Demographic Research a free, expedited, online journal
of peer-reviewed research and commentary
in the population sciences published by the
Max Planck Institute for Demographic Research
Konrad-Zuse Str. 1, D-18057 Rostock · GERMANY
www.demographic-research.org

DEMOGRAPHIC RESEARCH

VOLUME 20, ARTICLE 29, PAGES 721-730
PUBLISHED 18 JUNE 2009
http://www.demographic-research.org/Volumes/Vol20/29/
DOI: 10.4054/DemRes.2009.20.29

Formal Relationships 4

The age separating early deaths from late deaths

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This article is part of the Special Collection "Formal Relationships".
Guest Editors are Joshua R. Goldstein and James W. Vaupel.

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The age separating early deaths from late deaths

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Abstract

Averting deaths may either increase or reduce life disparity, as measured by $e^\dagger$. The sign of the effect depends on the age at which the deaths are averted. We show that there may, depending on the entropy of the life table, be an age such that averting deaths before that age reduces disparity, but averting deaths after that age increases disparity. We say that this age separates "early" from "late" deaths (in terms of their effect on disparity) and investigate the factors determining this threshold age. If life table entropy is less than one, a unique threshold age separating early and late deaths always exists. If life table entropy is greater than one, averting deaths at any age increases disparity. If entropy equals one, averting deaths at age zero has no effect, but averting a death at any age after zero increases disparity.

1. Relationship

We measure life disparity by life expectancy lost due to death, $e^\dagger = \int_0^\infty e(a)\,d(a)\,da$, where $e(a) = \left(\int_a^\infty \ell(x)\,dx\right)/\ell(a)$ is remaining life expectancy at age a and $d(a) = \ell(a)\ \mu(a)$ is the life table distribution of deaths. Note that $\ell(a) = \exp(-\int_0^a \mu(x)\,dx)$, with $\ell(0) = 1$, gives the probability of survival to age a, that $\mu(a)$ denotes the age-specific hazard of death, and that $H(a) = \int_0^a \mu(x)\,dx$ is the cumulative hazard function with $H(0) = 0$.

Consider a perturbation that alters the cumulative hazard function by a step function with a single negative step of size s. Let $e_s^\dagger$ be the corresponding value of life disparity with the perturbation. To study the impact on $e^\dagger$ of a concentrated decrease in mortality at age a, it is useful to consider the function

$$k(a) = \frac{1}{\ell(a)} \frac{de_s^\dagger}{ds}\bigg|_{s=0} = e^\dagger(a) + e(a)\big(H(a) - 1\big), \tag{1}$$

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where $e^\dagger(a) = \left(\int_a^\infty e(x)\,d(x)\,dx\right)/\ell(a)$ is life expectancy lost due to death among people surviving to age a. Mortality reductions at ages with $k < 0$ decrease life disparity, while those at ages with $k > 0$ increase life disparity.

We prove below that if $e^\dagger/e(0) < 1$, then there exists an age $a^\dagger$ greater than zero such that $k > 0$ for a greater than $a^\dagger$ and $k < 0$ for values of a less than $a^\dagger$. We also prove that if $e^\dagger/e(0) = 1$, then averting a death at age zero does not change life disparity, but averting a death at any higher age increases disparity. Finally we prove that if $e^\dagger/e(0) > 1$, then averting a death at any age increases life disparity. In the proof we make use of the fact that $k(0) = e^\dagger - e(0)$. Note that the "entropy of the life table" equals $e^\dagger/e(0)$ (Keyfitz 1977; Mitra 1978; Goldman and Lord 1986; Vaupel 1986), which differs from the standard statistical definition by a term equal to $\ln e(0)$. If life table entropy is less than one, then $k(0) < 0$; if entropy is one, then $k(0) = 0$; and if entropy is greater than one, then $k(0) > 0$.

2. Proof

We consider three cases.

Case 1: $k(0) < 0$ (entropy < 1)

Because life spans are finite, $\ell(a)$ approaches zero and $H(a)$ approaches infinity. Therefore at an old enough age x, $H(x) > 1$ and $k(x) > 0$. Since $k(0) < 0$ and $k(a)$ is continuous, the intermediate value theorem implies that there exists at least one point, say z, in $[0, \infty]$ such that $k(z) = 0$. We prove that at any such root the derivative of k is positive. This implies the root is unique, as shown in the Appendix.

