NBER WORKING PAPER SERIES

IDIOSYNCRATIC PRODUCTION RISK, GROWTH AND THE BUSINESS CYCLE

George-Marios Angeletos
Laurent E. Calvet

Working Paper 9764
http://www.nber.org/papers/w9764

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
June 2003

This manuscript is a thoroughly revised version of an earlier working paper, "Incomplete Markets, Growth and the Business Cycle". We are especially grateful to Daron Acemoglu, Fernando Alvarez, Abhijit Banerjee, Robert Barro and John Campbell for detailed feedback and extensive discussions. We also received helpful comments from P. Aghion, A. Alesina, J. Benhabib, A. Bisin, R. Caballero, C. Carroll, O. Galor, J. Geanakoplos, M. Gertler, J. M. Grandmont, R. King, N. Kiyotaki, P. Krusell, J. Leahy, N. G. Mankiw, M. Mobius, H. Polemarchakis, T. Sargent, K. Shell, R. Townsend, and seminar participants at Boston University, Brown, Harvard, Maryland, MIT, Northwestern, NYU, Rochester, Stony Brook, Yale, the 2000 Conference of the Society of Economic Dynamics, the 2001 Minnesota Summer Macro Workshop, and the 2002 NBER Summer Institute. The views expressed herein are those of the authors and not necessarily those of the National Bureau of Economic Research.

Idiosyncratic Production Risk, Growth and the Business Cycle
George-Marios Angeletos and Laurent E. Calvet
NBER Working Paper No. 9764
June 2003
JEL No. D5, D9, E3, O1

ABSTRACT

We introduce a neoclassical growth economy with idiosyncratic production risk and incomplete markets. Each agent is an entrepreneur operating her own neoclassical technology with her own capital stock. The general equilibrium is characterized in closed form. Idiosyncratic production shocks introduce a risk premium on private equity and reduce the demand for investment. The steady state is characterized by a lower capital stock due to entrepreneurial risk and a lower interest rate due to precautionary savings as compared to complete markets. The private equity premium is endogenously countercyclical: the anticipation of low savings and high interest rates in the future feed back to high risk premia and low investment in the present. Countercyclicality in risk taking slows down convergence to the steady state and amplifies the magnitude and persistence of the business cycle. These results, which contrast sharply with those obtained in Bewley models, highlight the macroeconomic significance of missing markets in production and investment risk.

George-Marios Angeletos
Department of Economics
MIT
50 Memorial Drive, E21-251
Cambridge, MA 02142
and NBER
angelet@mit.edu

Laurent E. Calvet
Department of Economics
Harvard University
Littauer Center
Cambridge, MA 02138
and NBER
laurent_calvet@harvard.edu

1. Introduction

Undiversifiable entrepreneurial and investment risks are paramount not only in the developing world, but also in the most advanced economies. In a recent study of US private equity, Moskowitz and Vissing-Jørgensen (2002) document that entrepreneurs and private investors face a "dramatic lack of diversification" and an extreme dispersion in returns.[1] The survival rate of private firms is only 34 percent over the first 10 years, and returns on investment vary widely among surviving firms. These undiversifiable risks are potentially important for macroeconomic performance because entrepreneurs and firm-owners control a large fraction of savings and investment. For instance in the United States, private companies account for about half of production, employment, and corporate equity.

Idiosyncratic production risks are not limited to owners of private firms. In publicly traded corporations, the tenure and compensation of executives are closely tied to the outcome of the investment decisions they make on behalf of shareholders. Similarly, labor income often includes returns to education, learning-by-doing, or some form of human or intangible capital. Idiosyncratic risks thus influence a large class of investment decisions and potentially have substantial aggregate effects.

The standard neoclassical growth model (Cass, 1965; Koopmans, 1965; Brock and Mirman, 1972) assumes complete markets, implying that private agents can fully diversify any idiosyncratic risk in their labor and production choices. Following Bewley (1977), a large literature has investigated the macroeconomic impact of uninsurable shocks in exogenous income but not in investment returns.[2] The precautionary or buffer-stock motive leads agents to *over*accumulate capital in the steady state as compared to complete markets (Aiyagari, 1994) and has no quantitatively important effect on business-cycle dynamics (Krusell and Smith, 1998). Bewley models also predict that improvements in borrowing limits or risk sharing reduce capital accumulation and medium-run growth, which contradicts the positive correlation between financial sophistication and economic development documented in a large literature (e.g. King and Levine, 1993; Levine, 1997).

This paper departs from the Bewley paradigm by considering idiosyncratic risk in investment. We introduce a neoclassical economy with decentralized production and incomplete insurance markets, in which the equilibrium dynamics are characterized in

[1] Moskowitz and Vissing-Jørgensen (2002) also observe: "About 75 percent of all private equity is owned by households for whom it constitutes at least half of their total net worth. Furthermore, households with entrepreneurial equity invest on average more than 70 percent of their private holdings in a single private company in which they have an active management interest."

[2] Examples include Aiyagari (1994), Calvet (2001), Huggett (1993, 1997), Krusell and Smith (1998), Ríos-Rull (1996), and Scheinkman and Weiss (1986). See Ríos-Rull (1995) and Ljungqvist and Sargent (2000, ch. 14) for a review.

closed form. Each agent is an entrepreneur operating her own neoclassical technology with her own capital stock. Production is subject to firm-specific uncertainty, which generates idiosyncratic risk in entrepreneurial income.

Missing markets have strikingly different implications than in Bewley-type models. Idiosyncratic production shocks introduce a risk premium on private equity, which reduces the demand for investment at any given interest rate. Uninsurable income risks also encourage the precautionary supply of savings, implying a lower interest rate in the steady state as compared to complete markets. The reduction in investment demand dominates the effect of a lower interest rate and leads to the *under*accumulation of capital for reasonable elasticities of intertemporal substitution.[3] The negative effect of idiosyncratic risk on investment thus contrasts with the *over*accumulation of capital obtained in Bewley-type models from precautionary savings. Our model also predicts that improvements in entrepreneurial risk sharing, induced for instance by financial liberalization or the introduction of new hedging instruments, will have a positive effect on savings and medium run growth.

Undiversifiable production risks generate a novel amplification and propagation mechanism over the business cycle. This result hinges on the property that the risk premium on private capital is endogenously countercyclical. The expectation of a recession in the near future implies a low willingness to take risk in the present, which in turn discourages current investment, amplifies the recession and hinders the recovery. This partially self-fulfilling mechanism increases the persistence and magnitude of the business cycle. Any model in which risk premia increase in anticipation of an economic slowdown is likely to produce similar effects.

In our economy, the countercyclicality of the risk premium originates in the negative effect of future borrowing costs on current risk taking. Figure 1 illustrates this mechanism in an economy hit at date $t = 0$ by an unanticipated negative shock to aggregate wealth. The solid lines represent transmission under complete markets. Agents smooth consumption by reducing current investment, which results in low wealth, low savings and high interest rates in later periods. This is the fundamental propagation mechanism in the RBC paradigm. In the presence of idiosyncratic production risk, the traditional channel is complemented by the endogenous countercyclicality of risk premia, as illustrated by the dashed arrows in the figure. Anticipating high interest rates in the near future, agents become less willing to engage in risky production activities and further reduce investment at date $t = 0$, which amplifies the recession in later periods. Countercyclicality in the risk premium thus amplifies the impact of the shock and slows down convergence to the steady state.

The mechanism identified in this paper originates in the incomplete risk sharing of

[3] Our model thus complements earlier research investigating the impact of *aggregate* production risk on investment and diversification (e.g. Acemoglu and Zilibotti, 1997).

private investment and entrepreneurial income. Our approach thus complements, but also differs from, the literature examining the macroeconomic effects of credit-market imperfections and wealth heterogeneity (e.g. Bernanke and Gertler, 1989, 1990; Banerjee and Newman, 1993; Galor and Zeira, 1993; Kiyotaki and Moore, 1997; Aghion, Banerjee and Piketty, 1999). This earlier research has focused on the effect of wealth and borrowing constraints on the individual *ability* to invest in productive capital. We show that incomplete markets also affect the *willingness* to invest, and has novel implications for capital accumulation and business cycles. To illustrate these points, we derive our results in an economy in which agents face no borrowing constraints and wealth heterogeneity has no impact on aggregate dynamics. In particular, we assume that agents have constant absolute risk aversion (CARA), which renders equilibrium prices and macro aggregates independent of the wealth distribution. This allow us to overcome the "curse of dimensionality" and characterize the general equilibrium in closed form, which, to the best of our knowledge, is new to the incomplete-market growth literature.

The paper is organized as follows. Section 2 introduces the economy and Section 3 analyzes the individual decision problem. In Section 4 we characterize the general equilibrium in closed form. Section 5 analyzes the steady state and Section 6 discusses the propagation mechanism arising from idiosyncratic production risk. All proofs are in the Appendix.

2. A Ramsey Economy with Idiosyncratic Production Risk

Time $t \in \{0, 1, ...\}$ is discrete and infinite. The economy is populated by a continuum of agents, indexed by $j \in [0, 1]$, who are born at $t = 0$ and live forever. Each individual is a producer, or entrepreneur, who operates her own production scheme using her own capital stock. In every period, agents can borrow and lend from each other at the market interest rate but cannot diversify their idiosyncratic risks.

2.1. Technology and Idiosyncratic Risks

In each period t, the gross output of agent j is given by $A_t^j f(k_t^j)$, where k_t^j is her capital stock at the beginning of the period, A_t^j is her random total factor productivity (TFP), and f is a neoclassical production function. The function f is common across households and satisfies $f' > 0$, $f'' < 0$, $\lim_{k \to 0} f'(k) = +\infty$, and $\lim_{k \to +\infty} f'(k) = 0$. The individual controls k_t^j at date $t - 1$, but only observes the idiosyncratic productivity A_t^j at date t. The return on capital investment is thus subject to idiosyncratic uncertainty.

