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ABSTRACT

We explore the consequences for asset pricing of admitting a bequest motive into an otherwise standard overlapping generations model where agents trade equity and perpetual debt securities. Prices of securities are seen to be approximately 50% higher in an economy with bequests as compared to an otherwise identical one where bequests are absent. Robust estimates of the equity premium are obtained in several cases where the desire to leave bequests is modest relative to the desire for old age consumption.

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1. Introduction

This paper explores the implications of bequests for the statistical pattern of equilibrium stock and bond returns. It does so in the context of a "behavioral style" model in which households make their consumption and savings decisions not only to smooth consumption over their saving and dis-saving years, but also to provide for "indirect consumption" in their old age in the form of inter-vivos transfers and bequests. In this paper these two terms are used interchangeably, as the generality of our model precludes distinguishing between them.[1] We model the elderly as being motivated by a well defined "joy of giving".

There are two primary motivations for this study:

1. Over the next thirty years the "baby-boom" generation will grant to its heirs many trillions of dollars of economic property, including a majority of the stock market's total capitalization. It is thus of interest to explore – in anticipation of the aforementioned event – the implications of a model with an explicit bequest motive for the profile of security prices and returns. We find that the explicit incorporation of bequests – even at a high level of generality – has substantial impact on these profiles, frequently in unexpected ways.

2. Intuition suggests that bequests may provide a possible route to the resolution of some of the most celebrated anomalies in financial economics; viz., the risk free and

[1] Our model construct presumes that gifts of either sort can occur only in the final period of an agent's life. Since in basic discrete time models we may assume consumption occurs at any time within a period,

equity premium puzzles. The logic with respect to the equity premium and risk free rate puzzles is particularly straightforward. Within the context of the representative consumer, time separable preferences paradigm of, e.g., Grossman and Shiller (1981), Hansen and Singleton (1983), and Mehra and Prescott (1985), it is the very low covariance of aggregate consumption growth with equity returns that constitutes a major stumbling block to explaining the mean equity premium; vis-à-vis consumption risk, stocks are simply too good a hedging instrument to command a return much in excess of that on risk free securities.

In the model to be considered here, however, the magnitude of a household's bequests – and the indirect utility thereby derived – are perfectly positively correlated with the prices of and returns to securities.[2] With regard to "bequest risk", equity securities, in particular, constitute an especially poor hedge, a fact that suggests high equilibrium equity and low risk free returns. Confirming this basic intuition, our benchmark cases indeed display high equity premia in conjunction with low risk free returns. It is not the case, however, that an increased preference for bequests necessarily results in a higher premium.

These explorations entail significant methodological innovations in the nature of the economy's fundamental asset pricing relationships. No longer are asset prices benchmarked solely to consumption and the standard inter-temporal consumption trade-

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it may be viewed either as preceding the gift (in which case the gift effectively constitutes a bequest) or in simultaneity with it (in which case the gift qualifies as an inter-vivos transfer).

[2] And, of course, real estate. Our model does not attempt to explicitly model real estate as a differential asset.

off. In effect, the consumption cost to an investor of acquiring one more unit of an asset is significantly reduced by the amount of the bequest he can rationally expect to receive. In a stationary equilibrium, the more investors wish to bequeath, the more wealth they receive – in the form of bequests – with which with to do so. Equilibrium asset prices are thus higher than they would be otherwise in an identically parameterized standard pure consumption-savings context.

What motivates the bequeathing of economic property? While a casual consideration of bequests naturally assumes that they exist because of parents' altruistic concern for the economic status of their offspring, results in Hurd (1989) and Kopczuk and Lupton (2004), among others (see also Wilhelm (1996), Laitner and Juster (1996), Altonji et al. (1997), and Laitner and Ohlsson (2001)), suggest otherwise: households with children do not in general exhibit behavior more in accord with a bequest motive than childless households. As a result, the literature is presently largely agnostic as to bequest motivation, attributing bequests to general idiosyncratic, egoistic reasons.[3] The model we will explore, however, is sufficiently general to be consistent both with purely egoistic and purely altruistic, concern-for-offspring based motivations.

Although the motivation for bequests is not yet well understood, there is little dispute as to their pervasiveness and significance for household capital accumulation. Kotlikoff and Summers (1981) present evidence that roughly 46% of household wealth arises from intergenerational transfers, although Modigliani's (1988) analysis points to a more modest 20% estimate.[4] Other studies place inherited wealth as a proportion of household wealth in the range of 15% – 31%.[5] Using a more general statistical methodology, Kopczuk and Lupton (2004) estimate that 70% of the elderly population has a bequest motive, which directly motivates 53% of the wealth accumulation in single person, elderly U. S. households. Among wealthy households, those that own the vast majority of stocks and are most likely to trade financial instruments, Hurd and Mundaca (1989) report that between 44% and 60% of household wealth is attributable to gifts and inheritances. None of these estimates is so small as to imply that bequests can be ignored in a discussion of asset pricing regularities. Yet, to our knowledge, the implications of bequests for such regularities have not yet been explored in the applied literature.

A consideration of bequests mandates that our study be undertaken in an OLG context. Agents live for three periods. In the first period, while young, they consume their income and neither borrow nor lend. We adopt this convention as a parsimonious device for acknowledging that, with a steep expected future income profile, the young do not wish to lend and cannot borrow because they have no assets to offer as collateral. In the second, high wage, middle-aged period of their lives they consume, save for old

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[3] These empirical results will lead us to eschew the perspective of Barro and Becker (1988), who postulate that each generation receives utility from the consumption of the generations to follow, in favor of a more general formulation.

[4] We discuss the basis of this wide discrepancy in estimates in the calibration section of the paper. The estimates themselves come from converting flows of bequests into stocks of capital. Alternatively, one may estimate life cycle savings and compare this with accumulated wealth. Under this latter method, the estimates of Kotlikoff and Summers (1981) and Modigliani (1988) become, respectively, 81% and 20%.

[5] This range of estimates is drawn from Menchick and David, (1983), Modigliani (1988), Hurd and Mundaca (1989), Gale and Scholz (1994), and Laitner and Juster (1996).

age and receive bequests of securities from the then old who were born one period earlier. In the third and final period of their lives, as elderly, they consume out of their pension income and savings and themselves leave the residual as a bequest of securities, the value of which is modeled as directly providing them utility.

We further refine the behavior of the elderly in a number of alternative ways. In the simplest version of the model, the consumption of the old is fixed, with the entire residual value of savings going to bequests. For the old aged, the only source of risk is therefore bequest risk. In making this assumption we appeal to the fact that a substantial component of old aged spending is medically determined. It is thus related to the state of a person's health and uncorrelated with the business cycle. Other components of old aged consumption, such as vacations, entertainment and housing, are also largely a function of the state of an elderly person's health. Particularly for the well-to-do, fluctuations in the value of their wealth invested in the stock market play but a secondary role in determining overall spending, a fact that is confirmed by the low empirical correlation between the direct consumption of the old and the return on the stock market. As a first approximation, it is thus reasonable to exclude the direct consumption of the old consumers from consideration in examining the relevant Euler equations. Fixing old age direct consumption has this effect. Subsequent versions of the model jointly endogenize the choice between old-age direct consumption versus indirect consumption in the form of bequests.

While our discussion thus far has stressed the motivation for bequests, there is also the issue of who receives them. Many of our results that most accurately replicate the data require that a portion of bequests be generation-skipping; that is, granted to the young (grandchildren) rather than to the middle aged (children). More generally we can thus view our work as investigating the asset pricing implications of various family arrangements for bequeathing wealth. We do not consider, however, the consequences of alternative estate tax mechanisms.

1.1 Related Literature

The theoretical antecedents of this work are many. Since not all agents in our model hold securities, it is directly related to the literature emphasizing the limited participation of some households in the financial markets. Mankiw and Zeldes (1991) emphasize that it should be the risk preferences and consumption risk of the stockholding class that matter for equilibrium security returns. Although 52 percent of the U.S. adult population held stock directly or indirectly in 1998, as compared to 36 percent in 1989, substantial stock holdings remain largely concentrated in the portfolios of the wealthiest few. Brav, Constantinides, and Geczy (2002) and Vissing-Jorgenson (2002) find evidence that per capita consumption growth can explain the equity premium with a relatively high coefficient of relative risk aversion (CRRA) once we account for limited stock market participation.[6] In addition, wealthy investors may be

[6] Brav, Constantinides, and Geczy (2002) point out, however, that the statistical evidence is weak and the results highly sensitive to experimental design.

infra marginal in the equity markets if their wealth is tied up in private equity. See, for example, Blume and Zeldes (1993) and Haliassos and Bertaut (1995).

The presence of financial market incompleteness connects us to another well developed branch of the literature. Bewley (1982), Mankiw (1986) and Mehra and Prescott (1985) suggest the potential of enriching the asset pricing implications of the representative agent paradigm by relaxing the implicit complete markets paradigm. More recently, Constantinides and Duffie (1996) confirm that incomplete markets can substantially enrich the implications of the representative household. Their main result is a proposition demonstrating, by construction, the existence of a household income process, consistent with calibrated aggregate dividend and income processes such that equilibrium equity and bond price processes match the analogous observed price processes for the U.S. economy. Unlike the household-specific heterogeneity introduced in Constantinides and Duffie (1996), the OLG model considered here emphasizes only the heterogeneity across age cohorts. Whereas introducing household-specific heterogeneity may enhance the explanatory power of the model, we eschew this option in order to highlight the role of the indirect consumption of the old aged in the form of gifts and bequests. See Kocherlakota (1996) for an excellent review of the drawbacks to relying purely on incomplete-markets phenomena.

