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Decentralized Production and Public Liquidity with Private Information

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Abstract: Firms with private information about the outcomes of production under uncertainty may face capital (liquidity) constraints that prevent them from attaining efficient levels of investment in a world with costly and/or imperfect monitoring. As an alternative, we examine the efficiency of a simple pooling scheme designed to provide a public (cooperative) supply of liquidity that results in the first best outcome for economic growth. We show that if, absent aggregate uncertainty, the elasticity of scale of the production technology is sufficiently small, then efficient levels of investment and growth can always be supported. Finally, some results for a special case (constant elasticity of scale) are examined when investors face aggregate uncertainty. We show that, in addition to a low elasticity of scale for the production function, investors must have sufficiently optimistic prior beliefs if efficient growth is to be achieved regardless of the actual future state of the world.

JEL classification: D30, D50, D82, D92

Key words: private information, production economies, welfare analysis, liquidity

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Decentralized Production and Public Liquidity with Private Information

1. Introduction.

Numerous authors, in a wide variety of settings, have investigated conditions that are sufficient for efficient allocations when there is private information. For example, with fixed preferences and common knowledge concerning outcomes, Debreu (1959) has shown that complete securities markets are sufficient to implement Pareto optimal allocations in a given economy. However, it is generally the case that the social optimum cannot be achieved, with securities markets or otherwise, when there is asymmetric information regarding outcomes from private production. This follows from the fact that if information regarding output is private to each firm, any claims on the firm, public or private, can be repudiated. Thus, any monitoring scheme that is costly and/or less than fully revealing will result in, at best, a constrained Pareto Optimal allocation.

Atkeson and Lucas (1991) study the problem of efficient allocation in a world with a known level of aggregate output each period but private information regarding shocks to concave preferences. They develop an allocation rule that minimizes social costs subject to the requirement that individual anonymity be maintained and show, for a class of preferences, that this least cost rule results in ever increasing levels of income inequality over time. They go on to discuss decentralized allocation mechanisms, specifically a bond market, and conclude that it cannot be used to mimic the least cost allocation rule.

This paper addresses the question of efficient allocations in the context of a production economy when there is private information concerning output and participants are risk neutral. As noted earlier, the market for claims on private assets cannot function due to private information. Not surprisingly, when a firm's only supply of capital is its private storage or saving, the firm may face liquidity constraints that prevent it from attaining efficient levels of investment, even though in the aggregate there is sufficient capital to fund first best levels of economic growth.

Our approach to solving this problem involves leaving production to take place in a decentralized setting while at the same time introducing a simple pooling scheme where firms can save collectively, i.e., endogenously make their storage publicly observable. We introduce a planner that can coordinate the use of pooled liquidity by specifying the withdrawal requirements that respect firm anonymity.

In order to achieve efficient levels of investment and growth the pooling scheme must effect a transfer of liquidity from firms with good productive outcomes to firms with poor productive outcomes. We show that such a transfer may not be incentive compatible if the returns to scale of the technology are large. Specifically, when there is no aggregate uncertainty, a condition is identified on the elasticity of scale of the private technology that is sufficient for incentive compatibility to hold and hence for efficient levels of investment and growth to be implemented. We show feasibility of this condition in the context of the family of non-increasing elasticity of scale technologies.

When aggregate uncertainty is introduced we show, for the special case of constant elasticity production functions, that if firms are sufficiently optimistic and the elasticity of scale is sufficiently low, feasibility and incentive compatibility of the public liquidity system is maintained. However, if investors are very pessimistic ex ante and/or the scale elasticity is high, efficient levels of investment may not be feasible or incentive compatible.

Intuitively, if firm priors are poor, most of their endowment goes to storage. Such a strategy may a.) yield insufficient future aggregate output to fund efficient levels of investment and/or b.) cause the ex post prosperous firms to abandon the collective pooling scheme. This later action will occur when the expected payoff from excess levels of investment in the private technology exceed those from optimally investing in the private technology, paying the liquidity tax and leaving any remainder on personal balance in the liquidity pool. Thus, it is when the economy turns out to be prosperous relative to expectations that there are problems in keeping the system of collective liquidity management from unraveling.

The remainder of the paper is structured as follows. In section 2 we outline the basic model without aggregate uncertainty and show that investment when there is only private storage is strictly less than the full information optimum. Equivalently, this is a situation where too much liquidity is maintained relative to the social optimum. Section 3 contains the simple public pooling scheme for storage. Sufficient conditions for incentive compatibility for this case are provided in Section 4. The case of aggregate uncertainty is treated in Section 5 while Section 6 contains a summary and some additional discussion of the results as they relate to the extant literature.

2. The Basic Model.

Consider a one good, two period (three date) economy populated by a large number of risk neutral participants, we will call them firms, with identical productive opportunities; a risky production technology that displays decreasing marginal productivity and a storage technology yielding a zero rate of return. At date zero each firm is endowed with units of the single storable good. At date one output from the investment at date zero is realized. Adding this output to date zero storage gives the resources available for investment at date one. There is no intermediate endowment and at date two the output from the investment at date one is realized and the economy is over. We assume that the outcome from investment is the private information of the firm. Firms are interested in making expected wealth at date 2 as large as possible[1].

The production technology is given by the production function $f: \Re_+^2 \rightarrow \Re_+$, which is assumed to be as smooth as necessary for what follows. Denote invested capital by I, the realized value of the production shock by θ, and define realized output by $f(I, \theta)$. For simplicity, we assume that there are only two possible realizations of the production shock; 0 or $\theta \neq 0$. We also assume that $0 = f(I, 0) = f(0, \theta) < f(I, \theta)$, $I > 0$. With this specification, we will suppress the reference to the production shock and just write f(I) for $f(I, \theta)$, when it is convenient to do so. We assume that capital is productive but marginal productivity is decreasing, i. e., $f''(I) < 0 < f'(I)$, $I > 0$.

Initially, we assume that the production shocks $(\tilde{\theta}_1, \tilde{\theta}_2)$ are iid with p = Prob{ $\tilde{\theta}_i \neq 0$}, i = 1, 2, some p, $0 < p < 1$, which is the case where there is no aggregate uncertainty. We later drop this assumption and consider the question of efficient investment when there is aggregate uncertainty in the economy.

The one period productive optimum[2] is the level of I that achieves

$$\sup_{I \geq 0} \; pf(I) - I. \qquad (1)$$

The first order condition for (1) is

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[1] We rule out intermediate consumption although the model could be generalized to endogenize the consumption decision if some cost to consumption is incorporated into the framework. For example, if at any given date the investor can only consume or produce, it can be shown that public liquidity will remain optimal for some, but not all, values of the parameters of the problem.

[2] We drop the subscripts on the production shocks until they are needed to avoid confusion

$$pf'(I) = 1. \quad (2)$$

By concavity, (2) is both a necessary and a sufficient condition for (1), and if a solution exists it is unique. We assume that there is such a solution of (2) and that $I^* > 0$.

Each firm has initial resources of W_0 units of the single good such that

$$I^* < W_0 < 2I^*. \quad (3)$$

Thus <u>ex ante</u>, there is enough private liquidity to undertake the productive optimum in the first period, but no firm has enough resources to ensure that it can undertake the unconstrained productive optimum in each of two periods. This is the essence of the liquidity problem and we refer to the quantity $2I^* - W_0$ as the liquidity gap or liquidity shortfall.

Given this set-up it is straightforward to show that autarky (private storage and private production) leads to an over supply of liquidity relative to that carried if the unconstrained optimum is achievable. To see this, let I_0 denote the firm's investment at date zero. At date one, if the value of the production shock is $\tilde{\theta}_1 = \theta_1$, the firm will have resources of

$$W_1 = f(I_0, \theta_1) + W_0 - I_0. \quad (4)$$

Letting I_1 denote the firm's investment at date one, then if the value of the production shock at date two is $\tilde{\theta}_2 = \theta_2$, resources are given by

$$W_2 = f(I_1, \theta_2) + W_1 - I_1. \quad (5)$$

Since by assumption the firm chooses I_0 and I_1 to achieve

$$\sup \ E[\tilde{W}_2] \quad (6)$$

$$\text{subject to: } 0 \le I_0 \le W_0$$

$$0 \leq I_1 \leq \tilde{W}_1$$

(4), (5),

then, given I_0 and θ_1, the firm will choose I_1 so as to achieve

$$\sup\ E\left[\tilde{W}_2 \middle| \theta_1\right] \qquad (7)$$

subject to : $0 \leq I_1 \leq W_1$

(4), (5).

