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A LINEARIZED VERSION OF LUCAS'S NEUTRALITY MODEL

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ABSTRACT

The model developed in Robert Lucas's influential "Expectations and the Neutrality of Money" has not been widely used for extensions or modifications of the original analysis, in part because of its difficulty of manipulation. The present paper describes a linearized version that--unlike other models prominent in the rational expectations literature--retains the original's main features yet is comparatively easy to manipulate. Two examples of modifications facilitated by this linearization are included. These involve an autoregressive money growth specification and the assumption of lump-sum (rather than proportional) monetary transfers.

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I. Introduction

It is at least arguable that the most influential paper of the past decade in the field of macro and monetary economics has been Robert Lucas's "Expectations and the Neutrality of Money" (1972). The specific model developed in that paper has not, however, been widely used for extensions or modifications of the original analysis.[1/] Instead, most analysts have adopted alternative models suggested by Lucas (1973), Sargent and Wallace (1975), or Barro (1976) (1980) in which the supply and demand functions are not as well grounded in individual choice problems.[2/] A major reason for this practice is, of course, that the model in Lucas (1972)--henceforth, the ENM model--is not easily solved or manipulated. The object of the present paper is, accordingly, to describe a linearized version of the ENM model that retains the original's main features yet is comparatively easy to manipulate.[3/] Two examples of modifications facilitated by this linearization are included.

II. Individual Agents

The ENM model economy is populated with overlapping generations of agents who live for two periods, able to work when young but not when old. In period t a young agent expends N_t units of labor, producing a like number of units of perishable output. Some of this output will typically be exchanged for paper money, which is the only store of value. Old agents receive monetary transfers from the government, the magnitudes being stochastically proportional to existing money holdings.

In analysing the young agents' choice problem--old agents have none--we begin with a non-stochastic version of the model. Thus we assume that an agent born in t seeks to maximize $U(C_t^0, N_t) + V(C_t^1)$ subject to $N_t = C_t^0 + \Lambda_t/P_t$ and $X_{t+1}(\Lambda_t/P_t)(P_t/P_{t+1}) = C_t^1$, where C_t^0, C_t^1 are consumption quantities when young and old,[4/] Λ_t is the nominal money demanded when young, P_t is the money price of output in t, and X_{t+1} reflects transfers in t+1.[5/] Clearly the agents' choices of C_t^0, C_t^1, N_t, and Λ_t/P_t depend upon the single variable that is taken parametrically, namely, $X_{t+1}P_t/P_{t+1}$. And under Lucas's assumptions concerning the properties of U and V, both Λ_t/P_t and N_t are positively related to $X_{t+1}P_t/P_{t+1}$.[6/] Next, we revert to a stochastic setting but pretend that certainty-equivalence prevails. In particular, we assume that money demand and labor supply functions can be well approximated by the log-linear relations

$$(1) \quad \lambda_t - p_t = a_0 + a_1 E_t(x_{t+1} + p_t - p_{t+1}) \qquad a_1 > 0$$

$$(2) \quad n_t = b_0 + b_1 E_t(x_{t+1} + p_t - p_{t+1}) \qquad b_1 > 0$$

where lower-case letters denote logarithms. The notation in (1) and (2) recognizes that expectations formed in t of x_{t+1} and p_{t+1} are relevant for choices made in t. As in Lucas (1972), it is assumed that agents know the values of all past variables, but that the (local) value of p_t is the only variable observed contemporaneously. Thus $E_t x_{t+1} = E(x_{t+1} | p_t, \Omega_{t-1})$, where p_t is the local (log) price and Ω_{t-1} denotes values of all variables in t-1 and before.[7/]

III. Equilibrium

The ENM economy includes two informationally-distinct islands populated by agents of the type just described, the total number of which does not change over time. In each period old agents are allocated across islands so as to equate the start-of-period money stock on the two islands, while young agents are assigned randomly with the fraction $\theta_t/2$ going to island One. Monetary policy can be characterized by the stochastic behavior of $X_t \equiv M_t/M_{t-1}$, where M_t is the post-transfer money supply per old person in period t. The values of X_t and M_t are the same on both islands, but are currently unknown to individual agents. Taking logarithms we have

(3) $$m_t = m_{t-1} + x_t,$$

with the stochastic behavior of x_t yet to be specified.