1.2 Outline of the Paper

The outline of the paper is as follows: Section 2 details the simplest model formulation and presents the calibration. That agents receive utility directly from the magnitude of their bequests represents a departure from the standard Arrow-Debreu equilibrium construct: the level of consumption provided by the bequests simultaneously provides utility to two distinct agents, the old who bequeath the bequests and the middle-aged who receive them. Existence issues are addressed. In Section 3 we present the results of computing equilibrium security prices and returns for a wide class of reasonable parameterizations. Robustness issues are explored in Section 4 where we also generalize the model to allow the old to undertake a consumption-bequest choice. Section 5 concludes the paper.

2. The Model, Equilibrium and Calibration

2.1 Model Description

As in Constantinides et al. (2002), we consider an overlapping generations, pure exchange economy in which each generation lives for three periods, as young, middle aged and old. Each generation is modeled as a representative consumer, a choice that implicitly ignores consumer heterogeneity within a generation in favor of exploring the implications of heterogeneity across generations in as parsimonious a construct as possible.

Income (output) in this model is denominated in terms of a single consumption good, and may be received either as wages, dividends or interest payments. There are two types of securities in *positive net supply*, an equity claim and a consol bond, b. Each bond pays one unit of the consumption good every period in perpetuity (aggregate interest payments are thus b) and q_t^b denotes its period t, ex-coupon price. We view the bond as a proxy for long-term government debt.

The single equity security represents a claim to the stochastic aggregate dividend stream $\{d_t\}$. We interpret the dividend as the sum total of all private capital income including corporate dividends, corporate bond interest and net rents. The ex-dividend period t share price is denoted by q_t^e. In equilibrium, the stock and consol bond are the instruments by which the economic participants can seek to alter their income profiles across dates and states.

Lastly, we postulate the existence of a one period, risk free discount security, with period t price $q_t^{r_f}$ in *zero net supply.* The payoff profile associated with such a security issued in some arbitrary period t is

$$\begin{array}{cc} t & t+1 \\ -q_t^{r_f} & 1 \end{array}$$

While the formal presence or absence of this security does not alter the equilibrium allocations in any way, we include it in order to assess the economy's implied risk free rate. In what follows, we detail only the most basic version of the model; elaborations are detailed in subsequent sections.

Let $B_{t-2,2}$ be the total bequest in period t granted by the old generation born two periods previously. We hypothesize that they grant the fraction x to their grandchildren, those born in the current period t, and the fraction (1-x) to their children born in t-1. Under this arrangement each generation receives two bequests over the course of its life, one from its parents and another from its grandparents.

Accordingly, a representative consumer born in period t receives deterministic wage income W^0 and a bequest of securities $xB_{t-2,2}$ when young. We assume that he concludes the young period of his life with zero holdings of securities; in effect, $c_{t,0} = c_0 = W^0 + xB_{t-2,2}$, where $c_{0,t}$ denotes the consumption of a young agent born in period t. This requirement is a simple way of capturing the fact that wage income does not collateralize loans in modern economies, and that under our calibration, the wage cum wealth profile of a representative consumer is sufficiently steep that it is non-optimal for him to save.

In the second period of his life, as middle aged, the period-t-born agent receives a stochastic wage income, $\tilde{W}^1_{t+1}$, and a stochastic bequest of securities from the preceding generation born in period t-1; we denote the latter by $(1-x)\tilde{B}_{t-1,2}$. Out of this aggregate wealth, the middle aged agent chooses the number of equity securities, $z^e_{t,1}$, consol bonds, $z^b_{t,1}$, and risk free securities, $z^{r_f}_{t,1}$ he wishes to acquire in order to finance his old-age consumption and bequests, and his (residual) level of middle aged consumption. Accordingly, his budget constraint assumes the form

$$(1) \qquad c_{t,1} + q^e_{t+1}\, z^e_{t,1} + q^b_{t+1}\, z^b_{t,1} + q^{r_f}_{t+1}\, z^{r_f}_{t,1} \le \tilde{W}^1_{t+1} + (1-x)\tilde{B}_{t-1,2}$$

where $c_{t,1}$ denotes the consumption of a middle aged agent born in period t.

In the final period of his life, the period-t born-agent receives a pension income W^2. He fully consumes this quantity. He also consumes, by selling securities from his portfolio, and bequeaths his residual holdings:

$$(2) \qquad \tilde{B}_{t,2} = z^{e}_{t,1}\ (\tilde{q}_{t+2} + \tilde{d}_{t+2}) + z^{b}_{t,1}\ (\tilde{q}^{b}_{t+2} + 1) + z^{r_f}_{t,1}\ - \tilde{c}^{*}_{t,2}.$$

In effect, the elderly in this model sell a portion of their security holdings to the middle aged to supplement their old-age pension income. Their total consumption is therefore $W^2 + \tilde{c}^{*}_{t,2}$, and they pass down the residual value of their portfolio as a gift. We consider the case when $\tilde{c}_2$ is endogenously determined and when it is fixed: $\tilde{c}^{*}_{t,2} = \bar{c}_2$. As mentioned in the introduction, the later is a parsimonious device to capture the fact at old age consumption is uncorrelated with the return on securities and is largely governed by health status.

Taking prices as given, the decision problem faced by a representative agent (generation) born in period t is

$$\text{(3)} \qquad \underset{\left\{z^e_{t,1},\ z^b_{t,1},\ z^{r_f}_{t,1}\right\}}{\text{Max}} \ E\left\{\sum_{i=0}^{2}\beta^i u(c_{t,i}) + \beta^2 M v(\tilde{B}_{t,2})\right\}$$

$$\text{s.t.} \qquad c_{t,o} \le W^0 + x\tilde{B}_{t-2,2}$$

$$c_{t,1} + q^e_{t+1} z^e_{t,1} + q^b_{t+1} z^b_{t,1} + q^{r_f}_{t+1} z^{r_f}_{t,1} \le \tilde{W}^1_t + (1-x)\tilde{B}_{t-1,2}$$

$$\tilde{c}^*_{t,2} + \tilde{B}_{t,2} \le (\tilde{q}^e_{t+2} + \tilde{d}_{t+2})\, z^e_{t,1} + (\tilde{q}^b_{t+2} + 1)\, z^b_{t,1} + z^{r_f}_{t,1}$$

$$\tilde{c}_{t,2} \equiv \tilde{c}^*_{t,2} + W^2$$

$$0 \le z^e_{t,1} \le 1,\ 0 \le z^b_{t,1} \le b,\ 0 \le z^{r_f}_{t,1}.$$

In the above formulation, u (·) denotes the agent's utility-of-consumption function and v (·) his utility-of-bequests function. The constant M is the relative weight assigned to the utility of bequests. Both u (·) and v(·) are assumed to display all the basic properties sufficient for problem (3) to be well defined: they are continuously differentiable, strictly concave, increasing, and satisfy the Inada conditions. The postulated bequest function v(·) is sufficiently general to encompass both altruistic and egoistic bequest motivations. Notice that old agents are concerned only about their aggregate bequest and not its relative apportionment to their children and their grandchildren.

2.2 Optimality Conditions and Equilibrium

Let $\tilde{Y}_t$ denote the period t aggregate income. By construction, the economy's overall budget constraint satisfies:

$$\text{(4)} \qquad \tilde{Y}_t = W^0 + \tilde{W}_t^1 + W^2 + b + \tilde{d}_t = c_0 + \tilde{c}_{t-1,1} + \tilde{c}_{t-2,2} .$$

We first examine the case where old age consumption is fixed, that is $\tilde{c}^*_{2,t} = \overline{c}_2$ so that $\tilde{c}_{2,t} = \overline{c}_2 + W^2$. In equilibrium, the middle aged are the exclusive source of the demand for securities, and their optimal holdings are determined by the tradeoff between their marginal utility of consumption as middle aged and the expected discounted marginal benefit to granting one additional unit of indirect consumption in the form of a bequest. Taking prices as given, the middle aged agent's optimal holdings of equity, bonds, and risk free assets satisfy, respectively, the following three equations:

$$\text{(5)} \qquad z^e_{t,1} : u_1(c_{t,1})q^e_t = \beta E_t \left\{ M v_1(\tilde{B}_{t,2})[q^e_{t+1} + d_{t+1}] \right\}$$

$$\text{(6)} \qquad z^b_{t,1} : u_1(c_{t,1})q^b_t = \beta E_t \left\{ M v_1(\tilde{B}_{t,2})[q^b_{t+1} + 1] \right\}$$

$$\text{(7)} \qquad z^{r_f}_{t,1} : u_1(c_{t,1})q^{r_f}_t = \beta E_t \left\{ M v_1(\tilde{B}_{t,2}) \right\}$$

where (i) $\tilde{B}_{t,2}$ is defined as in (2) and, (ii), the (conditional) expectations are taken over all realizations of the economy's aggregate state variables, $\tilde{Y}_{t+1}$ and $\tilde{W}^1_{t+1}$.

Market clearing conditions for the three securities are as follows:

(8) $\quad z^{e}_{t,1} = 1,\ z^{b}_{t,1} = b$, and $z^{r_f}_{t,1} = 0$.