Now (7) is the one period productive optimum problem whenever $W_1 \geq I^*$. Furthermore, since $E\left[\tilde{W}_2 \middle| \theta_1\right]$ is increasing in I_1 the solution to (7) is given by

$$I_1^* = I^*, \text{if } W_1 \geq I^*, \qquad (8')$$

$$= W_1, \qquad \text{if } W_1 < I^*. \qquad (8'')$$

Let W_2^* denote (5) composed with (8). Then the value function for (7) is given by

$$E\left[\tilde{W}_2^* \middle| \theta_1\right] = pf\left(I^*\right) + W_1 - I^*, \qquad \text{if } W_1 \geq I^*, \qquad (9')$$

$$= pf\left(W_1\right), \qquad \text{if } W_1 < I^*. \qquad (9'')$$

Now, from (9) $E\left[\tilde{W}_2^* \middle| \theta_1\right]$ is continuous in I_0. Furthermore, we have that

$$\frac{\partial E\left[\tilde{W}_2^* \middle| \theta_1\right]}{\partial I_0} = \frac{\partial W_1}{\partial I_0}, \qquad \text{if } W_1 > I^*, \qquad (10')$$

$$= pf'\left(W_1\right)\frac{\partial W_1}{\partial I_0}, \qquad \text{if } W_1 < I^*, \qquad (10'')$$

i. e., $E\left[\tilde{W}_2^* \middle| \theta_1\right]$ is differentiable in I_0 whenever $W_1 \neq I^*$. For the case $W_1 = I^*$, (10') and (10") are the same since in this case, by (2), $pf'\left(W_1\right) = 1$. Thus (10') holds for $W_1 = I^*$, as well. Let m^* be defined by

$$m^* = 1, \qquad \text{if } W_1 \geq I^*, \qquad (11')$$

$$= pf'(W_1), \qquad \text{if } W_1 < I^*. \tag{11”}$$

By concavity and (2), $m^* > 1$ if $W_1 < I^*$. Thus $m^* \geq 1$, and (11) can be written as

$$\frac{\partial E\left[\tilde{W}_2^{\,*} \middle| \theta_1\right]}{\partial I_0} = m^* \frac{\partial W_1}{\partial I_0}. \tag{12}$$

Let I_0^* denote the value of I_0 that maximizes the unconditional expectation $E\left[\tilde{W}_2^{\,*}\right]$, and let $E\left[\tilde{W}_2^{\,**}\right]$ denote the expectation $E\left[\tilde{W}_2^{\,*}\right]$ evaluated at I_0^*.

Theorem 1. The optimal investment policy with private storage (autarchy) is (I_0^*, I_1^*), where $I_0^* < I^*$ and $I_1^* \leq I^*$.

Proof: See the Appendix

Theorem 1 shows that under autarky there is, relative to the unconstrained private optimum, too little invested in the productive technology and therefore an inefficient growth rate in the economy. In order to show that this unconstrained private optimum is in fact the social optimum as well, assume that there is a continuum of identical firms indexed in [0, 1], each with iid shocks $(\tilde{\theta}_1, \tilde{\theta}_2)$, independent across firms. Recall that p = Prob{$\tilde{\theta}_i \neq 0$}, i = 1, 2, for some p, $0 < p < 1$. At date one, a fraction p of the firms will have

$$W_1 = f(I_0) + W_0 - I_0, \tag{4’}$$

and a fraction 1 - p will have

$$W_1 = W_0 - I_0. \tag{4”}$$

The first order condition is (2) and is satisfied by $I = I^*$, and of course we must have that

$$pf(I^*) > I^*, \tag{13}$$

the condition that expected profit (NPV) is positive at the optimum. We assume (13) henceforth. Denote the efficient level of investment in this economy as the pair $\left(\hat{I}_0, \hat{I}_1\right)$ that solves the program

$$\sup \ pf(I_0) + pf(I_1) + W_0 - I_0 - I_1 \qquad (14)$$

$$\text{subject to: } 0 \leq I_0 \leq W_0,$$

$$0 \leq I_1 \leq W_0 - I_0 + pf(I_0).$$

Note that the objective function in (14) is the same as the objective function in (6) but the wealth constraint on I_1 differs from the constraint in (6). The constraint on I_1 in (14) is aggregate wealth in the economy at date one whereas the constraint in (6) is realized firm wealth. Now, by (13) and (3), the pair $\left(I^*, I^*\right)$ is feasible for (14). We therefore have

Theorem 2. The pair $\left(I^*, I^*\right)$ solves (14).

Theorem 2 shows that the unconstrained private optimal level of investment over time is indeed the same as the optimum for the economy. However, with private information concerning risky output there is no way to costlessly implement efficient levels of investment for this economy by pooling private production; either through an intermediary or through a securities market. Moreover, we have already shown, in the proof of Theorem 1, that there will be an excess supply of liquidity and suboptimal levels of expected growth when liquidity is carried privately. However, in the next section of the paper we explore the possibility of achieving the social optimum by separating production and storage arrangements. In essence, we wish to investigate the welfare properties of leaving production decentralized while at the same time giving firms the opportunity to invest in a “public” liquidity pool. We show that under certain circumstances the social optimum can indeed be achieved by separating economic activity into a decentralized production sector and a pooled liquidity sector.

3. A Simple Pooling Scheme

Suppose that firms can agree to make their private storage public so that any wealth not invested in the risky technology is observable, verifiable, and can be confiscated. This is an intuitively reasonable assumption to the extent that it is known by all that the outcome from storage is not random. Moreover, we note that even under full information there is no incentive to pool the production of firms since the risky production technology displays decreasing returns to scale[3].

We also assume that if storage is made public at date zero it remains so at date 1, i. e., once storage is made public, private storage is not an option[4]. We can now discuss ways to implement the efficient level of investment. In particular, consider a pooling arrangement whereby each firm contributes the amount $S_0 \equiv W_0 - I_0$ to a pool at date zero. The realization of the production shocks at date one will leave p firms with resources given by (4') and 1 - p firms with resources given by (4"). If $I_0 = I^*$, then from (3) we know that these firms, acting alone, will be unable to undertake the productive optimum at date one.

One way to remedy this liquidity gap is to devise a scheme to redistribute resources at date 1 from those firms having resources given by (4') to those with

[3] The solution to this problem under full information is for there to be no trade among firms at date 0. At date 1 liquidity poor firms will issue securities whose expected rate of return in equilibrium is zero. Prosperous risk neutral participants will be indifferent between purchasing these securities and storing.

[4] Without this assumption all firms that were successful would, ex post, choose to privately carry their liquidity from date 1 to date 2. In essence we are assuming that once mechanisms for transferring endowment through time are set, they remain so over the life of economy.

resources given by (4”). Call these type R (rich) and P (poor) firms respectively and imagine a planner (or public liquidity manager) who chooses an allocation (I_0, I_1) to solve (14) with some redistribution of aggregate wealth at date 1; ideally from type R firms to type P firms. Of course this reallocation rule is subject to incentive-compatibility constraints which we study later. For now we assume

Hypothesis (A). Every type R firm will do the same thing and every type P firm will do the same thing.

We will consider (measure zero) deviations from this rule in the next section.

The pool authority, or planner, observes the declaration of storage decisions of every firm but does not know the outcome of firm investment at date one (firm type). Thus the planner can only observe and redistribute the resulting aggregate public storage. Let $S_1'(S_1'')$ denote the amount of storage made by a type R (P) firms, with resources at date 1 given by (4’)((4”)). We allow S_1' and S_1'' to be of any sign. Having observed each choice S_1, the pool authority can determine the aggregate amount of this storage (roughly it can count the number of firms storing this quantity), call it $A(S_1)$. So long as $S_1' \neq S_1''$, the pool authority computes the aggregate storage for the economy $B(S_1', S_1'') \equiv A(S_1')$ + $A(S_1'')$. If S_1' = S_1'', the pool authority observes S_1 = S_1' = S_1'' and computes the aggregate storage as $B(S_1', S_1'') = A(S_1) = S_1$.

If $S_1' \neq S_1''$ and $S_1' \neq 0$, the pool authority can compute

$$q_1' \equiv A(S_1')/S_1'. \tag{15a}$$

Similarly, if $S_1' \neq S_1''$ and if $S_1'' \neq 0$, then the pool authority can compute

$$q_1'' \equiv A(S_1'')/S_1''. \tag{15b}$$

Under Hypothesis (A), we have that $q_1' = p$ and $q_1'' = 1 - p$. Note, however, that the pool authority is not required to know what p is or to be able to distinguish firms except by information provided by the numbers S_1' and S_1''.