Given the foregoing assumptions, market clearing on island One requires that $\Lambda_t^{(1)} = M_t/\Theta_t$ or

(4) $$\lambda_t^{(1)} = m_{t-1} + x_t - \theta_t,$$

where θ_t is the log of Θ_t, while the corresponding condition on island Two is [8/]

(5) $$\lambda_t^{(2)} = m_{t-1} + x_t + \theta_t.$$

The random variables θ_t are independent and identically distributed (iid) with mean zero and variance σ_θ^2.

To complete the model, we must specify how the stochastic policy variable x_t is generated. Following Lucas, we assume in our basic exposition that the x_t values are independent of θ_t and iid with mean zero and variance σ_x^2.

IV. Solution

In the specified economy, the behavior of the variables λ_t and p_t on island One is described by equations (1) and (4). Given the linearity of these relations and our stochastic assumptions regarding x_t and θ_t, it is clear that the[9/] solution for p_t on this island will be of the form

(6) $$p_t = \pi_0 + \pi_1 m_{t-1} + \pi_2 x_t + \pi_3 \theta_t$$

and consequently that

(7) $$E_t p_{t+1} = \pi_0 + \pi_1 E_t m_t = \pi_0 + \pi_1(m_{t-1} + E_t x_t).$$

To evaluate $E_t x_t$, we note that agents can, by way of (4), observe the value of $x_t - \theta_t$. Their optimal linear predictor of x_t is then $\beta(x_t - \theta_t)$ with $\beta = E[x_t(x_t - \theta_t)]/E(x_t - \theta_t)^2 = \sigma_x^2/(\sigma_x^2 + \sigma_\theta^2)$. And, given current stochastic assumptions, $E_t x_{t+1} = E_t \theta_{t+1} = 0$.

Substitution of (4), (6), and (7) into (1) then yields

$$(8) \qquad m_{t-1} + x_t - \theta_t = a_0 + (1 + a_1)\,[\pi_0 + \pi_1 m_{t-1} + \pi_2 x_t + \pi_3 \theta_t] - a_1[\pi_0 + \pi_1 m_{t-1} + \pi_1 \beta(x_t - \theta_t)]$$

which implies undetermined-coefficient identities that are readily solved, giving $\pi_0 = -a_0$, $\pi_1 = 1$, $\pi_2 = (1 + a_1\beta)/(1 + a_1)$, and $\pi_3 = -(1 + a_1\beta)/(1 + a_1)$. Using these values with (6) and (7) we find that

$$(9) \qquad E_t(x_{t+1} + p_t - p_{t+1}) = \pi_2 x_t + \pi_3 \theta_t - \pi_1 \beta(x_t - \theta_t) = [(1 - \beta)/(1 + a_1)]\,[x_t - \theta_t]$$

on island One. Since the relationships are the same on island Two except that $-\theta_t$ appears in place of θ_t, this is also true of the expression for $E_t(x_{t+1} + p_t - p_{t+1})$. Using these expressions in (2) and summing over the two islands we then find that aggregate employment/output equals

$$(10) \qquad n_t^{(1)} + n_t^{(2)} = 2b_0 + 2b_1[(1 - \beta)/(1 + a_1)]\, x_t.$$

This shows, since $1 - \beta = \sigma_\theta^2/(\sigma_\theta^2 + \sigma_x^2)$, that aggregate employment/output responds positively to monetary shocks x_t, with a slope coefficient that is negatively related to the variance of these shocks.

V. First Modification

The usefulness of this linearized version of Lucas's model--apart from any pedagogic merits--results from its ease of modification. In order to exemplify the latter, we now consider two variations. In the first of these we assume that the x_t policy process is autoregressive, i.e., that

$$(11)\qquad x_t = \rho x_{t-1} + \epsilon_t \qquad\qquad |\rho| < 1$$

where ϵ_t is iid with $E(\epsilon_t) = 0$ and $E(\epsilon_t^2) = \sigma_\epsilon^2$.