Imposing these market clearing conditions on the first order conditions (5)-(7) and recognizing that all the constraints in problem (3) will be satisfied with equality, we define a Stationary Bequest Equilibrium as follows:

<u>Definition</u>: A Stationary Equilibrium for the economy described by problem (3) and market clearing conditions (8) is a triple of time stationary security pricing functions $q^{e}(Y_t, W^1_t)$, $q^{b}(Y_t, W^1_t)$ and $q^{r_f}(Y_t, W^1_t)$ which satisfy equations (9) – (11):

$$(9) \quad u_1(W^1_t + (1\text{-}x)d_t + (1\text{-}x)\ b - (1\text{-}x)\,\bar{c}_2 - xq^{e}(Y_t, W^1_t) \text{-}\ xbq^{b}(Y_t, W^1_t))\ q^{e}(Y_t, W^1_t)$$

$$= \beta \int Mv_1\ (q^{e}(Y_{t+1}, W^1_{t+1}) + d(Y_{t+1}, W^1_{t+1}) + bq^{b}(Y_{t+1}, W^1_{t+1}) + b - \bar{c}_2)$$

$$[q^{e}(Y_{t+1}, W^1_{t+1}) + d(Y_{t+1}, W_{t+1})]\ dF(Y_{t+1}, W^1_{t+1}; Y_t, W^1_t)$$

$$(10) \quad u_1(W^1_t + (1\text{-}x)d_t + (1\text{-}x)\ b - (1\text{-}x)\,\bar{c}_2 - xq^{e}(Y_t, W^1_t) \text{-}\ xbq^{b}(Y_t, W^1_t))\ q^{b}(Y_t, W^1_t)$$

$$= \beta \int Mv_1\ (q^{e}(Y_{t+1}, W^1_{t+1}) + d(Y_{t+1}, W^1_{t+1}) + bq^{b}(Y_{t+1}, W^1_{t+1}) + b - \bar{c}_2)$$

$$[q^{b}(Y_{t+1}, W^1_{t+1}) + 1]\ dF(Y_{t+1}, W^1_{t+1};\ Y_t, W^1_t),$$

and

$$(11) \quad u_1(W^1_t + (1\text{-}x)d_t + (1\text{-}x)\ b - (1\text{-}x)\,\bar{c}_2 - xq^{e}(Y_t, W^1_t) \text{-}\ xbq^{b}(Y_t, W^1_t))\ q^{r_f}(Y_t, W^1_t)$$

$$= \beta \int Mv_1\ (q^{e}(Y_{t+1}, W^1_{t+1}) + d(Y_{t+1}, W^1_{t+1}) + bq^{b}(Y_{t+1}, W^1_{t+1}) + b - \bar{c}_2)]$$

$$dF(Y_{t+1}, W^1_{t+1};\ Y_t,\ W^1_t),$$

where dF(;) denotes the conditional density function on the economy's aggregate state variables.

Specializing the economy even further, we assume that the joint stochastic evolution of ($\tilde{Y}_t$, $\tilde{W}^1_t$) is governed by a discrete Markov process with no absorbing states. Our benchmark calibration recognizes that output and the total wage bill are highly positively correlated in the U.S. economy. A number of variations are considered which differ only with respect to the assumed correlation structure between $\tilde{Y}_t$ and $\tilde{W}^1_t$.

As was argued in the introduction, asset prices are higher in the presence of bequests than in a standard consumption-savings setting and the basis for this assertion is directly apparent in equations (9) – (11): there is a reduced (by the factor (1-x)) middle aged utility cost of paying more for a security since higher prices only mean greater offsetting bequests in our stationary equilibrium (see also Geanakoplos et al. (2003))[7]. As a result, prices are bid up to higher levels.

To varying degrees, all three agents receive utility from the same portfolio of securities: the young, whose consumption is enhanced when they sell their share of the bequest to the middle-aged; the middle aged who receive the bulk of the inheritance which thereby allows them to save for their own bequests with a much diminished reduction in

[7] Note that in the special case when x=0, that is when there is no bequest to the young, q^e_t, q^b_t, and $q^{r_f}_t$ do not appear in the marginal utility expressions on the left hand side of, respectively, equations (9), (10), and (11). This is unlike in a standard OLG setting. As the "auctioneer" calls out an increasing set of prices, the marginal utility of period t consumption does not increase to reduce demand. The effect of price increases on the suppression of demand is thus greatly reduced, a fact that suggests the possibility of explosive price behavior. That prices are likely to be higher under a bequest equilibrium relative to a

consumption; and the old who receive utility directly from the bequests they bequeath. As such, equations (9) – (11) represent a fundamental departure from the standard CCAPM based asset pricing relationships and are unique to the 'behavioral finance' literature.

Following Constantinides et al. (2002), we specify four admissible states representing two possible values of output in conjunction with two possible values of the wage endowment of the middle aged. The two preference functions are assumed to be of the standard form, $u(c_{t,i}) = \frac{(c_{t,i})^{1-\gamma_C}}{1-\gamma_C}$, i = 0, 1, 2, and $v\left(B_{t-1,2}\right) = \frac{(B_{t-1,2})^{1-\gamma_B}}{1-\gamma_B}$. In general we impose $\gamma_c = \gamma_B$ for the benchmark cases, though subsequently we explore $\gamma_C \neq \gamma_B$ ($\gamma_C > \gamma_B$ is intuitively the more plausible case). With these specifications, the equations defining the equilibrium functions may be simplified as follows:

(9′)
$$\frac{q^e(j)}{(W^1(j)+(1-x)d(j)+(1-x)b-(1-x)\bar{c}_2-xq^e(j)-xbq^b(j))^{\gamma_C}}$$

$$=\beta\sum_{k=1}^{4}\frac{M(q^e(k)+d(k))\ \pi_{jk}}{(q^e(k)+d(k)+bq^b(k)+b-\bar{c}_2)^{\gamma_B}}$$

(10′)
$$\frac{q^b(j)}{(W^1(j)+(1-x)d(j)+(1-x)b-(1-x)\bar{c}_2-xq^e(j)-xbq^b(j))^{\gamma_C}}$$

$$=\beta\sum_{k=1}^{4}\frac{M(q^b(k)+1)\pi_{jk}}{(q^e(k)+d(k)+bq^b(k)+b-\bar{c}_2)^{\gamma_B}}$$

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pure consumption savings context says nothing about relative return behavior, however. An explicit solution for equations (9) - (11) is therefore required.

(11′)
$$\frac{q^{r_f}(j)}{(W^1(j)+(1-x)d(j)+(1-x)b-(1-x)\overline{c}_2-xq^e(j)-xbq^b(j))^{\gamma_C}}$$

$$=\beta\sum_{k=1}^{4}\frac{M\ \ \pi_{jk}}{(q^e(k)+d(k)+bq^b(k)+b-\overline{c}_2)^{\gamma_B}}$$

where the states are indexed j = 1,2,3,4 and $d(j) = Y(j) - W^1(j) - W^0 - b$, and π_{jk} represents the probability of passing from state j to k.

2.3 Existence of Equilibrium and its Properties

Generically (that is, for all reasonable parameterizations of $Y(j)$, $W^1(j)$, W^0, $d(j)$, $\overline{c}_2$, β, and π_{ij}) equilibrium does not exist for this bequest driven model. In particular, if M is "too small," securities are insufficiently valued for bequests to be strictly positive in all states. As a consequence, there is no solution to (9) – (11) with positive real prices. If M is extremely large, middle aged investors, in their desire to leave more generous bequests, bid up security prices all the while receiving simultaneously more resources with which to do so. This scenario gives rise to equilibria where prices are so high that returns are absurdly low (even extremely negative in the risk free asset case). These latter equilibria are of little interest. Taken together these considerations suggest a fairly narrow range of M values $0 < \mathbf{M}_1 < M < \mathbf{M}_2 < \infty$ for which relevant equilibria are likely to result[8]. These thoughts are confirmed in the numerical solutions to follow. See appendix 1 for proof of existence.

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[8] For the parameterization considered here the interval $[M_1, M_2]$ is roughly $\left[\frac{1}{100}, 1\right]$.

By the homogeneity property of our utility specification, the numerical search for the equilibrium price functions can be substantially simplified: if $\{(q^e(j), q^b(j), q^{r_f}(j))$ j = 1,2,3,4$\}$ constitutes an equilibrium for an economy defined by $\{(Y(j), W^1(j), W_0, b, \bar{c}_2)$: j = 1,2,3,4$\}$, then for any $\lambda>0$, $\{(\lambda q^e(j), q^b(j), q^{r_f}(j))$: j=1,2,3,4$\}$ is an equilibrium for the economy defined by $\{(\lambda Y(j), \lambda W^1(j), \lambda W_0, \lambda b, \lambda \bar{c}_2)$: j=1,2,3,4$\}$.[9] Returns are thus unaffected if the economy is scaled up or down.

2.4 The Comparison Pareto Optimum

One can approach the notion of a Pareto optimum for a bequest driven model from a number of perspectives. On the one hand, with utility defined over wealth (prices) the standard notion of a Pareto optimal allocation as one resulting from the actions of an all powerful central planner empowered to reallocate real resources does not apply. We therefore must appeal to the notion of a constrained Pareto optimum, constrained by the participation of the market (trading) mechanism. That is, either before or after the planner reallocates "something," trade must be permitted.

There does not seem to be, a priori, an obviously unique way to do this. For example, one could postulate the "planner" as choosing the value of x such that the expected welfare of a representative cohort is maximized. Alternatively, one could propose a Pareto optimum allocation as that arising from the application of a system of

[9] If $\gamma_C \neq \gamma_B$, then the economy with scaled output, wages, interest payments and old aged consumption will have the same prices as the unscaled economy but with M altered to $M\lambda^{\gamma_C - \lambda_B}$, where λ is the scaling factor.

wage taxes and wage subsidies such that, again, the welfare of a representative cohort is maximized. Since we are principally interested in security price and return behavior, however, we maintain our emphasis on the bequest equilibrium economy alone.[10]

3. Calibration

In this section we select parameter values for the period utility and bequest function while also specifying the joint stochastic process on Y_t and W_t^1. Our calibration closely follows Constantinides et al. (2002).

