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*Formal Relationships 2*

**Life lived and left: Carey's equality**

**James W. Vaupel**

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Guest Editors are Joshua R. Goldstein and James W. Vaupel.

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## **Life lived and left: Carey's equality**

**James W. Vaupel**<sup>1, 2</sup>

### **Abstract**

In a stationary population, age composition and the distribution of remaining lifespans are identical. This equivalence can be used to estimate age structure if information is available on time to death.

### **1. Relationship**

If an individual is chosen at random from a stationary population with a positive force of mortality at all ages, then the probability the individual is one who has lived  $a$  years equals the probability the individual is one who has that number of years left to live. For example, it is as likely the individual is age 80 as it is the individual has 80 years to live—not 80 years of remaining life expectancy but a remaining lifetime of precisely 80 years. In continuous time for a population of infinite size, this can be more formally expressed as:

$$(1) \quad c(a) = g(a).$$

where the probability density function  $c(a)$  describes the age composition of the population whereas the probability density function  $g(a)$  gives the distribution of remaining lifespans.

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## 2. Proof

In a stationary population

$$(2) \quad c(a) = \ell(a) / e(0) ,$$

where  $\ell(a)$  is the chance of surviving to age  $a$  and  $e(0)$  is life expectancy. The proportion of the population with  $n$  years of life remaining is equal to the proportion dying  $n$  years in the future. The probability that an individual age  $a$  will die in  $n$  years is given by

$$(3) \quad f(n|a) = \mu(a+n) \frac{\ell(a+n)}{\ell(a)} ,$$

where  $\mu$  is the force of mortality. Hence,

$$(4) \quad g(n) = \int_0^\omega c(a) f(n|a) da = \int_0^\omega c(a) (\mu(a+n) \frac{\ell(a+n)}{\ell(a)}) da .$$

Substituting the right-hand side of (2) for  $c(a)$  and simplifying yields

$$(5) \quad g(n) = \int_n^\omega \mu(a) \frac{\ell(a)}{e(0)} da = \frac{\ell(n)}{e(0)} .$$

Therefore, when  $n = a$ , then  $c(a) = g(a)$ .

Q.E.D.

## 3. History and related results

James R. Carey (Müller et al. 2004, 2007) discovered the basic concept underlying this result, although he did not express it as above. The relationship also can be derived from the distributional equality of backward and forward recurrences in renewal theory: see Cox (1962).

In a stationary population births equal deaths:  $c(0)$ , the birth rate, equals  $g(0)$ , the death rate. Carey's equality generalizes this result by showing that the proportion of individuals younger than  $a$  equals the proportion whose remaining lifespan is less than  $a$ . Similarly, the proportion of individuals  $a$  or older is equal to the proportion of individuals who will still be alive in  $a$  years.

Note that Carey's equality implies that the average number of years lived by the individuals in a stationary population equals the average remaining lifespan of these individuals:

$$(6) \quad \int_0^\omega a c(a) da = \int_0^\omega a g(a) da .$$

In *Formal Relationships 1* (Goldstein 2009), it was shown that

$$\int_0^{\omega} a c(a) da = \int_0^{\omega} e(a) c(a) da,$$

where  $e(a)$  denotes remaining life expectancy at age  $a$ . An individual's remaining lifetime is not the same as the individual's remaining life expectancy. But the right-hand sides of these two equalities are equivalent: mean remaining lifespan equals mean remaining life expectancy. Hence the main result in *Formal Relationships 1* can be viewed as a special case of the main result presented here in *Formal Relationships 2*.

#### 4. Applications

Carey (Müller et al. 2004, 2007) proposed capturing wild medflies and recording their residual lifespans to estimate the age-structure of the wild population. More generally, individuals of unknown age could be identified and then followed to death. The individuals could be humans in a population lacking reliable information about age or baboons in a carefully followed population or sessile organisms such as plants or barnacles. As discussed by Müller et al. (2004, 2007), statistical adjustments can be made if the population is not stationary.

In the 2005 lifetable for the United States (males and females combined, available at [www.mortality.org](http://www.mortality.org)), more than 48% of hypothetical individuals are 41 years old or older, implying that nearly half of the lifetable population will be alive in 2050, a date 41 years from now (2009) that is sometimes considered as being in the distant future. The U.S. population is younger than the corresponding lifetable population and age-specific death rates are declining. Hence it is likely that substantially more than half of the actual current U.S. population will reach mid-century.

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