WORKING PAPER

Strongly rational expectations equilibria with endogenous acquisition of information

by

Gabriel Desgranges and Maik Heinemann

University of Lüneburg
Working Paper Series in Economics

No. 9

March, 2005

www.uni-lueneburg.de/vwl/papers
ISSN 1860–5508

Strongly rational expectations equilibria with endogenous acquisition of information

Gabriel Desgranges [1] Maik Heinemann [2]

17 March 2005

This paper analyzes conditions for existence of a strongly rational expectations equilibrium (SREE) in models with private information, where the amount of private information is endogenously determined. It is shown that the conditions for existence of a SREE known from models with exogenously given private information do not change as long as it is impossible to use the information transmitted through market prices. In contrast, these conditions are too weak, when there is such learning from prices. It turns out that the properties of the function which describes the costs that are associated with the individual acquisition of information are important in this respect. In case of constant marginal costs, prices must be half as informative than private signals in order for a SREE to exist. An interpretation of this result that falls back on the famous Grossman–Stiglitz–Paradox is also given.

Key words: Eductive Learning, Private Information, Rational Expectations, Strongly Rational Expectations Equilibrium

JEL-Classification: D 82, D 83.

---

[1] University of Cergy-Pontoise, France.

[2] University of Lüneburg, Institute of Economics, D–21332 Lüneburg, Germany, heineman.

1. Introduction

The concept of a rational expectations equilibrium (REE) is indeed quite ambitious if the underlying severe requirements on agent's information gathering and processing capabilities are considered. It is therefore not surprising that many attempts have been made in order to justify this concept and to state a clear set of assumptions that imply rational expectations on the side of the agents. One such attempt is the concept of a strongly rational expectations equilibrium (SREE) proposed by Guesnerie (1992, 2002). This concept asks, whether an REE can be educed by rational agents, meaning that the REE is the solution of some kind of mental process of reasoning of the agents. A SREE is then a REE that is learned by agents using this 'eductive' mental process (equivalently, the REE is said to be eductively stable). As shown by Guesnerie (1992, 2002), eductive learning of rational expectations is possible, if based on a suitably specified game–form of the model, agent's use an iterative process to eliminate non–best responses from their strategy sets and if this process converges to the REE. It turns out that an REE is not necessarily a SREE, but that additional restrictions have to met for a SREE to exist. Guesnerie (2002) provides an overview over the conditions for existence of SREE that have been derived in various economic contexts.

Among other things, the concept of a SREE has been successfully applied to models with private information, which usually exhibit quite complex rational expectations equilibria. Conditions for existence of a SREE have been derived for models, where agents are unable to use the information transmitted through current market prices (cf. Heinemann (2003)), as well as for models, where this information can be used (cf. Desgranges et al. (2003), Desgranges (1999), Heinemann (2002)). However, a common feature of all these studies is that they assume an exogenously given amount of private information. This means that so far not only the question how this private information comes into the market has been ignored. It also means that by now it has not been analyzed, whether the endogenization of private information acquisition causes additional restrictions an REE must fulfill in order to be eductively stable.

The present paper tries to fill this gap. We will introduce endogenous information acquisition into a simple market model and are able to derive conditions for existence of a SREE given this endogenously acquired information. Regarding the introduction of endogenous information acquisition, we follow the seminal work by Grossman and Stiglitz (1980) and more precisely Verrecchia (1982) who has analyzed rational expectations equilibria with endogenous acquisition of information in a quite similar economic environment. The present analysis considers two different equilibrium concepts that are both reasonable in the framework underlying our analysis. Initially, we will look at equilibria without learning from prices, where agents are not able to use the information transmitted trough current market prices for their own decisions. This model corresponds for example to a situation where every agent makes an irreversible production decision before he/she knows the price (this is the case of the cobweb models). After that, we will consider a more demanding equilibrium concept, where agents are able to use the information revealed by prices. This model is inspired from the well-known literature about REE under asymmetric information à la Grossman (1976).

The central results of the paper can be summarized as follows: As long as equilibria without learning from prices are considered, the opportunity to acquire private information endogenously leads to no conditions for existence of a SREE beyond that known for the case with exogenously given information. This, however, is not true for equilibria with learning from prices. Here, the conditions for existence of a SREE turn out to be stronger than the respective conditions for the case with exogenously given information.

This striking difference between these two results is driven by the following intuition: In the first model where no information is extracted from the price by agents, endogenous acquisition of private information does not create additional difficulties of coordinating expectations. In other words, at the time where he/she makes his/her decision, every agent needs to guess the price to make an optimal choice. To this purpose, it is enough to guess the shape of the supply and demand curves. In particular, no agent is concerned by the precision of the information acquired by others. Hence, the conditions for stability of the REE (i.e. existence of a SREE) are not affected by endogenous acquisition of private information and they depend on the relative slope of the demand and supply curves only.

In the second model where agents use the informational content of the price, the problem is quite different, and endogenous acquisition of private information does create additional difficulties of coordinating expectations. Namely, every agent needs to know the precision of information acquired by others in order to correctly understand the informational content of the price. The condition for existence of a SREE in the model with information transmitted by the price deserves some more comments. This condition states that the price must not be too informative, with respect to the informativeness of the private information acquired by agents. The underlying intuition is that the informational content of the price is determined by the correlation between private information and agents' decisions. As long as the price is not very informative, this correlation is easy to predict. But, when the price is very informative, agents' decisions mainly depend on the beliefs about the informational content of the price, and agents' decisions are therefore not easy to predict. This condition for existence of a SREE is quite analogous to the condition in the case with exogenously given precision of private information. Still, it is a more demanding condition. The fact that endogenous information acquisition makes it more difficult for a SREE to exist can be explained as follows: As already explained above, when the price is very informative, the REE is not eductively stable because every agent reacts less to his private information than to his beliefs about the information revealed by the price. In this case, given that private information is not very useful to agents, the precision of the private information acquired decreases. This last fact reinforces the stability problem. Namely, agents become much less reactive to their private information. Hence, agents' decisions depend more on their beliefs, which corresponds to a greater instability problem.

Lastly, an interesting feature of this stronger condition for existence of a SREE is that it ensures that the problem described by the the Grossman–Stiglitz–Paradox (cf. Grossman and Stiglitz (1980)) cannot occur. This famous paradox claims that existence of informationally efficient markets is impossible, since it is impossible to explain how information comes into the market in the first place. Namely, as long as the price publicly reveals all the relevant information, there is no incentive to acquire costly private information. But, if no one acquires information in order to make an accurate decision, the price cannot aggregate any information. Our 'solution' of the GS paradox builds on two points. Firstly, our condition for existence of a SREE implies that, as soon as a SREE exists, each firm can educe that there is always a positive amount of private information in the market, because the incentive to free–ride on others' information must be bounded from above. Secondly, when a SREE does not exist (meaning that there is a REE that is not a SREE), the 'eductive theory' is not meant to make an accurate prediction of the market outcome. It only states that the REE is not a plausible outcome (or, at least, is not more plausible than many other outcomes). Although we give no formal content to this claim, we conclude that this additional uncertainty should create incentives to acquire private information.

2. A competitive market model

The model that builds the framework of our analysis is a model of a competitive market with a continuum of risk neutral firms in $I = [0,1]$. Market demand X is random, but the inverse demand function is known to the firms:

$$p = \beta - \frac{1}{\phi} X + \varepsilon$$

Here, ε is a normally distributed demand shock with zero mean and precision τ_ε. $\beta > 0$ and $\phi > 0$ are known constants. Every firms faces increasing marginal costs that are affected by the parameter θ. With $x(i)$ denoting the output of firm i, her costs are $c(i) = \theta x(i) + \frac{1}{2}\frac{1}{\psi}x(i)^2$, where $\psi > 0$. The cost parameter θ is unknown to the firms. The firms, however, know that this parameter is drawn from a normal distribution with mean $\bar{\theta}$ and precision τ.

Private information on the side of the firms regarding the unknown parameter is introduced into the model by allowing for endogenous acquisition of information as in Verrecchia (1982) (generalizing the seminal framework of Grossman and Stiglitz (1980)). It is assumed that each firm is able to perform an experiment (independent from experiments of other firms) that reveals additional but costly information regarding the unknown parameter θ. In particular, it is assumed that each firm $i \in I$ can acquire a costly private signal $s(i)$ that reveals additional private information. The private signal is given by $s(i) = \theta + u(i)$, where the signal's noise $u(i)$ is normally distributed with mean zero and precision $\tau(i)_u$. The costs of acquiring a signal with precision $\tau(i)_u$ are given by $K(\tau(i)_u)$ and we let $\kappa(\tau(i)_u)$ denote the respective marginal costs. The objective of a firm is to maximize the expected profit, where ex-ante profit $\pi(i)$ of firm i is given by:

$$\pi(i) = [p - \theta]\,x(i) - \frac{1}{2}\frac{1}{\psi}\,[x(i)]^2 - K(\tau(i)_u), \tag{1}$$

Costs are are assumed to be increasing and convex: $\kappa(\tau(i)_u) \geq 0$ and $\kappa'(\tau(i)_u) \geq 0$ for all $\tau(i)_u \geq 0$.

Throughout the following analysis it will always be assumed that the average of the firm's private signals reveals the unknown value of the unknown parameter by the law of large numbers, such that $\int_0^1 s(i)\,di = \theta$ because $\int_0^1 u(i)\,di = 0$.

In what follows, we will first consider equilibria of this simple market model, where the firms are unable to use the information transmitted trough prices. This simply means, that every firm must decide on her profit maximizing output, before the actual market price becomes known and is unable to condition her supply decision on the market price. An equilibrium concept, where such learning from prices is possible because the information transmitted through prices can be used, will be analyzed in section 4.

3. SREE without learning from prices

3.1. *Description of the REE*

We will start here with a brief description of the kind of REE that appears, when decisions are made before the actual market price becomes known. Because of the distributional assumptions made above, this REE takes a quite simple form: In equilibrium, each firm's supply decision $x(i)$ will be a linear function of the estimator for the unknown parameter θ

based on public information and — if the firm chooses to acquire private information — the private signal $s(i)$ the firm observes. The decision to acquire information altogether, in turn depends on the marginal costs and benefits associated with private information acquisition. The next result summarizes the properties of the REE:

Proposition 1. *Let* $\alpha = -\psi/\phi < 0$. *The model then possesses an unique linear REE with the following properties:*

(i) *Each firm* $i \in I$ *will acquire the same level of precision* $\tau(i)_u^* = \tau_u^* = \max\{0, \tilde{\tau}_u\}$ *of her private signal* $s(i)$. $\tilde{\tau}_u$ *is the solution of the equation:*

$$\frac{\psi}{2} \frac{1}{(\tau + (1-\alpha)\,\tilde{\tau}_u)^2} = \kappa(\tilde{\tau}_u) \tag{2}$$

Furthermore, a positive amount of information is acquired in equilibrium, i.e., $\tau_u^* > 0$, *iff* $\frac{\psi}{2\tau^2} > \kappa(0)$.

(ii) *Each firm* $i \in I$ *will use the same supply function* $x(i) = \psi\,[\gamma_0^* + \gamma_1^*\, s(i)]$, *where the weights* γ_0^* *and* γ_1^* *are functions of the model parameters:*

$$\gamma_0^* = \frac{\beta}{1-\alpha} - \frac{1}{1-\alpha}\frac{\tau}{\tau + (1-\alpha)\,\tau_u^*}, \qquad \gamma_1^* = -\frac{\tau_u^*}{\tau + (1-\alpha)\,\tau_u^*}$$

Proof. See Appendix. □

Existence and uniqueness of a linear equilibrium in various cases of CARA/Gaussian settings is a very common result, and this result deserves few comments only. A market equilibrium with private information acquisition (i.e., $\tau_u^* > 0$) will therefore exist only if the marginal benefit of information acquisition at zero (i.e., $\frac{\psi}{2\tau^2}$), is greater than the marginal cost of information acquisition at zero (i.e., $\kappa(0)$). In what follows, we assume that this condition is satisfied. Thus, there always exists a nontrivial REE, where individual acquisition of information takes place. For simplicity, we will also sometimes make the assumption that the marginal costs of information acquisition are constant, such that $\kappa(\tau(i)_u)) = \bar{\kappa} > 0$ for all $\tau(i)_u > 0$. Under this assumption, a REE with information acquisition (i.e., $\tau(i)_u^* = \tau_u^* > 0$ for all $i \in I$) exists if and only if $Q \equiv \sqrt{\frac{\psi}{2\bar{\kappa}}} > \tau$. From the equilibrium condition (2) we obtain that in this case the equilibrium amount of information acquisition is $\tau_u^* = \frac{Q-\tau}{1-\alpha}$.