There are eleven parameter values to be selected: $\{(Y(j), W^1(j)$: $j=1,2,3,4\}$, W^0, b, $\bar{c}_2$, β, M, γ_C and $\gamma_B\}$. In light of the homogeneity property, for an arbitrary choice of E(Y), $\{Y(j), W^1(j)$: $j=1,2,3,4\}$, W^0, b, and $\bar{c}_2$ can be chosen to replicate the fundamental ratios

$$\sigma_{\tilde{Y}}\big/E(\tilde{Y}),\ \sigma_{\tilde{W}^1}\big/E(\tilde{W}^1),\ \ E(W^0)/E(\tilde{Y})\ ,\ \ E(W^0+\tilde{W}^1+W^2)/E(\tilde{Y})\ ,\ \ E(b)/E(\tilde{Y}) \text{ and } E(\bar{c}_2)/E(\tilde{Y})$$

With a period corresponding to 20 years, and a maximum of five or six reliable non-overlapping 20 year periods in U.S. real GDP and aggregate wage data, it is difficult to conclusively fix the output and middle aged wage coefficients of variation. Following the discussion in our earlier paper, both are chosen to be about 0.20[11] (see Constantinides et al. (2002) for an elaboration).

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[10] Our benchmark calibrations will call for x = .25. By evaluating the expected utility of a representative cohort for a variety of x, we find that x = .25 is not far from the Pareto optimum.

[11] The exact values are 0.18 for the former and 0.23 for the latter.

The remaining ratios, however, can be established with more confidence. Consistent with U.S. historical experience, we fix the share of income to interest on U.S. government debt, $b / E(\tilde{Y})$, at 0.03. Depending on the historical period and the manner by which single proprietorship income is imputed, the average share of income to wages, $E(W^0 + \tilde{W}^1 + W^2) / E(\tilde{Y})$ is generally estimated (U.S. data) to lie in the range (.60, .75). For most of our examples, we match the ratio $E(W^0 + \tilde{W}^1 + W^2) / E(\tilde{Y}) = 0.69$

We choose W^0, W^2 and $\bar{c}_2$ in order to replicate the U.S. age-consumption expenditure profile in Fernandez-Villaverde and Krueger (2002; Figure 4.1.1), where we interpret our three period lifetimes as corresponding roughly to the 0-20, 20-60 and 60-80 age cohorts detailed there. For our benchmark calibration, in particular, their data suggest $\frac{W^2 + \bar{c}_2}{E(\tilde{Y})} \approx 0.2$ and $\frac{W^0}{E(\tilde{Y})} \approx 0.2$ [12]. We satisfy these conditions by choosing $W^0 = 18{,}000$, $W^2 = 8{,}000$ and $\bar{c}_2 = 10{,}000$. Lastly, we fix $\beta = .55$ (corresponding to a $\beta^{\text{annual}} = .97$) for all cases and, in all benchmark calibrations, $\gamma_C = \gamma_B = 5$, which is within the acceptable range of estimates provided by micro studies.

None of the aforementioned expectations and standard deviations can be computed without specifying the Markov chain governing the evolution of the $\tilde{Y}_t$ and state variables. Following Constantinides et al. (2002) we postulate a transition matrix Π of the form:

[12] Fernanadez-Villaverde and Krueger (2002) present data on per capita consumption on a quarterly basis from year 20 to year 80. Aggregating these quantities into the 20-60 and 60-80 age ranges plus adopting

$$\{\pi_{ij}\} = \begin{bmatrix} \phi & \Pi & \sigma & H \\ \Pi+\Delta & \phi-\Delta & H & \sigma \\ \sigma & H & \phi-\Delta & \Pi+\Delta \\ H & \sigma & \Pi & \phi \end{bmatrix}$$

Choices of ϕ, Π, σ, H and Δ determine the following important correlations $\rho(Y_t, Y_{t-1})$, $\rho(Y_t, W_t^1)$, $\rho(W_t^1, W_{t-1}^1)$.

Taking all these requirements into account yields the following benchmark calibration: $Y_t \in \{126, 200, 86{,}850\}$, $W_t^1 \in \{57, 850, 26, 450\}$, $\bar{c}_2 = 10{,}000$, $W^0 = 18{,}000$ and $W^2 = 8000$ with these quantities employed in conjunction with any of the four probability structures detailed in Table 1. All the major ratios detailed earlier are thereby replicated.

Table 1

Correlation Structures and Associated Parameter Values

$\mathrm{corr}(Y_t, Y_{t-1})$ and $\mathrm{corr}(W_t^1, W_{t-1}^1)$	$\mathrm{corr}(Y_t, W_t^1)$	φ	Π	σ	H	Δ
0.1	0.1	.5298	.0202	.0247	.4253	.01
0.8	0.1	.8393	.0607	.0742	.0258	.03
0.1	0.8	.5496	.0004	.0034	.4466	.03
0.8	0.8	.8996	.0004	.0034	.0966	.03

It remains to calibrate the parameter M.

the convention that quarterly consumption in years 1-20 coincides with year 20 first quarter consumption yields the indicated proportions.

3.1 Choosing a Value for the Bequest Parameter M

The parameter M, by governing the extent to which the middle-aged desire to bequeath, substantially influences both the relative and absolute level of equilibrium security price. Given this setting we select a value for M in order that the share of existing wealth that is being gifted, $\left(B_{t-1,2} \Big/ \left(q_t^e + d_t + b(q_t^b + 1)\right)\right)$, roughly respects the data.

As noted in the introduction, Summers and Kotlikoff (1981) estimate that intergenerational transfers (inter-vivos gifts and bequests), as a fraction of private wealth accumulation can be as much as 80%, while Modigliani (1988) concludes that a reasonable lower bound on this same fraction is 20%. These estimates differ because of the inconsistent treatment of durable goods valuation, college tuition payments and the assumed fraction of inheritances not spent. The average of these extreme estimates suggests that intergenerational transfers may account for as much as 50% of private wealth accumulation, a figure consistent with estimates in Hurd and Mundaca (1989) for high income families. In terms of absolute quantities, Gale and Scholz (1992) estimate (for the year 1983) that the flow of bequests was on the order of \$30 – \$40 billion, with inter-vivos transfers ranging to \$56 billion. If college tuition expenses are included, the latter rises to \$88 billion. Unfortunately, none of these studies separates out bequests and gifts of marketable securities from aggregate totals (which include real estate, undoubtedly the largest component of smaller estates).

A more useful estimate of the desired ratio can be obtained directly from estate tax data which provides the aggregate market value of bequeathed equity. As a fraction of CRSP aggregate Equity Market Value, this latter quantity gives a rough approximation to the $\left(B_{t-1,2} \Big/ \left(q_t^e + d_t + b(q_t^b + 1)\right)\right)$ ratio under a number of simplifying assumptions. Since equity bequests include private equity we need to argue that the latter is small. McGrattan and Prescott (2000), for the year 2000 estimate that more than 90% of business capital is publicly traded equity capital, an estimate that supports this assertion. Consistent with the figures in the prior paragraph we will also assume that inter-vivos transfers of stock alone may be conservatively estimated as having value equal to stock transfers as elements of bequests.[13]

Under these assumptions, the ratio of twice the value of equity bequests as a proportion of CRSP aggregate market value is roughly analogous to our quantity $\left(B_{t-1,2} \Big/ \left(q_t^e + d_t + b(q_t^b + 1)\right)\right)$. Table 2 below supplies the relevant data for a selection of the years for which data is available.

[13] Most equity is owned by the wealthiest segment of the population who holds an above average fraction of their total wealth in stock. We are simply asserting here that for this segment of the population, the fraction {(inter vivos transfers of equity/value of bequested equity} is approximately the same as the ratio of {inter vivos transfers/ bequests} for the population as a whole.

Table 2

U.S. Equity Bequests as a Proportion of U.S. Equity Market Value

Select Years[ii]

(A) Year	(B)[i] Value of Equity Component of Bequest	(C) CRSP Aggregate Equity Market Value	(D) $\frac{(B)}{(C)}$
1931	1.909	21.577	.0885
1938	1.273	40.680	.0313
1950	1.773	85.701	.0207
1961	6.766	383.720	.0180
1970	10.495	643.326	.0163
1977	12.483	1002.450	.0124
1991	27.087	4072.320	.0067
1996	44.151	8497.241	.0052
2001	77.343	14,419.260	.0055

(i) all values measured in billions of dollars

(ii) Source: IRS Estate Tax Returns, Publication 764; indicated years.

The value of annually bequeathed stock generally declined as a percentage of aggregate stock market value until the 1990s when it stabilized at roughly 0.6%. On the basis of a 20 year time horizon, and assuming stationary-in-levels asset values, this represents a total equity bequest equal to 12% of aggregate stock market valuations. If 1977 is used as the base, the ratio rises to 25%; in 1950 the fraction was around 45% while in 1931 it was 160%. These figures suggest a wide of estimates[14]. Doubling these figures to include inter-vivos transfers, in any event, encourages us to conclude that a reasonable value of M should result in a ratio $\left(B_{t-1,2} \big/ \left(q_t^e + d_t + b(q_t^b + 1)\right)\right)$ lying in the range [0.5, 1] for postwar data. This is easily attained given our parameterization.

In what follows we numerically solve equations (9) – (11) for the indicated parameterizations. In order to gauge model sensitivity, we allow M, and $\gamma_C = \gamma_B$ to vary. Since the results depend very little, either qualitatively or quantitatively, on the choice of transition matrix, we typically only report results for cases corresponding to $\phi = .5298$.