The price in market One is now determined by (1),(3),(4), and (11) so the relevant state variables are m_{t-1}, x_{t-1}, ϵ_t, and θ_t. Accordingly we write

$$(12)\qquad p_t = \pi_0 + \pi_1 m_{t-1} + \pi_2 x_{t-1} + \pi_3 \epsilon_t + \pi_4 \theta_t$$

and note that

$$(13)\qquad E_t p_{t+1} = \pi_0 + \pi_1(m_{t-1} + E_t x_t) + \pi_2 E_t x_t.$$

Furthermore,

$$(14)\qquad E_t x_t = \rho x_{t-1} + E_t \epsilon_t = \rho x_{t-1} + \beta(\epsilon_t - \theta_t)$$

with $\beta = \sigma_\epsilon^2/(\sigma_\epsilon^2 + \sigma_\theta^2)$. Substitution into (1) then gives

$$(15)\qquad m_{t-1} + \rho x_{t-1} + \epsilon_t - \theta_t = a_0 + (1 + a_1)[\pi_0 + \pi_1 m_{t-1} + \pi_2 x_{t-1} + \pi_3 \epsilon_t + \pi_4 \theta_t]$$
$$+ a_1\rho[\rho x_{t-1} + \beta(\epsilon_t - \theta_t)] - a_1\{\pi_0 + \pi_1 m_{t-1} + (\pi_1 + \pi_2)[\rho x_{t-1} + \beta(\epsilon_t - \theta_t)]\},$$

and the undetermined-coefficient identities imply

that $\pi_0 = -a_0$, $\pi_1 = 1$, $\pi_2 = \rho$, $\pi_3 = (1 + a_1\beta)/(1 + a_1)$,

and $\pi_4 = -(1 + a_1\beta)/(1 + a_1)$. Consequently, we have

$$(17)\qquad E_t(x_{t+1} + p_t - p_{t+1}) = \rho E_t x_t + \pi_2 x_{t-1} + \pi_3 \epsilon_t + \pi_4 \theta_t$$
$$-(\pi_1 + \pi_2)E_t x_t = [(1 - \beta)/(1 + a_1)][\epsilon_t - \theta_t].$$

Proceeding as before, then, we find that employment/output again obeys expression (10) but with ϵ_t now replacing x_t as the monetary surprise. Thus the magnitude of the policy parameter ρ has no effect on the behavior of output in the ENM model.

VI. Second Modification

For a second and somewhat more substantial variation, we now assume that transfers to the old are of the lump-sum, rather than proportional, variety. That is, we assume that each agent believes that the size of the transfer to be received when old is independent of the quantity of money which that agent carries into old age. In this case, the constraint on second-period consumption for an agent born in t is

$$(18) \qquad C_t^1 = \frac{\Lambda_t}{P_{t+1}} + \frac{M_{t+1} - M_t}{P_{t+1}} = \frac{\Lambda_t}{P_t}\frac{P_t}{P_{t+1}} + \frac{X_{t+1}M_t}{P_{t+1}} - \frac{M_t}{P_{t+1}}$$

instead of $C_t^1 = X_{t+1}(\Lambda_t/P_t)(P_t/P_{t+1})$. Thus each young person's choices of C_t^0, C_t^1, N_t, and Λ_t/P_t depend (under perfect foresight) on three variables faced parametrically: P_t/P_{t+1}, X_{t+1}, and M_t/P_{t+1}. Accordingly, when we log-linearize and revert to a stochastic setting, the decision rules in (1) and (2) must be replaced with the following:

$$(19) \qquad \lambda_t - p_t = a_0 + a_1E_t(p_t - p_{t+1}) + a_2E_tx_{t+1} + a_3E_t(m_t - p_{t+1})$$

$$(20) \qquad n_t = b_0 + b_1E_t(p_t - p_{t+1}) + b_2E_tx_{t+1} + b_3E_t(m_t - p_{t+1}).$$

In these formulations, the signs of the a_j and b_j coefficients are determined by the direction of response of Λ_t/P_t and N_t to the three parametric variables in the agent's decision problem. Under Lucas's assumptions, both Λ_t/P_t and N_t depend positively on P_t/P_{t+1} as before--this is the intertemporal substitution phenomenon--but now the response to X_{t+1} is negative: additional old-age income depresses optimal money demand (saving) and labor supply when young.[10/] Consequently, we have a_1, $b_1 > 0$ and a_2, $b_2 < 0$. From (18) we see that the effect of M_t/P_{t+1} on the magnitude of the real transfer in t+1 is positive when $X_{t+1} > 1$ and negative when $X_{t+1} < 1$. Thus a_3 and b_3 should be specified as negative constants when the average money growth rate is positive ($EX_{t+1} > 1$), as positive when $EX_{t+1} < 1$, and as zeros when $EX_{t+1} = 1$.

