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Formal Relationships 8

Total incremental change with age equals average lifetime change

James W. Vaupel

This article is part of the Special Collection "Formal Relationships".
Guest Editors are Joshua R. Goldstein and James W. Vaupel.

Table of Contents

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Abstract

In a stationary population, the change with age in some characteristic at a point in time, summed over all the individuals in the population, equals the change in this characteristic, from the start to the end of the lifetime of each individual, averaged over all lifetimes of the individuals in the cohort.

1. Relationship

Let $u(a)$ denote some continuously varying, differentiable characteristic, such as remaining life expectancy, chance of death or average weight, of individuals age a in a cohort or, alternatively, a lifetable or stationary population. Let $\dot{u}(a) \equiv du(a)/da$. Let $\ell(a)$ be the chance of surviving from initial age 0 to age a; $\ell(a)$ is proportional to the number of individuals alive at age a. Let $d(x)$ be the probability density function of life spans x; it is proportional to the number of deaths at age x. Then

$$\int_0^\infty \dot{u}(a)\,\ell(a)\,da = \int_0^\infty \Big(u(x) - u(0)\Big)\,d(x)\,dx. \tag{1}$$

The left-hand expression in this equality can be interpreted as total incremental change, whereas the right-hand expression gives average lifetime change.

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[2] The author thanks Annette Baudisch, Joel Cohen, Joshua Goldstein, Trifon Missov, Zhen Zhang, Peter Wagner and an anonymous reviewer.

2. Proof

Because everyone alive will die,

$$\ell(a) = \int_a^\infty d(x)\,dx. \tag{2}$$

Substituting this in the left of (1) and then reversing the order of integration yields:

$$\begin{aligned}\int_0^\infty \dot{u}(a) \int_a^\infty d(x)\,dx\,da &= \int_0^\infty d(x) \int_0^x \dot{u}(a)\,da\,dx \\ &= \int_0^\infty \Big(u(x) - u(0)\Big)\,d(x)\,dx.\end{aligned} \tag{3}$$

Q.E.D.

3. Discussion

The proof relies on two standard stratagems in formal demography: use of (2) and reversing the order of integration. The extra fillip comes from the basic relationship in calculus that

$$\int_0^x \frac{du}{da}\,da = u(x) - u(0). \tag{4}$$

Another way of proving the relationship is to integrate by parts:

$$\begin{aligned}\int_0^\infty \dot{u}(a)\,\ell(a)\,da = \int_0^\infty \ell(a)\frac{du(a)}{da}\,da &= \ell(a)\,u(a)\Big|_0^\infty - \int_0^\infty u(a)\frac{d\ell(a)}{da}\,da \\ = \int_0^\infty u(a)\,d(a)\,da - u(0) &= \int_0^\infty \Big(u(a) - u(0)\Big)\,d(a)\,da.\end{aligned} \tag{5}$$

The left side of (1) can be interpreted as the total incremental change with age at a point in time. Because in a stationary population, demographic rates vary by age but are constant over time, the left side of (1) can, alternatively, be interpreted as the total incremental change with age for a cohort. The right side of (1) can be interpreted as average lifetime change for a cohort – or for a synthetic cohort at a point in time. Hence, (1) can be applied to a stationary population, a lifetable or a cohort.

4. Applications

Suppose $u(a) = a$. Then (1) reduces to

$$\int_0^\infty \ell(a)\,da = \int_0^\infty x\,d(x)\,dx, \tag{6}$$

the two alternative formulas demographers use for life expectancy.

Suppose $u(a) = a^2$. In this case (1) becomes

$$2\int_0^\infty a\,\ell(a)\,da = \int_0^\infty a^2\,d(a)\,da. \tag{7}$$

The expression on the left-hand side of (7) can be written as

$$\frac{2\int_0^\infty a\,\ell(a)\,da}{\int_0^\infty \ell(a)\,da}\int_0^\infty \ell(a)\,da = 2\bar{a}e_0, \tag{8}$$

where $\bar{a}$ is the average age of the living and life expectancy e_0 is the average age at death. Hence the variance of the age at death is given by

$$\sigma^2 = \int_0^\infty a^2\,d(a)\,da - e_0^2 = 2\bar{a}e_0 - e_0^2. \tag{9}$$

Defining the coefficient of variation as $CV = \sigma/e_0$, this implies that

$$CV^2 = \frac{2\bar{a}}{e_0} - 1, \tag{10}$$

which in turn implies the interesting result (see, e.g., Preston 1976; Keyfitz 1977; Goldstein 2009)

$$\frac{\bar{a}}{e_0} = \frac{1 + CV^2}{2}. \tag{11}$$

If everyone dies at the same age, then the living are half as old, on average, as those who just died. If individuals die at different ages, then the average age of the living exceeds half of life expectancy. If the force of mortality is constant over age, then $CV = 1$ and the living and dead are equally old. If the force of mortality tends to decline with age such that $CV > 1$, then the living are older on average than the dead.

When mortality is described by the Gompertz curve $\mu(x) = \alpha \exp(\beta x)$, then $\frac{d\mu(x)/dx}{\mu(x)} = \beta$. Because (1) and (5) imply

$$\text{(12)} \qquad \int_0^\infty \dot{\mu}(x)\,\ell(a)\,da = \int_0^\infty \frac{\dot{\mu}(a)}{\mu(a)}\,d(a)\,da = \int_0^\infty \mu(x)\,d(x)\,dx - \mu(0),$$

it follows that

$$\text{(13)} \qquad \int_0^\infty \mu(x)\,d(x)\,dx = \alpha + \beta.$$

If $\beta \gg \alpha$, which is the case for humans in countries with high life expectancy, then the level of mortality, on average at the ages when death strikes, approximately equals the rate of increase in mortality with age.

Suppose $u(a) = e(a)$, remaining lifespan at age a. Then (1) implies

$$\text{(14)} \quad \int_0^\infty \dot{e}(a)\,\ell(a)\,da = \int_0^\infty \Big(e(x) - e_0\Big)\,d(x)\,dx = \int_0^\infty e(x)\,d(x)\,dx - e_0 = e^\dagger - e_0,$$

where $e^\dagger$ gives remaining life expectancy lost due to death (Zhang and Vaupel 2009). The integral on the left-hand side of (14) can be interpreted as a measure of senescence. The idea is that the size of the population at age a is proportional to $\ell(a)$ and the change in remaining life expectancy is a measure of how much individuals are senescing. The three equivalent expressions in the middle and right side of (14) provide an alternative measure of how much senescence a species suffers. The idea here is that senescence can be measured by the lifetime change in remaining life expectancy from the initial age until age at death. That these seemingly different measures are equal can also be shown by substituting

$$\text{(15)} \qquad \begin{aligned} \dot{e}(a) &= \frac{d}{da}\left(\frac{\int_a^\infty \ell(x)\,dx}{\ell(a)}\right) = \frac{\frac{d\int_a^\infty \ell(x)\,dx}{da}}{\ell(a)} - \frac{\int_a^\infty \ell(x)\,dx}{\ell(a)}\,\frac{d\ell(a)/da}{\ell(a)} \\ &= -1 + e(a)\,\mu(a). \end{aligned}$$

in the expression on the left side of (14). The equality in (14) is remarkable because e_0 plays a starring role in the expression on the right, and a hidden, background role in the expression on the left.

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