3.2. *Existence of a SREE*

Since detailed descriptions of of the concept of a SREE are already available in the literature (cf. Guesnerie (2002)), it is adequate to limit the present analysis to an informal and pragmatic treatment of this concept and the game–theoretical issues that are involved here. The fundamental question associated with the concept of a SREE is whether the assumptions of individual rationality and common knowledge are sufficient to predict a particular REE as an outcome of a model. Therefore it is necessary to look at a suitable game–form of the model and to analyze the best responses of the individual firms to actions taken by other firms in order to derive conditions for eductive stability. If we confine our analysis to linear supply functions, such that an individual firm's supply is given by $x(i) = \psi\,[\gamma(i)_0 + \gamma(i)_1\, s(i)]$, the respective best response mapping can be summarized by the equations listed in the following Lemma: [3]

Lemma 1. *If aggregate behavior is summarized by the coefficients* $\gamma_0 = \int_0^1 \gamma(j)_0\, dj$ *and* $\gamma_1 = \int_0^1 \gamma(j)_1\, dj$*, the best response* $\gamma(i)_0$*,* $\gamma(i)_1$ *as well as* $\tau(i)_u$ *of a firm* $i \in I$ *is uniquely defined by the following equations:*

$$\gamma(i)_0 = \beta + \alpha\gamma_0 + (\alpha\gamma_1 - 1)\,\frac{\tau}{\tau + \tau(i)_u}\,\bar{\theta} \tag{3a}$$

$$\gamma(i)_1 = (\alpha\gamma_1 - 1)\,\frac{\tau(i)_u}{\tau + \tau(i)_u} \tag{3b}$$

$$0 = \psi\,\frac{1}{2}\left[\frac{\gamma(i)_1}{\tau(i)_u}\right]^2 - \kappa(\tau(i)_u) \tag{3c}$$

This Lemma (that is central to the study of stability, as will soon be clear) calls for several comments:

(i) The best response mapping defined by the above Equations (3a)–(3c) map the three real parameters $(\gamma_0, \gamma_1, \tau_u)$ into the three real parameters $(\gamma(i)_0, \gamma(i)_1, \tau(i)_u)$ characterizing the best response of firm i (where the aggregate value $\tau_u = \int_0^1 \tau(j)_u\, dj$ is defined analogously to γ_0 and γ_1).

(ii) Notice that $(\gamma(i)_0, \gamma(i)_1, \tau(i)_u)$ is not affected by τ_u, i.e., the precision of the information acquired by others. Intuitively, firm i makes his supply decision considering (1) his information on θ (that is s_i only as there is no learning from the price), and (2) his information on the price, that consists in the market clearing equation $p = \beta + \alpha\,[\gamma_0 + \gamma_1 \int_0^1 s(j)\, dj] + \varepsilon$, where ε and $\int_0^1 s(j)\, dj = \theta$ are unknown. Thus, given that the precision of the aggregate information $\int_0^1 s(j)\, dj$ on θ does not depend on the individual precisions $\tau(j)_u$ (it is infinite), the decision made by firm i does not depend on the $\tau(j)_u$ either.

(iii) Furthermore, the decision of firm i can be separated into two successive problems. To see this, notice first that equation (3c) defines $\tau(i)_u$ as a function of $\gamma(i)_1$ and rewrite

---

[3] Optimal output of a firm is given by $x(i) = \psi\, \mathrm{E}[p - \theta\,|\, s(i)]$. Hence, this linear supply rule assumes that $\mathrm{E}[p - \theta\,|\, s(i)] = \gamma(i)_0 + \gamma(i)_1\, s(i)$.

$\tau(i)_u = F(\gamma(i)_1)$. Then, given F, equations (3a)–(3b) define $(\gamma(i)_0, \gamma(i)_1)$ as a function of (γ_0, γ_1), namely:

$$\gamma(i)_0 = \beta + \alpha\gamma_0 + (\alpha\gamma_1 - 1)\,\frac{\tau}{\tau + F(\gamma(i)_1)}\,\bar{\theta} \tag{4a}$$

$$\gamma(i)_1 = (\alpha\gamma_1 - 1)\,\frac{F(\gamma(i)_1)}{\tau + F(\gamma(i)_1)} \tag{4b}$$

It follows that, on the one hand, the supply function of firm i is determined by the expected aggregate supply curve (characterized by (γ_0, γ_1)), and, on the other hand, the precision of the private information is determined on the basis of $\gamma(i)_1$ through the map F. This last remark will have essential consequences to the stability problem.

We can now turn attention to the question of the strong rationality (or stability) of the REE. By definition, the REE γ_0^*, γ_1^* and τ_u^* is a fixed point of the best response mapping (3a)–(3c). In particular, equation 3b) implies that $\gamma_1^* \leq 0$. Again, a detailed account of the analytical characterization of SREE is given in Guesnerie (2002). We just recall here that this REE is a SREE (or, equivalently is "eductively stable") if and only if it is a locally stable stationary point of the dynamical system made up from this best response mapping. Now, with respect to this dynamical system, the eigenvalues λ_1, λ_2 and λ_3 of the Jacobian matrix at the equilibrium point can be computed as follows: [4]

$$\lambda_1 = 0, \quad \lambda_2 = \alpha, \quad \lambda_3 = \frac{\alpha\,\left(\gamma_1^2\,\psi + \kappa'(\tau_u)\,\tau_u^3\right)}{\gamma_1^2\,\psi + \kappa'(\tau_u)\,\tau_u^2(\tau + \tau_u)}$$

Since we have assumed that marginal costs of information acquisition are increasing or at least constant, we have $\kappa'(\tau_u) \geq 0$ for all τ_u. Therefore, we always have $\lambda_3 < \alpha$ and the condition $|\alpha| < 1$ is necessary and sufficient for local stability ands thus for existence of a SREE:

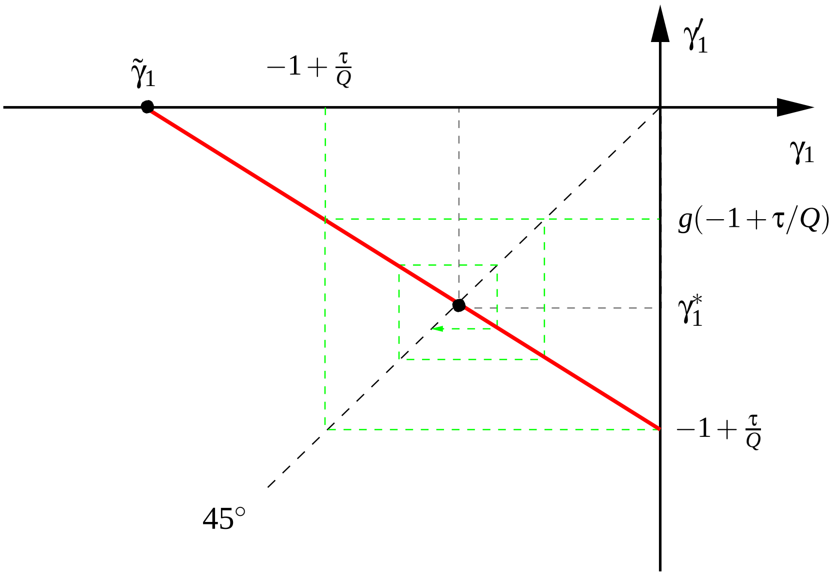
Proposition 2. *The linear REE with private information acquisition is locally eductively stable if and only if* $|\alpha| < 1$

Basically, this condition requires $-1 < \alpha < 0$, since $\alpha = -\psi/\phi$, where ψ and ϕ are positive constants. Interestingly, this condition is exactly identical to the respective stability condition for the case with exogenously given private information (cf. Heinemann (2003)). As long as equilibria without learning from prices are considered, we therefore get no change with

---

[4] Details on the computation of the eigenvalues of this dynamical system are given in the appendix at the end of the proof of Lemma 1.

Fig. 1. *Graphical representation of the SREE condition*



regard to the conditions for existence of a SREE, even if we allow for endogenous acquisition of information.

To understand the intuition for this surprising result, we go back to the above comments to Lemma 1. In these comments, we have explained that the first ‘supply’ decision is made independently of the precisions of private information, and that the ‘precision’ decision does not add further ‘expectational difficulties’, i.e. the choice of $\gamma(i)_1$ requires to guess other’s behavior, but, once $\gamma(i)_1$ has been chosen, $\tau(i)_u$ is simply $F[\gamma(i)_1]$, it does not depend on others’ behavior $(\gamma(i)_0, \gamma(i)_1)$. Summing up, (1) the decision of firm i is not affected by expectation about others’ precision of information $\int_0^1 \tau(j)_u \, dj$, and (2) the choice of $\tau(i)_u$ is unambiguously made, once $\gamma(i)_1$ has been chosen. Thus, as far as the question of expectations coordination is concerned, it is enough to look at the system (γ_0, γ_1), and it is useless to look at τ_u. This intuition is clearly emphasized in the specific case with constant marginal costs described below.

Under the assumption of constant marginal costs of information acquisition, it is quite easy to give a graphical representation of the stability condition and the iterative process that leads to the REE. Under the assumption that $\kappa(\tau_u) = \overline{\kappa}$ such that $\kappa' = 0$, Equation (3c) gives $\tau(i)_u = -Q\gamma(i)_1$, where Q was already defined above. Then, the best response mapping (3a)-(3b) rewrites as the following linear system:

$$\gamma(i)_0 = \beta + \alpha\gamma_0 - \frac{\tau}{Q}\bar{\theta} \tag{5a}$$

$$\gamma(i)_1 = \alpha\gamma_1 - 1 + \frac{\tau}{Q} \equiv g(\gamma_1) \tag{5b}$$

where the variables τ_u and $\tau(i)_u$ do not appear, as explained above. The first equation characterizes the dynamics of γ_0, while the second describes the dynamics of γ_1.

For example, consider the dynamics of γ_1 as depicted in figure 1. To draw the figure, denote γ_1^* the equilibrium weight of private information (such that $\gamma_1^* = g(\gamma_1^*)$) and $\tilde{\gamma}_1 = -\frac{1}{\alpha}\left(-1+\frac{\tau}{Q}\right)$ the root of g, and notice that existence of an REE with $\tau_u^* > 0$ implies $Q > \tau$ such that $g(0) < 0$. Thus, whenever the stability condition stated in Proposition 2 is satisfied, such that $-1 < \alpha < 0$, we have $\tilde{\gamma}_1 < -1 + \frac{\tau}{Q}$. Figure 1 can then be used to describe the iterative process of elimination of non-best responses that converges to this REE. This process is illustrated in the figure, starting from the assumption that it is common knowledge that no firm uses a weight $\gamma(i)_1$ greater than zero. [5] This necessarily implies that $\gamma_1 \leq 0$ and from the figure it can be seen that in this case no firm will ever choose a weight $\gamma(i)_1$ which smaller than $-1 + \frac{\tau}{Q}$. [6] From this, however, it in turn follows that γ_1 must be greater than $-1 + \frac{\tau}{Q}$, which implies that no firm i will use a weight $\gamma(i)_1 > g(-1 + \tau/Q)$. It is easily verified that this process converges to the equilibrium γ_1^*, whenever $-1 < \alpha < 0$.

4. Eductive stability with learning from prices

4.1. *The case of exogenously given information*

Let us now turn to the second equilibrium concept, where learning from current prices is possible. It is reasonable to start this analysis with a brief discussion of a version of the model, where the amount of private information is given. This enables us to build on some known results and to illustrate, where these known results have to be modified if endogenous acquisition of information is allowed for. The analysis is based on the initially considered model with risk neutral firms and it is assumed that each firm's signal has precision $\tau_u > 0$. When there is learning from prices, the firms are able to use the information transmitted through the actual market price for their own decisions. Hence, profit maximizing output for a firm $i \in I$ is now given by $x(i) = \psi\,[p - \mathrm{E}[\theta\,|\,s(i), p]]$. In analogy to the financial market models considered by Desgranges (1999) and Heinemann (2002), it can then be established that there exists an unique linear REE in this model with learning from prices.