We can now sketch how analysis proceeds with the autoregressive policy specification (11). The price in market One is in this case determined by equations (19), (3), (4), and (11) so expressions (12), (13), and (14) again apply. Substitution into (19) results in the following replacement for (15):[11/]

$$\begin{aligned}(21)\quad m_{t-1} + \rho x_{t-1} + \epsilon_t - \theta_t &= a_0 + (1+a_1)[\pi_0 + \pi_1 m_{t-1} + \pi_2 x_{t-1} + \pi_3 \epsilon_t + \pi_4 \theta_t] \\ &+ a_2\rho[\rho x_{t-1} + \beta(\epsilon_t - \theta_t)] - a_1\{\pi_0 + \pi_1 m_{t-1} + (\pi_1+\pi_2)[\rho x_{t-1} + \beta(\epsilon_t - \theta_t)]\}.\end{aligned}$$

Solution of the implied undetermined-coefficient identities then yields $\pi_0 = -a_0$, $\pi_1 = 1$, $\pi_2 = \rho(1+a_1-a_2\rho)/(1+a_1-a_1\rho)$, $\pi_3 = (1+a_1\beta-a_2\rho\beta+a_1\beta\pi_2)/(1+a_1)$, and $\pi_4 = -\pi_3$. These expressions can be used, in combination with corresponding expressions for market Two, to determine how prices and quantitities behave in the aggregate. A significant feature of the results, which we can recognize without further manipulation, is that ρ appears in the expressions for π_2, π_3, and π_4. Consequently, this policy parameter also appears in the solution for n_t that is the counterpart of (10). Complete independence of employment from monetary policy parameters in Lucas's model thus requires--as Lucas has recognized (1975, p.1119)--the proportional-transfer feature of the original specification.

VII. Conclusions

The foregoing paragraphs demonstrate constructively that it is possible to devise a linearized version of Lucas's ENM model that retains the original's main properties yet is comparatively easy to manipulate and modify. The strategy used in effecting the simplification involves linearization of relationships implied by the original model under conditions of perfect foresight or certainty equivalence, with the list of relevant variables and the informational structure retained intact. This type of procedure could, it would appear, be similarly applied in the analysis of other nonlinear stochastic models.

Appendix

The object here is to outline the comparative-static analysis that leads, under Lucas's (1972) assumptions, to sign restrictions on the coefficients in equations (19) and (20). Under constraint (18), the agent's perfect-foresight problem is to maximize $U(N-\Lambda,N) + V(\Lambda R+T)$ where Λ, R, and T are used to denote Λ_t/P_t, P_t/P_{t+1}, and the transfer variable $(X_{t+1}-1)M_t/P_{t+1}$. The first-order conditions are $U_1 + U_2 = 0$ and $U_1 = RV'$, and their differentials are

(A-1) $$U_{11}(dN-d\Lambda) + U_{12}dN + U_{21}(dN-d\Lambda) + U_{22}dN = 0$$

(A-2) $$U_{11}(dN-d\Lambda) + U_{12}dN = RV''(\Lambda dR+Rd\Lambda+dT) + V'dR$$

From (A-1) we see that

(A-3) $$(U_{11}+U_{12}+U_{21}+U_{22})dN = (U_{11}+U_{21})d\Lambda.$$

Thus, since $U_{11} + U_{12} < 0$ and $U_{22} + U_{21} < 0$ by assumption (Lucas, 1972, p.106), dN and $d\Lambda$ are of the same sign.

First letting dT = 0, we find that

(A-4) $$[(U_{11}+U_{21})(U_{11}+U_{12}) -(U_{11}+V''R^2)(U_{11}+U_{12}+U_{21}+U_{22})]d\Lambda$$
$$= (R\Lambda V''+V')(U_{11}+U_{12}+U_{21}+U_{22})dR.$$

But since $U_{11}U_{22} - U_{12}U_{21} > 0$ by strict concavity, the term in brackets on the left-hand side is unambiguously negative. And since Lucas assumes $R\Lambda V''+V' > 0$ (1972, p.107), The product on the right-hand side is also negative. This shows that $d\Lambda/dR > 0$ and thus that $dN/dR > 0$; consequently $a_1,b_1 > 0$.