For comparison with production shocks, we find it useful to also introduce endowment risks. The individual receives an exogenous idiosyncratic income e_t^j, which is outside her control and does no effect investment or production opportunities. For instance, e_t^j corresponds to acts of God, or to the wages that the agent receives for renting labor to an unmodeled outside firm.[4] The overall non-financial income of household j in period t is

$$y_t^j = A_t^j f(k_t^j) + e_t^j. \tag{2.1}$$

Note that A_t^j and e_t^j respectively introduce a multiplicative and an additive shock to income. A_t^j captures idiosyncratic *entrepreneurial or capital-income risk*, whereas e_t^j captures *endowment risk*.

We assume for simplicity that idiosyncratic production and endowment risks are Gaussian, mutually independent, and IID across time and individuals:[5]

$$A_t^j \sim \mathcal{N}(1, \sigma_A^2), \qquad e_t^j \sim \mathcal{N}(0, \sigma_e^2).$$

The averages $\mathbb{E}A_t^j = 1$ and $\mathbb{E}e_t^j = 0$ are simple normalizations. The standard deviations σ_A and σ_e parsimoniously parameterize the magnitude of the uninsurable production and endowment shocks. Under complete markets, σ_A and σ_e are both equal to zero. Bewley economies consider idiosyncratic labor-income risk but no idiosyncratic capital-income risk, which here corresponds to $\sigma_e > 0$ but $\sigma_A = 0$. This paper focuses instead on the case $\sigma_A > 0$.

2.2. Financial Markets and Preferences

Agents can buy and sell a riskless asset or short-term bond. One unit of the bond purchased at date t yields $1 + r_t$ units of the good with certainty at date $t + 1$. In equilibrium, the interest rate r_t clears the period-t bond market. We rule out default, borrowing constraints, and any other credit-market imperfections. Without loss of generality, the riskless bond is in zero net supply.[6] We assume that agents can trade no other financial assets, and interpret A_t^j and e_t^j as the undiversifiable components of idiosyncratic risks.[7]

Let c_t^j, i_t^j and θ_t^j denote the consumption, capital investment, and bond purchases of agent j in period t. The individual is subject to the budget constraint

$$c_t^j + i_t^j + \theta_t^j = y_t^j + (1 + r_{t-1})\theta_{t-1}^j, \tag{2.2}$$

[4] Introducing an explicit labor market would slightly complicate the exposition, but would not alter our qualitative results.

[5] Although we assume for tractability that idiosyncratic shocks are serially independent, we will mimic persistence in numerical simulations by adjusting the length of the time period.

[6] Ricardian equivalence holds in our model because agents have infinite horizons and can freely trade the riskless bond. Therefore, as long as public debt is financed by lump-sum taxation, there is no loss of generality in assuming that the riskless bond is in zero net supply.

[7] When agents can also trade risky financial assets, σ_A and σ_e are the standard deviations of the *uninsurable* components of the entrepreneurial and endowment risks. See Angeletos and Calvet (2000) for details. The endogenization of the asset structure is beyond the scope of this paper, and could be pursued following the methodologies of Townsend (1982), Banerjee and Newman (1991), Kocherlakota (1996), and Alvarez and Jermann (2000).

where y_t^j is the non-financial income defined in (2.1). Capital depreciates at a fixed rate $\delta \in [0,1]$, and is determined by the accumulation equation $k_{t+1}^j = (1-\delta)k_t^j + i_t^j$. To simplify notation, we conveniently rewrite the budget set in terms of stock variables. The agent's total *wealth* or *cash-in-hand* at date t is

$$w_t^j \equiv A_t^j f(k_t) + (1-\delta)k_t^j + e_t^j + (1+r_{t-1})\theta_{t-1}^j. \tag{2.3}$$

We then restate the budget constraint (2.2) as

$$c_t^j + k_{t+1}^j + \theta_t^j = w_t^j. \tag{2.4}$$

The model is most tractable when agents have identical expected utility $\mathbb{E}_0 \sum_{t=0}^{+\infty} \beta^t u(c_t)$, where

$$u(c) = -\Psi \exp(-c/\Psi).$$

In numerical simulations, it will be useful to distinguish between intertemporal substitution and risk aversion. For this reason, we assume more generally that agents have identical non-expected utility of the Kreps-Porteus/Epstein-Zin type. The function $u(c) = -\Psi \exp(-c/\Psi)$ characterizes intertemporal smoothing, and $v(c) = -\exp(-\Gamma c)/\Gamma$ specifies preference over risky choices.[8] A high Ψ corresponds to a strong willingness to substitute consumption through time, while a high Γ implies a high degree of risk aversion. When $\Gamma = 1/\Psi$, the Bernoulli utilities u and v are equal and the preference reduces to the expected utility $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$.

2.3. Equilibrium

An *incomplete-market equilibrium* is an interest rate sequence $\{r_t\}_{t=0}^{\infty}$ and a collection of state-contingent plans $(\{c_t^j, k_{t+1}^j, \theta_t^j\}_{t=0}^{\infty})_{j\in[0,1]}$ such that: (*i*) the plan $\{c_t^j, k_{t+1}^j, \theta_t^j\}_{t=0}^{\infty}$ maximizes the utility of each agent j; and (*ii*) the bond market clears in every date and event: $\int_j \theta_t^j = 0$.

Idiosyncratic risks are independent in the population and cancel out in the aggregate. As a result, we will show that there exists an equilibrium in which the aggregate dynamics are deterministic. The next section characterizes optimal individual behavior for an exogenous interest-rate sequence. In Section 4, we aggregate across individuals and characterize the general equilibrium of the economy.

3. Decision Theory

For notational simplicity, we drop in this section the index j from all decision variables. Given a deterministic interest rate sequence $\{r_t\}_{t=0}^{\infty}$, the household chooses a contingent

[8] A stochastic consumption stream $\{c_t\}_{t=0}^{\infty}$ generates a stochastic utility stream $\{x_t\}_{t=0}^{\infty}$ according to the recursion $u(x_t) = u(c_t) + \beta u[\mathbb{CE}_t(x_{t+1})]$, where $\mathbb{CE}_t x_{t+1} \equiv v^{-1}[\mathbb{E}_t v(x_{t+1})]$ is the certainty equivalent of x_{t+1} under v conditional on period t information.

plan $\{c_t, k_{t+1}, \theta_t\}_{t=0}^{\infty}$ that maximizes expected life-time utility subject to (2.4). Since idiosyncratic risks are uncorrelated over time, individual wealth w_t fully characterizes the idiosyncratic state of the household in period t. For expositional simplicity, we temporarily assume that the agent has an expected utility function ($\Gamma = 1/\Psi$). The value function $V_t(w)$ satisfies the Bellman equation:

$$V_t(w_t) = \max_{(c_t, k_{t+1}, \theta_t)} u(c_t) + \beta \mathbb{E}_t V_{t+1}(w_{t+1}),$$

where the maximization is subject to the budget constraint (2.4). Given the CARA-normal specification, an educated guess is to consider an exponential value function and a linear consumption rule:

$$V_t(w) = u(a_t w + b_t), \qquad c_t = \widehat{a}_t w + \widehat{b}_t, \tag{3.1}$$

where $a_t, \widehat{a}_t > 0$ and $b_t, \widehat{b}_t \in \mathbb{R}$ are non-random coefficients to be determined.

By (2.3), individual wealth is Gaussian with conditional mean and variance:

$$\mathbb{E}_t w_{t+1} = f(k_{t+1}) + (1-\delta)k_{t+1} + (1+r_t)\theta_t,$$
$$\mathrm{Var}_t(w_{t+1}) = \sigma_e^2 + f(k_{t+1})^2 \sigma_A^2.$$

The value function has expectation $\mathbb{E}_t V_{t+1}(w_{t+1}) = V_{t+1}[\mathbb{E}_t w_{t+1} - \frac{\Gamma_t}{2}\mathrm{Var}_t(w_{t+1})]$, where $\Gamma_t \equiv \Gamma a_{t+1}$ measures absolute risk aversion in period t with respect to wealth variation in period $t+1$. We henceforth call Γ_t the *effective* degree of risk aversion at date t.

Taking the first-order conditions (FOCs) with respect to k_{t+1} and θ_t, we obtain the key condition for optimal investment:

$$r_t + \delta = f'(k_{t+1})[1 - \Gamma_t f(k_{t+1})\sigma_A^2]. \tag{3.2}$$

In the absence of idiosyncratic production shocks ($\sigma_A = 0$), the agent equates the net marginal product of capital with the interest rate: $r_t = f'(k_{t+1}) - \delta$. This is a familiar result in complete-market or Bewley-type economies. On the other hand when $\sigma_A > 0$, the return on investment is adjusted for risk. The expected marginal product of capital differs from the risk-free rate by $\Gamma_t f(k_{t+1}) f'(k_{t+1})\sigma_A^2$, which measures the *risk premium on private equity.* Note that the premium is proportional to the effective risk aversion Γ_t and the marginal contribution of investment to wealth risk.

The optimal level of savings is determined by the Euler equation $u'(c_{t+1}) = \beta R_t$ $\mathbb{E}_t u'(c_{t+1})$, or equivalently:

$$\mathbb{E}_t c_{t+1} - c_t = \Psi \ln[\beta(1+r_t)] + \frac{\Gamma}{2}\mathrm{Var}_t(c_{t+1}), \tag{3.3}$$

where $\mathrm{Var}_t(c_{t+1}) = (\widehat{a}_{t+1})^2 \left[\sigma_e^2 + f(k_{t+1})^2 \sigma_A^2\right]$. The precautionary motive implies that expected consumption growth increases with the variance of future consumption (Leland, 1968; Sandmo, 1970; Caballero, 1990; Kimball, 1990).