The derivative of (1) is given by

$$\begin{aligned}\frac{dk(a)}{da} &= \frac{de^\dagger(a)}{da} + \Big(H(a) - 1\Big)\frac{de(a)}{da} + e(a)\frac{d\big(H(a)-1\big)}{da} \\ &= -\mu(a)e(a) + \mu(a)e^\dagger(a) + \big(H(a)-1)\big)\big(\mu(a)e(a) - 1\big) + e(a)\mu(a) \\ &= \mu(a)e^\dagger(a) + \big(H(a)-1)\big)\big(\mu(a)e(a) - 1\big).\end{aligned} \tag{2}$$

When $k(a) = 0$ it follows from (1) that $H(a) - 1 = -e^\dagger/e(a)$. Substituting this into (2) yields

$$\left.\frac{dk(a)}{da}\right|_{a=z} = \mu(a)e^\dagger(a) - \frac{e^\dagger(a)}{e(a)}\big(\mu(a)e(a) - 1\big) = \frac{e^\dagger(a)}{e(a)} > 0. \tag{3}$$

Case 2: $k(0) = 0$ (entropy $= 1$)

In this case, $z = 0$. We need to show that this is the only value of z, i.e., the only age when $k = 0$. It follows from (2) that if $k(0) = 0$ then

$$\left.\frac{dk(a)}{da}\right|_{a=0} = 1.$$

Hence k becomes positive as age increases from zero. If there were an age above zero when $k(z) = 0$, then the derivative of k at this age would have to be zero or negative. But as shown in (3), the derivative has to be positive at any age when $k(z) = 0$. This contradiction implies that the value of $a^\dagger$ equal to zero is unique in the case when $k(0) = 0$.

Case 3: $k(0) > 0$ (entropy > 1)

As noted above in Case 2, if there were an age when $k(a) = 0$ then the derivative of k at this age would have to be zero or negative. But as shown in (3), the derivative has to be positive at any age when $k(a) = 0$. This contradiction implies that there is no age that separates early from late deaths when $k(0) > 0$: averting a death at any age would increase life disparity.

Q.E.D.

3. Numerical findings

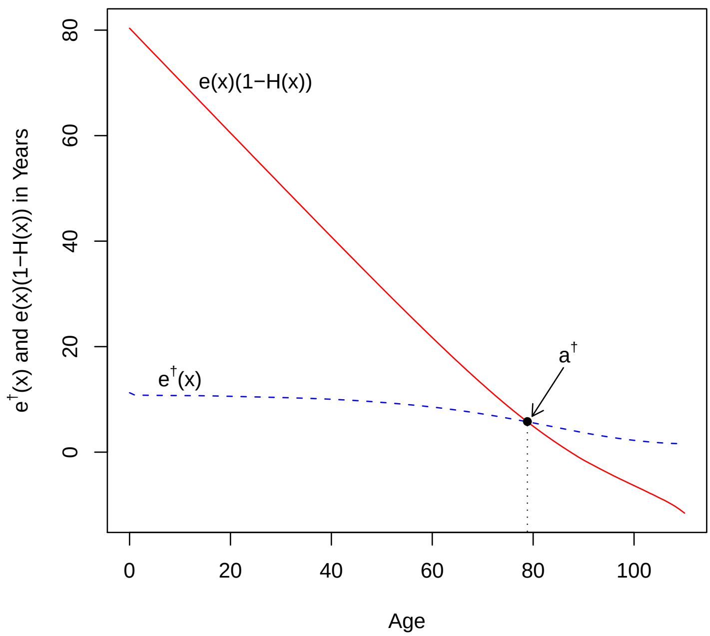
In Case 1, when entropy is less than one, to calculate $a^\dagger$ it is necessary to find the age at which $e^\dagger(a) = e(a)\big(1 - H(a)\big)$. Using interpolation routines widely available in statistical software, $a^\dagger$ can be readily obtained, as illustrated in Figure 1. Changes in $a^\dagger$ over time are shown in Figure 2. The value of $a^\dagger$ in Japan increased from 65 in 1950 to 84 in 2005. Note that a decline in mortality at age 75 before 1972 increased life disparity while such a decline after 1972 decreased life disparity.