---

[5] From equation (3c) it follows that this is equivalent to the assumption that it is common knowledge that $\tau(i)_u \geq 0$ for all i.

[6] Using equation (3c) it can be shown that with respect to the amount of information that is acquired this means that no firm will acquire information with precision greater than $\tau_u = (Q - \tau)$.

Proposition 3. *Let again* $\alpha = -\psi/\phi < 0$. *The model with learning from prices then possesses an unique linear REE, where every firm uses a linear supply function* $x(i) = \psi\,[(1-\gamma_2^*)\,p - \gamma_0^* - \gamma_1^*\,s(i)]$. *The coefficients* γ_0^* *and* γ_1^* *and* γ_2^* *are solutions to the equations:*

$$\gamma_0^* = \frac{\beta\alpha\gamma_1^*\tau_\varepsilon + \tau\bar{\theta} - \alpha^2\gamma_1^*\gamma_0^*\tau_\varepsilon}{\tau+\tau_u+\alpha^2{\gamma_1^*}^2\tau_\varepsilon}, \quad \gamma_2^* = -\frac{\gamma_1^*\alpha(1-\alpha(1-\gamma_2^*))\tau_\varepsilon}{\tau+\tau_u+\alpha^2{\gamma_1^*}^2\tau_\varepsilon}$$

and γ_1^* *is the unique solution of the polynomial* $H(\gamma_1^*) \equiv \gamma_1^*\left[(\gamma_1^*)^2\alpha^2\tau_\varepsilon + \tau + \tau_u\right] = \tau_u$.

Conditions for existence of a SREE in this model with exogenously given private information are derived by Heinemann (2003). For convenience the respective conditions are reproduced in the following proposition:

Proposition 4.

(i) *The rational expectations equilibrium* $\gamma_0^*, \gamma_1^*, \gamma_2^*$ *is a SREE if and only if* $\alpha^2\,(\gamma_1^*)^2\,\tau_\varepsilon < \tau_u$.

(ii) *The condition (i) for existence of a SREE is equivalent to the condition that in the rational expectations equilibrium the market price* p *is less informative regarding* θ *than the private signals.*

4.2. *Conditions for existence of a SREE with endogenous acquisition of information*

Starting from the above described rational expectations equilibrium with exogenously given information, it is quite easy to derive the respective equilibrium conditions for the model with endogenous information acquisition. The reason is, that all the conditions stated in Proposition 3 remain essentially valid. The only modification consists in an additional condition which describes the optimal equilibrium amount of private information acquisition:

Proposition 5. *In the model with learning from prices and endogenous information acquisition exists an unique linear REE, where every firm uses a linear supply function* $x(i) = \psi\,[(1-\gamma_2^*)\,p - \gamma_0^* - \gamma_1^*\,s(i)]$.

(i) *Each firm* $i \in I$ *will acquire the same level of precision* $\tau(i)_u^* = \tau_u^* = \max\{0, \tilde{\tau}_u\}$ *of her private signal* $s(i)$. $\tilde{\tau}_u$ *is the solution of the equation:*

$$\frac{\psi}{2}\left(\frac{\gamma_1^*}{\tilde{\tau}_u}\right)^2 = \kappa(\tilde{\tau}_u)$$

(ii) *The coefficients* γ_0^* *and* γ_1^* *and* γ_2^* *are given as in Proposition 3, that is* γ_1^* *is the unique solution of the polynomial* $H(\gamma_1^*) \equiv \gamma_1^*\left[(\gamma_1^*)^2\alpha^2\tau_\varepsilon + \tau + \tau_u^*\right] = \tau_u^*$ *and:*

$$\gamma_0^* = \frac{\beta\alpha\gamma_1^*\tau_\varepsilon + \tau\bar{\theta} - \alpha^2\gamma_1^*\gamma_0^*\tau_\varepsilon}{\tau + \tau_u^* + \alpha^2\gamma_1^{*2}\tau_\varepsilon}$$

$$\gamma_2^* = -\frac{\gamma_1^*\alpha(1-\alpha(1-\gamma_2^*))\tau_\varepsilon}{\tau + \tau_u^* + \alpha^2\gamma_1^{*2}\tau_\varepsilon}$$

Proof. See Appendix. □

Obviously, with endogenous acquisition of information, we end up with one additional equation, which requires that the marginal costs of the acquired information are equal to the marginal benefits from this information.

We now again ask, whether the assumptions of individual rationality and common knowledge are sufficient for a justification of this REE. In order to derive the respective conditions for existence of eductive stability, we have again to look a the best responses of the individual firms to actions taken by other firms. As in the preceeding section, we confine our analysis to linear supply functions, such that an individual firm's supply is given by $x(i) = \psi[(1-\gamma(i)_2)p - \gamma(i)_0 - \gamma(i)_1 s(i)]$. The respective best response mapping is then as summarized in the following Lemma:

Lemma 2. *If aggregate behavior is summarized by the coefficients* $\gamma_0 = \int_0^1 \gamma(j)_0\,dj$, $\gamma_1 = \int_0^1 \gamma(j)_1\,dj$ *and* $\gamma_2 = \int_0^1 \gamma(j)_2\,dj$, *the best response of a firm* $i \in I$ *is:*

$$\gamma'(i)_0 = \frac{\beta\alpha\gamma_1\tau_\varepsilon + \tau\bar{\theta} - \alpha^2\gamma_1\gamma_0\tau_\varepsilon}{\tau + \tau_u'(i) + \alpha^2\gamma_1^2\tau_\varepsilon} \tag{6a}$$

$$\gamma'(i)_1 = \frac{\tau_u'(i)}{\tau + \tau_u'(i) + \alpha^2\gamma_1^2\tau_\varepsilon} \tag{6b}$$

$$\gamma'(i)_2 = -\frac{\gamma_1\alpha(1-\alpha(1-\gamma_2))\tau_\varepsilon}{\tau + \tau_u'(i) + \alpha^2\gamma_1^2\tau_\varepsilon} \tag{6c}$$

$$\psi\frac{1}{2}\left(\frac{\gamma'(i)_1}{\tau_u'(i)}\right)^2 = \kappa(\tau_u'(i)) \tag{6d}$$

Proof. See Appendix. □

Notice, that possible nonnegativity constraints on the individually acquired precision $\tau(i)_u$ are ignored in the formulation of this Lemma. As implied by definition of the REE in Proposition 5 it might be the case that no information at all will be acquired in a REE. Similarly, it might wll happen, that a best response of a firm $i \in I$ is to acquire no private information. Since we are interested in conditions for eductive stability of a REE, where private acquisition of information actually takes place and since the conditions we look for are basically conditions for local stability, this restriction is not too severe. However, later on we will also discuss the best response dynamics under consideration of the nonnegativity constraint on individual precisions $\tau(i)_u$.

The REE is strongly rational, if the map defined by equations (6a)–(6d) is contracting. The required stability analysis can be simplified, however, since a closer look at this system reveals, that equations (6b) and (6d) can be analyzed independently from equations (6a) and (6c). This means that given stability of the subsystem (6b) and (6d) around γ_1^* and τ_u^*, it would be sufficient to find stability conditions for the remaining two linear equations (6a) and (6c). Indeed, with respect to these latter two equations this is a simple task, because the respective equations are the same as for the case with exogenously given information such that the corresponding stability condition coincides with the condition stated in Proposition 4 above.[7] Indeed, the system formed by the two equations (6a) and (6c) is stable if and only if,

$$\left| \frac{-\alpha^2 \gamma_1^* \tau_\varepsilon}{\tau + \tau_u^* + \alpha^2 \gamma_1^{*2} \tau_\varepsilon} \right| < 1$$

As $\frac{\tau_u^*}{\tau+\tau_u^*+\alpha^2\gamma_1^{*2}\tau_\varepsilon} = \gamma_1^*$ at the REE, this rewrites $\alpha^2 \gamma_1^{*2} \tau_\varepsilon < \tau_u^*$.

It therefore remains to analyze the subsystem (6b) and (6d) in order to check whether there are any consequences of endogenous acquisition of information on the conditions for existence of a SREE. Now, the eigenvalues λ_1, λ_2 of the Jacobian matrix of this system evaluated at the REE are:

$$\lambda_1 = 0, \qquad \lambda_2 = -\frac{2\alpha^2 \gamma_1^{*2} \tau_\varepsilon}{\tau + \tau_u^* + \alpha^2 \gamma_1^{*2} \tau_\varepsilon - (1-\gamma_1^*)W}$$

where $W \equiv \frac{\psi \frac{\gamma_1^*}{\tau_u^{*2}}}{\kappa' + \psi \frac{\gamma_1^{*2}}{\tau_u^{*3}}} \in [0, \frac{\tau_u^*}{\gamma_1^*}]$, depending on the value of κ'. Some computations show that $\lambda_2 < 0$. To see this point, notice first that the equilibrium condition (6b), $\tau + \tau_u^* + \alpha^2 \gamma_1^{*2} \tau_\varepsilon = \frac{\tau_u^*}{\gamma_1^*}$ implies that the intersection point of the two curves $\tau + \tau_u^* + \alpha^2 \gamma_1^{*2} \tau_\varepsilon$ and $\frac{\tau_u^*}{\gamma_1^*}$ satisfies $\gamma_1^* > 0$ and $\tau + \tau_u^* < \frac{\tau_u^*}{\gamma_1^*}$, that is $\gamma_1^* < \frac{\tau_u^*}{\tau+\tau_u^*} < 1$. Then, using again the equilibrium condition $\tau + \tau_u^* + \alpha^2 \gamma_1^{*2} \tau_\varepsilon = \frac{\tau_u^*}{\gamma_1^*}$,

$$\lambda_2 = -\frac{2\alpha^2 \gamma_1^{*2} \tau_\varepsilon}{\frac{\tau_u^*}{\gamma_1^*} - (1-\gamma_1^*)W}$$

---

[7] Heinemann (2003) shows that the stability conditions stated in Proposition 4 is in fact necessary and sufficient for stability of the subsystem (6a) and (6c) when γ_1 is fixed. For convenience, the complete stability analysis is given in the appendix as a proof of Proposition 6.

As $W \in [0, \frac{\tau_u^*}{\gamma_1^*}]$, the denominator of the above expression is greater than

$$\frac{\tau_u^*}{\gamma_1^*} - (1-\gamma_1^*)\,W \geq \frac{\tau_u^*}{\gamma_1^*} - (1-\gamma_1^*)\,\frac{\tau_u^*}{\gamma_1^*} = \tau_u^* > 0$$

Hence, $\lambda_2 < 0$.

Thus, strong rationality is equivalent to $\lambda_2 > -1$, that rewrites (using again the equilibrium condition (6b)):

$$2\,\alpha^2 \gamma_1^{*2}\,\tau_\varepsilon < \frac{\tau_u^*}{\gamma_1^*} - (1-\gamma_1^*)\,W$$

As $\frac{\tau_u^*}{\gamma_1^*} - (1-\gamma_1^*)\,W \geq \tau_u^*$, it follows that $\alpha^2 \gamma_1^{*2} \tau_\varepsilon < \tau_u^*/2$ is a sufficient condition for stability. It can be shown that informativeness τ_p of the market price p in a REE (i.e., $1/Var(\theta|p) - 1/Var(\theta)$) is given by $\tau_p^* \equiv \alpha^2\,(\gamma_1^*)^2\,\tau_\varepsilon$ (cf. Heinemann (2002)). Thus, we have exactly shown that a sufficient condition for stability is $\tau_p^* < \tau_u^*/2$: the precision of the information revealed by the equilibrium price is less than half the precision of the private signal ($\tau_u^* = 1/Var(\theta|s(i)) - 1/Var(\theta)$). A necessary and sufficient condition for stability is

$$2\,\tau_p^* < \tau_u^* \frac{\frac{\kappa'}{\gamma_1^*} + \psi \frac{\gamma_1^{*2}}{\tau_u^{*3}}}{\kappa' + \psi \frac{\gamma_1^{*2}}{\tau_u^{*3}}}$$

One sees that, given that $0 < \gamma_1^* < 1$, the condition $2\,\tau_p^* < \tau_u^*$ is necessary only when $\kappa' = 0$, that is: marginal costs are constant.