4 Equilibrium Results

4.1 Benchmark Economy

Much of the intuition provided by this model is evident from the fixed old age consumption case. This perspective was justified earlier by arguing that the consumption of the old aged is governed by their health status, a circumstance that is likely to be unrelated to the business cycle, especially for those with large equity holdings. Fixing old-age consumption at a constant level reflects this viewpoint in a parsimonious way.

Table 3 provides a basic set of results for an uncontroversial set of parameters. The risk aversion parameter γ_C is fixed at $\gamma_C = 5$, and M is chosen to be $M = \frac{1}{10}$. It seems intuitively reasonable that agents would value their bequests less highly than their own consumption.

[14] The substantially lower figures for more recent years are interesting and may reflect either an increased use of tax avoidance schemes (e.g., generation-skipping trusts) by the very wealthy who own the lion's share of equity in the U.S. or the broader ownership of stocks in small estates exempt from taxation.

Table 3

Basic Financial Statistics: First Benchmark Parameterization

	U.S. Data		Benchmark Model $\overline{c}_2$=10,000, M=$\frac{1}{10}$, ϕ=.5298, $\gamma_C=\gamma_B$=5, x=0.25	
	(a)[ii]	(b)	(a)	(b)
return on equity	7.0	16.5	6.1	17.1
risk free return	.80	5.7	1.2	21.9
equity premium	6.2	16.7	5.0	11.7
		Range		Range
bequests/assets[iii]		0.5 to 1		0.69 to 0.93

(i) For this set of parameters, the corresponding middle aged consumption and bequests in states j=1,2,3,4 are: $c_1(1)$ = 68,084; $c_1(2)$ = 45,182; $c_1(3)$ = 56,155; $c_1(4)$ = 43,006; B(1) = 88,465; B(2) = 22,672; B(3) = 136,181; B(4) = 31,375.

(ii) (a) is the unconditional mean while (b) is the unconditional standard deviation annualized in the manner described in Footnote (10). All returns are real. U.S. data from Mehra and Prescott (1985).

(iii) This ratio is defined as $\frac{q^e(j)+d(j)+b(q^b(j)+1)-\overline{c}_2}{q^e(j)+d(j)+b(q^b(j)+1)}$ with the range defined in reference to this quantity across the four states.

The benchmark economy displays considerable success in replicating the mean return on equity (6.1) and its standard deviation (17.1).[15] The equity premium is a robust 5.0%, attributable in large measure to a relatively low risk free rate (1.2%). Also, the bequests/assets ratio falls comfortably within the range of empirical estimates.

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[15] The reader is cautioned to keep in mind how these returns are computed and the consequent qualifications to any of the interpretations. For the equity security the annualized mean return was computed as $\frac{1}{20}\left\{\sum_{j=1}^{4}\varphi_j\sum_{k=1}^{4}\pi_{jk}\ \log\left(\frac{q^e(k)+d(k)}{q^e(j)}\right)\right\}$ with the mean returns of the other securities computed analogously. In the above expression φ_j denotes the stationary probability of state j. The 20 year standard deviation of the equity return was computed as

$$\left\{\sum_{j=1}^{4}\varphi_j\left(\sum_{k=1}^{4}\pi_{jk}\ \log\left(\frac{q^e(k)+d(k)}{q^e(j)}\right)-\sum_{j=1}^{4}\varphi_j\ \sum_{k=1}^{4}\pi_{jk}\ \log\left(\frac{q^e(k)+d(k)}{q^e(j)}\right)\right)^2\right\}^{1/2}$$

while the corresponding annualized standard deviation satisfied $SD^{equity}_{annuity}=\frac{1}{\sqrt{20}}SD^{equity}_{20\ year}$. Again, the return standard deviations for the other securities were computed in an identical fashion.

The standard deviation of the risk free return, however, is too high (21.9) and exceeds the standard deviation of the equity return. To understand this, consider the special case $x = c_2 = b = 0$ while appealing to continuity arguments for wider applicability. The Euler equations of consumption for the prices of equity and the one-period bond under this specialization are as follows:

$$q^e(j) = \beta M \left(W^1(j) + d(j)\right)^{\gamma_C} \sum_{k=1}^{4} \frac{\pi_{jk}}{\left(q^e(k) + d(k)\right)^{\gamma_B - 1}} \tag{4.1}$$

and

$$q^{r_f}(j) = \beta M \left(W^1(j) + d(j)\right)^{\gamma_C} \sum_{k=1}^{4} \frac{\pi_{jk}}{\left(q^e(k) + d(k)\right)^{\gamma_B}}. \tag{4.2}$$

The Euler equation of equity is isomorphic to that of the one-period bond except that the degree of bequest risk aversion is lower by one. This follows directly from the fact that the equity's next period pre-dividend value partially offsets variation in its marginal utility of wealth (for log utility the offset is perfect), making it effectively the less risky security in utility-of-bequest terms.

In the context of the consumption-based asset pricing model, higher risk aversion typically leads to higher return volatility because consumers have a greater incentive to smooth consumption. Their demand for securities is thus higher in high-income states and lower in low-income ones. Ceteris paribus, security price volatility and return volatility are higher. Similar reasoning applies to our bequest economy. Equity is effectively priced in a less risk averse environment and consequently displays lower return volatility, as observed.

Our results suggest that if a particular security (under our parameterizations, equity) provides the overwhelming majority of bequest utility, that security will display the greater relative price stability irrespective of the volatility of its dividend. In a world where agents derive utility directly from bequests (wealth), the notion of risk is blurred.[16] High variability of the risk free rate is sometimes a problem in certain multi-sector real business cycle models, as in Boldrin, Christiano and Fisher (2001). Alternative specifications that may reduce the variability of the risk free rate include state-dependent risk aversion, as in Campbell and Cochrane (1999).

Each period two cohorts receive utility from the same portfolio of bequeathed securities: the middle aged through an increase in their wealth and the old through the joy of giving. This feature represents a departure from the standard Arrow-Debreu economy. The prices of both equity and bonds are higher in the presence of bequests because two cohorts receive utility from the same portfolio of bequeathed securities.[17] In the benchmark case, the average equity price is more than twice what is observed in the pure consumption-savings analogue for an otherwise identical parameterization.[18]

4.2 Sensitivity to the Bequest Weight

Table 4 illustrates the effect of increasing the bequest weight M. As bequests become more important, security prices are bid up.[19] Since security payments are unaltered, rates of return decrease

[16] Cass and Pavlova (2000) illustrate analogous ambiguity in a standard Lucas (1978) asset pricing model with log utility where the representative agent trades a risk free bond and a stock. They introduce a simple linear transformation by which the stock becomes the risk free asset and the bond becomes the risky one in the sense that its payment is now uncertain. While their model context is very different from the one considered here, they present a similar instance of the more variable return security having the lesser associated payment variation.

[17] See also Gaenakoplos et al. (2003).

[18] In the special case x=0, that is when there is no bequest to the young, q_t^e, q_t^b and q_t^{rf} do not appear in the marginal utility of the middle-aged. This is unlike in a standard OLG setting. As the "auctioneer" calls out an increasing set of prices, the marginal utility of period t consumption does not increase to reduce demand. The effect of price increases on the suppression of demand is thus greatly reduced, a fact that suggests the possibility of explosive price behavior.

[19] With prices rising yet c_2 fixed, the E(B/A) ratio will naturally approach one, as observed.

Table 4

Effects of Changes in *M* on Equilibrium Security Prices, Bequests and Returns

M	0.1	0.5	1
$q^e(1)$	42,754	64,246	73,343
$q^e(2)$	3,111	6,684	8,796
$q^e(3)$	7,927	15,820	20,162
$q^e(4)$	5,325	10,523	13,357
$q^{r_f}(1)$	1.40	1.70	1.78
$q^{r_f}(2)$	0.91	0.92	0.93
$q^{r_f}(3)$	2.21	2.10	2.05
$q^{r_f}(4)$	0.17	0.28	0.33
$B(1)$	88,465	112,010	122,062
$B(2)$	22,672	29,955	33,651
$B(3)$	136,181	148,203	153,997
$B(4)$	31,375	37,430	40,719
$\bar{r}^e$	6.1 %	4.5%	4.0%
σ_{r_e}	17.1%	14.9%	14.0%
$\bar{r}^f$	1.2%	0.28%	0.03%
σ_{r_f}	21.9%	18.0%	16.7%
$\bar{r}^p$	5.0%	4.2%	3.9%
σ_{r_p}	11.7%	8.2%	7.1%
Range B/A	0.69 - 0.93	0.75 - 0.94	0.77 - 0.94

We note that the standard deviations of the returns to all securities also decline with an increase in M and the origin of this result is less obvious and merits discussion. As M rises, investors become increasingly concerned about bequest volatility. Their only recourse is to attempt to acquire more securities, thereby bidding up prices but in a state by state fashion so as to to diminish price and wealth variation (rational expectations). As noted, security returns uniformly decline towards zero. Adding to this effect is reduced MRS volatility: as M increases B(j) increases with the result that $\left(\frac{c(j)}{B(k)}\right)^{\gamma}$ declines dramatically for all j, k state pairs. In the case of $x = 0$, c(j) is unaffected by *M* and thus only the denominator, $B(k)$, increases. The net effect is a decline in volatility. Note also that as M increases, the equity premium declines from the high benchmark level of 5%. This phenomena is directly attributable to the enormous increase in security prices which place the investor on a less concave portion of his bequest utility function. In effect, as he becomes wealthier the agent becomes less bequest risk averse, a result that acts as a break on the ability of bequest parameter M to generate arbitrarily high equity premia. It is thus not at all the case that the introduction of a bequest motive allows for a facile and contrived resolution of the equity premium or risk free rate puzzles.