The envelope and Euler conditions imply after simple manipulation that $\widehat{a}_t = a_t$ and $1/a_t = 1 + 1/[a_{t+1}(1 + r_t)]$. By forward iteration, the effective risk aversion $\Gamma_t \equiv \Gamma a_{t+1}$ satisfies

$$\Gamma_t = \frac{\Gamma}{1 + \sum_{s=1}^{+\infty} \frac{1}{(1+r_{t+1})...(1+r_{t+s})}}. \tag{3.4}$$

Effective risk aversion is thus inversely proportional to the price of a perpetuity delivering one unit of the consumption good in every period $s \geq t + 1$. In particular, we note that *effective* risk aversion is an increasing function of *future* interest rates. Higher interest rates in the future raise the cost of smoothing intertemporally any uninsurable adverse consumption shock, which increases effective risk aversion and discourages risk taking in the present.

Proposition 1 (Individual Choice) Given any interest rate path $\{r_t\}_{t=0}^{\infty}$, the demand for investment is given by

$$r_t + \delta = f'(k_{t+1}) \left[1 - \Gamma_t f(k_{t+1}) \sigma_A^2\right]. \tag{3.5}$$

Consumption and savings are characterized by the Euler equation

$$\mathbb{E}_t c_{t+1} - c_t = \Psi \ln[\beta(1 + r_t)] + \Gamma_t^2 [\sigma_e^2 + f(k_{t+1})^2 \sigma_A^2]/(2\Gamma). \tag{3.6}$$

Conditions (3.4) and (3.5) define optimal investment as a function of the idiosyncratic production risk and current and future interest rates: $k_{t+1} = k(\sigma_A, r_t, r_{t+1}, ...)$. An increase in the contemporaneous interest rate raises the cost of capital and reduces the demand for investment. Under incomplete markets, investment demand is independent of endowment risk σ_e but is negatively affected by the production risk σ_A. Moreover when $\sigma_A > 0$, an increase in future interest rates raises the risk premium on private equity and thus reduces investment.

The supply of savings is characterized by the Euler condition (3.6). The sensitivity of the growth rate to consumption risk is governed by the risk aversion Γ, while sensitivity to the interest rate depends on the intertemporal substitution Ψ. Endowment risk unambiguously increases the wealth variance $\text{Var}_t(w_{t+1}) = \sigma_e^2 + f(k_{t+1})^2 \sigma_A^2$. The effect of production shocks, however, seems to be generally ambiguous because the optimal investment k_{t+1} and thus risk exposure fall with σ_A. We check this intuition in the Appendix by showing that an increase in the standard deviation σ_A can *reduce* the variance of wealth and thus precautionary savings.

4. General Equilibrium

We now characterize in closed form the general equilibrium of the economy. Let C_t and K_t denote respectively the population averages of consumption and capital in period

t. Because agents have CARA preferences and face no borrowing constraints, private investment is independent of individual wealth. As a result, the wealth distribution does not affect the aggregate dynamics and the equilibrium path is easily characterized:[9]

Proposition 2 (General Equilibrium) There exists an incomplete-market equilibrium in which the macro path $\{C_t, K_{t+1}, r_t\}_{t=0}^{\infty}$ is deterministic and agents choose identical levels of productive investment. For all $t \geq 0$, the equilibrium path satisfies

$$C_t + K_{t+1} = f(K_t) + (1-\delta)K_t, \tag{4.1}$$

$$r_t + \delta = f'(K_{t+1}) \left[1 - \Gamma_t f(K_{t+1})\sigma_A^2\right], \tag{4.2}$$

$$C_{t+1} - C_t = \Psi \ln[\beta(1+r_t)] + \Gamma_t^2 \left[\sigma_e^2 + f(K_{t+1})^2 \sigma_A^2\right] / (2\Gamma), \tag{4.3}$$

where effective risk aversion is inversely proportional to the contemporaneous price of a perpetuity: $\Gamma_t = \Gamma / [1 + \sum_{s=1}^{+\infty} \frac{1}{(1+r_{t+1})...(1+r_{t+s})}]$.

Condition (4.1) is the resource constraint of the economy; it is equivalent to the sum of individual budget constraints when the bond market clears. Conditions (4.2) and (4.3) characterize the aggregate demand for investment and the aggregate supply of savings.

Under complete markets, equations (4.2) and (4.3) reduce to the familiar conditions $r_t + \delta = f'(K_{t+1})$ and $u'(C_t)/\ u'(C_{t+1}) = \beta(1+r_t)$. In the presence of uninsurable risks, the precautionary motive increases the aggregate consumption growth and the supply of savings. This tends to reduce the risk-free rate and stimulate investment, as is well-known in Bewley models (e.g., Aiyagari, 1994). Idiosyncratic production and capital-income shocks, however, introduce a risk premium on private equity and reduce aggregate investment for any given risk-free rate. When $\sigma_A > 0$, this may lead to both a *lower* capital stock and a lower risk-free rate than under complete markets. Moreover, an anticipated increase in future rates raises the premium on private equity and thereby decreases the aggregate demand for investment. This feedback, which is absent from Bewley-type models, may induce persistence and amplification in the transitional dynamics.

[9] Aggregation is further simplified by the following assumptions: (i) idiosyncratic shocks are serially uncorrelated, and (ii) preferences, technology and the structure of risks are identical across agents. The first hypothesis implies that individual investment and precautionary savings are independent of contemporaneous idiosyncratic shocks; the second makes investment and precautionary savings identical across agents. Both assumptions could be relaxed without altering our results, but at the cost of complicating the solution of the model.

5. Steady State

We now analyze how undiversifiable idiosyncratic production risks affect the capital stock in the medium run.[10] A *steady state* is a fixed point $(C_\infty, K_\infty, r_\infty)$ of the dynamic system (4.1)-(4.3). We easily show

Proposition 3 (Steady State) The consumption level is $C_\infty = f(K_\infty) - \delta K_\infty$. The interest rate and the aggregate capital stock satisfy

$$r_\infty + \delta = f'(K_\infty)\left[1 - \Gamma_\infty f(K_\infty)\sigma_A^2\right], \tag{5.1}$$

$$\ln[\beta(1+r_\infty)] = -\tfrac{\Gamma_\infty^2}{2\Psi\Gamma}\left[\sigma_e^2 + f(K_\infty)^2\sigma_A^2\right], \tag{5.2}$$

where $\Gamma_\infty = \Gamma r_\infty/(1+r_\infty)$. When the steady state is unique,[11] the capital stock K_∞ increases with the endowment risk σ_e, but is ambiguously affected by the production risk σ_A.

The first equation corresponds to the aggregate demand for productive investment, and the second to the aggregate supply of savings. We note that $1 + r_\infty = 1/\beta$ under complete markets ($\sigma_A = \sigma_e = 0$), but $1 + r_\infty < 1/\beta$ in the presence of undiversifiable idiosyncratic risks ($\sigma_A > 0$ or $\sigma_e > 0$). The property that the risk-free rate is below the discount rate under incomplete markets has been proposed as a possible solution to the low risk-free rate puzzle (e.g., Weil, 1992; Huggett, 1993).

Endowment and production risks have very different effects on the steady state. Consider first the case $\sigma_e > 0$ and $\sigma_A = 0$, as in a Bewley economy. A higher σ_e implies a higher consumption risk, increases the precautionary supply of savings, and reduces the interest rate. Since investment is still determined by the equation $r_\infty + \delta = f'(K_\infty)$, the capital stock necessarily increases with σ_e. This is precisely the effect considered by Aiyagari (1994). Consider next the case $\sigma_A > 0$. Production risk affects both the savings supply and the investment demand. As previously, a higher σ_A tends to encourage the precautionary savings and stimulate investment. On the other hand, a higher σ_A also increases the private risk premium and reduces the demand for investment at any level of the interest rate. There is thus a conflict between the savings and the investment effect.

Intuition suggests that the investment channel dominates in two cases. First, when agents have a weak precautionary motive, a higher production risk increases the variance of consumption but has little effect on savings. Second, when real returns have

[10] Consistent with a large literature, we view medium-run growth as the steady state of the economy for a given level of technology. The analysis of long run growth caused by endogenous technological progress is outside the scope of this paper.

[11] The steady state is unique when markets are complete and, by continuity, when σ_A and σ_e are sufficiently small. We checked that uniqueness holds in the numerical simulations of Sections 5 and 6.

a strong impact on long-run savings (the steady-state supply of savings is very elastic with respect to r_∞), an increase in precautionary savings only has a small effect on the equilibrium interest rate. In either case, the new steady state is mainly determined by the reduction in investment demand. Note that these arguments hinge on the sensitivities of savings to consumption risk and interest rates. In our framework, they are both valid when the coefficient Ψ is sufficiently high. The numerical simulations will demonstrate that K_∞ decreases with σ_A unless the elasticity of intertemporal substitution is implausibly low. Therefore, while Aiyagari (1994) suggests that the economy *over*invests under incomplete insurance, we instead conclude that *under*investment is the most likely scenario.

Calibration and Numerical Simulation

We obtained a tractable model by assuming serial uncorrelation in idiosyncratic shocks. Empirically, however, production and investment risks are highly persistent – especially under broad definitions that include education, human-capital formation and R&D. To capture persistence in the simulations, we interpret the length of a time period, T, as the horizon of an investment project or the average life of an idiosyncratic shock. The production function is Cobb-Douglas: $f(K) = K^\alpha$ with $\alpha \in (0, 1)$. The standard deviations of the uninsurable risks are measured as percentages of GDP.[12] We finally choose parameters Γ and Ψ that match a given relative risk aversion γ and a given elasticity of intertemporal substitution ψ at the complete-markets steady state. This allows us to control for the dependence of the local transitional dynamics on the consumption-capital ratio (see Appendix for the details). Overall, a *calibrated economy* is parameterized by $\mathcal{E}^{cal} = (T, \beta, \gamma, \psi, \alpha, \delta, \sigma_A, \sigma_e)$, where T is the length of an idiosyncratic shock, β the yearly discount factor, γ the relative risk aversion, ψ the elasticity of intertemporal substitution, α the income share of capital, δ the yearly depreciation rate, and σ_A and σ_e the idiosyncratic production and endowment risks as percentages of GDP.