Using again $\frac{\tau_u^*}{\gamma_1^*} = \tau + \tau_u^* + \tau_p^*$, the stability condition becomes:

$$2\,\tau_p^* < \frac{\kappa'\left(\tau + \tau_u^* + \tau_p^*\right) + \psi \frac{1}{\left(\tau+\tau_u^*+\tau_p^*\right)^2}}{\kappa' + \psi \frac{1}{\tau_u^*\left(\tau+\tau_u^*+\tau_p^*\right)^2}}$$

$$\kappa' < \psi \frac{1 - 2\frac{\tau_p^*}{\tau_u^*}}{(\tau + \tau_u^* + \tau_p^*)^2(\tau_p^* - \tau - \tau_u^*)}$$

This mainly says that κ' must not be too large, nor ψ too small. The fact that κ' must not be too large can be easily understood: a small κ' implies that this is not very costly to adjust $\tau_u(i)$ for firm i so that this quantity cannot be easily predicted by others.

The implications of these results for existence of a SREE are summarized in the next Proposition:

Proposition 6.

(i) If private information is endogenously acquired, a sufficient condition for the rational expectations equilibrium $d^* = \left(\gamma_0^*, \gamma_1^*, \gamma_2^*\right)$ *to be a SREE is* $\alpha^2 \left(\gamma_1^*\right)^2 \tau_\varepsilon < \frac{1}{2}\tau_u^*$.

(ii) If marginal costs are constant such that $\kappa(\tau(i)_u) = \overline{\kappa}$*, the rational expectations equilibrium* $d^* = \left(\gamma_0^*, \gamma_1^*, \gamma_2^*\right)$ *is strongly rational if and only if* $\alpha^2 \left(\gamma_1^*\right)^2 \tau_\varepsilon < \frac{1}{2}\tau_u^*$.

(iii) The condition for existence of a SREE is equivalent to the condition that in the rational expectations equilibrium the market price p is at most half as informative regarding θ *than the private signals.*

Proof. See Appendix. □

The condition stated in Proposition 6 is obviously stronger than the respective condition for existence of a SREE with exogenously given information which is stated in Proposition 4. Thus, contrary to the above considered case without learning form prices, the presence of endogenous information acquisition in the model with learning from prices implies that conditions for existence of a SREE have to be qualified. As usual, our condition for existence of a SREE is based on local stability of the best response mapping. Thus, without further restrictions on the set of strategies used by the firms, even this condition might not be sufficient for convergence of the eductive process towards the REE. This is the reason why Guesnerie (1992), discusses the imposition of 'credible restrictions' on the set of strategies which guarantee that this convergence in fact takes place. Basically, the underlying problem is one of global versus local stability of the REE under the dynamics induced by the best response mapping. As we will demonstrate soon for the case of constant marginal costs of information acqusition, i.e $\kappa' = 0$, our condition for existence of a SREE is in fact necessary and sufficient for convergence of the eductive process towards the REE.

In the remainder of the paper, we will look at the case of constant marginal costs of information acquisition in more detail. We will first present some numerical examples which serve to illustrate the so far derived results and then discuss the issue of global convergence of the eductive process towards the REE.

The assumption of constant marginal costs, i.e. $\kappa(\tau_u(i)) = \overline{\kappa}$ makes it possible to analyze the best response dynamics with the help of a single equation, which results from the substitution of equation (6b) into (6d). Equation (6d), which implies that $\gamma_1^2 = \frac{2\overline{\kappa}}{\psi}\tau_u^2$, can then be used to eliminate γ_1.[8] With $Q \equiv \sqrt{\frac{\psi}{2\overline{\kappa}}}$, the resulting equation then can be interpreted as the individual best response $\tau_u'(i) = T(\tau_u)$ to the '*amount of information in the market*' τ_u:

---

[8] Equation (6d) implies $\gamma(i)_1 = \tau_u(i)/Q$. Integrating over all $i \in I$ then gives $\gamma_1 = \tau_u/Q$, where $\tau_u = \int_0^1 \tau_u(i)\, di$.

$$\tau'(i)_u = T(\tau_u) = \max\left\{0, Q - \tau - \frac{\alpha^2 \tau_\varepsilon}{Q^2}\tau_u^2\right\} \tag{7}$$

Equation (7) describes the best response dynamics for the endogenously acquired amount of private information and implicitly also the best response dynamics for the weight $\gamma(i)_1$, which is given to this private information in the individual supply function. The consequences of these best response dynamics for the remaining weights $\gamma(i)_0$ as well as $\gamma(i)_2$ will be discussed later. Notice, that the best response function (7) now explicitly accounts for the above mentioned nonegativity constraint on the individually acquired amount of precision $\tau'(i)_u$.

We will now illustrate the properties of the best response mapping (7) with some numerical examples.

Example 1: Consider a numerically specified version of the model where $\alpha = -0.85$, $\psi = 1$, $\tau = 0.1$ and $\tau_\varepsilon = 1$. Marginal costs of information acquisition are constant and given by $\overline{\kappa} = 0.5$. From equation (6b) and (6d), the values for γ_1 and τ_u in a rational expectations equilibrium can be computed as: $\gamma_1^* = 0.621197$ as well as $\tau_u^* = 0.621197$.[9] Now let us look first at the case where the amount of private information is exogenously given and equal to τ_u^*. According to Proposition 4, the condition for existence of a SREE is $\alpha^2 \tau_\varepsilon \gamma_1^{*2} < \tau_u^*$, which is satisfied since $\alpha^2 \tau_\varepsilon \gamma_1^{*2} = 0.278803$.

If we now consider the case where information is endogenously acquired, the stronger stability condition according to Proposition 6 is $\alpha^2 \tau_\varepsilon \gamma_1^{*2} < \frac{1}{2}\tau_u^*$, which is also satisfied, since $\tau_u^*/2 = 0.310599$. Thus, in this case the fact that information acquisition is endogenously determined is not relevant for existence of a SREE.

Figure 2 shows how this function $T(\tau_u)$ looks like in case of the underlying numerical specification. Since τ_u is necessarily nonnegative, it is common knowledge that $\tau_u \geq 0$. As the figure reveals, it is always individually optimal to acquire information, as long as no other firm does it. If $\tau_u = 0$, the corresponding maximum amount of private information a firm will ever acquire is given by $T(0) = Q - \tau > 0$. Since it is therefore also common knowledge that $\tau_u \leq T(0)$, it follows that no firm will ever choose $\tau(i)_u < T(T(0)) = T(Q - \tau)$. This means that no firm expects the amount of information in the market to be such large that it becomes optimal to stop the individual acquisition of information. As indicated in the figure, the dynamics that result if this kind of reasoning is iterated, are similar to the well known cobweb–dynamics. The condition stated in Proposition 6 then ensures that these dynamics

---

[9] The respective values for γ_0^* and γ_2^* are omitted here. They are not of interest, since they do not appear in the stability conditions.

Fig. 2. *Best response mapping* $T(\tau_u)$ *for example 1*

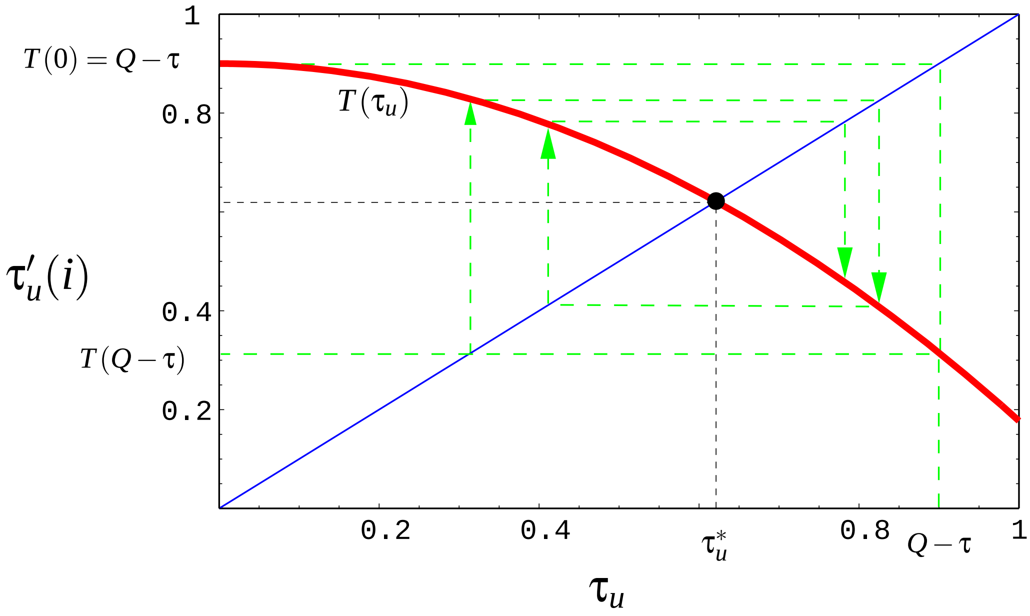
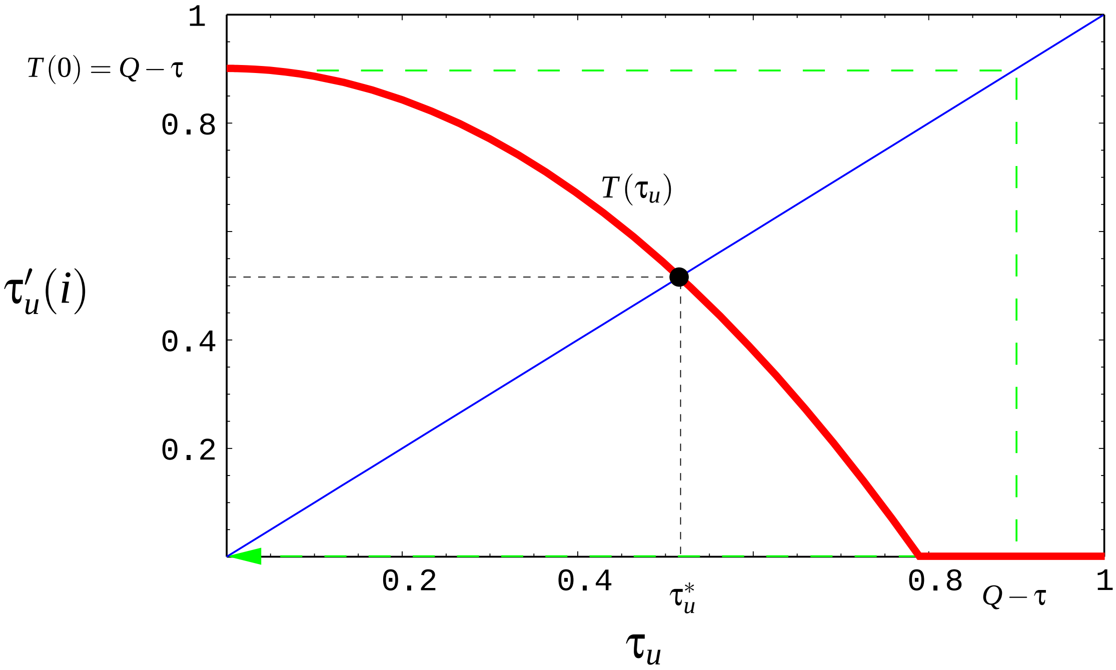


Fig. 3. *Best response mapping* $T(\tau_u)$ *for example 2*



converge to the REE precision τ_u^*. In this case each firm can educe that only the precision REE $\tau_u^* = 0.621197$ constitutes possible solution under the assumptions of individual rationality and common knowledge. [10]