4.3 Sensitivity to the RRA Coefficient on Consumption and Bequests

In Table 5, we consider the effect of an increase in the RRA coefficient of both consumption and bequests.

Table 5

Effects of Changes in RRA on Security Prices, Returns and Bequests

RRA	1	3	5
$q^e(1)$	5,297	18,642	42,754
$q^e(2)$	3,691	2,846	3,111
$q^e(3)$	5,012	5,972	7,927
$q^e(4)$	2,911	3,980	5,325
$q^{r_f}(1)$	0.17	0.61	1.40
$q^{r_f}(2)$	0.65	0.87	0.91
$q^{r_f}(3)$	0.77	1.73	2.21
$q^{r_f}(4)$	0.10	0.14	0.17
$\overline{r}^e$	9.4%	7.4%	6.1 %
σ_{r_e}	10.6%	13.0%	17.1%
$\overline{r}^f$	6.4%	3.0%	1.1%
σ_{r_f}	19.4%	21.1%	21.9%
$\overline{r}^p$	3.0%	4.5%	5.0%
σ_{r_p}	9.4%	13.2%	11.7%
Range B/A	0.19 - 0.88	0.60-0.92	0.69-0.93

We see that equity and bond prices increase in all states as γ increases for reasons similar to an increase in *M*. The average equity/output ratio naturally increases and the average bequest-over-assets ratio asymptotically approaches one. Equity returns decrease less rapidly than risk free returns, giving rise to an increasing premium as γ increases. The volatility of returns increases as well. Collectively, these phenomena are consistent with behavior of standard CCAPM models (e.g., Mehra and Prescott (1985)).

4.4 Sensitivity to Changes in the Allocation of Bequests, x

In Table 6, we present the effect of changing the allocation of bequests between the young and the middle aged.

Table 6

Effect of Changes in x on Security Prices, Returns and Bequests

x	0	.10	.25	.50
$q^e(1)$	111,162	66,126	42,754	25,829
$q^e(2)$	3,745	3,456	3,111	2,701
$q^e(3)$	56,371	17,238	7,927	3,132
$q^e(4)$	7,878	6,772	5,325	3,384
$q^{r_f}(1)$	3.39	2.08	1.40	0.90
$q^{r_f}(2)$	0.91	0.91	0.91	0.90
$q^{r_f}(3)$	13.18	4.35	2.21	0.99
$q^{r_f}(4)$	0.24	0.21	0.17	0.12
$\bar{r}^e$	3.5%	4.9 %	6.1 %	8.0%
σ_{r_e}	27.7%	20.5%	17.1%	16.25%
$\bar{r}^f$	-2.5%	-0.4%	1.1%	3.2%
σ_{r_f}	33.3%	25.4%	21.9%	20.5%
$\bar{r}^p$	6.1%	5.3%	5.0%	4.8%
σ_{r_p}	10.6%	10.7%	11.7%	12.8%

The general effect of changes in the allocation of bequests is unambiguous. As the fraction of bequests passed to the young increases, all security prices decline and returns rise and the premium declines. As x increases, more securities pass to the young, which they sell. The middle aged receive smaller bequests and must, in equilibrium, buy more securities. In effect, the supply of securities (vis-à-vis the middle aged) increases and, ceteris paribus, equilibrium prices decline. This is reinforced by the fact that the wealth of the middle aged also declines, thereby diminishing demand across the board. Faced with declining resources it is also unsurprising that the middle aged investors should slightly shift their portfolio holdings in favor of high payoff securities, the stocks, a fact accounts for the diminished premium.

The other unambiguous phenomena is the greater equity and risk free return volatility, as x

diminishes. This reflects more pronounced wealth effects for the middle-aged investors: as x declines there is a progressively diminished consumption cost to the middle aged of assembling their own bequest portfolios. As a result, their demand for securities tends to react more dramatically to changes in their wealth with the ensuing heightened price and return volatility.

4.5 Endogenous Consumption of the Old

Unlike the benchmark case in which the consumption of the old is fixed, we now endogenize the consumption of the old in economies with and without bequests. Bequests and old age consumption are thus jointly determined by the added requirement that

$$u_1\left(c_2(j)\right) = M v_1\left(B(j)\right)$$

for all states j. Once B(j) is determined in this way, the fraction x is bequeathed to the young and the fraction (1-x) to the middle aged, as before.

The results are presented in Table 7.

Table 7 Exogenous vs. Endogenous Consumption of the Old

Consumption of the Old	Exogenous with x =0.25	Endogenous Bequests, x =0.25	Endogenous w/o Bequests
$c_1(1)$, $c_2(1)$	68084, 18000	41755, 57392	43,456, 64,763
$c_1(2)$, $c_2(2)$	45182, 18000	35716, 28620	37,155, 31,694
$c_1(3)$, $c_2(3)$	56155, 18000	33851, 64219	24,367, 83,833
$c_1(4)$, $c_2(4)$	43006,18000	30786, 32878	25325, 43,525
B(1)	88,465	36,212	0
B(2)	22,672	18,058	0
B(3)	136,181	40,519	0
B(4)	31,375	20,745	0
$q^e(1)$	42,754	37,442	13,057
$q^e(2)$	3,111	14,384	15,444
$q^e(3)$	7,927	9,571	1,561
$q^e(4)$	5,325	9,706	1,019
$q^{r_f}(1)$	1.40	0.91	0.44
$q^{r_f}(2)$	0.91	0.90	1.75
$q^{r_f}(3)$	2.21	0.57	0.17
$q^{r_f}(4)$	0.17	0.24	0.04
$\bar{r}^e$	6.1 %	5.4%	12.1%
σ_{r_e}	17.1%	11.4%	29.6%
$\bar{r}^f$	1.1%	2.9%	10.1%
σ_{r_f}	21.9%	12.5%	28.5%
$\bar{r}^p$	5.0%	2.5%	1.9%
σ_{r_p}	11.7%	5.3%	10.7%
Range B/A	0.69-0.93	0.42-0.47	NA

In the first column, we present the benchmark case with exogenous consumption for purposes of comparison. In the second column, the consumption and bequests of the old are endogenously determined. In the last column, there are no bequests; the consumption of the old is endogenously determined by pure consumption and savings considerations.[20] Note that for each security type, the associated payments are invariant across the three cases.

As we move across the table from left to right bequests progressively recede in importance. Asset prices decline dramatically when bequests are eliminated entirely, a fact directly attributable to the large influence bequests have on the equilibrium steady state security prices: unlike saving for old age consumption which entails an actual (steady state) cost for the middle aged, bequests do not impinge upon middle aged consumption (at least to the extent of the (1-x) fraction they receive). As a further consequence of declining bequests, old age consumption increases, but not by the full magnitude of the bequest reduction because asset prices are lower.

A number of other idiosyncratic features of Table 7 are worth exploring. For one, the equity price is consistently highest in state one. It is this state that corresponds to the highest output level and the highest possible middle-aged wage level. While not the highest attained value, dividends in this state are much higher than in a majority of the other states. With a relatively persistent dividend steam and a high level of income (wages) with which to purchase securities, it is not surprising that these two effects conspire to bid equity prices up to uniquely high levels. Although state

[20] This corresponds to the constrained problem detailed in Constantinides et al. (2002): middle aged agents accumulate securities purely to finance their retirement consumption (no bequests). The latter is accomplished by selling their security accumulation ex dividend to the then middle aged agents. More formally, the maximization problem of the period-t-born agent is:

$$\text{Max E}\left(\sum_{j=0}^{2}\beta^j u(c_{t,j})\right)$$

$$c_{t.0} \leq W_0$$

$$c_{t,1} + q^e_{t+1} \; z_{t,1} + \; q^b_{t+1} \; z^b_{t,1} + q^{r_f}_{t+1} \; z^{r_f}_{t,1} \leq \tilde{W}^1_t$$

$$c_{t,2} \leq q^e_{t+2} z_{t,1} + q^b_{t+2} z^b_{t,1} + q^{r_f}_{t+2} z^{r_f}_{t,1}$$

$$0 \leq z^e_{t,1} \leq 1$$

$$0 \leq z^b_{t,1} \leq b$$

$$0 \leq z^{r_f}_{t,1}$$

three experiences the highest dividend, resources for purchasing securities are much lower.

Comparing the endogenous bequest and no bequest cases, it is also interesting to observe that middle aged consumption is higher in the former and old age consumption higher in the latter. This is not surprising as bequests provide more resources to the middle aged. Furthermore, consumption appears to be less smooth intertemporally under the no bequest regime: comparing the endogenous and no bequest cases, in every state, middle-aged consumption is lower and old-age consumption higher in the latter case. This phenomenon follows again from the observation that the effect of bequests is to shift consumption principally to the middle aged; they do not have to save fully for old age consumption, and thus can more easily enjoy more consumption as middle aged. In effect, bequests are equivalent to costless borrowing.[21] As a result, middle aged investors have much higher wealth in the bequest case and bid up securities prices to much higher levels as observed.

4.6 Exploring Changes in M and other parameters

The data for various M and various γ in an environment of endogenous bequests is presented in Tables 8 and 9, respectively. Broadly speaking, almost all of the qualitative relationships detailed for the fixed old age consumption case, and their underlying justifications, carry over to this more general setting.

[21] We have to be careful of this interpretation in that there is no agency or individual in the model from whom the middle aged might borrow. It is intended to be construed in the sense that a gift is equivalent to a loan that never needs repayment.