Let $R_2'(S_1', S_1'')$ be the return to storage at date two for a type R firm. Similarly, let $R_2''(S_1', S_1'')$ be the return to storage at date two for a type P firm. Finally, let $T_1'(S_1', S_1'')$ and $T_1''(S_1', S_1'')$ denote the transfer at date 1 given to the type P and type R firms, respectively. These returns and transfers satisfy the following rules:

Table 1

				$S_0 - I^* < 0$				
Case	1	2	3	4	5	6	7	8
S_1'	≥ 0	≥ 0	≥ 0	≥ 0	$< S_0 - I^*$	$S_0 - I^* \leq S_1' < 0$	$S_0 - I^* \leq S_1' < 0$	≤ 0
S_1''	≥ 0	$< S_0 - I^*$	$S_0 - I^* \leq S_1'' < 0$	$S_0 - I^* \leq S_1'' < 0$	≥ 0	≥ 0	≥ 0	≤ 0
B	≥ 0		≥ 0	< 0		≥ 0	< 0	≤ 0
R_2'	S_1'	S_1'	B/q_1'	S_1'	0	0	0	0
T_1'	0	0	$(1-q_1')S_1''/q_1'$	0	0	$-S_1'$	0	0
R_2''	S_1''	0	0	0	S_1''	B/q_1''	S_1''	0
T_1''	0	0	$-S_1''$	0	0	$(1-q_1'')S_1'/q_1''$	0	0
I_1'	$(4') - S_1'$	$(4') - S_1'$	$(4') - S_1'$	$(4') - S_1'$	$(4')$	$(4') + T_1'$	$(4')$	$(4')$
I_1''	$(4'') - S_1''$	$(4'')$	$(4'') + T_1''$	$(4'')$	$(4'') - S_1''$	$(4'') - S_1''$	$(4'') - S_1''$	$(4'')$

Table 2

		$S_0 - I^* \geq 0$		
Case	1	2	3	4
S_1’	≥ 0	≥ 0	≤ 0	≤ 0
S_1"	≥ 0	≤ 0	≥ 0	≤ 0
B	≥ 0			≤ 0
R_2’	S_1’	S_1’	0	0
T_1’	0	0	0	0
R_2"	S_1"	0	S_1"	0
T_1"	0	0	0	0
I_1’	(4’) – S_1’	(4’) – S_1’	(4’)	(4’)
I_1"	(4") – S_1"	(4")	(4") – S_1"	(4")

Table 1 treats the situation when S_0 - I^* < 0, while Table 2 treats the situation when S_0 - I^* ≥ 0. The last seven rows of these tables are evaluated at the (S_1', S_1'') pair specified in the third and fourth rows. The last two rows specify the investment decisions implied by the fact that public storage and investment in the productive technology are the only two possible uses of resources at date 1. These rows are (4’) - S_1’ and (4”) - S_1", respectively, when the authority accepts the storage quantities S_1’ and S_1’’, respectively.

The situation in Table 2 is easiest to explain. In this latter situation, every firm has enough resources to undertake the efficient level of investment at date 1 so the pool authority in this situation just provides a means of storing resources in excess of I^*. Therefore, either both types of firms store some non-negative amount (Case 1) and there are no transfers or some firm demands resources from the planner (Cases 2, 3, and 4) and these demands are ignored. In both cases the returns on the liquidity provided is zero.

The situation in Table 1 is the one in which type P firms will have insufficient resources to undertake the efficient level of investment at date 1. When both type firms choose positive storage (Case 1) there are no transfers. Case 2 in Table 1 is when type R firms supply resources but the demand for liquidity by type P firms exceeds I*. In this case the storage is accepted (again at a zero rate of return) but the demands are rejected and no transfers are made.

Case 3 in Table 1 is the interesting case when storage is supplied and demanded, but where the demands are not excessive and where in the aggregate, there is enough supplied to cover the demand. This condition entails that $S_1'' < 0 < S_1'$ and hence the pool authority can differentiate types of firms and specify returns and transfers accordingly. In this case each supplying firm is taxed $(1 - q_1')|S_1''|/q_1'$ so that in the aggregate $(1 - q_1')|S_1''|$ is transferred, i. e., each type P firm gets $|S_1''|$. The remainder remains on account for type R firms on a pro rata basis.

Case 4 in Table 1 is identical to Case 3, except that the resources supplied in the aggregate are insufficient to cover the demand. In this case, the storage is accepted and returned one-to-one, but the demands are rejected and no transfers are made. Cases 5, 6, and 7, are mirror images of Cases 2, 3, and 4, with the roles of S_1' and S_1'' reversed since the pool authority only observes the pair of numbers (S_1', S_1''), not the primes. Note that in Case 6 the fraction of supplying firms is q_1''. In the last case (Case 8 in Table 1), there is no storage, no transfers and no returns.

To illustrate this public storage concept, imagine that the pool authority is trying to implement the efficient level of investment (I^*, I^*). Suppose also that at date zero each firm makes the investment of I^*. Then $S_0 = W_0 - I^*$, and by (3), we are in the situation in Table 1. At date t = 1, p of the firms will have resources given by (4') and 1 - p will have resources given by (4"). Suppose also that all firms choose storage quantities that implement investment of I^* at date 1. Then $S_1' = f(I^*) + S_0 - I^* > 0$ and $S_1'' = S_0 - I^* < 0$, by (3) and (13). Aggregate storage is

$$B(S_1', S_1'') = p\left(f(I^*) + S_0 - I^*\right) + (1 - p)\left(S_0 - I^*\right)$$

$$= pf(I^*) + \left(S_0 - I^*\right) > 0, \qquad (16)$$

again by (3) and (13). Thus we are in Case 3 in Table 1. Each type R firm would then have a return to storage at date two of $R_2'(S_1', S_1'') = \left(pf(I^*) + S_0 - I^*\right)/p > 0$, while each type P firm would have a return to storage at date 2 of $R_2''(S_1', S_1'') = 0$, with $I_1' = I_1'' = I^*$. The problem of course is making such storage choices incentive-compatible.

We analyze these incentives, as in Section 2, via backward induction. Moreover, since the firm knows p and how the authority determines q_1' and q_1'', we substitute in p and 1 – p freely. A strategy for a firm is a triple (I_0, S_1', S_1''). The wealth of a firm at date 2 following the strategy (I_0, S_1', S_1'') is

$$W_2' = f(I_1', \theta_2) + R_2'(S_1', S_1''), \qquad (17')$$

$$W_2'' = f(I_1'', \theta_2) + R_2''(S_1', S_1''). \qquad (17'')$$

Therefore, the conditional expectation at date one of wealth at date 2 is given by

$$E\left[\tilde{W}_2 \middle| \theta_1 = \theta\right] = pf(I_1') + R_2'(S_1', S_1''), \qquad (18')$$

$$E\left[\tilde{W}_2 \middle| \theta_1 = 0\right] = pf(I_1'') + R_2''(S_1', S_1''). \qquad (18'')$$

It is easy to see from (18") and Tables 1 and 2 what a type P firm will do in terms of choosing S_1''. Let $\vec{S}_1''$ denote the value of S_1'' that maximizes (18").

Lemma 1. If $S_0 - I^* < 0$, then

$$\vec{S}_1'' = \max\left[-pS_1'/(1-p), S_0 - I^*\right], \qquad \text{if } S_1' \geq 0, \qquad (19')$$

$= 0,$ if $S_1' < 0.$ (19")

If $S_0 - I^* \geq 0$, then

$$\vec{S}_1'' = S_0 - I^*. \tag{20}$$

Proof: See the Appendix.

The results in Lemma 1 make sense. The first part treats the situation of a firm that has invested more at date zero than allows for the efficient level of investment at date 1 with certainty. In that situation, the incentives of a type P firm will be to demand all that is available from the pool authority. The binding constraints are that (15) hold and that demands be at least $S_0 - I^*$. These constraints produce (19'). In essence, the individual type P firm is concerned about aggregate storage staying non-negative, because if it demanded more, then by Table 1, no transfer would be made. The second part of Lemma 3 treats the situation of a firm that has invested at date zero such that at date one the efficient level of investment can be undertaken with certainty. In this situation, given the structure of the pool authority in Table 2, it is optimal for the type P firm to do just that.

The incentives of type R firms are a little more complicated. Let $\vec{S}_1'$ denote the value of S_1' that maximizes (18'). Also let $\breve{S}_1' = \left(S_0 + f(I_0) - I^*\right)^+$ and let $\widehat{S}_1' = \max\left[S_0 - I^*, -(1-p)S_1''/p\right]$.

Lemma 2. If $S_0 - I^* < 0$, then

$$\vec{S}_1' = \widehat{S}_1', \text{ if } S_1'' \geq 0 \text{ and } pf\left(S_0 + f(I_0) - \widehat{S}_1'\right) > pf\left(S_0 + f(I_0) - \breve{S}_1'\right) - \breve{S}_1' \tag{21'}$$

$$= \breve{S}_1', \text{ otherwise.} \tag{21''}$$

If $S_0 - I^* \geq 0$, then

$$\vec{S}_1' = S_0 + f(I_0) - I^*. \qquad (22).$$

Proof: See the Appendix .