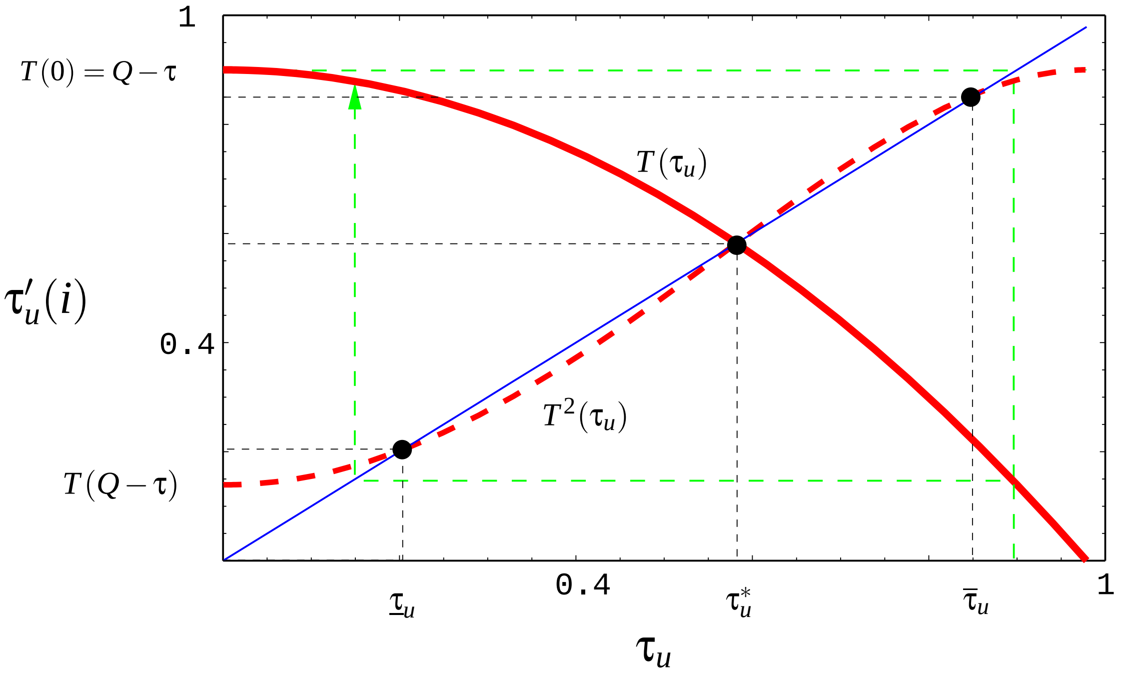
---

[10] The formal proof that the dynamics indeed converge towards the REE will be given below.

Example 2: In this example, the precision of noise is given by $\tau_\varepsilon = 2.0$ and, hence, larger than in example 1. In the REE we have $\gamma_1^* = 0.515703$ and $\tau_u^* = 0.515703$. Since $\alpha^2 \tau_\varepsilon \gamma_1^{*2} = 0.384297$, this REE is still eductively stable, if information is assumed to be exogenously given, but not (since $\tau_u^*/2 = 0.257851$), when information acquisition is endogenous. Thus, we have here an example where a SREE exists if the amount of private information is exogenously given, but does not exist if there is endogenous acquisition of information. The best response function $T(\tau_u)$ depicted in figure 3 reveals that in this example we have $T(Q-\tau) = 0$, i.e. now the nonnegativity constraint on $\tau'(i)_u$ becomes relevant. This example then gives rise to an interpretation which is quite similar to the famous Grossman–Stiglitz–Paradox (cf. Grossman and Stiglitz (1980)). As the figure shows, it is again individually optimal to acquire private information, as long there is no information in the market, i.e. $T(0) = Q - \tau > 0$. If, however, each firm acquires this amount of private information such that $\tau_u = T(0) = Q - \tau$, there is so much information in the market, that it is individually optimal to stop the acquisition of information, i.e. $T(Q-\tau) = 0$. This is indeed quite similar to the Grossman–Stiglitz–Paradox where the impossibility of informationally efficient markets is claimed. The underlying problem there is that the information revealed by prices makes it unattractive to spend any resources for the acquisition of private information. A similar problem appears here, where, because of nonexistence of a SREE, it is not possible by rational agents to rule out the possibility that no firm acquires information privately, since it might be that prices are too informative. Starting from the fact that it is common knowledge that $\tau_u(i) \geq 0$ for all i, the firms therefore are able to educe that no firm will ever acquire information with precision larger than $Q - \tau$, but they are not able restrict this set of possible precisions any further. Thus, in this case the whole set $[0, Q-\tau]$ is compatible with the assumptions of individual rationality and common knowledge,

Example 3: In this final expample, we look at a specification, where the precision of the noise is given by $\tau_\varepsilon = 1.3$, which is between the two values considered in the above two examples. From equations (6b) and (6d), the values for γ_1 and τ_u in a rational expectations equilibrium can be computed as $\gamma_1^* = 0.58193$ and $\tau_u^* = 0.58193$. If the case, where the amount τ_u^* of private information is assumed to be exogenously given, is again considered first, it turns out that a SREE exists, since $\alpha^2 \tau_\varepsilon \gamma_1^{*2} = 0.31807$, which is smaller than τ_u^*. However, if information is endogenously acquired, the stronger stability condition is $\alpha^2 \tau_\varepsilon \gamma_1^{*2} < \frac{1}{2}\tau_u^*$, which is not satisfied, since $\tau_u^*/2 = 0.290965$. Again, a SREE exists if the amount of private information is exogenously given, but does not exist if information is endogenously acquired.

Fig. 4. *Best response mapping* $T(\tau_u)$ *for example 3*



This numerical example is particuslarly interesting because it gives rise to a special kind of best response dynamics. To see this look at figure 4 where again the function $T(\tau_u)$ is plotted. In addition, however, we have now also plotted the second iterate of this function $T^2(\tau_u) \equiv T(T(\tau_u))$. As can be seen, this function possesses two additional fixed points, denoted $\underline{\tau}_u$ and $\overline{\tau}_u$. Notice too that the associated 2–cycle is stable. If we repeat the argumentation used in the discussion of the first two examples, we therefore get a process which converges to this 2–cycle: $T(0) = Q - \tau > 0$ implies that an individual firm will always acquire information, as long as there is no information at all in the market. As from this it follows that it is common knowledge that $\tau_u \leq Q - \tau$, no firm will ever aquire a precision smaller than $T(Q - \tau) = T^2(0)$. Hence, no firm ever expects so much information in the market that it becomes optimal to stop the individual acquisition of information. Iterating this argument allows only to eliminate precisions outside the interval $[\underline{\tau}_u, \overline{\tau}_u]$ as being incompatible with individual rationality and common knowledge. However, since the REE τ_u^* is not a SREE, all precisions in the set $[\underline{\tau}_u, \overline{\tau}_u]$ still constitute possible solutions under individual rationality and common knowledge. As the above discussed example 2 revealed, this kind of dynamics not necessarily emerges if the stability condition of Proposition 6 is violated. Moreover, existence of a SREE excludes the possibility of such a 2–cycle in the best response dynamics such that the associated best response dynamics indeed converge towards the REE precision.

Bearing in mind that the ‘amount of information in market’ τ_u is necessarily non–negative and $Q-\tau$ is the maximum precision ever acquired, we can restrict the analysis of the best response dynamics described by the mapping $T(\tau_u)$ to the set $S=[0,Q-\tau]$. Given this, the results regarding rationalizable precisions unformally described in the above three examples can then be gathered in the next proposition:

Proposition 7. *Consider the case with constant marginal costs of information acqusisition described by best response mapping $T(\tau_u)$ according to (7) with τ_u restricted to the set $S=[0,Q-\tau]$. Let S^* denote the set of rationalizable precisions which therefore represent outcomes of an eductive learning process on the side of the firms:*

(a) If a SREE exists, $S^*=\tau_u^*$, i.e. τ_u^* is the unique stable fixed point of the mapping $\tau_u'=T(\tau_u)$.

(b) If no SREE exits, the following two cases can be distinguished:

  (b.1) If $Q-\tau \geq \frac{Q^2}{\alpha^2 \tau_\varepsilon}$ we have $S^*=[0,Q-\tau]$, i.e. all precisions in S are compatible with individual rationality and common knowledge.

  (b.2) If $\frac{3}{4}\frac{Q^2}{\alpha^2 \tau_\varepsilon} \leq Q-\tau < \frac{Q^2}{\alpha^2 \tau_\varepsilon}$, we have $S^*=[\underline{\tau}_u,\overline{\tau}_u] \subset S$, i.e. the set of precisions compatible with individual rationality and common knowledge is a strict subset of S.

Proof. See Appendix. □

Notice that Proposition 7 connects the above derived condition for existence of a SREE, which requires prices in a REE to be at most half as informative than private signals directly with the parameters of the model. This is possible in case of constant marginal costs of information acqusition, because in this special case $\alpha^2\,(\gamma_1^*)^2\tau_\varepsilon < \frac{1}{2}\tau_u^*$ is equivalent to $\frac{3}{4}\frac{Q^2}{\alpha^2 \tau_\varepsilon} < Q-\tau$.

An immediate consequence of Proposition 7 is that the REE is the unique rationalizable solution of our market model, whenever it is strongly rational. This must be true, since existence of a SREE implies that the REE is a stable stationary point of the whole dynamical system given by equations (6a)—(6d). Thus, given the restriction of constant marginal costs of information acquisition, we get the result that individual rationality and common knowledge are indeed sufficient in order to justify the REE as an outcome of our market model with learning from price. However, this is true only if the price in the REE is at most half as informative as private signals. If this condition for existence of an SREE fails to hold, individual rationality and common knowledge alone are not sufficient to justify the REE. In the following section we will discuss this latter case in more detail.

4.3. *Nonexistence of an SREE and the Grossman–Stiglitz–Paradox*

While nonexistence of an SREE implies that the hypotheses of individual rationality and common knowledge are not sufficient to predict the REE as a reasonable outcoem of our model, Proposition 7 also reveals that it might still be possible to restrict the set of precisions in such a case.

Figure 5 displays the implications of this Proposition. The figure shows the set of rationalizable precisions τ_u dependend on the precision of the noise τ_ε. Notice, that an increase in the precision of the noise implies an increase in the precision of the market price even though the amount of information acquired in the REE becomes smaller when τ_ε grows.[11] Hence, as τ_ε increases when we move along the horizontal axis in the figure, the informativeness τ_p^* of the market price in the REE increases too. The solid line in the figure represents the amount of information acquisition in the REE, which decreases as τ_ε increases.

Now, as long $\tau_\varepsilon < \frac{3}{4}\frac{Q^2}{\alpha^2\,[Q-\tau]}$, which is equivalent to $\tau_p^* < \tau_u^*/2$, a SREE exists. Thus, the corresponding part of the solid line also represents the set ao rationalizable precisions, which coincides with the REE precision. If $\tau_\varepsilon \geq \frac{3}{4}\frac{Q^2}{\alpha^2\,[Q-\tau]}$, no SREE exists. The shaded area in the figure represents all precisions that are rationalizable in such a case. As can be seen, whenever the precision of prices is not too large, i.e. if $\tau_\varepsilon < \frac{Q^2}{\alpha^2\,[Q-\tau]}$, the best response dynamics exhibit a two–cycle and the hypotheses of individual rationality and common knowledge lead to restrictions on the set of rationalizable precisions. Even this is impossbile, when price become too informative, i.e. if $\tau_\varepsilon \geq \frac{Q^2}{\alpha^2\,[Q-\tau]}$.

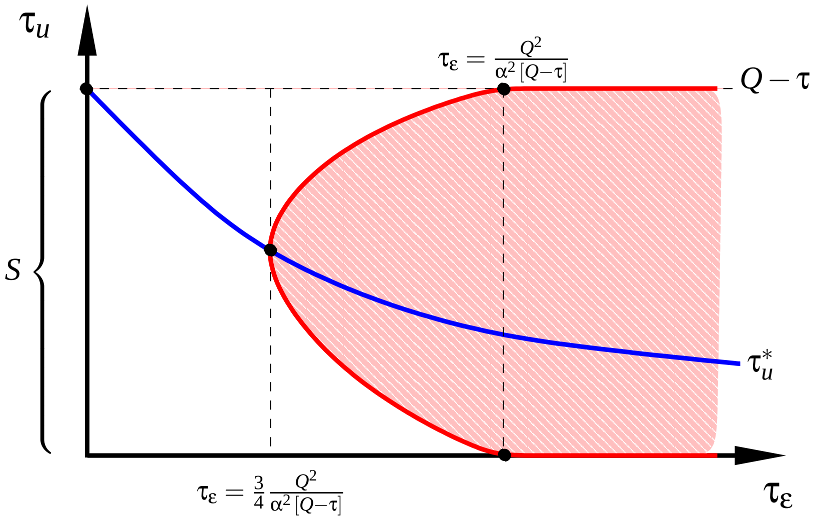
As prices become fully informative regarding the unknown parameter, which would happen in our model if $\tau_\varepsilon \to \infty$, the famous Grossman-Stiglitz-Paradox appears: In such a case, no firm has an incentive to acquire costly the information, prices will reveal anyway. If, however, no firm acquires any information, the market price cannot be revealing. In this case no REE exists.

