky asset price, $\lambda\beta$, shall not exceed the "opportunity costs" for investment in the risky asset, $r_2 - r_1$, too much. Then, the household is not willing to borrow more than its total wealth $a + w/r_2$ to finance risky investment.

Finally, we can now use theorem 5.5 in Sennewald (2005) to deduce that the optimal Markov controls (34) and (35) are even optimal within the class of general controls. The assumptions required in this theorem — "$\geq$" in the HJB equation (25) is satisfied, and limiting inequality (27) holds for all general controls — are implicitly shown by steps 1.) and 2.) above.

3.4.3 Economic insights

Equation (34) shows that the optimal consumption is a constant fraction of total wealth, $a + w/r_2$, consisting of physical wealth plus the present value of all current and future labor income. If wealth is small then consumption exceeds wealth and the household runs into debt. Future wages are used to repay this debt. This behavior embodies the consumption smoothing motive of a risk avers household.

Equation (35) shows that the optimal share of wealth invested into the risky asset is a constant times total wealth divided by physically wealth. The lower the physical wealth the higher this share. If a is very low, then the optimal behaving household borrows to finance risky investment, i.e., $\theta^* > 1$.[19]

Since the absolute investment in the risky asset, $\theta^* a$, is a constant fraction of total wealth, $a + w/r_2$, the optimal share θ^* goes to infinity as a tends toward 0, namely according to

$$\lim_{a \to 0} \theta^* a = \left[\left(\frac{\lambda\beta}{r_2 - r_1} \right)^{\frac{1}{\sigma}} - 1 \right] \frac{w}{r_2}.$$

If a is negative, i.e., the household is in debt, then θ^* is negative as well. Hence, $\theta^* a$ is positive, which implies that the optimal behaving household, once it is in debt, borrows to finance more risky investment.[20]

At first view, it might appear paradox with respect to the household's risk aversion that with lower wealth the share held in the risky asset increases and that the household even borrows to buy more from the risky asset. But read equation (35) like this: The absolute investment into the risky asset, $\theta^* a$, is a constant fraction of total wealth, $a + w/r_2$. Hence, the lower total wealth the lower the risky investment. This is consistent with the results found in, e.g., Merton (1969, 1971). But since here θ^* is expressed as a fraction of physical and not of total wealth, it must be decreasing in a if the income w remains constant.

[19] Borrowing in this context means short-selling the risk-free asset.

[20] The risk-free investment amounts to $(1 - \theta^*) a$, which is for negative θ^* lower than the debts.

4 Conclusion

This paper has given examples of how the CVF and the HJB equation can be used to analyze optimal behavior in an optimal control setup of Poisson uncertainty. When a closed form solution for optimal behavior is available, further analysis is straightforward. When only a Keynes-Ramsey rule can be derived, further analysis can use, e.g., phase diagrams to understand properties of optimal behavior.

The presented derivations and results should apply in different setups with Poisson processes as well. The principles of deriving a Keynes-Ramsey rule or closed form solutions remain the same.

A Heuristic derivation of HJB equation (5)

Assuming that an optimal consumption process exists, we derive from

$$0 = \max_{c(s)\geq 0} E_t \int_t^\infty e^{-\rho(s-t)} u\left(c\left(s\right)\right) ds - V(t, a\left(t\right))$$

for some small $h > 0$

$$0 = \max_{c(s)\geq 0} \left\{ \begin{array}{c} E_t \int_t^{t+h} e^{-\rho(s-t)} u\left(c\left(s\right)\right) ds \\ +E_t \left[e^{-\rho h} E_{t+h} \int_{t+h}^\infty e^{-\rho(s-(t+h))} u\left(c\left(s\right)\right) ds \right] \end{array} \right\} - V(a\left(t\right)),$$

where E_{t+h} denotes the expectation operator conditional on wealth in $t+h$. This conditional expectation is nothing else than the expected lifetime utility for a household starting with wealth $a\left(t+h\right)$ at time $t+h$. Therefore, for any control $c\left(s\right)$, $s \geq t+h$,

$$E_{t+h} \int_{t+h}^\infty e^{-\rho(s-(t+h))} u\left(c\left(s\right)\right) \leq V\left(t+h, a\left(t+h\right)\right),$$

where equality holds for the optimal consumption process $c^*\left(s\right)$. Hence,

$$0 = \max_{c(s)\geq 0, t\leq s<t+h} \left\{ \begin{array}{c} E \int_t^{t+h} e^{-\rho(s-t)} u\left(c\left(s\right)\right) ds \\ +E\left[e^{-\rho h} V\left(t+h, a\left(t+h\right)\right)\right] \end{array} \right\} - V(a\left(t\right)).$$