We now verify the intuition that idiosyncratic production risks lead to the underaccumulation of capital in the medium run unless the elasticity of intertemporal substitution is very low.

Proposition 4 (Comparative Statics) As we move away from complete markets, an increase in production risk σ_A reduces the interest rate r_∞. The capital stock K_∞ also decreases if and only if $\psi > \underline{\psi}$, where $\underline{\psi} \equiv \frac{1}{2}\left[1 + \frac{(1-\alpha)\beta^T[1-(1-\delta)^T]}{(1-\beta^T)}\right]^{-1}$.

For all parameter values, the lower bound $\underline{\psi}$ is smaller than $1/2$, and is thus substantially lower than the elasticity of intertemporal substitution $\psi \approx 1$ suggested by empirical

[12] For instance, $\sigma_A = 0.25$ and $\sigma_e = 0.50$ imply that the standard deviation of gross output and the standard deviation of the endowment represent 25% and 50% of mean production.

evidence. The most likely scenario therefore is that uninsurable production risks have a negative impact on capital accumulation.

Although accurate estimates of σ_A are not readily available, idiosyncratic risks in production, entrepreneurial, and investment are known to be very substantial. Moskowitz and Vissing-Jørgensen (2002) show that households with entrepreneurial equity invest more than 70% of their private holdings in a single company, which suggests to calibrate σ_A on the returns of a single firm. The empirical evidence points towards high values of σ_A. The survival rate of a new private firm is only 34% after 10 years. The distribution of returns to entrepreneurial activity is extremely wide even conditionally on survival. For example, Moskowitz and Vissing-Jørgensen report that the average and median returns of a single private differ by as much as 121%. They also assess that a *portfolio* of private firms has a standard deviation well above 40%. Since the portfolio diversifies away much of the idiosyncratic risk, we conclude that a value for σ_A above 50% and perhaps close to 100% is a plausible guess.

In Figure 2, we illustrate the impact of production risk on the steady-state capital stock and interest rate for plausible parameter values.[13] The length of an idiosyncratic shock is 5 years, the discount and depreciation rates are 5% per year, relative risk aversion is 4, and the elasticity of intertemporal substitution is 1. Panel A corresponds to a broad definition of capital ($\alpha = 70\%$) and Panel B corresponds to a narrow definition ($\alpha = 35\%$). The effect of σ_A on K_∞ is very strong. At $\sigma_A = 100\%$, the capital stock is 30% of its complete-market value if $\alpha = 0.35$ (Panel B), and only 15% if $\alpha = 0.70$ (Panel A). Hence, in contrast to Aiyagari (1994), incomplete markets imply both a low risk free rate and low aggregate savings. Furthermore, the risk-free rate can be a very poor proxy for the marginal productivity of capital. In Panel B of Figure 2, for example, the marginal productivity of capital is 18% per year at $\sigma_A = 100\%$, as compared to a yearly interest rate of 4%.

The simulations also provide useful insights on the interaction between endowment and production risks. In Figure 2, the dashed lines correspond to $\sigma_e = 50\%$, whereas the solid ones to $\sigma_e = 0$. The value $\sigma_e = 50\%$ probably represents an upper bound for labor income risk (e.g. Aiyagari, 1994), but is useful for comparison with σ_A. The steady state becomes less sensitive to σ_e as σ_A increases. This is because when σ_A is large, individuals already hold a buffer stock that can be used to self-insure against either production or endowment risks. We also note that when σ_e and σ_A are equal, the impact of production risk on capital stock is prevalent. In Figure 2, for example, K_∞ is well below its complete-markets value for $\sigma_A = \sigma_e = 50\%$. Since 50% is a lower bound

[13] The numerical results of Figures 2 and 3 are not very sensitive to changes in ψ and γ. A higher ψ weakens the effect of σ_A on r_∞ and strengthens its impact on K_∞, because it increases the interest elasticity of savings. On the other hand, γ tends to have a small ambiguous effect, since a higher γ increases both the precautionary motive and the risk premium on investment.

for private production risk and an upper bound for labor-income risk, we conclude that under-accumulation of capital is the most likely scenario.

6. Transitional Dynamics and Propagation

Idiosyncratic production risks have novel implications for the business cycle and medium-run growth. We illustrate these effects in Figure 1 for an economy hit at date $t = 0$ by an unanticipated negative shock to aggregate wealth. Consider the impact of the shock under complete markets. Consumption and investment fall, interest rates rise, and the economy converges monotonically back to the steady state, as illustrated by the solid lines in the figure. The transition takes some time under complete markets only because agents seek to smooth consumption.

When production is subject to uninsurable idiosyncratic shocks, the standard transitional dynamics are complemented by the countercyclicality of risk-taking. The anticipation of high interest rates at $t = 1$ leads to an increase in the risk premium at $t = 0$ and hence a further reduction of investment at $t = 0$. Similarly, the anticipation of higher interest rates in any period $\tau > 1$ feeds back to even higher risk premia and even lower investment in earlier periods. The feedback from interest rates to risk premia and investment is illustrated by the dashed lines in Figure 1.

The combination of the consumption-smoothing channel (solid lines) and the risk-premium channel (dashed lines) generates a dynamic multiplier. The fall in current investment further reduces aggregate income and increases interest rates in the future, which in turn amplifies the fall in current investment, and so on. The endogenous countercyclicality of the private risk premium thus amplifies the impact of the exogenous shock on investment and output and slows down convergence to the steady state as compared to complete markets.

We make several remarks on this mechanism. First, it hinges on the effect of idiosyncratic uncertainty on production and investment, and is thus not present in Bewley-type economies (e.g., Aiyagari, 1994; Krusell and Smith, 1998) that only consider endowment risks. This explains why this earlier research did not find the effect identified in this paper.

Second, incomplete markets generate a particular form of pecuniary externality. In the presence of uninsurable production shocks, risk-taking depends on future interest rates. When private agents decide how much to save and invest in a future period, they do not internalize the impact of their choices on future interest rates and therefore on current investment.

Third, the pecuniary externality generates a dynamic macroeconomic complementarity. Because interest rates are endogenous and influence risk-taking, the anticipation of low aggregate investment in the future feeds back into low aggregate investment in the present. Low levels of investment can thus be self-sustaining for long periods of time. This dynamic macroeconomic complementarity is the basis of amplification and persistence over the business cycle. The reader may be familiar with a standard example of macroeconomic complementarity – the production externalities considered by Bryant (1983) and Benhabib and Farmer (1994).[14] In this literature, an individual's marginal productivity is assumed to increase in the aggregate stock of capital, which generates a complementarity in investment. Note that this type of production externality is exogenous and *ad hoc*. In contrast, the complementarity in our model is a genuine general-equilibrium implication of a market imperfection.

Fourth, the macroeconomic complementarity between future and current investment may generate multiple steady states and endogenous cycles. An earlier version of the paper (Angeletos and Calvet, 2000) establishes that our model can generate self-fulfilling equilibria, such as poverty traps and endogenous fluctuations. Our work thus contributes to the literature on equilibrium indeterminacy (e.g., Benhabib and Farmer, 1994). Such phenomena, however, only occur when idiosyncratic risks are very large or agents are very impatient. Thus, they do not arise for the plausible parameter values considered in the simulations. For this reason, this paper focuses on the effects of incomplete markets in economies with a unique stable steady state.

Finally, the propagation mechanism is likely to extend to any framework where the private risk premium on investment is higher at the onset of an economic downturn. This premise is obviously much more general than our specific model. Consider for instance an economy with both incomplete insurance and credit-market imperfections. We expect that a risk-averse agent will take less risk and thus invest less in the present when he anticipates a higher borrowing rate, a higher probability to use credit, or a higher probability to face a binding borrowing constraint at a *future* date. As long as credit conditions worsen during recessions, the anticipation of a downturn in the near future will raise risk premia and discourage investment in the present, thus making the recession partially self-fulfilling. Note that this effect occurs whether or not *current* investment is financially constrained.

Local Dynamics and Numerical Simulation

The local dynamics around the steady state can be approximated by $(K_{t+1} - K_\infty) = \lambda(K_t - K_\infty)$, where λ is the stable eigenvalue of the Jacobian of the dynamic system (4.1)-(4.3) evaluated at the steady state. The quantity $1-\lambda$ gives the *rate of convergence* to the steady state. Incomplete insurance slows down convergence if $1 - \lambda$ decreases with σ_A.

In Figure 3 we illustrate the impact of idiosyncratic production risk on the convergence rate for the same numerical example as in Figure 2. We also report the half-life of

[14] Cooper (1999) provides an overview of macroeconomic complementarities.

an aggregate wealth shock.[15] The convergence rate decreases rapidly with σ_A. With a narrow definition of capital ($\alpha = 0.35$, Panel B), the half-life of a shock almost doubles as σ_A increases from 0 to 100%. The effect is even stronger when incompleteness affects both physical and human capital ($\alpha = 0.7$, Panel A).[16]

These results suggest that production risk could generate additional persistence over the business cycle in standard RBC models with aggregate uncertainty (e.g. Kydland and Prescott, 1982). Furthermore, the magnitude of uninsurable productivity shocks appears as a potential determinant of both the steady state and conditional convergence. Cross-country variation in the degree of risk sharing may thus help explain the large diversity of productivity levels and growth rates around the world (e.g. Barro, 1997; Jones, 1997).

7. Concluding Remarks

This paper examines a standard neoclassical growth economy with heterogeneous agents, decentralized production, and uninsurable production and endowment risks. Under a CARA-normal specification for preferences and risks, we obtain closed-form solutions for individual choices and aggregate dynamics. Uninsurable production shocks introduce a risk premium on private equity and reduce the aggregate demand for investment. As a result, the steady-state capital stock tends to be lower under incomplete markets, despite the low risk-free rate induced by the precautionary motive. Moreover, the endogenous countercyclicality of private risk premia amplify the impact of an exogenous aggregate shock on output and investment, slows down convergence to the steady state, and increases the persistence of the business cycle.