This problem cannot arise, whenever the precision τ_ε of the noise is bounded from above, because then the precision of the market price is bounded from above too and thus, a REE exists. However, while this means that the Grossmann–Stiglitz–Paradox in its original shape does not show up in this case, the essentiell problem underlying it is still present. If the τ_ε is bounded from above but greater than $\frac{Q^2}{\alpha^2\,[Q-\tau]}$, individual rationality and common knowledge are not sufficient to exclude the possibility that it is indivually optimal to acquire no private information because there is already much information in the market. Since

---

[11] From Proposition 5 we get that in a REE $\gamma_1^* = \tau_u^*/Q$, such that τ_p^* is given by $\tau_p^* = \alpha^2\,(\tau_u^*)^2\,\tau_\varepsilon$.

Fig. 5. *Set of rationalizable strategies.*

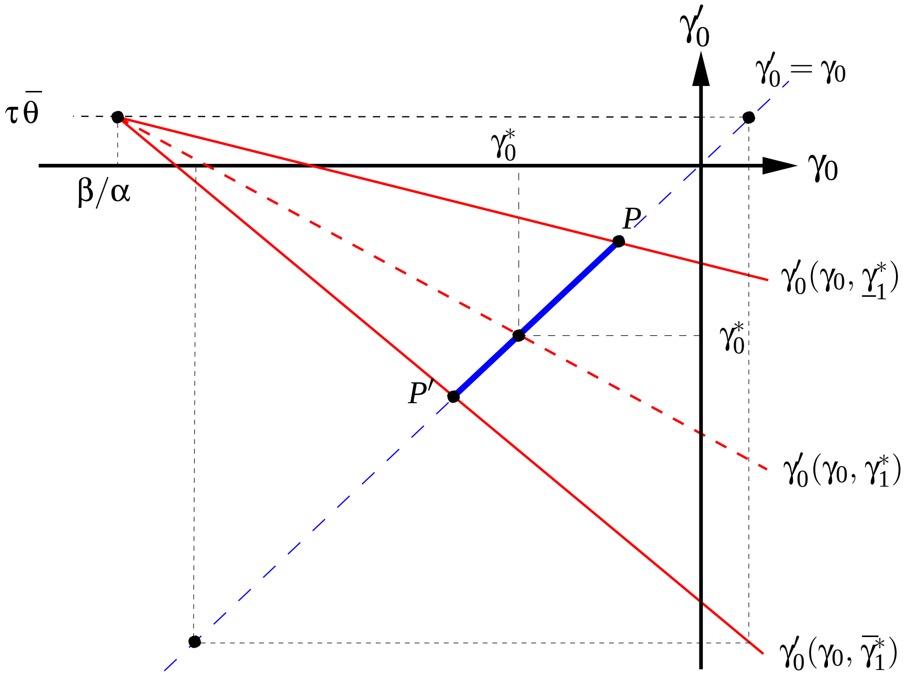


the corresponding best response mapping implies that it is optimal to acquire information individually if no other firm does it, but to stop the acquisition of information if every firms behaves like this, it is not possible to restrict the set S of precisions any further. This problem disappears only if prices are even less informative, which is the case if $\tau_\varepsilon < \frac{Q^2}{\alpha^2\,[Q-\tau]}$. In such a case the hypotheses of individual rationality and common knowledge indeed provide further restrictions on the set of precisions, but even then the set of rationalizable precisions not necessarily coincides with the REE precision.

Viewed from this perspective, the main result of the paper can be summarized as follows: Whenever prices are fully informative, no REE exists, since there is no incentive to acquire information. As the informativeness of prices becomes smaller, e.g. because τ_ε becomes smaller, a REE exists, but it may be impossible to justify this REE using the assumptions of rationality and common knowledge, since no SREE exist. Only if the informational content of prices falls short of a certain upper bound, it is at least possible to predict that rational individual behavior will be restricted to a particular set of actions and only for an even lower informativeness of prices a SREE exists.

Let us finally analyse, what we can say regarding the remaining two weights γ_0, γ_1 and γ_2 of the individual supply function in case of nonexistence of a SREE. Let us begin with the case where the mapping $T(\tau_u)$ possesses a stable two–cycle. This means that the assumptions of common knowledge and rationality restrict the set of possible individual precisions to the set $[\underline{\tau}_u^*, \overline{\tau}_u^*]$. Since $\gamma(i)_1 = \sqrt{Q}^{-1}\tau(i)_u$, set of individual weights for this private information is accordingly restricted to the set $[\underline{\gamma}_1^*, \overline{\gamma}_1^*] = [\sqrt{Q}^{-1}\underline{\tau}_u^*, \sqrt{Q}^{-1}\overline{\tau}_u^*]$.

Fig. 6. *Best response dynamics for the weight* γ_0.



It remains to ask whether it is also possible to restrict the remaining two weights γ_0 and γ_2 to particular sets. This requires to analyze the dynamical properties of the two equations (6a) and (6c) for all values for γ_1 that are rationalizable, i.e., for all $\gamma_1 \in [\underline{\gamma}_1^*, \bar{\gamma}_1^*]$. The next proposition establishes the respective result:

Proposition 8.

(i) Assume $\frac{3}{4}\frac{Q^2}{\alpha^2\tau_\varepsilon} \leq Q - \tau < \frac{Q^2}{\alpha^2\tau_\varepsilon}$*, such that* $S^* = [\underline{\tau}_u, \bar{\tau}_u]$ *and* $\gamma(i)_1$ *can be restricted to the set* $[\underline{\gamma}_1, \bar{\gamma}_1] = [\sqrt{Q}^{-1}\underline{\tau}_u, \sqrt{Q}^{-1}\bar{\tau}_u]$*. Let* $[\underline{\gamma}_0, \bar{\gamma}_0]$ *as well as* $[\underline{\gamma}_2, \bar{\gamma}_2]$ *denote the set of fixed points of equations (6a) and (6c) given* $\gamma_1 \in [\underline{\gamma}_1, \bar{\gamma}_1]$ *and* $\tau_u \in S^*$*. These sets then represent all values for the weights* γ_0 *as well as* γ_2 *which are compatible with common knowledge and rationality.*

(ii) If $S^* = [0, Q - \tau]$, $\gamma(i)_1$ *can be restricted to the set* $[0, \bar{\gamma}_1] = [0, \sqrt{Q}^{-1}(Q - \tau)]$ *and the hypotheses of rationality and common knowledge impose no further restrictions on the weights* γ_0 *and* γ_2

Proof. See Appendix. □

Figure 6 serves to illustrate this result. The figure shows the best response function for the weight γ_0 according to equation (6a) in case of a two–cycle for all values of γ_1 within the set $[\underline{\gamma}_1^*, \bar{\gamma}_1^*]$. The result stated in Proposition 8 builds on the fact that the maximum of slopes of these best responses (i.e. the slope of the straight line $\gamma_0'(\gamma_0, \overline{\gamma}_1^*)$) is less then one in absolute value, whenever a stable two–cycle exists. Given this it is possible to restrict the set of weights γ_0 that are compatible with rationality and common knowledge to values corresponding to the line segment between the points P and P' in the figure.[12] In a similar fashion it can be shown that regarding the weight γ_2 there exists a set $[\underline{\gamma}_2^*, \overline{\gamma}_2^*]$ of weights such that all γ_2 within this set are compatible with rationality and common knowledge. Thus, even if there may exist no SREE, the assumptions of rationality and common knowledge allow to restrict the set of possible supply functions that will be used by rational firms, when a stable two–cycle exists. If even this is not the case, that is, if not even a stable two–cycle exists, it is still possible to restrict the weights γ_1 and the precisions τ_u of the privately acquired information, but the best response mappings (6a) and (6c) are unstable for some of the reasonable values for γ_1 and τ_u. This means any values for γ_0 and γ_2 are compatible with rationality and common knowledge in this case.

5. Conclusions

In the present paper, we have shown how known results for existence of SREE must be modified, if models with endogenously acquired private information are considered. While this assumptions does not lead to modifications of the respective conditions for existence when there is no learning from prices, it turns out we arrive at stronger conditions if there is such learning. In particular, it was shown that prices in a REE need to be half as informative than private signals for a SREE to exist in case of learning from prices, whereas it is sufficient for prices to be less informative than private signals without such learning. It was also possible to give an interpretation of the result that falls back on the well known Grossman–Stiglitz Paradox of the impossibility of informationally efficient markets. Viewed from this perspective, our result says that for existence of a SREE markets have to show a minimum level of informational inefficiency.

Future work on this subject will analyze the case of increasing marginal costs of information acquisition in more detail in order to check the robustness of the results obtained for the case of constant marginal costs. Moreover, it should be analyzed whether the results carry over to financial market models with learning from current prices, where risk aversion of traders is allowed for.

---

[12] The underlying argument is quite similar to the one presented in section 3 in the discussion of figure 1.

References

Desgranges, G., (1999), Common knowledge and information revealed by prices: An example, mimeo.

Desgranges, G., Geoffard, P. and Guesnerie, R., (2003), Do prices transmit rationally expected information?, Journal of the European Economic Association 1, 124–153.

Grossman, S., (1976), On the efficiency of competitive stock markets where traders have diverse information, Journal of Finance 31, 573–585.

Grossman, S. and Stiglitz, J., (1980), On the impossibility of informationally efficient markets, American Economic Review 70, 393–408.

Guesnerie, R., (1992), An exploration of the eductive justifications of the rational–expectations hypothesis, American Economic Review 82, 1254–1278.

Guesnerie, R., (2002), Anchoring economic predictions in common knowledge, Econometrica 70, 439–480.

Heinemann, M., (2002), Existence of strongly rational expectations equilibria on asset markets with asymmetric information, Schumpeter Working Paper 2002-2, Humboldt–University Berlin, Berlin.

Heinemann, M., (2003), Are rational expectations equilibria with private information eductively stable?, forthcoming: Journal of Economics.

Verrecchia, R., (1982), Information acquisition in a noisy rational expectations economy, Econometrica 50, 1415–1430.