That means, assumed optimal behavior from time $t+h$ on, the optimal consumption has only to be determined until $t+h$ and not on the whole infinite time horizon. Dividing by h and applying the limit $h \searrow 0$, the last equation becomes

$$0 = \max_{c(t)} \left\{ \begin{array}{c} \lim_{h\searrow 0} E\frac{1}{h} \int_t^{t+h} e^{-\rho(s-t)} u\left(c\left(s\right)\right) ds \\ +\lim_{h\searrow 0} E\frac{1}{h}\left[e^{-\rho h} V\left(a\left(t+h\right)\right) - V(a\left(t\right))\right] \end{array} \right\}. \quad (38)$$

The second expression on the right-hand side is the derivation of $e^{-\rho h} EV(a(t+h))$ with respect to h for $h=0$. Hence, using the product rule this derivation becomes

$$-\rho V(t, a(t)) + \frac{d}{dh} EV(t+h, a(t+h)).$$

Since $\frac{d}{dh} EV(t+h, a(t+h))$ in $h=0$ is equal as $\frac{d}{dt} EV(t, a(t))$ and today's wealth is independent on today's consumption choice, we may rewrite (38) as

$$\rho V(t, a(t)) = \max_{c(t)} \left\{ \lim_{h \searrow 0} E \frac{1}{h} \int_t^{t+h} e^{-\rho(s-t)} u(c(s))\, ds + \frac{d}{dt} EV(t, a(t)) \right\}.$$

Under certain conditions the theorem of bounded convergence allows to interchange limit and expectation (and thus differentiation and expectation). Then the latter equation becomes

$$\rho V(t, a(t)) = \max_{c(t)} \left\{ u(c(t)) + \frac{EdV(t, a(t))}{dt} \right\},$$

which is the general HJB equation (5).

B Deriving budget constraint (9): The self-financing approach

An other approach to derive the budget constraint is the self-financing concept, taken from finance, where the change of a portfolio value is only due to stock price changes. In our example it means that, if dividend payments are not taken into account, the only source for a change in the household's wealth are price changes of the stocks held by the household, labor income and consumption expenditure. We can thus describe the evolvement of wealth by

$$da(t) = n_1(t_-)\, dp_1(t) + n_2(t_-)\, dp_2(t) + (w - c(t))\, dt,$$

where $n_1(t)$ and $n_2(t)$ denote the number of stocks hold from the risky and the safe asset, respectively. Then, inserting the differentials for the asset prices, (7) and (8), yields

$$\begin{aligned} da(t) &= [r_1 n_1(t) v(t) + r_2 n_2(t) v(t) + w - c(t)]\, dt + \beta n_1(t_-) v_1(t_-)\, dq(t) \\ &= \{[\theta(t) r_1 + (1-\theta(t)) r_2] a(t) + w - c(t)\}\, dt + \beta\theta(t_-) a(t_-)\, dq(t), \end{aligned}$$

which gives already budget constraint (9).

C A special case: $u(c) = \ln c$

If we let the risk aversion parameter in (10), σ, tend toward 1, utility becomes

$$u(c) = \ln c.$$

For this special case, the Keynes-Ramsey rule reads

$$\frac{dc^*(t)}{c^*(t_-)} = \left[r_2 + \frac{r_2 - r_1}{\beta} - \lambda - \rho\right] dt + \left[\frac{\lambda\beta}{r_2 - r_1} - 1\right] dq(t).$$

The optimal consumption is given by

$$c^* = \rho\left[a + \frac{w}{r_2}\right],$$

and the optimal investment behavior is

$$\theta^* = \left[\frac{\beta\lambda}{r_2 - r_1} - 1\right]\frac{a + \frac{w}{r_2}}{\beta a}.$$

The value function reads

$$V(a) = \frac{\ln\left(a + \frac{w}{r_2}\right)}{\rho} + \frac{\ln\rho}{\rho} + \frac{\lambda\left(\ln\lambda + \ln\frac{\beta}{r_2 - r_1}\right) - \left(\rho + \lambda - r_2 - \frac{r_2 - r_1}{\beta}\right)}{\rho^2}.$$