The second part of Lemma 2 treats the situation of a firm that has invested at date zero such that at date one the efficient level of investment can be undertaken with certainty. In this situation, given the structure of the pool authority in Table 2, it is optimal for the type R firm to do just that. The first part of Lemma 2 treats the situation of a firm that has invested more at date zero than allows for the efficient level of investment at date one with certainty. It follows that, if there are resources available for redistribution ($S_1'' \geq 0$), the type R firm has to choose between storage (Case 1 of Table 1) and demanding resources from the pool authority (Cases 5-7 in Table 1). The relevant comparison is the conditioning inequality in (21'). It is this tradeoff that is at the heart of the incentive compatibility problem analyzed in the following section. But under Hypothesis (A), the conditioning inequalities in (21') are inconsistent. This is established in the following result.

Lemma 3. The storage strategy $(\vec{S}_1', \vec{S}_1'')$ satisfies

$$\vec{S}_1' = S_0 + f(I_0) - I^*, \qquad (23)$$

$$\vec{S}_1'' = S_0 - I^*. \qquad (24)$$

Proof: See the appendix.

Lemma 3 gives the optimal storage strategy under the pool authority. Regardless of the choice of the initial date zero investment, at date one rich firms will store the quantity $S_0 + f(I_0) - I^*$, and poor firms will demand a transfer of $S_0 - I^*$. This makes the optimal initial investment easy to determine.

Theorem 3. Under the pooling authority specified in Tables 1 and 2, and under Hypothesis (A), the strategy $\left(I^*, \vec{S}_1', \vec{S}_1''\right)$ maximizes $E\left[\tilde{W}_2\right]$ and yields the supremum in (14).

Proof: See the appendix.

The result of Theorem 3 is not surprising given the behavior imposed by Hypothesis (A). In the next section, we examine if it is in the interest of individual firms of each type to act in this way[5]

4. Incentive Compatibility

In this section we examine the validity of Hypothesis (A). A necessary condition for Hypothesis (A) to be valid is that it must be in any firm's interest to act as required by Hypothesis (A), given that all other firms act in this way. It is clear from the result in Lemma 1 and the discussion that follows it that this is the case for type P firms. This conclusion does not extend to rich firms. Given that type P firms can receive the transfer, it may be in the interest of a rich firm to make the same demand, rather than supply positive amounts of storage. Thus, we are asking if it is in the interest of a rich firm to behave as a poor firm since such behavior could not be detected in any way by the pool authority.

Since none of this capital can be stored, a rich firm receiving a transfer would have to invest it all in the productive technology. Its conditional expected date two wealth at date one is therefore $pf\left(I^* + f\left(I^*\right)\right)$. Using the results from the proof of Theorem 3, it is straightforward to show that storage is incentive compatible for a rich firm if and only if

[5] We note that if incentive compatibility holds for individual firms then the allocation is coalition proof as well. Since output is unobservable and storage cannot be made private once it is public, at date one there is

$$\mathrm{p}f\left(I^* + f\left(I^*\right)\right) \leq \mathrm{p}f\left(I^*\right) + \frac{\mathrm{p}f\left(I^*\right) - I^*}{\mathrm{p}} + \frac{W_0 - I^*}{\mathrm{p}}, \qquad \text{(IC)}$$

which after rearranging terms becomes

$$\mathrm{p}^2\left(f\left(I^* + f\left(I^*\right)\right) - f\left(I^*\right)\right) \leq \mathrm{p}f\left(I^*\right) - I^* + W_0 - I^*. \qquad (26)$$

By strict concavity of f, Mangasarian (1994, Theorem 6.2.1),

$$\mathrm{p}^2\left(f\left(I^* + f\left(I^*\right)\right) - f\left(I^*\right)\right) < \mathrm{p}^2 f'\left(I^*\right)\left(I^* + f\left(I^*\right) - I^*\right)$$

$$= \mathrm{p}f'\left(I^*\right)\mathrm{p}f\left(I^*\right)$$

$$= \mathrm{p}f\left(I^*\right),$$

by the first order condition (2). It follows from this strict inequality that (26) holds for W_0 sufficiently close to $2I^*$, and hence the condition (IC) is not vacuous.

At the other extreme, under the resource restriction (3), (26) holds if

$$\mathrm{p}^2\left(f\left(I^* + f\left(I^*\right)\right) - f\left(I^*\right)\right) \leq \mathrm{p}f\left(I^*\right) - I^*. \qquad (27)$$

If (27) holds, then (IC) holds for all W_0 satisfying (3). Now, (27) can be written as

$$f\left(I^* + f\left(I^*\right)\right) - f\left(I^*\right) \leq f'\left(I^*\right)\left(f\left(I^*\right) - I^* f'\left(I^*\right)\right). \qquad (28)$$

The inequality (28) shows that something more than concavity is required for (27) to hold. In order to see what more than concavity is sufficient for (IC) to hold, let $\varepsilon(I)$ denote the elasticity of scale of the production function f (Varian, 1992, p. 16), at I. Then $\varepsilon(I) = If'(I)/f(I)$, and (28) becomes

$$f\left(I^* + f\left(I^*\right)\right) - f\left(I^*\right) \leq f'\left(I^*\right)f\left(I^*\right)\left(1 - \varepsilon\left(I^*\right)\right). \qquad (29)$$

which can be rewritten as

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no mechanism such that a group of rich firms will find it in their interest to contract with one another outside of the pooling arrangement.

$$\frac{f\left(I^* + f\left(I^*\right)\right) - f\left(I^*\right)}{f'\left(I^*\right)f\left(I^*\right)} \leq \left(1 - \varepsilon\left(I^*\right)\right), \tag{30}$$

If (30) holds for all values of I*, then the incentive compatibility condition holds for all W_0 satisfying (3) and, by Theorem 6, the pool authority implements the efficient level of investment.

We now show that there does indeed exist a critical level of scale elasticity such that for every production function that has decreasing (which includes the constant case) elasticity of scale below this level, (29) and (30) hold. We first derive this critical level in the case of constant elasticity and then prove that it holds for all decreasing returns to scale functions.

Constant elasticity of scale production function. The production function $f(I) = I^{\alpha}\theta/\alpha$, $0 < \alpha < 1$, $\theta > 0$, has constant elasticity of scale $\varepsilon(I) = \alpha$, $I \geq 0$. In this case, $I^* = (p\theta)^{\frac{1}{1-\alpha}}$. Using this result and some algebra, condition (30) can be written as

$$p\left[\left(1 + \frac{1}{p\alpha}\right)^{\alpha} - 1\right] \leq 1 - \alpha. \tag{31}$$

It can be shown that the term on the left hand side of (31) is increasing in p. It follows that

$$p\left[\left(1 + \frac{1}{p\alpha}\right)^{\alpha} - 1\right] < \left(1 + \frac{1}{\alpha}\right)^{\alpha} - 1. \tag{32}$$

Using L'Hospital's rule (Rudin (1976, Theorem 5.13)), it follows that

$$\lim_{\alpha \downarrow 0}\left(1 + \frac{1}{\alpha}\right)^{\alpha} = 1.$$

Let $\hat{\alpha}$ denote the unique value of α such that

$$\left(1+\frac{1}{\alpha}\right)^{\alpha} + \alpha = 2.$$

Then $\hat{\alpha}$ = .3735495 to seven decimal places. Choosing $\alpha < \hat{\alpha}$, it follows from (32) that (31) holds for all values of p.

The result for constant elasticity of scale is easily extended to the case of decreasing elasticity of scale (e.g., Frisch (1965), chapter 8). Rewrite (29) by adding $f(I^*)$ to both sides of the equation. This yields

$$f\left(I^* + f\left(I^*\right)\right) \le f\left(I^*\right) + f'\left(I^*\right)f\left(I^*\right)\left(1 - \varepsilon\left(I^*\right)\right). \tag{33}$$

Theorem 4. If f exhibits decreasing elasticity of scale and $\varepsilon(I^*) < \hat{\alpha}$, then (33) holds. In particular, if f exhibits decreasing elasticity of scale and $\varepsilon(0) < \hat{\alpha}$, then (33) holds for all p, $0 \le p \le 1$.

Proof: See the appendix.

5. Aggregate Uncertainty

In the previous sections of the paper we analyzed the liquidity problem under the assumption that the likelihood of a productive shock was known with certainty and constant through time. In this section we extend our analysis to the case where p is initially uncertain. The basic model of Section 2 is maintained, i. e., the production shocks $(\tilde{\theta}_1, \tilde{\theta}_2)$ are iid with p = Prob$\{\tilde{\theta}_i \neq 0\}$, i = 1, 2, except that now we assume that firms have a common prior distribution on p over the interval [0, 1].