Appendix

Proof of Proposition 1. We will first show, that there exists an unique linear equilibrium with <u>given</u> precisions $\tau_u(i) \geq 0$ for all $i \in I$. After that, we derive the optimal individual amount of information acquisition in such an equilibrium. Assume that each $i \in I$ firm uses the linear strategy $x^*(i) = \psi\,[\gamma(i)_0 + \gamma(i)_1\, s(i)]$. With $\int_0^1 \gamma(i)_0\, di = \gamma_0$, and $\int_0^1 \gamma(i)_1\, di = \gamma_1$ as well as $x^s = \int_0^1 x^*(i)\, di = \psi[\gamma_0 + \gamma_1\,\theta]$ such strategies result in the market price:

$$p = \beta + \alpha\,[\gamma_0 + \gamma_1\,\theta] + \varepsilon$$

where $\alpha = -\psi/\varphi$. From this it follows $p - \theta = \beta + \alpha\gamma_0 + (\alpha\gamma_1 - 1)\,\theta + \varepsilon$ and the respective conditional expectation of a firm $i \in I$ results as:

$$\begin{aligned}
\mathrm{E}[p-\theta\,|\,s(i),w] &= \beta + \alpha\gamma_0 + (\alpha\gamma_1 - 1)\ \mathrm{E}[\theta\,|\,s(i),w] \\
&= \beta + \alpha\gamma_0 + (\alpha\gamma_1 - 1)\ \left[\frac{\tau(i)_u}{\tau + \tau(i)_u}\, s(i) + \frac{\tau}{\tau + \tau(i)_u}\,\hat{\theta}\right] \\
&= \beta + \alpha\gamma_0 + (\alpha\gamma_1 - 1)\,\frac{\tau}{\tau + \tau(i)_u}\,\hat{\theta} + (\alpha\gamma_1 - 1)\,\frac{\tau(i)_u}{\tau + \tau(i)_u}\, s(i) \qquad \text{(A.1)}
\end{aligned}$$

Therefore, in an equilibrium the individual coefficients must satisfy the following two equations:

$$\gamma(i)_0 = \beta + \alpha\gamma_0 + (\alpha\gamma_2 - 1)\,\frac{\tau}{\tau + \tau(i)_u}\,\hat{\theta} \tag{A.2a}$$

$$\gamma(i)_1 = (\alpha\gamma_1 - 1)\,\frac{\tau(i)_u}{\tau + \tau(i)_u} \tag{A.2b}$$

Now assume $\tau(i)_u = \tau_u^*$ for all $i \in I$. Eqs. (A.2a) and (A.2b) can then be solved for the equilibrium coefficients:

$$\gamma_0^* = \frac{\beta}{1-\alpha} - \frac{1}{1-\alpha}\,\frac{\tau}{\tau + (1-\alpha)\,\tau_u^*} \tag{A.3a}$$

$$\gamma_1^* = -\,\frac{\tau_u^*}{\tau + (1-\alpha)\,\tau_u^*} \tag{A.3b}$$

It remains to derive the optimal individual amount of information acquisition. Assume that the costs associated with information acquisition are given by $K(\tau(i)_u)$ and let $\kappa(\tau(i)_u)$ denote the respective marginal costs. Profit $\pi(i)$ of firm i in an equilibrium is then given by:

$$\begin{aligned}
\pi(i) &= [p - \theta]\,x(i) - \frac{1}{2}\,\frac{1}{\psi}\,[x(i)]^2 - K(\tau(i)_u) \\
&= \psi\left[\beta + \alpha\gamma_0 + (\alpha\gamma_1 - 1)\,\theta + \varepsilon\right]\left[\gamma(i)_0 + \gamma(i)_1\,s(i)\right] - \frac{1}{2}\,\psi\left[\gamma(i)_0 + \gamma(i)_1\,s(i)\right]^2 - K(\tau(i)_u)
\end{aligned} \tag{A.4}$$

We can write Eq. (A.4) as follows:

$$\begin{aligned}
\pi(i) = {} & \psi\,\left[\beta + (\alpha\gamma_1 - 1)\,[\theta - \bar{\theta}] + (\alpha\gamma_1 - 1)\bar{\theta} + \alpha\gamma_0 + \varepsilon\right]\,\left[\beta + \gamma(i)_0 + \gamma(i)_1\,[s(i) - \bar{\theta}] + \gamma(i)_1\,\bar{\theta}\right] \\
& - \frac{1}{2}\,\psi\left[\beta + \gamma(i)_0 + \gamma(i)_1\,[s(i) - \bar{\theta}] + \gamma(i)_1\,\bar{\theta}\right]^2 - K(\tau(i)_u)
\end{aligned}$$

Taking expectations then yields:

$$\begin{aligned}
\mathrm{E}[\pi(i)] = {} & \psi\,\left(\beta + (\alpha\gamma_1 - 1)\bar{\theta} + \alpha\gamma_0\right)\left(\beta + \gamma(i)_0 + \gamma(i)_1\,\bar{\theta}\right) + \psi\gamma(i)_1\,(\alpha\gamma_1 - 1)\,\frac{1}{\tau} \\
& - \frac{\psi}{2}\,\left(\gamma(i)_0 + \gamma(i)_1\,\bar{\theta}\right)^2 - \frac{\psi}{2}\,\gamma(i)_1^2\left(\frac{\tau + \tau(i)_u}{\tau(i)_u\,\tau}\right) - K(\tau(i)_u)
\end{aligned} \tag{A.5}$$

The first order conditions with respect to $\gamma(i)_0$, $\gamma(i)_1$ and $\tau_u(i)$ are:

$$\begin{aligned}
\frac{\partial\,\mathrm{E}[\pi(i)]}{\partial\gamma(i)_0} &= \psi\,\left(\beta + (\alpha\gamma_1 - 1)\bar{\theta} + \alpha\gamma_0\right) - \psi\,\left(\gamma(i)_0 + \gamma(i)_1\,\bar{\theta}\right) \\
\frac{\partial\,\mathrm{E}[\pi(i)]}{\partial\gamma(i)_1} &= \psi\bar{\theta}\left(\beta + (\alpha\gamma_1 - 1)\bar{\theta} + \alpha\gamma_0\right) + \psi\,(\alpha\gamma_1 - 1)\,\frac{1}{\tau} - \psi\bar{\theta}\,\left(\gamma(i)_0 + \gamma(i)_1\,\bar{\theta}\right) - \psi\gamma(i)_1\,\frac{\tau + \tau(i)_u}{\tau(i)_u\,\tau} \\
\frac{\partial\,\mathrm{E}[\pi(i)]}{\partial\tau(i)_u} &= \psi\,\frac{1}{2}\,\gamma(i)_1^2\,\frac{1}{\tau(i)_u^2} - \kappa(\tau(i)_u)
\end{aligned}$$

We obtain the following solutions:

$$\gamma(i)_0 = \beta + \alpha\gamma_0 + (\alpha\gamma_1 - 1)\,\frac{\tau}{\tau + \tau(i)_u}\,\hat{\theta} \tag{A.6a}$$

$$\gamma(i)_1 = (\alpha\gamma_1 - 1)\,\frac{\tau(i)_u}{\tau + \tau(i)_u} \tag{A.6b}$$

$$0 = \psi\,\frac{1}{2}\left[\frac{\gamma(i)_1}{\tau(i)_u}\right]^2 - \kappa(\tau(i)_u) \tag{A.6c}$$

Under the assumption of an equilibrium with $\gamma(i)_1 = \gamma_1^*$ and $\tau(i)_u = \tau_u^* > 0$, substitution of Eq. (A.6b) into (A.6c) gives:

$$\frac{\psi}{2}\,\frac{1}{(\tau + (1-\alpha)\,\tilde{\tau}_u)^2} = \kappa(\tilde{\tau}_u) \tag{A.7}$$

Eq. (A.7) will not necessarily possess a solution with $\tau_u^* > 0$. In such a case, the respective solution is $\tau_u^* = 0$. Together with the above derived Eqs. (A.3a) and (A.3b) the REE is then completely described. □

Proof of Lemma 1. The best response mapping has already been derived while proving Proposition 1. It is given by Eqs.(A.6a)–(A.6c).

The total differentials of these equations evaluated at the REE are given by:

$$d\gamma'(i)_0 + (\alpha\gamma_1^* - 1)\frac{\tau}{[\tau + \tau_u^*]^2}\,\hat{\theta}\,d\tau'(i)_u = \alpha\,d\gamma_0 + \alpha\,\frac{\tau}{\tau + \tau_u^*}\,\hat{\theta}\,d\gamma_1 \tag{A.8a}$$

$$d\gamma'(i)_1 - (\alpha\gamma_1^* - 1)\frac{\tau_u^*}{[\tau + \tau_u^*]^2}\,d\tau'(i)_u = \alpha\,\frac{\tau_u^*}{\tau + \tau_u^*}\,d\gamma_1 \tag{A.8b}$$

$$W\,d\gamma'(i)_1 - d\tau_u'(i) = 0\,, \tag{A.8c}$$

where $W \equiv \frac{\psi\frac{\gamma_1}{\tau_u^2}}{\kappa' + \psi\frac{\gamma_1{}^2}{\tau_u^3}}$. Using matrices, this system can be formulated as follows:

$$\begin{bmatrix} 1 & 0 & \frac{(\alpha\gamma_1^* - 1)\,\tau}{[\tau + \tau_u^*]^2}\,\hat{\theta} \\ 0 & 1 & -\frac{(\alpha\gamma_1^* - 1)\,\tau_u^*}{[\tau + \tau_u^*]^2} \\ 0 & W & -1 \end{bmatrix} \begin{pmatrix} d\gamma'(i)_0 \\ d\gamma'(i)_1 \\ d\tau'(i)_u \end{pmatrix} = \begin{bmatrix} \alpha & \frac{\alpha\tau}{\tau + \tau_u^*}\,\hat{\theta} & 0 \\ 0 & \frac{\alpha\tau_u^*}{\tau + \tau_u^*} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} d\gamma_0 \\ d\gamma_1 \\ d\tau_u \end{pmatrix}$$

We write this system as $x' = Px$, where $P = A^{-1}B$. Since it turns out that P is a triangular matrix, its eigenvalues are equal the elements on its main diagonal. The respective eigenvalues λ_1, λ_2 and λ_3 are:

$$\lambda_1 = 0\,, \quad \lambda_2 = \alpha\,, \quad \lambda_3 = -\frac{\alpha\left(\gamma_1^2\,\psi + \kappa'(\tau_u)\,\tau_u^3\right)}{\gamma_1^2\,\psi + \kappa'(\tau_u)\,\tau_u^2(\tau + \tau_u)}$$

□

Proof of Proposition 5. The proof parallels the proof of Proposition 1. Starting point again is that profit $\pi(i)$ of firm i in an equilibrium is then given by:

$$\pi(i) = [p - \theta]\,x(i) - \frac{1}{2}\,\frac{1}{\psi}\,[x(i)]^2 - K(\tau(i)_u) \tag{A.9}$$

Since we assume a linear supply rule and are only interested in the optimal choice of $\tau(i)_u$, only the last two terms of the profit equation are relevant. With $C(i)$ denoting these two terms, we get:

$$\begin{aligned} C(i) &= -\frac{1}{2}\frac{1}{\psi}\,[x(i)]^2 - K(\tau(i)_u) \\ &= -\frac{1}{2}\,\psi\left[(1-\gamma(i)_2)\,p - \gamma(i)_0 - \gamma(i)_2\,s(i)\right]^2 - K(\tau(i)_u) \end{aligned}$$

Since expectations are conditioned on the price p, taking expectations and differentiating with respect to $\tau_u(i)$ yields

$$\frac{\partial\,\mathrm{E}[C(i)\,|\,p]}{\partial\tau(i)_u} = \psi\,\frac{1}{2}\,\gamma(i)_2^2\,\frac{1}{\tau(i)_u^2} - \kappa(\tau(i)_u) \tag{A.10}$$

Thus, the necessary condition for the optimal amount of information acquisition is the same as in the case, where there is no learning from prices. □

Proof of Lemma 2. The best response mapping for the case with a given amount of private information (i.e., Eqs. (6a)– (6c) is derived in Heinemann (2003). The additional Eq. (6d) has been derived in the above given proof of Proposition 5. □

Proof of Proposition 6. The relevant dynamical system is given by Eqs. (6a) – (6d). The total differentials of these equations evaluated at the REE are given by:

$$d\gamma'(i)_0 + \frac{\gamma_0^*}{Z}\,d\tau'(i)_u = -\frac{\alpha^2\gamma_1^{*2}\,\tau_\varepsilon}{\tau_u^*}\,d\gamma_0 + \frac{[\beta\tau_\varepsilon - \alpha\gamma_0^* - 2\alpha\gamma_1^*\,\gamma_0^*]\,\alpha\,\tau_\varepsilon}{Z}\,d\gamma_1 \tag{A.11}$$

$$d\gamma'(i)_1 - \frac{1-\gamma_1^*}{Z}\,d\tau'(i)_u = -\frac{2\,\alpha^2\gamma_1^{*2}\,\tau_\varepsilon}{Z}\,d\gamma_1 \tag{A.12}$$

$$d\gamma'(i)_2 + \frac{\gamma_2^*}{Z}\,d\tau'(i)_u = \frac{\alpha\,\tau_\varepsilon(1-\alpha(1-\gamma_2^*)) + 2\,\alpha^2\gamma_1^*\gamma_2^*\,\tau_\varepsilon}{Z}\,d\gamma_1 - \frac{\alpha^2\gamma_1^{*2}\,\tau_\varepsilon}{\tau_u^*}\,d\gamma_2 \tag{A.13}$$

$$W\,d\gamma'(i)_1 - d\tau_u'(i) = 0\,, \tag{A.14}$$

where $Z = \tau + \tau_u^* + \alpha^2\,\gamma_1^{*2}\,\tau_\varepsilon$ and $W \equiv \frac{\psi\frac{\gamma_1}{\tau_u^2}}{\kappa' + \psi\,\frac{\gamma_1{}^2}{\tau_u^3}}$. Using matrices, this system can be formulated as follows:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{\gamma_0^*}{Z} \\ 0 & 1 & 0 & -\frac{1-\gamma_1^*}{Z} \\ 0 & 0 & 1 & \frac{\gamma_2^*}{Z} \\ 0 & W & 0 & -1 \end{bmatrix} \begin{pmatrix} d\gamma'(i)_0 \\ d\gamma'(i)_1 \\ d\gamma'(i)_2 \\ d\tau'(i)_u \end{pmatrix} =$$

$$\begin{bmatrix} -\frac{\alpha^2\gamma_1^{*2}\,\tau_\varepsilon}{\tau_u^*} & \frac{[\beta\tau_\varepsilon - \alpha\gamma_0^* - 2\alpha\gamma_1^*\,\gamma_0^*]\,\alpha\,\tau_\varepsilon}{Z} & 0 & 0 \\ 0 & -\frac{2\,\alpha^2\gamma_1^{*2}\,\tau_\varepsilon}{Z} & 0 & 0 \\ 0 & \frac{\alpha\,\tau_\varepsilon[(1-\alpha(1-\gamma_2^*)) + 2\,\alpha\gamma_1^*\gamma_2^*]}{Z} & -\frac{\alpha^2\gamma_1^{*2}\,\tau_\varepsilon}{\tau_u^*} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} d\gamma_0 \\ d\gamma_1 \\ d\gamma_2 \\ d\tau_u \end{pmatrix}$$