Table 8

Effects of Changes in M on Equilibrium Prices, Returns, and Bequests

Endogenous Old Age Consumption

$\phi = 0.5298,\ \gamma = 5$[(i)]

	M=0.1	M=0.5	M=1
$(c_1(1), c_2(1))$	41755, 57392	40660,55468	40093,54486
$(c_1(2), c_2(2))$	35716, 28620	34247,27914	34415,27548
$(c_1(3), c_2(3))$	33851, 64219	34247,60735	34271,59143
$(c_1(4), c_2(4))$	30786, 32878	30837,31219	30743,30485
B(1)	36,212	48,287	54486
B(2)	18,058	24,300	27548
B(3)	40,519	52,873	59143
B(4)	20,745	27,177	30485
$q^e(1)$	37,442	46,411	51,014
$q^e(2)$	14,384	18,412	20,554
$q^e(3)$	9,571	14,636	17,482
$q^e(4)$	9,706	13,894	16,153
$q^{r_f}(1)$	0.91	1.01	1.06
$q^{r_f}(2)$	0.90	0.91	0.91
$q^{r_f}(3)$	0.57	0.69	0.75
$q^{r_f}(4)$	0.24	0.31	0.34
$\bar{r}^e$	5.4%	4.5%	4.1%
σ_{r_e}	11.4%	9.8%	9.2%
$\bar{r}^f$	2.9%	2.1%	1.8%
σ_{r_f}	12.5%	10.8%	10.1%
$\bar{r}^p$	2.5%	2.3%	2.3%
σ_{r_p}	5.3%	4.3%	3.9%
Range B/A	0.42-0.47	0.5 – 0.55	0.54 -0.58

(i) All other parameters as in Table 4.

Table 9

Effects of Changes in γ on Equilibrium Prices, Returns, and Bequests

Endogenous Old Age Consumption
ϕ=.5298, M=0.1[(i)]

	$\gamma=1$	$\gamma=3$	$\gamma=5$
(c1(1), c2(1))	41154,65411	44737,56865	41755, 57392
(c1(2), c2(2))	40074,28075	38100,27552	35716, 28620
(c1(3), c2(3))	21765,84327	33039,67346	33851, 64219
(c1(4), c2(4))	19741,47911	29022,35687	30786, 32878
B(1)	6,541	26,394	36,212
B(2)	2,807	12,789	18,058
B(3)	8,433	31,259	40,519
B(4)	4,791	16,564	20,745
qe(1)	19,823	29,404	37,442
qe(2)	16,826	18,143	14,384
qe(3)	9,533	10,762	9,571
qe(4)	9,432	8,779	9,706
$q^{r_f}(1)$	0.41	0.65	0.91
$q^{r_f}(2)$	0.54	0.82	0.90
$q^{r_f}(3)$	0.27	0.45	0.57
$q^{r_f}(4)$	0.20	0.20	0.24
$\overline{r}^e$	5.9%	5.5%	5.4%
σ_{r_e}	8.1%	10.8%	11.4%
$\overline{r}^f$	5.6%	3.9%	2.9%
σ_{r_f}	8.4%	12.0%	12.5%
$\overline{r}^p$	0.3%	1.7%	2.5%
σ_{r_p}	1.1%	4.1%	5.3%
Range B/A	0.10 – 0.12	0.35 – 0.40	0.42-0.47

(i)

[(i)] All other parameters as in Table 4.

As M increases, in particular, asset prices and the value of bequests rise, while expected returns decline. As in the fixed old age consumption case, return volatilities decline over the indicated range. In general, the range of security prices across states is less in the endogenous consumption case. This is manifest in lower standard deviations of return across all securities. Bequests are also smaller since proportionally more securities are offered for sale. However, the volatility of the risk free return exceeds that of the equity security, as in Table 3, and for the same fundamental reasons. That bequests and old age consumption coincide in real terms for the right most case (Table 8) is a direct implication of constraint (13), since M = 1 and $\gamma_C = \gamma_B$.

The comparative results (Tables 5 vs. 9) for an increase in risk aversion are in the same spirit. As in the fixed old age consumption case, greater risk aversion coincides with lower expected returns and higher return volatilities. The equity premium also increases with gamma. For $\gamma \geq 2$, E(B/A) uniformly lies nearly within the acceptable range. As in the previous case, prices and return statistics are muted relative to their fixed old age consumption counterparts. There are no issues of the non-existence of equilibrium for any of these cases, however.

Substantial differences can be found in the level and variation in the price and bequest series. Comparing Tables 9 with 5, there is seen to be much less variation in bequest levels or asset prices across the four states, a fact that is also manifest in the means and standard deviations of returns across all the securities. This is to be expected: in the former case quantities can adjust more freely. There is thus less need for prices themselves to adjust.

4.7 Changes in the Bequest Parameter x

Table 10 is the endogenous counterpart to Table 6. Most of the intuition comes over from that latter case: an increase in "x" restricts the flexibility of the middle aged and, necessarily increases the supply of securities which the middle aged, in equilibrium, must purchase. Prices necessarily decline with the resultant increase in expected return. Notice also that, for any choice of x, return volatilities are higher under the fixed old age consumption regime. This follows from the countervailing force at work in the endogenous consumption case which is otherwise absent in the exogenous old age consumption setting. Under the former setting, the investor also wishes to stabilize his old age consumption, a fact that leads him to seek more strongly to acquire securities in low dividend (low price) states than in higher ones. This behavior, per se , tends to stabilize prices and is absent in the fixed old age consumption case. Thus price and return volatilities are lower.

The pattern of volatilities as x increases also varies from Table 6 to Table 10, declining with x in the former case and rising in the later. With only a bequest motive (Table 6), as the wealth of the middle aged declines (x increases) the price and return effects resulting from their desire to stabilize their future wealth are more muted. In the exogenous case, this is offset by the middle aged generation's desire to smooth its old age consumption; apparently the former force predominates in Table 10. In either case the effects are not large.

Table 10

Effect on Equilibrium Security Prices and Returns of Changes in x

$M = 0.1,\ \phi = .5298,\ Y(1),\ Y(2),\ W^1(1),\ W^1(2)$ as in Table 9

$\gamma_C = \gamma_B = 5$

	x = 0	x= 0.10	x= 0.25	x= 0.50
qe(1)	44,861	41,720	37,442	31,252
qe(2)	15,699	15,163	14,384	13,130
qe(3)	13,490	11,822	9,571	6,393
qe(4)	12,495	11,329	9,706	7,306
$q^{r_f}(1)$	1.02	0.97	0.91	0.80
$q^{r_f}(2)$	0.91	0.90	0.90	0.89
$q^{r_f}(3)$	0.75	0.68	0.57	0.41
$q^{r_f}(4)$	0.29	0.27	0.24	0.19
$\bar{r}^e$	4.7%	4.9 %	5.4%	6.2%
σ_{r_e}	10.2%	10.6%	11.4%	13.3%
$\bar{r}^f$	2.1%	2.4%	2.9%	3.7%
σ_{r_f}	11.5%	11.8%	12.5%	14.1%
$\bar{r}^p$	2.5%	2.5%	2.5%	2.5%
σ_{r_p}	5.0%	5.1%	5.3%	5.6%
Range B/A	0.42-0.46	0.42-0.46	0.42-0.47	0.39-0.39

5. Concluding Remarks

We have examined the influence of bequests on equilibrium security prices and returns. Generally speaking, the effect of bequests is to dramatically increase security prices. In a standard consumption-savings model, the purchase of securities to finance future consumption reduces consumption today, thereby raising the marginal utility of consumption, which acts as a discouragement to further savings. This latter effect is not present in a bequest-driven model of the type considered here, at least in the steady state, leading to much more powerful income effects. Both asset prices and price volatility tend to be substantially higher. We are able to keep the prices low and generate realistic values of the mean risk free rate, the mean equity premium, the variance of the equity premium and the ratio of bequests to wealth by stipulating that a portion of the bequests skips a generation.

Two key parameters of the model are the weight on the utility of bequests and the fraction of the bequests that skips the generation of the middle-aged and is received by the young. It is possible that a judicious choice of these parameters may lower the observed unrealistically high relative variance of the risk free rate.

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Appendix 1:
Existence of Equilibrium

In all cases we set x = 0 for transparency. Our argument is cast as a series of propositions.

Proposition 1: Suppose that $u(\cdot) = v(\cdot)$ is of the CRRA family of utility functions with common CRRA parameter γ and that $(Y(j), W^1(j))$ follows a level stationary N state Markov chain. Suppose also that $\theta(j) \equiv d(j) + b - \bar{c}_2 > 0 \; \forall j$ and that $d(j) > 1 \; \forall j$. Let $\phi \geq 1$ be an arbitrarily chosen constant. Define

$$\Psi = \phi(\max_{1 \leq j \leq N} d(j)) \text{, and}$$

$$L = \max_{1 \leq j \leq N} \sum_{k=1}^{N} \pi_{jk} \left(\frac{w^1(j) + \theta(j)}{\theta(k)} \right)^{\gamma}$$

Then there exists a solution to (9) − (11) in $A \subseteq R_+^{2N}$ where $A = \{x(1), \ldots, x(N), y(1), \ldots, y(N)) : 0 \leq x(i) \leq \Psi, 0 \leq y(i) \leq \Psi\}$,

provided $M\,L \left(\dfrac{1}{1 + \dfrac{1}{\phi}} \right) < 1.$[22]

Proof: Define the operator $T : A \mapsto R_+^{2N}$ by

$$T(x_1, \ldots, x_N, y_1, \ldots, y_N) =$$

$$\left(\beta M \sum_{k=1}^{N} \pi_{1k} \left(\frac{W^1(1) + \theta(1)}{x(k) + b\,y(k) + \theta(k)} \right)^{\gamma} \big(x(k) + d(k)\big), \ldots, \right.$$

$$\beta M \sum_{k=1}^{N} \pi_{Nk} \left(\frac{W^1(N) + \theta(N)}{x(k) + b\,y(k) + \theta(k)} \right)^{\gamma} \big(x(k) + d(k)\big),$$

$$\beta M \sum_{k=1}^{N} \pi_{1k} \left(\frac{W^1(1) + \theta(1)}{x(k) + b\,y(k) + \theta(k)} \right)^{\gamma} \big(y(k) + 1\big), \ldots,$$

$$\left. \beta M \sum_{k=1}^{N} \pi_{Nk} \left(\frac{W^1(N) + \theta(N)}{x(k) + b\,y(k) + \theta(k)} \right)^{\gamma} \big(y(k) + 1\big) \right).$$

---

[22] Note that once the existence of $q^e(j)$ and $q^b(j)$ $j = 1, 2, 3, 4$, is guaranteed, $q^{r_f}(j)$ follows from (11′) directly.