Let p_0 denote the mean of this prior distribution. From Section 2, for any $p \in [0, 1]$, let I^* denote the solution to (1) with I_0^* being the solution to (1) for $p = p_0$. If the full information optimum is to be achieved it must be the case that

$$I_0^* < W_0. \qquad (3')$$

Turning to the optimal investment at date one, we note that if the firm did not learn any more about the value of p than what could be inferred from the observation of its productivity shock θ_1, then the firm will invest $I^*(p(\theta_1))$, which is the solution to (1) for the posterior mean $p(\theta_1)$. The firm is constrained at t = 1 if $W_1 < I^*(p(\theta_1))$, where W_1 is given by (4). Assuming that the firm makes the optimal investment of $I_0 = I_0^*$ at date zero, this constraint can be written as $W_0 - I_0^* < I^*(p(\theta_1 = 0))$, for P firms, or as $f(I_0^*) + W_0 - I_0^* < I^*(p(\theta_1 \neq 0))$, for R firms. The efficiency question is only interesting when the firm with the bad production shock is constrained at date one under autarky. Furthermore, we want the first best to be attainable. A necessary condition for this is that firms with good production shocks not be constrained under autarky. Thus, we assume that

$$W_0 - I_0^* < I^*(p(\theta_1 = 0)) \qquad (3'a)$$

$$I^*(p(\theta_1 \neq 0)) < f(I_0^*) + W_0 - I_0^*. \qquad (3'b)$$

Along the lines of Theorem1 we would expect that the optimal investment policy in autarchy would be (I_0^*, I_1^*) with $I_1^* \leq I(p(\theta_1))$ and $I_0^* < I(p_0)$, while the non-liquidity constrained optimal investment policy is $I_0 = I_0^*$ and $I_1 = I^*(p(\theta_1))$.

While the investment policy $(I_0^*, I^*(p(\theta_1)))$ is the unconstrained optimum for the individual firm that learns only the value of its private shock θ_1, it is not clear that this policy is even feasible, much less socially efficient. First of all, this policy depends on the private information of the firm. In principal the pool authority might be able to engage in the redistribution of liquidity in a manner similar to that when there is no aggregate uncertainty. In particular, aggregate wealth in the economy at date one is given by $pf(I_0^*) + W_0 - I_0^*$, where p is the realized fraction of resource rich firms. If transfers are made so that true P firms invest $I^*(p(\theta_1 = 0))$ and true R firms invest $I^*(p(\theta_1 \neq 0))$, then aggregate use of wealth for investment must be $pI^*(p(\theta_1 \neq 0)) + (1-p)I^*(p(\theta_1 = 0))$. It follows from (3'a) that for p sufficiently small, $pI^*(p(\theta_1 \neq 0)) + (1-p)I^*(p(\theta_1 = 0)) > pf(I_0^*) + W_0 - I_0^*$, and the policy $(I_0^*, I^*(p(\theta_1)))$ is not feasible. If the inequality (3'b) is reversed, then for realizations of p sufficiently close to 1, the policy $(I_0^*, I^*(p(\theta_1)))$ is infeasible. Since either (3'a) or the reverse of (3'b) is essential to having a liquidity problem, the infeasibility of the policy $(I_0^*, I^*(p(\theta_1)))$ is unavoidable.

This concern about feasibility of the policy $(I_0^*, I^*(p(\theta_1)))$ stems from its optimality when firms know only the value of its private shock θ_1 and is of some social concern only if the social planner cannot know more. Note that by design, the pool authority observes both individual and aggregate storage decisions. If the pool authority can announce this information to all firms at t = 1, then for every firm p = $E[\tilde{p}|\theta_1, \text{announcement}]$ regardless of the value of θ_1, and the unconstrained optimal investment at t = 1 would be I^*. Analogous to the argument in Section 2, the socially efficient investment policy solves (14) subject to the computation of expectations using the prior distribution[6]. Thus, the pool authority provides fully revealing information regarding the economy from its storage function even in situations where there turns out

[6] This additional qualification is needed because firms will not learn the value of p until date one. Having learned this p, however, the pool authority may be able to redistribute liquidity to achieve investment of I^*.

to be no liquidity ex post. To see why there may be no liquidity problem, recall that, by assumption, I^* is increasing and continuous in p and $I^* = 0$ at p = 0. Intuitively, if the realized value of p is sufficiently small then I^* may be small enough that there is no need for transfers. To get some intuition for this, assume further that the residual $W_0 - I_0^*$ is in the range of I^*. Then by (3') there is a value $\bar{p}_0 \in (0, 1]$, such that

$$\bar{I}_0^* = W_0 - I_0^*, \tag{34}$$

where $\bar{I}_0^*$ is I^* for $p = \bar{p}_0$. Then in a low productivity economy, i. e., when $p \leq \bar{p}_0$, by monotonicity, $I^* \leq \bar{I}_0^*$, and every firm would have enough resources to undertake the optimal investment I^* if it knew p.

In a high productivity economy, $p > \bar{p}_0$, and poor firms would be resource constrained because $I^* > W_0 - I_0^*$. This is really the counterpart of the inequality on the right hand side of equation (3) and it comes naturally without any assumption other than there is positive probability that $p > \bar{p}_0$. On the other hand (4') firms would have wealth of $f\left(I_0^*\right) + W_0 - I_0^*$, and the question is, is this enough to undertake the optimal investment I^*? This brings up the feasibility of the policy $\left(I_0^*, I^*\right)$ for the program (14) with $I_0 = I_0^*$. In order to have a chance to implement this constrained first best investment policy, at the very least we would need to have that

$$pf\left(I_0^*\right) + W_0 - I_0^* \geq I^*, \; 0 < p \leq 1, \tag{35}$$

The previous question regarding the resources of P firms is answered by (35) at the end point p = 1. We will assume that (35) holds and seek modifications of the simple pooling scheme of Section 3 that will implement the policy $\left(I_0^*, I^*\right)$.

Recall from Section 3 that a strategy for a firm is a triple (I_0, S_1', S_1''), where $S_1'(S_1'')$ denotes the amount of storage made by a rich (poor) firm at date one. As in Section 4, we invoke Hypothesis (A). Having observed the storage choice S_1, the pool authority can compute the aggregate of this amount of storage (roughly it can count the number of firms storing this quantity) call it $A(S_1)$. Having observed S_1' and S_1'', as long as $S_1' \neq S_1''$, the pool authority can compute $A(S_1')$ and $A(S_1'')$, and check the resource conservation condition (15). We now assume that the pool authority announces to all firms the pairs $(S_1', A(S_1'))$ and $(S_1'', A(S_1''))$. If $S_1' = S_1''$, the pool authority announces the pair $(S_1, A(S_1))$, where $S_1 = S_1' = S_1''$.

In the case $S_1' \neq S_1''$, every firm can infer the realized fraction of poor firms, for either $S_1' \neq 0$ or $S_1'' \neq 0$. If $S_1' \neq 0$, every firm can compute $q_1' = A(S_1')/S_1'$ and set p = q_1'. Here R firms know that q_1' is the value of p because they know their storage decision S_1'. Also P firms know q_1' is the value of p because they know their storage decision S_1'' and hence they know that S_1', which is different from S_1'', is the storage decision of rich firms. Similarly, when $S_1'' \neq 0$, every firm can compute $q_1'' = A(S_1'')/S_1''$ and set p = 1 - q_1''.

The case when $S_1' = S_1''$ is more difficult. In this case, firms learn nothing about the realized fraction of R firms and must value outcomes using their posterior $E[\tilde{p}|\theta_1]$. The difficulty is that no firm can decide at date one whether or not to choose $S_1' \neq S_1''$ or $S_1' = S_1''$, since this later choice would have to be based on knowledge of p. But p cannot be known if the firm chooses $S_1' = S_1''$. To avoid this difficulty, we make the following assumption.

Hypothesis (B). Firms choose among strategies (I_0, S_1', S_1'') for which $S_1' \neq S_1''$.

We can now consider a modified version of the pooling scheme of Section 4. Under Hypothesis (B), the pool authority, modified to make the announcements $(S_1', A(S_1'))$ and $(S_1'', A(S_1''))$, is described by Tables 1 and 2. The results of Lemma 5 hold, i.e., any optimal strategy is of the form $(I_0, \vec{S}_1', \vec{S}_1'')$, where $\vec{S}_1'$ is given in (23) and $\vec{S}_1''$ is given in (24). Notice that in this policy $\vec{S}_1' > \vec{S}_1''$, for all $I_0 > 0$, consistent with Hypothesis (B). We also have

Theorem 5. Under Hypotheses (A) and (B) and (45), the strategy $(I_0^*, \vec{S}_1', \vec{S}_1'')$ maximizes $E[\tilde{W}_2]$.

Proof: See the appendix.