We have computed the value of $k(0)$ for all 5830 life tables since 1840 in the (Human Mortality Database 2009). We have also computed the value of $k(0)$ for the 3404 life tables in the (Human Life-Table Database 2009). In every case $k(0) < 0$. The closest approach to zero was found for females in 1911-1921 in India: for this population $e(0) = 23.33$ and $e^\dagger = 23.08$, so that $k(0) = -0.22$ and life table entropy $= 0.99$. Goldman and Lord (1986), however, provide two examples of life tables for which $k(0)$ is positive because entropy exceeds one. Both pertain to selected populations in rural areas of China in the period 1929-31. One is for females (Barclay et al. 1976) and the other is for males

(Coale, Demeny, and Vaughan 1983). For the Chinese women $e(0) = 21.00$, $e^{\dagger} = 21.73$, and entropy $= 1.03$. For the Chinese males $e(0) = 17.43$, $e^{\dagger} = 22.17$, and entropy $= 1.27$. These cases may be artifacts of bad data.

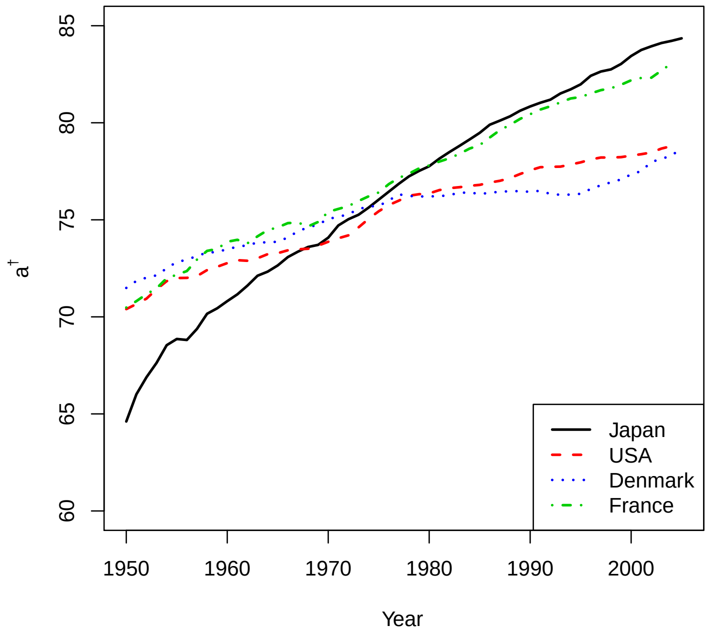
It would be interesting to determine the conditions such that the life table entropy is greater than, equal to, or less than one. Although life table entropy may rarely exceed one for human populations, this may be common for some non-human species or for some kind of machines or equipment.

Figure 1: A graphical depiction of the calculation of the threshold age ($a^{\dagger}$), for US females, 2005



Note: $e^{\dagger}(0) = 11.3$ and $e^{\dagger}(100) = 2.2$

Figure 2: The threshold age ($a^\dagger$) over time in four countries, females 1950-2005



4. History and related results

Based on the notion of life table entropy (Keyfitz 1977), Mitra (1978), Goldman and Lord (1986) and Vaupel (1986) independently derived the mathematical expression for $e^\dagger$ and showed that life table entropy equals $e^\dagger/e(0)$. Vaupel and Canudas-Romo (2003) later showed that the derivative of life expectancy over time is given by the product of $e^\dagger$ and the rate of progress in reducing age-specific death rates.

Vaupel (1986) found that if the force of mortality follows a Gompertz curve, $\mu(a) = \mu(0)e^{ba}$, then $e^\dagger \approx 1/b$. The parameter b has traditionally been interpreted as the rate of aging, but this result indicates that an alternative interpretation of b is that it is an inverse function of life disparity.