The tractability of our setup easily generalizes to multiple sectors. For instance, each agent can have access to two private technologies: one with high risk and high mean return, and another with low risk and low return. We can also consider several forms of investment, such as physical, human or intangible capital. In such an environment, incomplete risk sharing distorts not only the aggregate levels of savings and investment, but also the cross-sectoral allocation of capital and labor. This additional effect reduces aggregate productivity and implies a further reduction in steady-state capital and income.[17] The anticipation of stringent future credit conditions in the near

[15] The half-life τ of a deviation from the steady state is defined by $\lambda^\tau = 1/2$, or $\tau = -\log_2 \lambda$.

[16] Figure 3 also demonstrates the asymmetry between production and endowment risk. While endowment risk does not introduce a dynamic macroeconomic complementarity, the precautionary motive tends to boost savings above the steady state and thus speed up convergence in an initially poor economy. The convergence rate thus tends to *decrease* with σ_A but *increase* with σ_e. When σ_A and σ_e are equal, the investment effect dominates and the convergence rate is lower than under complete markets.

[17] A multi-sector extension of our paper would thus complement the endogenous-growth literature examining the effect of uninsurable investment risks on the allocation of savings across different invest-

future induces agents to forego the high-risk high-return investment opportunities. The economy thus shifts to safer but less productive technologies during downturns, which further increases the persistence and the amplitude of the business cycle.

Wealth heterogeneity, credit-market imperfections and non-convexities in production have been viewed by many authors as a source of macroeconomic persistence. Although these departures from the neoclassical growth model are not considered here, we find that incomplete risk sharing alone is sufficient to generate underinvestment and introduce a powerful propagation mechanism. The presence of uninsurable production risks reduces the individual's *willingness* to invest. Introducing borrowing constraints would in addition restrict the *ability* to undertake risky projects and increase the sensitivity of risk premia and investment demand to future credit conditions. While CARA preferences rule out wealth effects on risk taking, the private equity premium is likely to be even more countercyclical when risk taking increases with wealth.[18] The impact of credit constraints and wealth on risk premia could thus reinforce the steady-state and business-cycle effects identified in this paper.

In conclusion, we anticipate that the impact of risk premia on investment represents a source of amplification and persistence that is much more general than our specific model. The next step is to construct a full-fledged RBC model with isoelastic preferences, decentralized production, borrowing constraints, and both aggregate and idiosyncratic production uncertainty. This extension would permit a careful quantitative evaluation of the interaction between risk premia and business cycles. It would also help us reassess the impact of wealth heterogeneity on aggregate dynamics. Idiosyncratic entrepreneurial and capital-income risk may also have important implications for asset pricing.[19] We leave these questions open for future research.

---

ment opportunities, such as liquid and illiquid assets (Bencivenga and Smith, 1991), storage and risky production (Greenwood and Jovanovic, 1990; Obstfeld, 1994), or physical and human capital (Krebs, 2002).

[18] Angeletos (2002) considers a similar incomplete-market economy to the one developed in this paper, but assumes constant *relative* risk aversion (CRRA) preferences. The effect of future interest rates on investment is then complemented by the impact of future wealth.

[19] As long as some agents hold both private and public equity, anticipated financial conditions are likely to affect the market price of risk. This intuition is consistent with the empirical evidence in Heaton and Lucas (2000) that proprietary income risk has a strong impact on portfolio holdings.

Appendix

Proof of Proposition 1 (Individual Choice)

In the main text, we solve the decision problem of an agent with expected utility ($\Gamma = 1/\Psi$). We derive here the optimal choice choice of an Epstein-Zin agent. An educated guess is that the agent has linear value function $J_t(w) = a_t w + b_t$, and consumption policy $c_t = \hat{a}_t w_t + \hat{b}_t$. Since w_t is normal, the certainty equivalent of $J_{t+1}(w_{t+1})$ is $J_{t+1}(\mathbb{E}_t w_{t+1} - \frac{\Gamma_t}{2}\mathrm{Var}_t w_{t+1})$. The agent solves in period t the optimization problem:

$$u[J_t(w_t)] = \max_{(c_t^j, k_t^j, \theta_t^j)} u(c_t) + \beta u\left[J_{t+1}\left(\mathbb{E}_t w_{t+1} - \frac{\Gamma_t}{2}\mathrm{Var}_t w_{t+1}\right)\right]. \tag{7.1}$$

We know that $\mathbb{E}_t w_{t+1} = f(k_{t+1}) + (1-\delta)k_{t+1} + (1+r_t)\theta_t$ and $\mathrm{Var}_t w_{t+1} = \sigma_e^2 + f(k_{t+1})^2\sigma_A^2$. It is convenient to consider the function $G(k_{t+1}, \Gamma_t) \equiv f(k_{t+1}) + (1-\delta)k_{t+1} - \Gamma_t[\sigma_e^2 + f(k_{t+1})^2\sigma_A^2]/2$. The agent thus maximizes

$$u(c_t) + \beta u\{\, J_{t+1}[(1+r_t)\theta_t + G(k_{t+1}, \Gamma_t)]\,\}, \tag{7.2}$$

subject to $c_t + k_{t+1} + \theta_t = w_t$. The FOCs with respect to k_{t+1} and θ_t give

$$\begin{aligned} u'(c_t) &= \beta u'(J_{t+1})\, a_{t+1}\, \{1 - \delta + f'(k_{t+1})[1 - \Gamma_t f(k_{t+1})\sigma_A^2]\}, \\ u'(c_t) &= \beta u'(J_{t+1})\, a_{t+1}\, (1 + r_t). \end{aligned}$$

Dividing these equalities yields $1 + r_t = 1 - \delta + f'(k_{t+1})[1 - \Gamma_t f(k_{t+1})\sigma_A^2]$.

We now write the envelope condition: $u'[J_t(w_t)]a_t = u'(c_t)$, or equivalently $c_t = a_t w_t + b_t - \Psi \ln a_t$. We infer that $\widehat{a}_t = a_t$ and $\widehat{b}_t = b_t - \Psi \ln a_t$. We then rewrite the FOC with respect to θ_t as

$$\begin{aligned} u'(c_t) &= \beta a_{t+1}(1 + r_t)u'\left[J_{t+1}\left(\mathbb{E}_t w_{t+1} - \frac{\Gamma_t}{2}\mathrm{Var}_t w_{t+1}\right)\right] \\ &= \beta(1 + r_t)u'\left(a_{t+1}\mathbb{E}_t w_{t+1} - \frac{\Gamma}{2}a_{t+1}^2\mathrm{Var}_t w_{t+1} + b_{t+1} - \Psi \ln a_{t+1}\right). \end{aligned}$$

Since $\widehat{a}_{t+1} = a_{t+1}$ and $\widehat{b}_{t+1} = b_{t+1} - \Psi \ln a_{t+1}$, the FOC reduces to

$$u'(c_t) = \beta(1 + r_t)u'\left[\mathbb{E}_t c_{t+1} - \Gamma \mathrm{Var}_t(c_{t+1})/2\right],$$

which implies Euler condition (3.6).

The budget constraint and the consumption rule imply that $\theta_t = (1 - \widehat{a}_t)w_t - k_{t+1} - \widehat{b}_t$. Since $\widehat{a}_t = a_t$, we infer from the Euler condition (3.6) that

$$\begin{aligned} a_t w_t + b_t &= a_{t+1}(1 - a_t)w_t + \\ &\quad + a_{t+1}\left[G(k_{t+1}, \Gamma_{t+1}) - k_{t+1} - \widehat{b}_t\right] + b_{t+1} - \Psi \ln[\beta a_{t+1}(1 + r_t)/a_t]. \end{aligned}$$

Since this linear relation holds for every w_t, we conclude that $a_t = a_{t+1}R_t(1 - a_t)$ or equivalently $a_t = 1/[1 + (a_{t+1}R_t)^{-1}]$. Iterating forward yields (3.4).

We now turn to the comparative statics. Since $\text{Var}_t(w_{t+1}) = \sigma_e^2 + \sigma_A^2 f(k_{t+1})^2$, we infer that $\partial\text{Var}_t(w_{t+1})/\partial\sigma_e^2 > 0$. On the other hand, $\partial\text{Var}_t(w_{t+1})/\partial\sigma_A^2 = f(k_{t+1})^2 + [2\sigma_A^2 f(k_{t+1})f'(k_{t+1})](\partial k_{t+1}/\partial\sigma_A^2)$ has an ambiguous sign. Consider the special case $f(k) = \sqrt{k}$. The FOC $r_t + \delta = (2\sqrt{k_{t+1}})^{-1}(1 - \Gamma_t\sigma_A^2\sqrt{k_{t+1}})$ implies $k_{t+1} = [\Gamma_t\sigma_A^2 + 2(\delta + r_t)]^{-2}$. We conclude that $\text{Var}_t(w_{t+1}^j) = \sigma_e^2 + \sigma_A^2/[\Gamma_t\sigma_A^2 + 2(\delta + r_t)]^2$ is a single-peaked function of σ_A. ***QED***

Proof of Proposition 2 (General Equilibrium)

We now derive the equations characterizing general equilibrium. First, note that (3.4) implies $a_t^j = a_t$ and $\Gamma_t^j = \Gamma_t$ for all j, t. We infer from the optimality condition (3.5) that $k_{t+1}^j = K_{t+1}$ for all j. Equation (3.5) then reduces to (4.2) and the Euler equation (3.6) can be rewritten as

$$\mathbb{E}_t c_{t+1}^j - c_t^j = \Psi \ln(\beta R_t) + \frac{\Gamma a_{t+1}^2}{2}[\sigma_e^2 + f(K_{t+1})\sigma_A^2].$$

We aggregate these equalities across agents and infer (4.3). Finally, the aggregation of budget constraints yields (4.1). Existence and local determinacy are examined in Angeletos and Calvet (2000). ***QED***

Proof of Proposition 3 (Steady State)

A. Existence. The steady state is defined by the system (5.1) – (5.2). The second equation implies $r_\infty \leq \beta^{-1} - 1$. The transversality condition imposes that $r_\infty > 0$ and $\Gamma_\infty > 0$. The interest rate r_∞ therefore belongs to the intervals $(0, \beta^{-1} - 1]$. Since $r_\infty > 0$, the first equation implies $f'(K_\infty) > \delta$, or equivalently $K_\infty < \widehat{K} \equiv (f')^{-1}(\delta)$. The capital stock K_∞ is thus contained in the interval $[0, \widehat{K})$.