Next letting dR = 0, we obtain

(A-5) $$[(U_{11}+U_{21})(U_{11}+U_{12}) -(U_{11}+V''R^2)(U_{11}+U_{12}+U_{21}+U_{22})]d\Lambda$$
$$= RV''(U_{11}+U_{12}+U_{21}+U_{22})dT.$$

Here we see that $RV'' < 0$ appears in place of $R\Lambda V''+V' > 0$ in (A-4), so we conclude that $d\Lambda/dT < 0$ and $dN/dT < 0$. The restrictions in the text regarding a_2, b_2, a_3, and b_3 follow from the definition of T.

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Footnotes

1. A few modifications have, of course, appeared. Noteworthy examples include Azariadis (1981), Muench (1977), Polemarchakis and Weiss (1977), and Wallace (1980).

2. In this regard, it is worth mentioning that existing empirical tests of propositions suggested by the Lucas (1972) model--for example, the "policy ineffectiveness" hypothesis--have typically been conducted using specifications taken from these other papers. Leading examples are provided by Boschen and Grossman (1982), Gordon (1982), and Mishkin (1982).

3. Lucas (1981, pp. 12, 14-15) has suggested that a linearized version of the ENM model is provided by Lucas (1973). But the relative price variables in these papers are different--see fn. 7 below--and markets clear <u>locally</u> in the ENM model but not in Lucas (1973). Our linearization procedure is related to one employed by Lucas (1975) but relies more heavily on properties of the nonlinear model. Kydland and Prescott (1982) recently used a rather similar procedure in the context of a numerical analysis. These linearizations lose, of course, the effects of time-varying conditional variances and higher moments.

4. Our notation is related to Lucas's but is not identical.

5. Note that in this setup the nominal amount of each old agent's monetary transfer is proportional to that agent's nominal holdings of money. This feature of the specification gives the non-stochastic version of the model the property of superneutrality--see Barro and Fischer (1976, p.140). If transfers were of the lump-sum type, the model's properties would be different, as will be shown below.

6. These properties include non-inferiority of leisure and consumption when young, plus differentiability, strict concavity, and a condition on V which implies that the substitution effect of intertemporal price changes will dominate the income effect. For details, see Lucas (1972, pp. 106-109).

7. It might parenthetically be noted that the variable $E_t(x_{t+1} + p_t - p_{t+1})$ can be interpreted as the expected real rate of return on money holdings. In particular, $x_{t+1} = \log M_{t+1} - \log M_t$ is the effective nominal interest rate on money carried from t into t+1, given that transfers are proportional to existing holdings, while $p_{t+1} - p_t$ is the corresponding inflation rate. In addition, we note that this rate-of-return variable differs from the single relative-price variable $p_t - E(p_t | \Omega_{t-1})$ that appears in the supply functions used by Lucas (1973) and Sargent-Wallace (1975). Barro (1980) uses a different rate-of-return variable while Barro (1976) uses the same but includes a wealth variable that differs from x_{t+1}. Thus these specifications cannot be tightly rationalized by reference to Lucas's ENM model.

8. Note that the fraction of young agents allocated to island Two is $1 - \Theta/2$ so market clearing requires $\Lambda_t = M_t/(2-\Theta_t)$. Then the approximation $2 - \Theta_t = 1 + (1-\Theta_t) \doteq 1/[1 -(1-\Theta_t)] = 1/\Theta_t$ leads directly to (5).

9. As in most rational expectations models there may be a multiplicity of solutions. The one here described is the unique bubble-free solution. For relevant discussion, see McCallum (1983).

10. The validity of these assertions is demonstrated in the appendix.

11. Here since $Ex_{t+1} = 0$ we have used the special case restrictions $a_3 = b_3 = 0$ described in the previous paragraph. More generally the coefficient on the last term in (21) would be $-(a_1+a_3)$ and there would be an additional term, namely, $a_3[m_{t-1} + \rho x_{t-1} + \beta(\epsilon_t - \theta_t)]$. Of course $Ex_{t+1} = 0$ is not

equivalent to $EX_{t+1} = 1$ in the stochastic case, so our restrictions are only approximately appropriate. The main point of the example would continue to hold if a_3 and b_3 were nonzero.