In addition to $e^\dagger$, several other measures of lifespan variation in a life table have been proposed (Cheung et al. 2005). These include the variance of the age at death, the standard deviation, the standard deviation above age 10 (Edwards and Tuljapurkar 2005), the inter-quartile range (Wilmoth and Horiuchi 1999), the Gini coefficient (Shkolnikov, Andreev, and Begun 2003), and the entropy of the life table (Keyfitz 1977). These measures are highly correlated with each other. In particular, the correlation of $e^\dagger$ with the other measures is never less than 0.952, according to our calculations based on 5830 period life tables from 1840 to 2007 available from the Human Mortality Database (2009) (See table in Vaupel, Zhang, and van Raalte 2009). Hence $e^\dagger$ can be viewed as a surrogate for the other measures. Moreover, it turns out that the derivatives of some other measures (e.g., standard deviation of age at death, standard statistical entropy of age at death) with respect to the change in mortality have a similar form and all imply the existence of an age separating early and late deaths (Caswell, personal communication). We prefer $e^\dagger$ because of its desirable mathematical properties, used above and below, and because it can be readily explained and interpreted.

5. Applications

At the threshold age $a^\dagger$, the change in $e^\dagger$ resulting from mortality decline can be decomposed into two components

$$(4)\quad \dot{e}^\dagger(t) = \frac{de^\dagger(t)}{dt} = \int_0^{a^\dagger(t)} k(a,t)d(a,t)\rho(a,t)\ da + \int_{a^\dagger(t)}^{\omega} k(a,t)d(a,t)\rho(a,t)\ da,$$

where $\rho(a,t)$ is the rate of progress in reducing death rates. The first term on the right side of (4) represents the compression of mortality at younger ages, and the second term the expansion of mortality at older ages. The balance of the two components determines whether the population experiences mortality compression or expansion (Zhang and Vaupel 2008).

Analogously, the increase in life expectancy at birth $\dot{e}^\dagger(t)$ can be broken into two parts,

$$(5)\qquad \dot{e}(0,t) = \int_0^{a^\dagger(t)} e(a,t)d(a,t)\rho(a,t)\ da + \int_{a^\dagger(t)}^{\omega} e(a,t)d(a,t)\rho(a,t)\ da,$$

where the first term captures the contribution of averting early deaths to increases in $e(0)$, while the second that of decreasing mortality among the elderly or very elderly. In a recent study, Vaupel, Zhang, and van Raalte (2009) showed that the countries benefiting from the longest life expectancies are those that have succeeded in reducing disparities in how long individuals live by averting early deaths.

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Appendix

A simple lemma in calculus [4]

Let f be a real-valued differentiable function on $[0, \infty)$ with derivative f' and the property that $f'(x) > 0$ whenever $f(x) = 0$. Then f has at most one root.

Proof. Suppose to the contrary that there are at least two roots of f. First, we will show that between any two distinct roots of f there is another root of f. Pick any two roots of f, say a and b with $a < b$. Since f is differentiable at a,

$$\lim_{x \to a, x > a} \frac{f(x) - f(a)}{x - a}$$

exists and equals $f'(a)$. Since $f'(a) > 0$, there is an $\epsilon_a > 0$ such that $f(x) - f(a) > 0$, for all $x \in (a, a + \epsilon_a)$; that is, f is positive on the interval $(a, a + \epsilon_a)$. Analogously, there is an $\epsilon_b > 0$ such that f is negative on the interval $(b - \epsilon_b, b)$. Clearly, $a + \epsilon_a \leq b - \epsilon_b$. Thus $a + \epsilon_a/2 < b - \epsilon_b/2$, and $f(a + \epsilon_a/2) > 0$ and $f(b - \epsilon_b/2) < 0$. Note that, since f' exists, f is continuous on $[0, \infty)$. By the intermediate value theorem applied to f on $[a + \epsilon_a/2, b - \epsilon_b/2]$, f vanishes at some $c \in (a + \epsilon_a/2, b - \epsilon_b/2)$. Hence $f(c) = 0$, for some $c \in (a, b)$.

Having shown that between any two distinct roots of f there is another root of f, we will use the above setup with a, b, ϵ_a and ϵ_b to derive a contradiction. Let $X = \{x \in (a, b) : f(x) = 0\}$ be the set of all roots of f between a and b. Clearly, $X \neq \emptyset$ and $X \subset [a + \epsilon_a, b - \epsilon_b]$. So X has a supremum s with $s \in [a + \epsilon_a, b - \epsilon_b]$. Now $s = \lim_{n \to \infty} x_n$, for some $x_n \in X$. Since f is continuous at s, we have $f(s) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} 0 = 0$. Since $s < b$, f has a root $t \in (s, b)$, so that $t \in X$. This contradicts the fact that s is the supremum of X and finishes the proof. Q.E.D.

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