Each steady-state equation implicitly defines the interest rate as a function of the capital stock. Consider for instance equation (5.2). It is useful to define the functions $X : (0, \beta^{-1} - 1] \to [0, +\infty)$, $X(r) \equiv \frac{2\Psi}{\Gamma}(1 + \frac{1}{r})^2 \ln[\frac{1}{\beta(1+r)}]$, and $V : [0, \widehat{K}) \to [\sigma_e^2, \sigma_e^2 + f(\widehat{K})^2\sigma_A^2)$, $V(K) \equiv \sigma_e^2 + f(K)^2\sigma_A^2$. We observe that X is decreasing in r and V is increasing in K. The steady state equation (5.2) is equivalent to $X(r) = V(K)$. For each $K \in [0, \widehat{K})$, the equation $X(r) = V(K)$ has a unique solution, $r_2(K) \equiv X^{-1}[V(K)]$, which maps $[0, +\infty)$ onto $(0, X^{-1}(\sigma_e^2)] \subseteq (0, \beta^{-1} - 1]$. Similarly, the steady state equation (5.1) implicitly defines a decreasing function $r_1(K)$, which maps $(0, \widehat{K}]$ onto $[0, +\infty)$. The steady state K_∞ is given by the intersection of r_1 and r_2.

Consider the function $\Delta(K) \equiv r_2(K) - r_1(K)$. When $K \to 0$, we know that $r_1(K) \to +\infty$ and $r_2(K)$ is bounded, implying $\Delta(K) \to -\infty$. Since $\Delta(\widehat{K}) = r_2(\widehat{K}) > 0$, there exists at least one steady state for any (σ_A, σ_e). Under complete markets, the steady state is unique since the function r_2 is constant and r_1 is decreasing. By continuity, the steady state is also unique when σ_A and σ_e are sufficiently small.

B. Comparative Statics. Consider the functions r_1 and r_2 defined in the proof of Theorem 2. Observe that $r_1(K)$ and $r_2(K)$ are both decreasing. We know that $|r_1'(K_\infty)| > |r_2'(K_\infty)|$ when the steady state is unique. An increase in σ_e leaves the function $r_1(K)$ unchanged and pushes down the function $r_2(K)$. The steady state is therefore characterized by a lower interest rate and a higher capital stock. An increase in σ_A reduces both $r_1(K)$ and $r_2(K)$, reflecting the fact that σ_A enters in both the investment demand and the savings supply. σ_A can therefore have ambiguous effects, as verified in simulations. ***QED***

Calibrated Economies

We now examine in detail the calibration of Γ and Ψ, which allows the comparison between our CARA economy and the standard isoelastic setup used in RBC models. Relative risk aversion at the steady-state consumption level is ΓC_∞. We restrict the incomplete-market economy so that ΓC_∞ remains invariant at a fixed level γ.

We next consider Ψ. The elasticity of intertemporal substitution (EIS) is equal to Ψ/C_∞ at the steady state consumption level. Similar to the calibration of risk aversion, we could restrict Ψ/C_∞ to remain constant at a fixed level ψ. In Angeletos and Calvet (2000), we adopted this method and the additional restriction $\Psi = 1/\Gamma$ (expected utility). We found that typically idiosyncratic production risks strongly reduce the convergence rate to the steady state, confirming the predictions contained in Section 4.

In this paper, however, we propose a more elaborate calibration method that stems from the following observation. Consider a *complete-market* Ramsey economy with intertemporal utility $\sum_{t=0}^{+\infty} \beta^t u(c_t)$, where u is a smooth strictly concave function. Gross output is $\Phi(K) = f(K) + (1-\delta)K$. The local dynamics around the steady state are approximated by $(K_{t+1} - K_\infty) \approx \lambda(K_t - K_\infty)$, where λ is the stable eigenvalue of the linearized system. It is easy to show that[20]

$$\lambda = \frac{1}{2}\left\{1 + \beta(\beta^{-1} - 1 + \delta)M_\infty + \frac{1}{\beta} - \sqrt{[1 + \beta(\beta^{-1} - 1 + \delta)M_\infty]^2 - \frac{4}{\beta}}\right\},$$

where M_∞ quantifies the relative curvatures of the production and utility functions:

$$M_\infty = \frac{f''(K_\infty)/f'(K_\infty)}{u''(C_\infty)/u'(C_\infty)}.$$

The eigenvalue λ is thus fully determined by (β, δ) and M_∞. With a Cobb-Douglas production $f(K) = K^\alpha$ and a CARA utility $u(C) = \Psi \exp(-C/\Psi)$, the ratio M_∞

[20] Cass (1965) derives a similar result for continuous time economies.

reduces to $(1-\alpha)\Psi/K_\infty$. Under complete markets, the convergence rate $g = 1-\lambda$ is thus fully determined by the parameters (α, β, δ) and the ratio Ψ/K_∞.

When we move from complete to incomplete markets, two phenomena affect the eigenvalue λ. First, the transitional dynamics are affected by new terms in (4.2) and (4.3): the risk premium in the investment-demand equation and the consumption variance in the Euler equation. Second, changes in the steady state affect the relative curvature Ψ/K_∞ and thereby the eigenvalue λ. This second effect reflects the shift of the steady-state to different points on the production and utility functions. It is thus purely mechanical and sheds little light on the impact of incomplete risk sharing on the transitional dynamics. For this reason, we prefer to neutralize this effect by keeping Ψ/K_∞ (or equivalently M_∞) invariant at a prespecified level as we vary σ_A and σ_e.[21] This in turn requires an appropriate calibration of Ψ/K_∞. When markets are complete, we impose that the intertemporal elasticity Ψ/C_∞ be equal to a given coefficient ψ. This allows us to choose a value of ψ that matches empirical estimates of the EIS. A simple calculation also implies $C_\infty/K_\infty = Y_\infty/K_\infty - \delta$, where $Y_\infty/K_\infty = (\beta^{-1}-1+\delta)/\alpha$. Therefore, $\Psi/K_\infty = \psi C_\infty/K_\infty = \psi[(\beta^{-1}-1+\delta)/\alpha - \delta]$ under complete markets. When markets are incomplete, we keep Ψ/K_∞ invariant at its complete-market level. Our calibration thus disentangles the dynamic effect of financial incompleteness from purely mechanical changes in the relative curvatures of the production and utility functions.[22]

To summarize, a *calibrated economy* $\mathcal{E}^{cal} = (T, \beta, \gamma, \psi, \alpha, \delta, \sigma_A, \sigma_e)$ is thus an incomplete market economy $\mathcal{E} = (\beta', \Gamma, \Psi, f, \delta', \sigma'_A, \sigma'_e)$ such that $\beta' = \beta^T$, $1-\delta' = (1-\delta)^T$, $\Gamma C_\infty = \gamma$, $\Psi/K_\infty = \psi[(\beta'^{-1} - 1 + \delta')/\alpha - \delta']$, $f(k) = k^\alpha$, $\sigma'_A = \sigma_A$, and $\sigma'_e = \sigma_e f(K_\infty)$.

Proof of Proposition 4 (Calibrated Steady State)

We consider $\mathcal{E}^{cal} = (T = 1, \beta, \gamma, \psi, \alpha, \delta, \sigma_A, \sigma_e)$, and without loss of generality derive the proof in the case $T = 1$. Let $q_\infty = f(K_\infty)/K_\infty = K_\infty^{\alpha-1}$ denote the output-capital ratio in the steady state. The marginal productivity can then be rewritten as $f'(K_\infty) = \alpha q_\infty$, and the consumption-capital ratio as $C_\infty/K_\infty = q_\infty - \delta$. When markets are complete, the interest rate is $r^* = \beta^{-1} - 1$ and the output-capital ratio is $q^* = (\beta^{-1} - 1 + \delta)/\alpha$. For any economy, the calibration imposes that $\Gamma C_\infty = \gamma$ and

[21] The alternative calibration method, which keeps constant the EIS Ψ/C_∞ at ψ but lets Ψ/K_∞ vary, also implies a very substantial increase in persistence when σ_A increases from zero. But, because K_∞ typically decreases with σ_A, the change in Ψ/K_∞ tends to reduce persistence. For large production risks, the convergence rate $g = 1-\lambda$ is then slightly non-monotonic in σ_A in some simulations, but remains far below the complete market value.

[22] Our calibration method also has the following alternative interpretation. Instead of adjusting the EIS around the incomplete-markets steady state, we can set it at a predetermined level $\Psi/C_\infty = \psi$, but assume that the production function is exponential rather than Cobb-Douglas: $f(K) = 1 - \exp(-\phi K)$. We calibrate the coefficient ϕ by setting the income share of capital equal to α in the complete-market steady state. This specification exactly the same calibrated steady state and convergence rate.