The set A is compact in R^{2N}. Furthermore, since $\theta(j)>0\ \forall j$, T is continuous on A. Clearly, for every (x(1), . . . , x(N), y(1), . . . , y(N)) $\geq$ 0, T(x(1), . . . , x(N), y(1), . . . y(N)) $\geq$ 0. In order to apply Brower's Fixed Point Theorem we need only to show that each entry in the image of T falls short of Ψ ; i.e., that $T(A) \subseteq A$. For any x(j)

$$\beta M \sum_{k=1}^{N} \pi_{jk} \left(\frac{W^1(j)+\theta(j)}{\theta(k)} \right)^{\gamma} \left(x(k)+d(k) \right)$$

$$\leq \beta M\ (\Psi) \left(1+\frac{1}{\phi} \right) \sum_{k=1}^{N} \pi_{jk} \left(\frac{W^1(j)+\theta(j)}{\theta(k)} \right)^{\gamma}$$

$$< \left(1+\frac{1}{\phi} \right) \beta M L\ \Psi < \Psi$$

For any y(j)

$$\beta M \sum_{k=1}^{N} \pi_{jk} \left(\frac{W^1(j)+\theta(j)}{\theta(k)} \right)^{\gamma} \left(y(k)+1 \right)$$

$$< \beta M \left(1+\frac{1}{\phi} \right) \Psi\ L < \Psi$$

Thus there exists a fixed point $(\hat{x}(1),...,\hat{x}(N), \hat{y}(1),...,\hat{y}(N))$ of T on A. Identify

$$\hat{x}(j) \equiv q^e(j)$$

$$\hat{y}(j) \equiv q^b(j)$$

Then $\left(q^e(j), q^b(j) \right)$ solves (9) and (10) .

Note that since $\theta(j)>0$ and $d(j)>0\ \forall j$, $\left(q^e(j), q^b(j) \right)>0\ \forall j$. Finally, $q^{r_f}(j)>0$ is defined as per (11) once $q^e(j)$, $q^b(j)$ are determined.

Commentary: The critical assumption in Proposition 1 is that $\theta(j)>0\ \forall j$. This means that no matter how low asset prices may be, the value of assets cum dividends and interest payments is always sufficient to finance old age consumption $\bar{c}_2$. Without such an assumption, the constant M must be sufficiently large as to guarantee that asset prices are great enough to satisfy:

$$q^e(j)+d(j)+b(q^b(j)+1)-\bar{c}_2>0$$

We argue this fact because intuitively as $M \mapsto 0$, $q^e(j) \mapsto 0$ and $q^b(j) \mapsto \forall j$ (see also Proposition 2 to follow). Without the $\theta(j)>0\ \forall j$ requirement it is necessary to establish a lower bound on M in order for equilibrium to exist, a fact borne out repeatedly by the results of our numerical solutions to $(9')-(11')$.

Proposition 2: Suppose the conditions for the existence of equilibrium are satisfied, and assume furthermore that $\gamma_C = \gamma_B > 1$. Suppose also that the endowment process $\{Y(j), W^1(j)\}$ is i.i.d. through time. If $M_2 > M_1$,
then $q^e(j, M_2) > q^e(j, M_1)\ \forall j$, $q^b(j, M_2) > q^b(j, M_1)\ \forall j$ and $q^{r_f}(j, M_2) > q^{r_f}(j, M_1)\ \forall j$.

Proof: For simplicity, let us ignore the consol bond by setting its supply equal to zero.

The system of non-linear equations which define equilibrium is thus,

$j = 1, 2, ..., N$,

$$q^e(j) = \beta\theta(j)M\sum_{k=1}^{N}\pi_{jk}\frac{\left[q^e(k)+d(k)\right]}{\left[q^e(k)+d(k)-\bar{c}_2\right]^{\gamma_B}}$$

where $\theta(j) = \left(W^1(j)+d(j)-\bar{c}_2\right)^{\gamma_c} > 0 \quad \forall j$.

Define $\qquad Z(q^e(k)) =_{\text{define}} \dfrac{\left[q^e(k)+d(k)\right]}{\left[q^e(k)+d(k)-\bar{c}_2\right]^{\gamma_B}}$. We first consider a lemma.

Lemma 1: Let us maintain $\gamma_B > 1$. Since $d(k) > \bar{c}_2,\ \forall k$,

$$Z'(x) < 0, \text{ where } Z(x) = \frac{x+d(k)}{\left[x+d(k)-\bar{c}_2\right]^{\gamma_B}}.$$

<u>Proof</u>: Clearly $Z(x)$ is differentiable for $x > 0$, and

$$Z'(x) = \frac{\left[x+d(k)-\bar{c}_2\right]^{\gamma_B} - \left[x+d(k)\right]^{\gamma_B}\left[x+d(k)-\bar{c}_2\right]^{\gamma_B-1}}{\left[x+d(k)-\bar{c}_2\right]^{2\gamma_B}}$$

$$= \frac{1-\gamma_B\left[\frac{x+d(k)}{x+d(k)-\bar{c}_2}\right]}{\left[x+d(k)-\bar{c}_2\right]^{\gamma_B}}$$

The denominator is strictly positive and $\frac{x+d(k)}{x+d(k)-\bar{c}_2}>1\ \forall k$.

Thus, since $\gamma_B>1$,

$$\left[\frac{x+d(k)}{x+d(k)-\bar{c}_2}\right]>1$$

and $Z'(x) < 0$.

Continuation of Proof of Proposition 2: As noted in the Lemma, we may write the equilibrium conditions defining the equity price as, $\forall j$,

$q^e(j)=\beta M\theta(j)\sum_{k=1}^{N}\pi_{jk} Z(q^e(k))$, where Z (x) is differentiable with $Z'(x) < 0$ for x > 0. Differentiating the equilibrium condition yields

$$\frac{\partial q^e(j)}{\partial M}=\beta\theta(j)\left[\sum_{k=1}^{N}\pi_{jk}Z(q^e(k))+M\pi_{jj}Z'(q^e(j))\ \frac{\partial q^e(j)}{\partial M}\right].$$

Thus,

$$\frac{\partial q^e(j)}{\partial M}\left[1-\beta\theta(j)M\pi_{jj}Z'(q^e(j))\right]=\beta\theta(j)\sum_{k=1}^{N}\pi_{jk}Z(q^e(k)).$$

Equivalently,

$$\frac{\partial q^e(j)}{\partial M}=\left[\beta\theta(j)\sum_{k=1}^{N}\pi_{jk}Z(q^e(k))\right]\Big/\left[1-\beta\theta(j)M\pi_{jj}Z'(q^e(j))\right].$$

Since both numerator and denominator are strictly positive, $\frac{\partial q^e(j)}{\partial M}>0\ \ \forall j$.

It follows that if $M_2 > M_1$,

$$q^e(j,M_2)>q^e(j,M_1)\ \ \forall j.$$

Proposition 3: Again, consider the case of $b = 0$, and assume $\theta(j) > 0 \ \forall j$. Then if $\hat{q}^e(j)$ are the equilibrium equity prices for the standard consumption-savings problem and $q^e(j)$ are the equilibrium equity prices of the bequest economy of equations (9′) – (11′), then

$$q^e(j) > \hat{q}^e(j) \ \forall_j.$$

Proof: We know that equilibrium equity prices exist for both economies; let them be denoted as indicated. Then, for any state j,

$$\hat{q}^e(j) = \beta M(W^1(j) - \hat{q}^e(j))^\gamma \sum_{k=1}^{N} \frac{\pi_{jk}}{(\hat{q}^e(k) + d(k)^{\gamma-1}}$$

Since $\theta(j) > 0 \ \forall_j$,

$$\hat{q}^e(j) < \beta M(W^1(j) + d(j) - \bar{c}_2)^\gamma \sum_{k=1}^{N} \frac{\pi_{jk}}{(\hat{q}^e(k) + d(k) - \bar{c}_2)^{\gamma-1}}$$

or

$$\frac{\hat{q}^e(j)}{(W^1(j) + d(j) - \bar{c}_2)^\gamma} < \beta M \sum_{k=1}^{N} \frac{\pi_{jk}}{(\hat{q}^e(k) + d(k) - \bar{c}_2)^{\gamma-1}}.$$

Thus, for any j, at the prices $\hat{q}^e(j)$, the marginal utility cost of acquiring one share of the equity security is less than the expected marginal utility benefit in the bequest economy.

In order for equilibrium to be established, all prices must be bid up. Thus

$$q^e(j) > \hat{q}^e(j), \ \forall j.$$

The identical argument can be employed to demonstrate that

$$\hat{q}^{r_f}(j) < q^{r_f}(j), \ \forall j.$$