Armed with Theorem 5, we revisit the incentive compatibility of Hypothesis (A). As in Section 4, our intuition is that incentive compatibility will require that the elasticity of scale of the production function will have to be sufficiently small. In addition the prior mean p_0 will need to be sufficiently large. The analog here of the inequality (27) is the following:

$$p^2 \left(f\left(I^* + f\left(I_0^*\right)\right) - f\left(I^*\right)\right) \leq p\, f\left(I_0^*\right) - I^*, \tag{36}$$

which reduces to

$$f\left(I^* + f\left(I_0^*\right)\right) - f\left(I^*\right) \leq f'\left(I^*\right) f\left(I^*\right)\left(\frac{f\left(I_0^*\right)}{f\left(I^*\right)} - \varepsilon\left(I^*\right)\right), \tag{37}$$

by repeated use of the first order condition (2). Inequality (37) is analogous to (29). They differ only in the fact that the initial investment by firms is at the optimum for the prior mean I_0^*. The intuition about the scale elasticity follows from (37) just as in the case of

(29). The intuition about the size of the prior mean comes from the ratio $f\left(I_0^*\right)/f\left(I^*\right)$ on the right in (37). In particular, we see that the larger is p_0 the larger is I_0^* and the larger is the ratio $f\left(I_0^*\right)/f\left(I^*\right)$, which in turn makes (37) more likely to hold.

Generally speaking, one would have to check whether equation (37) holds for all p. However, the following result can be used to simplify the problem greatly. In particular, let $\vec{\varepsilon} = \sup\left\{\varepsilon\left(I^*\right): 0 < p < 1\right\}$. Then we can prove

Theorem 6. If $\vec{\varepsilon} < \varepsilon^* \leq 1/2$, $p_0 \geq \vec{\varepsilon}/\varepsilon^*$, and if $f\left(I^*\right)$ is concave in p, then (36) holds if it holds at p = 1.

Proof: See the Appendix:

The hypothesis that $\vec{\varepsilon} < \varepsilon^* \leq 1/2$ and $p_0 \geq \vec{\varepsilon}/\varepsilon^*$ expresses well our intuition that the elasticity of scale must be sufficiently small and the prior mean must be sufficiently large for (37) to hold. They do not guarantee that (37) holds at p = 1. However, we can show that for the constant elasticity of scale case the conditions in Theorem 6 are also sufficient for incentive compatibility to hold for all p.

Constant Elasticity of scale production function. Recall that $f(I) = I^{\alpha}\theta/\alpha$, $0 < \alpha < 1$, $\theta > 0$, with $\varepsilon(I) = \alpha$. Let $\hat{\alpha}$ be as in Section 5. For $f(I) = I^{\alpha}\theta/\alpha$, $0 < \alpha < \hat{\alpha}$, and $p_0 \geq \alpha/\hat{\alpha}$, (35) and (37) hold for all p. In order to show this we use the fact that $I^* = (p\theta)^{\frac{1}{1-\alpha}}$ and $f\left(I^*\right) = \dfrac{p^{\frac{\alpha}{1-\alpha}}\theta^{\frac{1}{1-\alpha}}}{\alpha}$, which is concave in p since $\alpha < 1/2$. We first check the feasibility condition (35). For the constant elasticity case this condition is that

$$W_0 - (p_0\theta)^{\frac{1}{1-\alpha}} + p\cdot\frac{p_0^{\frac{\alpha}{1-\alpha}}\theta^{\frac{1}{1-\alpha}}}{\alpha} - (p\theta)^{\frac{1}{1-\alpha}} \geq 0. \qquad (38)$$

By (3'), (38) holds at p = 0. It is straightforward to show that as a function of p, the left-hand side of (38) is concave. Hence (38) will hold if it holds at p = 1. A sufficient condition for this is that $p_0^{\frac{\alpha}{1-\alpha}} \geq \alpha$. By hypothesis, $p_0^{\frac{\alpha}{1-\alpha}} \geq p_0 \geq \alpha/\hat{\alpha} > \alpha$. Thus (35) holds.

Rewriting (37) as in (30), we have that in this case incentive compatibility holds if

$$p\left[\left(1+\left(\frac{p_0}{p}\right)^{\frac{\alpha}{1-\alpha}}\left(\frac{1}{p\alpha}\right)\right)^{\alpha} - 1\right] \leq \left(\frac{p_0}{p}\right)^{\frac{\alpha}{1-\alpha}} - \alpha. \qquad (39)$$

By Theorem 5, (39) holds if it holds for all p if it holds at p = 1. At p = 1, the inequality (39) becomes

$$0 \leq 1 - \left(1+\frac{p_0^{\frac{\alpha}{1-\alpha}}}{\alpha}\right)^{\alpha} + p_0^{\frac{\alpha}{1-\alpha}} - \alpha. \qquad (40)$$

The right hand side of (40) is increasing in p_0. By hypothesis, $p_0 \geq \alpha/\hat{\alpha} \geq (\alpha/\hat{\alpha})^{\frac{1-\alpha}{\alpha}}$. Evaluating the right hand side of (40) at $(\alpha/\hat{\alpha})^{\frac{1-\alpha}{\alpha}}$ yields the quantity

$$1 - \alpha + \frac{\alpha}{\hat{\alpha}} - \left(1+\frac{1}{\hat{\alpha}}\right)^{\alpha}. \qquad (41)$$

The quantity in (41) is zero at $\alpha = 0$. It is also zero at $\alpha = \hat{\alpha}$, by the definition of $\hat{\alpha}$. The quantity $\left(1+\frac{1}{\hat{\alpha}}\right)^{\alpha}$ is convex as a function of α, while the quantity $1 - \alpha + \frac{\alpha}{\hat{\alpha}}$ is affine in α. Since these quantities are equal at $\alpha = 0$ and at $\alpha = \hat{\alpha}$, $1 - \alpha + \frac{\alpha}{\hat{\alpha}}$ is greater than $\left(1+\frac{1}{\hat{\alpha}}\right)^{\alpha}$ for $0 < \alpha < \hat{\alpha}$. Thus (41) is nonnegative, and (40) holds.

Given the result for the constant elasticity of scale and our earlier results in Theorem 4, it seems reasonable to conjecture that if the production function f exhibits decreasing elasticity of scale and if $\varepsilon(0) < \hat{\alpha}$ and $p_0 \geq \varepsilon(0)/\hat{\alpha}$, then (35) and (37) would hold as well for this case.

Unfortunately, we have not been able at this point to generate a result as general as Theorem 4 for the case where there is aggregate uncertainty. However, it seems clear that the intuition from the constant elasticity of scale case should, in some form, carry over to more general functions. The basic point here is that, in addition to the possibility of incentive compatibility failing due a large value of the elasticity of scale, it may also fail if realized output at date 1 is much larger than anticipated at date 0. Thus, roughly speaking, it is when expectations are poor and the actual economy turns out to be good that incentive compatibility may fail.

6. Summary and Relationship to Other Work

In this paper we have developed of model in which the social optimum may be achieved by having decentralized production and a public liquidity sector. To our knowledge this is the first paper to generate such a result in a world where there is private information regarding output and allocation rules that depend only on announced liquidity needs. Thus the planner uses no more information than what can be surmised from the collection of individual liquidity reports by market participants. The results here can be contrasted with those in Atkeson and Lucas (1991) by noting that, unlike their model without storage, the storability of the commodity can itself be used to extract information that is capable of generating the first best allocation. Moreover, because our participants are risk neutral, liquidity taxes influence the welfare of those taxed only in a linear fashion. Thus, unlike Atkeson and Lucas, who model risk averse participants but do not include production and storage, we are able to generate Pareto Optimal solutions and are able to do so with a simple lump sum tax scheme. It is not clear that this result will be robust to the introduction of risk aversion. This extension is a topic for further research.

There are a number of other papers that deal with questions such as optimal taxation, insurance and liquidity provision when there is private information. Mirrlees (1997) provides a review of the work on trying to achieve a social optimum when there is private information concerning effort or outcomes. He focuses mainly on the optimal taxation problem with hidden effort and notes that, with some exceptions, achieving unconstrained Pareto Optimality is not generally possible in these models. Other authors have focused more specifically on optimal insurance or liquidity arrangements when there is private information. For example, Diamond and Dybvig (1983) posit a model whereby individuals can insure against shocks to preferences by forming an intermediary. However, since output is observable, taxes can be levied and other claims to production can be traded. For example, securities markets are also viable in this framework, and as Jacklin (1987) has shown, there are situations where a securities market can implement the first best allocation. It is obvious by design that securities markets cannot substitute for the liquidity arrangements developed in this paper.

Bhattacharya and Gale (1987) extend the Diamond and Dybvig model to analyze risk sharing between banks themselves in a world where there is no aggregate uncertainty. They show that Pareto Optimality cannot generally be achieved with an interbank lending market. One way to view our results in this context is that the liquidity pool authority is a central bank that provides firms with below market rate loans when they are in need of liquidity. Conversely, the central bank charges a tax to those who have excess liquidity, with the interest rate being –1.0 per dollar. Indeed, modern central banking seems to work very much along these lines. Firms with excess liquidity are usually not offered market rates while those in need of liquidity are subsidized. The results here are suggest that such arrangements may often be welfare improving.