We write this system as $x' = Px$, where $P = A^{-1}B$. Since it turns out that P is a triangular matrix, its eigenvalues are equal the elements on its main diagonal. The respective eigenvalues $\lambda_1 \dots \lambda_4$ are:

$$\lambda_1 = 0\,, \quad \lambda_2 = \lambda_3 = -\frac{\alpha^2 \gamma_1^{*2} \tau_\varepsilon}{\tau_u^*}\,, \quad \lambda_4 = -\frac{2\,\alpha^2 \gamma_1^{*2} \tau_\varepsilon}{Z - (1-\gamma_1^*)W}$$

The condition for stability of this dynamical system and, thus, the condition for existence of a SREE is that all eigenvalues are less than one in absolute value.

If we now assume that marginal costs of information acquisition are constant and equal to $\bar{\kappa}$, such that $\kappa' = 0$, we have $W = \frac{\tau_u}{\gamma_1}$. From equation (6b) it follows that in equilibrium $W = \tau + \tau_u \alpha^2 \gamma_1^2 \tau_\varepsilon$ and from this we get $(1-\gamma_1)W = \tau + \alpha^2 \gamma_1^2 \tau_\varepsilon$. Stability in this case requires:

$$\frac{\alpha^2 \gamma_1^{*2} \tau_\varepsilon}{\tau_u^*} < 1 \quad \text{and} \quad \frac{2\,\alpha^2 \gamma_1^{*2} \tau_\varepsilon}{\tau_u^*} < 1\,,$$

where the second inequality is obviously stronger, such that our stability condition in fact is:

$$2\,\frac{\alpha^2 \gamma_1^2 \tau_\varepsilon}{\tau_u} < 1 \quad \Rightarrow \quad \alpha^2 \gamma_1^2 \tau_\varepsilon < \frac{1}{2}\tau_u$$

□

Proof of Proposition 7. The proof proceeds in two steps. The first step is to derive some properties of the mapping $T^2(\tau_u)$ in order to find conditions for the existence of a 2–cycle in the best response mapping. The second step then draws the relevant conclusions.

(1) Consider the function $f(\tau_u) = Q - \tau - \frac{\alpha^2 \tau_\varepsilon}{Q^2} {\tau_u}^2$, which appears in Eq. (7) and let $f^2(\tau_u)$ denote its second interate, i.e. $f^2(\tau_u) \equiv f(f(\tau_u))$. It is straightforward to show that (a) $f^2(\tau_u)$ is monotone and increasing and that (b) $f^2(\tau_u)$ has exactly one inflection point for $\tau_u > 0$:

   a) With respect to the derivative with respect to τ_u, $f^{2'}(\tau_u)$, we get:

   $$f^{2'}(\tau_u) = f'(\tau_u)\, f'(f(\tau_u)) \geq 0$$

   because $f'(\tau_u) \leq 0$.

   b) The second derivative with respect to τ_u, $f^{2''}(\tau_u)$, is given by:

   $$f^{2''}(\tau_u) = 4\left(\frac{\alpha^2 \tau_\varepsilon}{Q^2}\right)^2 \left[\tau_u\, f'(\tau_u) + f(\tau_u)\right]$$

   From this it follows that $f^{2''}(\tau_u) = 0$, if $f(\tau_u) = \tau_u\, f'(\tau_u)$ which is equivalent to:

   $$(Q - \tau) = \tau_u^2\, \frac{\alpha^2 \tau_\varepsilon}{Q^2}$$

   With $Q - \tau > 0$, this equation possesses two real roots, such that there are two points of inflection where $f^{2''}(\tau_u) = 0$ and at most one such point in $S = [0, Q - \tau]$. Since $T(\tau_u) = \max\{0,\, f(\tau_u)\}$, it then follows, that $T^2(\tau_u)$ is monotone increasing on S with at most one point of inflection.

(2) Consider now the case where no SREE exists. This implies $|T'(\tau_u^*)| > 1$ and therefore $T^{2'}(\tau_u^*) > 1$. From the above derived properties of $T^2(\tau_u)$ it then follows that a 2–cycle with $0 < \underline{\tau}_u < \overline{\tau}_u <$

$Q-\tau$ and $T^2(\underline{\tau}_u) = T(\bar{\tau}_u) = \underline{\tau}_u$ exists if and only if $T^2(0) = T(Q-\tau) > 0$. Now, $T(Q-\tau)$ is given by:

$$T(Q-\tau) = \max\left\{0,\, [Q-\tau]\left(1 - \frac{\alpha^2\,\tau_\varepsilon}{Q^2}\,[Q-\tau]\right)\right\} \tag{A.15}$$

As an REE with $\tau_u^* > 0$ requires $Q-\tau > 0$, $T^2(0) > 0$ if and only if:

$$\tau_\varepsilon < \frac{\alpha^2\,[Q-\tau]}{Q^2} \tag{A.16}$$

If this condition is satisfied, a stable 2–cycle is a solution of the mapping $\tau_u' = T(\tau_u)$, and the best response dynamics converge to this 2–cycle. Thus, $S^* = [\underline{\tau}_u, \bar{\tau}_u]$ in this case. Otherwise, no sich cycle exists and because τ_u^* is unstable, we have $S^* = S$.

(3) Consider finally the case where $|T'(\tau_u^*)| < 1$ such that a SREE exists. In this case we have $T^{2\prime}(\tau_u^*) < 1$. Moreover, from $\tau_u^* = T(\tau_u^*)$ we get that:

$$\tau_u^* = \frac{Q^2}{2\,\alpha^2\,\tau_\varepsilon}\left\{\sqrt{4\,(Q-\tau)\,\frac{\alpha^2\,\tau_\varepsilon}{Q^2} + 1} - 1\right\}$$

With this, our condition for existence of a SREE becomes:

$$\alpha^2\,\gamma_1^{*2}\,\tau_\varepsilon = \frac{\alpha^2\,\tau_\varepsilon}{Q^2}\,\tau_u^{*2} < \frac{1}{2}\,\tau_u^* \quad\Leftrightarrow\quad \frac{\alpha^2\,\tau_\varepsilon}{Q^2}\,\tau_u^* < \frac{1}{2}$$

$$\Leftrightarrow\quad (Q-\tau)\,\frac{\alpha^2\,\tau_\varepsilon}{Q^2} < \frac{3}{4}$$

As this implies $T(0) > 0$ (cf. eq. (A.16)), a 2–cycle cannot exist in this case. Hence τ_u^* is the unique stable fixed point of the mapping $\tau_u' = T(\tau_u)$ and $S^* = \tau_u^*$.

□

Proof of Proposition 8. The slopes of the best responses (6a) and (6c) for a given value of γ_1 are given by:

$$\frac{\partial\gamma_0'}{\partial\gamma_0} = \frac{\partial\gamma_2'}{\partial\gamma_2} = -\frac{\alpha^2\,\gamma_1\,\tau_\varepsilon}{\tau + \tau_u' + \alpha^2\,\gamma_1^2\,\tau_\varepsilon} \equiv \Gamma$$

It must be shown that this slope is smaller than one in absolute value for the maximum value, the weight γ_1 can attain, if and only if $T^2(0) > 0$.

Let $\bar{\tau}_u$ denote the precision for which $T(\bar{\tau}_u) = 0$:

$$(Q-\tau)\,\frac{Q^2}{\alpha^2\,\tau_\varepsilon} = \bar{\tau}_u^2$$

This precision implies that $\gamma_1 = \bar{\tau}_u/Q$, which is the maximum value γ_1 can attain, as well as $\tau_u' = 0$. In this case, the slope is given by:

$$\begin{aligned}
\Gamma(\bar{\tau}_u) &= -Q\,\frac{\alpha^2\,\tau_\varepsilon\,\bar{\tau}_u}{Q^2\,\tau\alpha^2\,\tau_\varepsilon\,\bar{\tau}_u^2} = -Q\,\frac{\alpha^2\,\tau_\varepsilon\,\bar{\tau}_u}{Q^2\,\tau\alpha^2\,\tau_\varepsilon\left[(Q-\tau)\,\frac{Q^2}{\alpha^2\,\tau_\varepsilon}\right]} \\
&= -\frac{1}{Q^2}\,\alpha^2\,\bar{\tau}_u\,\tau_\varepsilon = -\frac{1}{Q^2\,\bar{\tau}_u}\,\alpha^2\,\bar{\tau}_u^2\,\tau_\varepsilon = -\frac{1}{Q^2\,\bar{\tau}_u}\,\alpha^2\,\tau_\varepsilon\,(Q-\tau)\,\frac{Q^2}{\alpha^2\,\tau_\varepsilon} \\
&= -\frac{Q-\tau}{\bar{\tau}_u}
\end{aligned} \tag{A.17}$$

From (A.17) it follows that $|\Gamma| < 1$ if and only if $Q-\tau < \bar{\tau}_u$, and because $T(\tau_u)$ is monotone decreasing, this requires $T^2(0) = T(T(0)) = T(Q-\tau) > 0$. □

Working Paper Series in Economics

(see www.uni-lueneburg.de/vwl/papers for a complete list)

No. 1: *Joachim Wagner:* Nascent and Infant Entrepreneurs in Germany. Evidence from the Regional Entrepreneurship Monitor (REM). March 2005

No. 2: *Ingrid Ott and Stephen Turnovsky:* Excludable and Non-Excludable Public Inputs: Consequences for Economic Growth, March 2005 (also published as CESifo Working Paper 1423)

No. 3: *Thomas Wein and Reimund Schwarze:* Is the Market Classification of Risk Always Efficient? - Evidence from German Third Party Motor Insurance, March 2005

No. 4: *Joachim Wagner:* Exports and Productivity: A Survey of the Evidence from Firm Level Data. March 2005

No. 5: *Christiane Clemens and Maik Heinemann:* Endogenous Redistributive Cycles — An Overlapping Generations Approach to Social Conflict and Cyclical Growth, March 2005

No. 6: *Christiane Clemens and Maik Heinemann:* On the Effects of Redistribution on Growth and Entrepreneurial Risk-Taking, March 2005

No. 7: *Thomas Wein:* Associations‘ Agreement and the Interest of the Network Suppliers — The Strategic Use of Structural Features, March 2005

No. 8: *Joachim Wagner:* Exports, Foreign Direct Investment, and Productivity: Evidence from German Firm Level Data. March 2005

No. 9: *Gabriel Desgranges and Maik Heinemann:* Strongly Rational Expectations Equilibria with Endogenous Acquisition of Information, March 2005

Universität Lüneburg
Institut für Volkswirtschaftslehre
Scharnhorststraße 1
D-21332 Lüneburg
Tel.: ++49 4131 78 2321
email: brodt@uni.lueneburg.de
www.uni-lueneburg.de/vwl/papers