$\Psi/K_\infty = \psi(q^* - \delta)$. The steady state system (5.1) – (5.2) can then be rewritten:

$$r_\infty + \delta = f'(K_\infty)(1 - \gamma x_\infty \sigma_A^2), \;\; \ln[\beta(1 + r_\infty)] = -\frac{\gamma x_\infty^2}{\psi z_\infty}(\sigma_A^2 + \sigma_e^2), \qquad (7.3)$$

where $x_\infty = \frac{q_\infty}{q_\infty - \delta}\frac{r_\infty}{1+r_\infty}$ and $z_\infty = 2\frac{q^* - \delta}{q_\infty - \delta}$. When $\sigma_A = \sigma_e = 0$ (complete markets), we know that $x_\infty = (1 - \beta)\frac{q^*}{q^* - \delta}$ and $z_\infty = 2$. When σ_A^2 is positive but close to 0, the first-order variations in r_∞ and K_∞ are obtained by differentiating the system (7.3). The second equation implies $dr_\infty = -\frac{\gamma x_\infty^2}{2\psi\beta} d(\sigma_A^2)$. We then infer from the first equation that $f''(K_\infty)dK_\infty = dr_\infty + \gamma x_\infty f'(K_\infty)d(\sigma_A^2)$ or equivalently $f''(K_\infty)dK_\infty = \gamma x_\infty f'(K_\infty)(1 - \underline{\psi}/\psi)d(\sigma_A^2)$, where

$$\underline{\psi} = \frac{1 - \beta}{2[1 - \beta + \beta\delta(1 - \alpha)]}.$$

It follows that $dK_\infty/d(\sigma_A^2) < 0$ if and only if $\psi > \underline{\psi}$. ***QED***

References

[1] Acemoglu, D., and Zilibotti, F. (1997), "Was Prometheus Unbound by Chance? Risk, Diversification, and Growth," *Journal of Political Economy* 105, 709-751.

[2] Aghion, P., Banerjee, A., and Piketty, T. (1999), "Dualism and Macroeconomic Activity," *Quarterly Journal of Economics* 114, 1359-1397.

[3] Aiyagari, S. R. (1994), "Uninsured Idiosyncratic Risk and Aggregate Saving," *Quarterly Journal of Economics* 109, 659-684.

[4] Alvarez, F., and Jermann, U. J. (2000), "Efficiency, Equilibrium and Asset Pricing with Risk of Default," *Econometrica* 68, 775-798.

[5] Angeletos, G. M., and Calvet, L. E. (2000), "Incomplete Markets, Growth, and the Business Cycle," MIT Working Paper #00-33 and Harvard Working Paper #1910 (Available at the SSRN website www.ssrn.com.)

[6] Banerjee, A., and Newman, A. (1991), "Risk Bearing and the Theory of Income Distribution," *Review of Economic Studies* 58, 211-235.

[7] Banerjee, A., and Newman, A. (1993), "Occupational Choice and the Process of Development," *Journal of Political Economy* 101, 274-298.

[8] Barro, R. (1997), *Determinants of Economic Growth*, Cambridge, Massachusetts: MIT Press.

[9] Bencivenga, A., and Smith, B. (1991), "Financial Intermediation and Endogenous Growth," *Review of Economic Studies*.

[10] Benhabib, J., and Farmer, R. (1994), "Indeterminacy and Increasing Returns," *Journal of Economic Theory* 63, 19-41.

[11] Bernanke, B., and Gertler, M. (1989), "Agency Costs, Net Worth, and Business Fluctuations," *American Economic Review* 79, 14-31.

[12] Bernanke, B., and Gertler, M. (1990), "Financial Fragility and Economic Performance," *Quarterly Journal of Economics* 105, 87-114.

[13] Bewley, T. (1977), "The Permanent Income Hypothesis: A Theoretical Formulation", *Journal of Economic Theory* 16, 252-292.

[14] Brock, W., and Mirman, L. (1972), Optimal Economic Growth and Uncertainty: The Discounted Case, *Journal of Economic Theory* 4, 497-513.

[15] Bryant, J. (1983), "A Simple Rational Expectation Keynes-Type Model," *Quarterly Journal of Economics* 97, 525-529.

[16] Caballero, R. (1990), "Consumption Puzzles and Precautionary Saving," *Journal of Monetary Economics* 25, 113-136.

[17] Calvet, L. E. (2001), "Incomplete Markets and Volatility," *Journal of Economic Theory* 98, 295-338.

[18] Campbell, J. (1999), "Asset Prices, Consumption and the Business Cycle", Chapter 19 in *Handbook of Macroeconomics* Vol. 1, J. Taylor and M. Woodford eds., Amsterdam: North Holland.

[19] Cass, D. (1965), Optimum Growth in an Aggregative Model of Capital Accumulation, *Review of Economic Studies* 32, 233-240.

[20] Cooper, R. (1999), *Coordination Games Complementarities and Macroeconomics*, Cambridge, UK: Cambridge University Press.

[21] Galor, O., and Zeira, J. (1993), "Income Distribution and Macroeconomics," *Review of Economic Studies* 60, 35-52.

[22] Greenwood, J., and Jovanovic, B. (1990), "Financial Development, Growth, and the Distribution of Income," *Journal of Political Economy* 98, 219-240.

[23] Heaton, J., and Lucas, D. J. (2000), "Portfolio Choice and Asset Prices The Importance of Entrepreneurial Risk," *Journal of Finance* 105, 1163-1198.

[24] Huggett, M. (1993), "The Risk Free Rate in Heterogeneous Agent, Incomplete-Insurance Economies," *Journal of Economic Dynamics and Control* 17, 953-969.

[25] Huggett, M. (1997), "The One-Sector Growth Model With Idiosyncratic Shocks," *Journal of Monetary Economics* 39, 385-403.

[26] Jones, C. I. (1997), "On the Evolution of the World Income Distribution", *Journal of Economic Perspectives* 11, 19-36.

[27] Kimball, M. S. (1990), "Precautionary Saving in the Small and in the Large," *Econometrica* 58, 53-73.

[28] King, R.G., and Levine, R., (1993), "Finance, Entrepreneurship, and Growth: Theory and Evidence," *Journal of Monetary Economics* 32, 513-542.

[29] Kiyotaki, N., and Moore, J. (1997), "Credit Cycles," *Journal of Political Economy* 105, 211-248.

[30] Kocherlakota, N. R. (1996), "Implications of Efficient Risk Sharing without Commitment," *Review of Economic Studies* 63, 595-609.

[31] Koopmans, T. C. (1965), On the Concept of Optimal Economic Growth, in *The Economic Approach to Development Planning*, Amsterdam: North-Holland.

[32] Krebs, T. (2002), "Human Capital Risk and Economic Growth," Working Paper, Brown University.

[33] Krusell, P., and Smith, A. (1998), "Income and Wealth Heterogeneity in the Macroeconomy," *Journal of Political Economy* 106, 867-896.

[34] Kydland, F., and Prescott, E. (1982), "Time to Build and Aggregate Fluctuations", *Econometrica* 50, 1345-1370.

[35] Leland, H. (1968), "Saving and Uncertainty The Precautionary Demand for Saving," *Quarterly Journal of Economics* 82, 465-473.

[36] Levine, R. (1997), "Financial Development and Economic Growth: Views and Agenda," *Journal of Economic Literature* 35, 688-726.

[37] Ljungqvist, L., and Sargent, T. J. (2000), *Recursive Macroeconomic Theory*, Cambridge, Mass.: MIT Press.

[38] Moskowitz, T., and Vissing-Jørgensen, A. (2002), "The Returns to Entrepreneurial Investment: A Private Equity Premium Puzzle?," *American Economic Review* 92, 745-778.

[39] Obstfeld, M. (1994), "Risk-Taking, Global Diversification, and Growth," *American Economic Review* 84, 1310-1329.

[40] Ríos-Rull, V. (1995), "Models with heterogenous agents," in Cooley ed. (1995).

[41] Ríos-Rull, V. (1996), "Life-Cycle Economies and Aggregate Fluctuations," *Review of Economic Studies* 63, 465-490.

[42] Sandmo, A. (1970), "The Effect of Uncertainty on Saving Decisions," *Review of Economic Studies* 37, 353-360.

[43] Scheinkman, J., and Weiss, L. (1986), "Borrowing Constraints and Aggregate Economic Activity", *Econometrica* 54, 23-45.

[44] Townsend, R. M. (1982), "Optimal Multiperiod Contracts and the Gain from Enduring Relationships Under Private Information," *Journal of Political Economy* 90, 1166-1186.

[45] Weil, P. (1992), "Equilibrium Asset Prices in Economies with Undiversifiable Labor Income Risk," *Journal of Economic Dynamics and Control* 16, 769-790.