Another important contribution to the literature on optimal liquidity management is the recent work by Holmstrom and Tirole (1998). They too study a production economy with risk neutral participants, albeit one without decreasing returns to investment. They are interested in the interaction between the supply of liquidity and optimal growth, or capital formation. Since output is observable while effort is not, they are able to show that, absent aggregate uncertainty, private liquidity arrangements can be used to implement second best solutions; there is no need for a public liquidity pool. They go on to show how the issuance of government securities may increase welfare if there is aggregate uncertainty but, of course, since output is observable, these bonds can be redeemed ex post via the collection of taxes.

Our interest is similar to that of Holmstrom and Tirole but our approach is to investigate under what conditions first best may be achieved without paying deadweight costs to the owners of capital. One way to view our results is that by introducing decreasing returns to scale we are able to generate the first best solution to what Holmstrom and Tirole call the case of exogenous liquidity. In equilibrium there is no moral hazard in our solution because there are no claims against risky production. This approach also allows us to show the benefits of pooling storage for purposes of intertemporal risk sharing without the need to bring in overlapping generations or distinguish between ex ante and ex post efficiency, as in Allen and Gale (1997).

Future work could proceed down a number of different paths. The first would be to investigate whether or not the first best allocation can be achieved with a similar liquidity pooling scheme in a production economy such as our when participants are risk averse. A second promising area of research would involve a comparison of a the welfare properties of a model with securities markets and costly state verification (Townsend (1979)) with the liquidity model developed here when first best cannot be achieved.

REFERENCES

Allen, F. and D. Gale, 1997, "Financial Markets, Intermediaries, and IntertemporaSmoothing", *Journal of Political Economy,* Vol. 105, 523-546.

Atkeson and Lucas, 1991, "Efficient Distribution with Private Information, *Review of Economic Studies*

Bhattacharya, S. and D. Gale (1987). "Preference Shocks, Liquidity, and Central Bank Policy," in *New Approaches to Monetary Economics*, W. Barnett and K. Singleton, eds., Cambridge University Press, Cambridge.

Debreau, G., 1959, *Theory of Value*

Diamond, D. and P. Dybvig, 1983, "Bank Runs, Deposit Insurance and Liquidity, *Journal of Political Economy,* Vol. 91, 401-419.

Fink, A. M., and M. Jodeit (1984). "On Chebyshev's Other Inequality," *in Inequalities in Statistics and Probability,* Y. L. Tong, ed., IMS Lecture Notes—Monograph Series Vol. 5, Institute of Mathematical Statics, Hayward, CA.

Jacklin, C. J., 1987, "Demand Deposits, Trading Restrictions, and Risk Sharing", in *Contractual Arrangements for Intertemporal Trade* (eds. E. Prescott and N. Wallace), Minneapolis, Univ. of Minnesota Press.

Frisch, R. (1965). *Theory of Production*, Rand McNally & Co., Chicago.

Holmstrom, B., and J. Tirole (1998). "Private and Public Supply of Liquidity," *Journal of Political Economy,* Vol. 106, No.1, 457-482.

Mangasarian, O. L. (1994). *Nonlinear Programming*, Society for Industrial and Applied Mathematics, Philadelphia.

Mirrlees, J.A., 1997, "Information and Incentives: The Economics of Carrots and Sticks", *Economic Journal;* Vol.107, 1311-1329.

Townsend, R. C., 1979, "Optimal Contracts and Competitive Markets with Costly State Verification", *Journal of Economic Theory*, Vol. 21, 265-293.

Varian, H. R. (1992). *Microeconomic Analysis*, 3rd ed., W. W. Norton & Co., New York.

Appendix

This appendix contains proofs of the Theorems and Lemmas in the text.

Proof of Theorem 1: By the arguments leading to equation (8) in the text, given I_0 and θ_1, $E\left[\tilde{W}_2^*|\theta_1\right] \bullet E\left[\tilde{W}_2|\theta_1\right]$, for every I_1, $0 \bullet I_1 \bullet W_1$. Thus for any choice of I_0, $0 \bullet I_0 \bullet W_0$, and for every I_1, $0 \bullet I_1 \bullet W_1$, $E\left[\tilde{W}_2^*\right] = E\left[E\left[\tilde{W}_2^*|\theta_1\right]\right] \bullet E\left[E\left[\tilde{W}_2|\theta_1\right]\right] = E\left[\tilde{W}_2\right]$. It follows that $E\left[\tilde{W}_2^{**}\right]$ is the supremum in (6) and thus (I_0^*, I_1^*) is an optimal investment policy.

By (8), $I_1^* \bullet I^*$. It follows that m^* is decreasing in θ_1, and strictly so for $W_1 < I^*$, and $\frac{\partial W_1}{\partial I_0}$ is increasing in θ_1. We then have by (12) that

$$\frac{dE\left[\tilde{W}_2^*\right]}{dI_0} = E\left[\frac{\partial E\left[\tilde{W}_2^*|\theta_1\right]}{\partial I_0}\right] \tag{A1}$$

$$= E\left[m^*\frac{\partial W_1}{\partial I_0}\right]$$

$$< E\left[m^*\right]E\left[\frac{\partial W_1}{\partial I_0}\right]$$

$$= E\left[m^*\right]\left(pf'(I_0)-1\right),$$

where the inequality follows by Chebyshev's (other) inequality (Fink and Jodeit (1984)). For $I_0 \bullet I^*$, $\left(pf'(I_0)-1\right) \bullet 0$. It follows from (A1) that $I_0^* < I^*$. []

Proof of Lemma 1: If S_0 - $I^* < 0$, then by (2), $-pf'(S_0 - S_1'')+1 < 0$, for any $S_1'' \bullet 0$. It follows from Table 1 that (18") is strictly decreasing in Cases 1, 5, 6, and 7. In Cases 2, 4, and 8, (18") is equal to $pf(S_0)$. In Case 3, (18") is strictly decreasing in S_1'', as well. Hence in Cases 1, 2, 3, and 4, the best that can be done is to set $S_1'' = \max\left[-pS_1'/(1-p), S_0 - I^*\right]$, which makes (18”) equal to $\min\left[pf\left(S_0 + pS_1'/(1-p)\right), pf\left(I^*\right)\right]$, which is at least $pf(S_0)$, when S_1' • 0. If $S_1' < 0$, in Case 5, the largest value of (18”) is $pf\left(S_0 + pS_1'/(1-p)\right)$, which is less than $pf(S_0)$. Thus the largest value of (18”) is $pf(S_0)$ when $S_1' < 0$. This obtains by the choice of $S_1'' = 0$, which results in Case 8 in Table 1.

If $S_0 - I^*$ • 0, then the maximum of (18”) in Cases 1 and 3 of Table 2 is $pf(I^*) + S_0 - I^*$, which is at least as great as $pf(S_0)$ since I^* solves (1). The maximum of (18”) in Cases 2 and 4 of Table 2 is $pf(S_0)$. This establishes (20).[]

Proof of Lemma 2: Assume that $S_0 - I^* < 0$ and that S_1'' • 0. Then Cases 1, 5, 6, and 7 in Table 1 apply. In Case 1, the best the type R firm can do is to choose $S_1' = \breve{S}_1'$, giving (18’) equal to $pf\left(S_0 + f(I_0) - \breve{S}_1'\right) - \breve{S}_1'$. In Cases 5-7, the best the type R firm can do is to choose $S_1' = \widehat{S}_1'$, giving (18’) equal to $pf\left(S_0 + f(I_0) - \widehat{S}_1'\right)$. This gives (21’) and the part of (21”) corresponding to when S_1'' • 0. If $S_1'' < 0$, then Cases 2-4, and 8 of Table 1 apply. In Case 8, (18’) is equal to $pf\left(S_0 + f(I_0)\right)$. Cases 2-4 are essentially the same as Case 1 and (18’) is equal to $pf\left(S_0 + f(I_0) - \breve{S}_1'\right) - \breve{S}_1'$, which is at least as large $pf\left(S_0 + f(I_0)\right)$, its value when $\breve{S}_1' = 0$. This establishes the other part of (21”).

If $S_0 - I^*$ • 0, then Table 2 applies and $\breve{S}_1' = \left(S_0 + f(I_0) - I^*\right)^+ = S_0 + f(I_0) - I^*$.