D Verification theorem: Deriving (32), (34) and (35)

Inserting the candidates for the value function and the optimal consumption and investment behavior, (24), (30), and (31), respectively, into the maximized HJB equation in (25) yields

$$\begin{aligned}
\rho\frac{K[a+L]^{1-\sigma} - M}{1-\sigma} &= \frac{K^{-(1-\sigma)/\sigma}[a+L]^{1-\sigma} - 1}{1-\sigma} \\
&+ \left\{\begin{array}{c}\left[\left(\left(\frac{\lambda\beta}{r_2 - r_1}\right)^{1/\sigma} - 1\right)\frac{a+L}{\beta a}(r_1 - r_2) + r_2\right] a \\ + w - K^{-1/\sigma}[a+L]\end{array}\right\} K[a+L]^{-\sigma} \\
&+ \lambda\left[\begin{array}{c}\frac{K\left\{\left[1 + \left(\left(\frac{\lambda\beta}{r_2 - r_1}\right)^{1/\sigma} - 1\right)\frac{a+L}{a}\right]a + L\right\}^{1-\sigma} - M}{1-\sigma} \\ - \frac{K[a+L]^{1-\sigma} - M}{1-\sigma}\end{array}\right]
\end{aligned}$$

Rearranging gives

$$\begin{aligned}
\rho\frac{K[a+L]^{1-\sigma}}{1-\sigma} &= \frac{K^{-(1-\sigma)/\sigma}[a+L]^{1-\sigma} - 1}{1-\sigma} - \frac{1 - \rho M}{1-\sigma} \\
&+ \left\{\begin{array}{c}\left(\left(\frac{\lambda\beta}{r_2 - r_1}\right)^{1/\sigma} - 1\right)\frac{r_1 - r_2}{\beta} \\ + r_2\frac{a + w/r_2}{a+L} - K^{-1/\sigma}\end{array}\right\} K[a+L]^{1-\sigma} \\
&+ \lambda\left[\frac{K\left(\frac{\lambda\beta}{r_2 - r_1}\right)^{(1-\sigma)/\sigma}(a+L)^{1-\sigma}}{1-\sigma} - \frac{K[a+L]^{1-\sigma}}{1-\sigma}\right].
\end{aligned}$$

Since this equation must hold for all $a > -w/r_2$, we conclude, $M = 1/\rho$. Hence, dividing the whole equation by $K\left[a+L\right]^{1-\sigma}$ and multiplying with $1-\sigma$ leads to

$$\begin{aligned}\rho &= K^{-1/\sigma} + (1-\sigma)\left[\left(\left(\frac{\lambda\beta}{r_2-r_1}\right)^{1/\sigma} - 1\right)\frac{r_1-r_2}{\beta} + r_2\frac{a+w/r_2}{a+L} - K^{-1/\sigma}\right] \\ &\quad + \lambda\left[\left(\frac{\lambda\beta}{r_2-r_1}\right)^{(1-\sigma)/\sigma} - 1\right].\end{aligned}$$

Then, again from the fact that this must hold for all $a > -w/r_2$, we obtain $L = w/r_2$. Further rearranging yields

$$\begin{aligned}\rho &= \sigma K^{-1/\sigma} + (1-\sigma)\left[-\lambda\left(\frac{\lambda\beta}{r_2-r_1}\right)^{(1-\sigma)/\sigma} + \frac{r_2-r_1}{\beta} + r_2\right] \\ &\quad + \lambda\left[\left(\frac{\lambda\beta}{r_2-r_1}\right)^{(1-\sigma)/\sigma} - 1\right],\end{aligned}$$

and, thus,

$$K^{-1/\sigma} = \frac{1}{\sigma}\left[\rho + \lambda - (1-\sigma)\left(\frac{r_2-r_1}{\beta} + r_2\right)\right] - \lambda\left(\frac{\lambda\beta}{r_2-r_1}\right)^{(1-\sigma)/\sigma}.$$

The expression on the right-hand side is equal as ψ in (33). Hence, $K^{-1/\sigma} = \psi$, and the explicit expressions for the candidates of the value function and optimal consumption and investment in (32), (34) and (35) follow.

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