Figure 1. Amplification and Propagation Mechanism

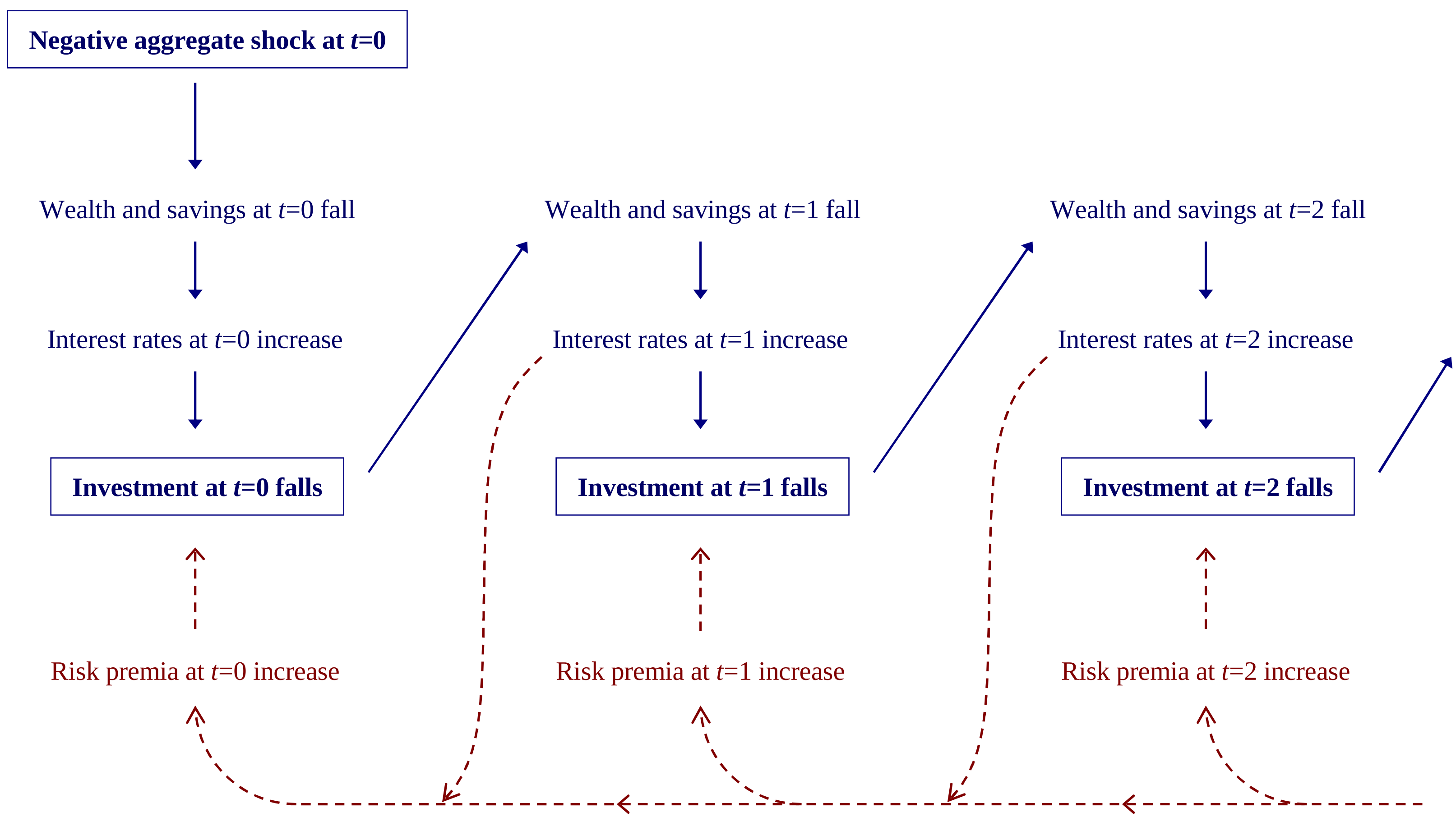

Figure 2.A (α = .70)

Capital Stock *Interest Rate and MPK*

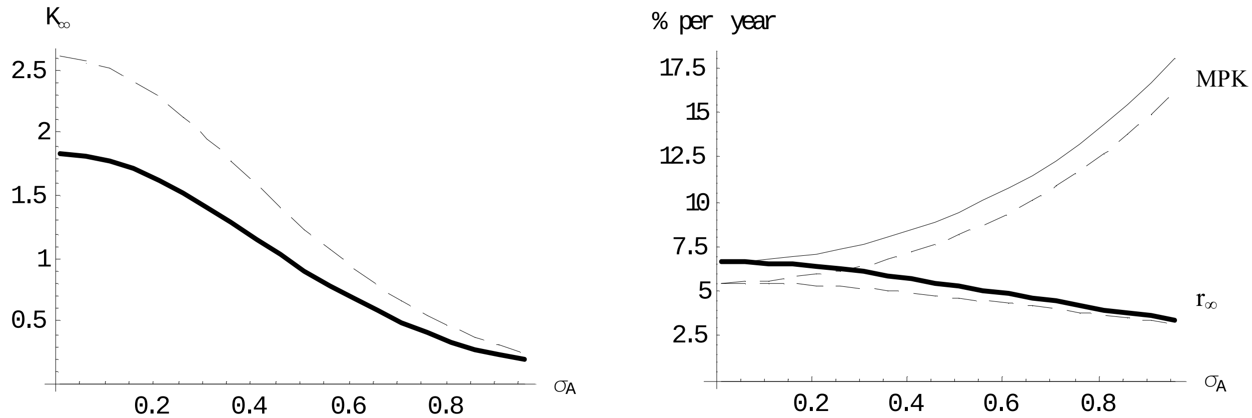

Figure 2.B (α = .35)

Capital Stock *Interest Rate and MPK*

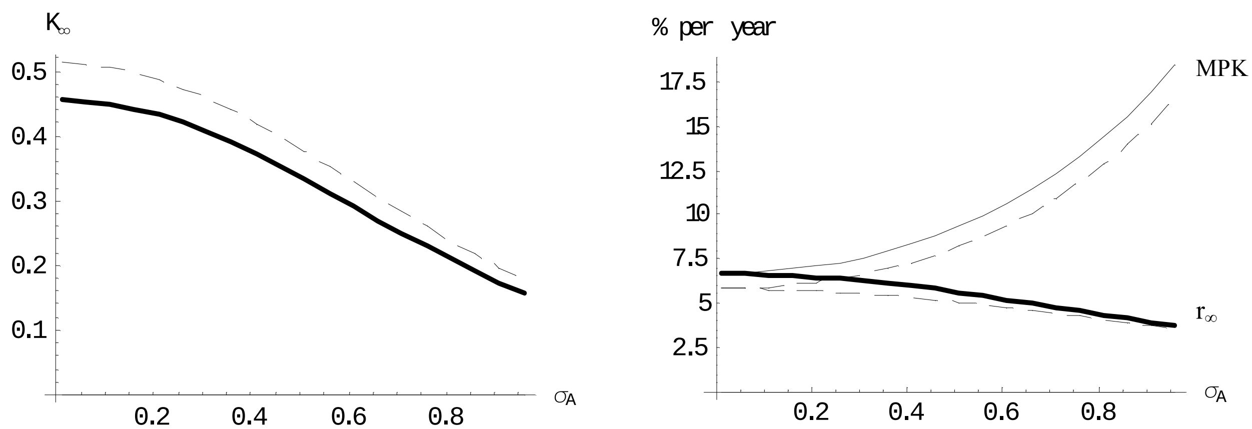

FIGURE 2. We present a numerical simulation of the steady state. The time period (the life of an idiosyncratic production shock) is five years, the discount and depreciation rates are 5% per year, the degree of relative risk aversion is 4, and the elasticity of intertemporal substitution is 1. The income share of capital is 70% in Panel A and 35% in Panel B. The solid lines correspond to $\sigma_e = 0$ (no idiosyncratic endowment risk) and the dashed ones to σ_e = 50% of steady-state GDP. We plot the steady-state levels of the capital stock, the interest rate, and the marginal product of capital (MPK), as idiosyncratic production risk σ_A varies between zero and 100% of steady-state GDP.

Figure 3.A (α = .70)

Convergence Rate *Half Life of a Shock*

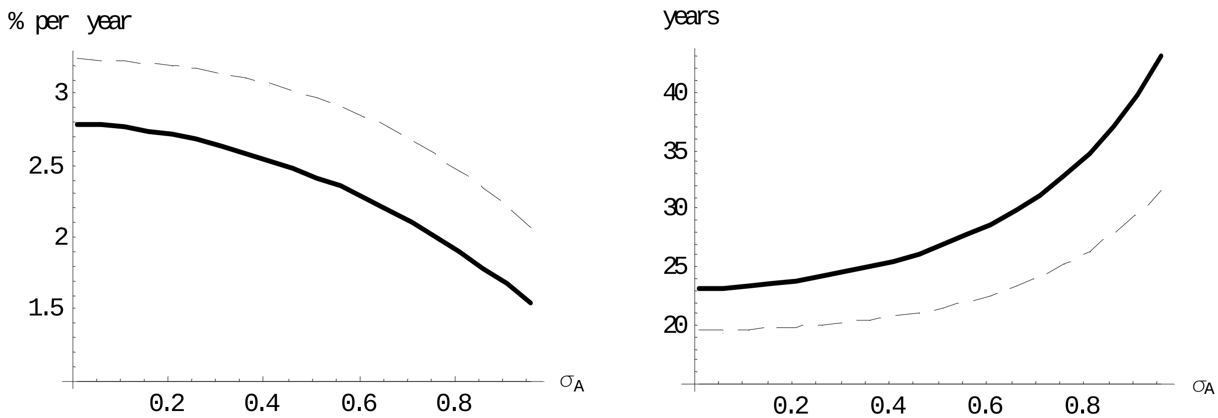

Figure 3.B (α = .35)

Convergence Rate *Half Life of a Shock*

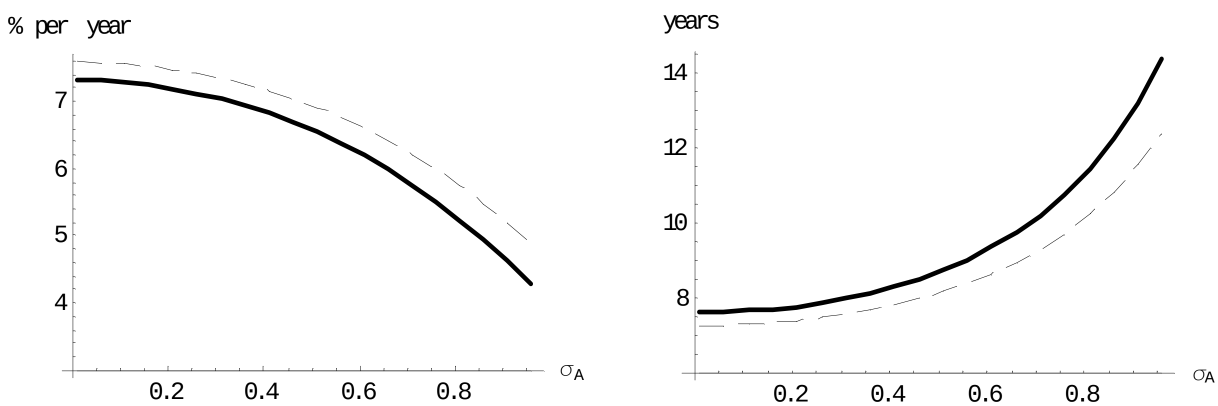

FIGURE 3. Assuming the same parameters as in Figure 2, we plot the convergence rate and the half-life of the deviation from the steady state as idiosyncratic production risk σ_A varies between zero and 100%. The solid lines correspond to $\sigma_e = 0$ and the dashed ones to $\sigma_e = 50\%$.