The maximum value of (18’) in Cases 1 and 2 in Table 2 is $pf\left(I^*\right) + S_0 + f(I_0) - I^*$, which is at least as great as $pf(S_0 + f(I_0))$ since I^* solves (1). The maximum of (18') in Cases 3 and 4 of Table 2 is $pf(S_0 + f(I_0))$. This establishes (22).[]

Proof of Lemma 3: If $S_0 - I^* \bullet 0$, then (23) and (24) are just (20) and (22), respectively. If $S_0 - I^* < 0$, then by (16), (23) and (24) are feasible. By (19), $\vec{S}_1''$ $\bullet$ 0, and $\vec{S}_1'' = 0$ only if $S_1' \bullet 0$. By (21), $\vec{S}_1' \bullet 0$ only when (21') obtains or in (21") when $\vec{S}_1' = 0$. When (22') obtains, then $\vec{S}_1'' = 0$ and $\vec{S}_1' = \hat{S}_1' = 0$. It follows that $\vec{S}_1' = 0$ if and only if $\vec{S}_1'' = 0$, the conditioning inequalities in (22') are mutually inconsistent, and hence $\vec{S}_1' = \breve{S}_1'$. For $I_0 = 0$, $\breve{S}_1' = (W_0 - I^*)^+ = W_0 - I^*$, by (3). For $I_0 = W_0$, $\breve{S}_1' = (f(W_0) - I^*)^+ = f(W_0) - I^*$, by monotonicity of f and (13). By concavity of f, $S_0 + f(I_0) - I^*$ is concave in I_0. It follows that $S_0 + f(I_0) - I^*$ is positive on the interval $[0, W_0]$. This gives (23).

By (19) and (23),

$$\vec{S}_1'' = \max\left[-p\vec{S}_1'/(1-p), S_0 - I^*\right] \tag{A2}$$

$$= S_0 - I^*,$$

$$\Leftrightarrow \qquad W_0 - I^* \bullet I_0 - pf(I_0). \tag{A3}$$

The right hand side of (A3) is convex in I_0. It is 0 at $I_0 = 0$ and hence (A3) holds strictly at $I_0 = 0$ by (3). When $I_0 = W_0$, (A3) also holds since by (14), $I^* < pf(I^*) < pf(W_0)$. It follows that (A3) holds with strict inequality on the interval $[0, W_0]$. This gives (24).[]

Proof of Theorem 3: By (23) and (24), evaluating (18') and (18") under the storage strategy $\left(\vec{S}_1', \vec{S}_1''\right)$ gives

$$E\left[\tilde{W}_2 \middle| \theta_1 = \theta\right] = pf\left(I^*\right) + \frac{pf\left(I_0\right) - I_0}{p} + \frac{W_0 - I^*}{p}, \tag{A4'}$$

$$E\left[\tilde{W}_2 \middle| \theta_1 = 0\right] = pf\left(I^*\right). \tag{A4''}$$

The unconditional expectation as a function of I_0 is then

$$E\left[\tilde{W}_2\right] = pf\left(I^*\right) + pf\left(I_0\right) - I_0 + W_0 - I^*. \tag{A5}$$

It follows that (A5) is maximized by the choice of $I_0 = I^*$, and the value of (A5) for this choice is the supremum in (14), by Theorem 2.[]

Proof of Theorem 4: Given f and p, let $\alpha = \varepsilon\left(I^*\right)$ and choose θ so that $p\theta = \left(I^*\right)^{1-\alpha}$. Let $h(I) = I^\alpha \theta / \alpha$, and let

$$g(I) = \frac{f(I)}{I^\alpha \theta / \alpha}, \tag{A6}$$

So that $f(I) = g(I)h(I)$. Clearly I^* is the optimal investment level for the constant elasticity production function h for this p. substituting in for α and θ, we get that

$$g\left(I^*\right) = \frac{f\left(I^*\right)}{\dfrac{\left(I^*\right)^\alpha \left(\left(I^*\right)^{1-\alpha} / p\right)}{\varepsilon\left(I^*\right)}} = \frac{f\left(I^*\right)}{\dfrac{I^*}{p\varepsilon\left(I^*\right)}} = \frac{pf\left(I^*\right)\varepsilon\left(I^*\right)}{I^*} = \frac{pf\left(I^*\right)}{I^*}\frac{I^* f'\left(I^*\right)}{f\left(I^*\right)}$$

$$= pf'\left(I^*\right) = 1, \tag{A7}$$

by the first order condition (2). It follows that $f(I^*) = h(I^*)$. It is straightforward to verify that the elasticity of scale $\varepsilon(I)$ of f is equal $\varepsilon_g(I) + \alpha$, where $\varepsilon_g(I) = Ig'(I)/g(I)$ is the elasticity of scale of the function g. Differentiating g with respect to I gives

$$g'(I) = \frac{I^{\alpha}f'(I) - \alpha I^{\alpha-1}f(I)}{I^{\alpha}h(I)}.$$

Thus

$$g'(I) \gtreqless 0 \quad \text{as} \quad \varepsilon(I) \gtreqless \alpha. \tag{A8}$$

In particular, we have that

$$g'(I^*) = 0, \text{ and g is decreasing for } I > I^*.$$

(A9)

By the analysis for the constant elasticity case, (33) holds for h and hence

$$h(I^* + h(I^*)) < h(I^*) + h'(I^*)h(I^*)(1-\alpha) = h(I^*) + h'(I^*)h(I^*)(1-\varepsilon(I^*)). \tag{A10}$$

Thus

$$f(I^* + f(I^*)) = g(I^* + f(I^*))h(I^* + f(I^*)) \bullet g(I^*)h(I^* + h(I^*))$$

$$< g(I^*)h(I^*) + g(I^*)h'(I^*)h(I^*)(1-\varepsilon(I^*))$$

$$= f(I^*) + g'(I^*)g(I^*)h(I^*)h(I^*)(1-\varepsilon(I^*)) + g(I^*)g(I^*)h'(I^*)h(I^*)(1-\varepsilon(I^*))$$

$$= f(I^*) + [g'(I^*)h(I^*) + g(I^*)h'(I^*)]g(I^*)h(I^*)(1-\varepsilon(I^*))$$

$$= f(I^*) + f'(I^*)f(I^*)(1-\varepsilon(I^*)),$$

where the first inequality follows from (A9) and the second from (A10), and we have used (A7) and (A9) repeatedly. So (33) holds for this f and p as well, proving the first statement of the theorem. The second statement follows directly from the first. []

Proof of Theorem 5: (A4') and (A4") of Theorem 3 still hold with the left-hand side of (A4') being $E\left[\tilde{W}_2 \middle| \theta_1 \neq 0, p\right]$ and the left-hand side of (A4") being $E\left[\tilde{W}_2 \middle| \theta_1 = 0, p\right]$. Expecting out θ_1 first yields (A5). Then taking the expectation with respect to p gives

$$E\left[\tilde{W}_2\right] = E\left[\tilde{p} f\left(I^*\right) - I^*\right] + p_0 f\left(I_0\right) - I_0 + W_0. \tag{A11}$$

It follows that (A11) is maximized by the choice of $I_0 = I_0{}^*$.[]

Proof of Theorem 6: Rewrite (36) in the form

$$f\left(I^* + f\left(I_0{}^*\right)\right) \bullet f\left(I^*\right) + f'\left(I^*\right)\left(f\left(I_0{}^*\right) - I^* f'\left(I^*\right)\right). \tag{A12}$$

The left hand side of (A12) is increasing in p. Taking the derivative of the right hand side and using the fact that I^* is increasing in p, the sign of the derivative of the right hand side is the same as the sign of the expression

$$f''\left(I^*\right)\left(f\left(I_0{}^*\right) - 2I^* f'\left(I^*\right)\right) + f'\left(I^*\right) - f'\left(I^*\right)^2. \tag{A13}$$

By the first order condition (2), $f'\left(I^*\right) - f'\left(I^*\right)^2 < 0$. By concavity of f, (A13) is negative if $f\left(I_0{}^*\right) \bullet 2I^* f'\left(I^*\right)$. Concavity of $f\left(I^*\right)$ in p implies that

$$p_0 f\left(I^*(1)\right) = p_0 f\left(I^*(1)\right) + \left(1 - p_0\right) f\left(I^*(0)\right) \bullet f\left(I^*\left(p_0 \cdot 1 + \left(1 - p_0\right) \cdot 0\right)\right) = f\left(I_0{}^*\right).$$

Since $p_0 \bullet \vec{\varepsilon}/\varepsilon^* \bullet \varepsilon(I^*)/\varepsilon^* = \frac{I^* f'(I^*)}{f(I^*)}\Big/\varepsilon^*$, it follows that

$$f(I_0^*) \bullet p_0 f(I^*(1)) \bullet \frac{I^* f'(I^*) f(I^*(1))}{f(I^*)}\Big/\varepsilon^* \bullet I^* f'(I^*)/\varepsilon^* \bullet 2I^* f'(I^*),$$

(A14)

where the third inequality in (A14) follows from the monotonicity of $f(I^*)$ in p and the fourth inequality follows from the hypothesis that $\varepsilon^* \bullet 1/2$. Thus (A13) is negative, implying that the smallest the right hand side of (A12) gets is at p = 1.[]