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Consumption and Asset Prices with Recursive Preferences: Continuous-Time Approximations to Discrete-Time Models

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Abstract: This paper presents tractable and efficient numerical solutions to general equilibrium models of asset prices and consumption where the representative agent has recursive preferences. It provides a discrete-time presentation of the approach of Fisher and Gilles (1999), treating continuous-time representations as approximations to discrete-time "truth." First, exact discrete-time solutions are derived, illustrating the following ideas: (i) The price-dividend ratio (such as the wealth-consumption ratio) is a perpetuity (the canonical infinitely lived asset), the value of which is the sum of dividend-denominated bond prices, and (ii) the positivity of the dividend-denominated asymptotic forward rate is necessary and sufficient for the convergence of value function iteration for an important class of models. Next, continuous-time approximations are introduced. By assuming the size of the time step is small, first-order approximations in the step size provide the same analytical flexibility to discrete-time modeling as Ito's lemma provides in continuous time. Moreover, it is shown that differential equations provide an efficient platform for value function iteration. Last, continuous-time normalizations are adopted, providing an efficient solution method for recursive preferences.

JEL classification: G12

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CONSUMPTION AND ASSET PRICES WITH RECURSIVE PREFERENCES: CONTINUOUS-TIME APPROXIMATIONS TO DISCRETE-TIME MODELS

MARK FISHER

ABSTRACT. This paper presents tractable and efficient numerical solutions to general equilibrium models of asset prices and consumption where the representative agent has recursive preferences. It provides a discrete-time presentation of the approach of Fisher and Gilles (1998), treating continuous-time representations as approximations to discrete-time "truth." First, exact discrete-time solutions are derived, illustrating the following ideas: (*i*) The price–dividend ratio (such as the wealth–consumption ratio) is a perpetuity (the canonical infinitely-lived asset), the value of which is the sum of dividend-denominated bond prices, and (*ii*) the positivity of the dividend-denominated asymptotic forward rate is necessary and sufficient for the convergence of value function iteration for an important class of models. Next, continuous-time approximations are introduced. By assuming the size of the time step is small, first-order approximations in the stepsize provide the same analytical flexibility to discrete-time modeling as Ito's lemma provides in continuous time. Moreover, it is shown that differential equations provide an efficient platform for value function iteration. Last, continuous-time normalizations are adopted, providing an efficient solution method for recursive preferences.

1. Introduction

This paper presents tractable and efficient numerical solutions to general equilibrium models of asset prices and consumption where the representative agent has recursive preferences and utility is derived solely from consumption. Models of this type have been developed in both discrete-time and continuous-time settings. This paper highlights the benefits—analytical and numerical—of using continuous-time models relative to discrete-time models. The analytical advantages of continuous-time modeling are attributable to Ito's lemma, which provides continuous-time modeling with great flexibility with respect to transforming expressions. Even so, Campbell (1993, p. 487) compares continuous-time techniques unfavorably with

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discrete-time techniques for analyzing asset pricing models of the sort discussed in this paper:

> ... Robert C. Merton ... suggested reformulating the consumption and portfolio-choice problem in continuous time. Doing this in effect linearizes by taking the decision interval infinitely small, so that the model becomes linear over this interval. However, this kind of linearity is only local, so it does not allow one easily to study longer-run aspects of intertemporal asset pricing theory.

Campbell instead log-linearizes a discrete-time intertemporal budget constraint and derives an impressive array of interesting results. Nevertheless, it turns out that continuous-time modeling is quite well-suited to to deal with these problems. In particular, Fisher and Gilles (1998) nest Campbell's results in a more general dynamic setting in a continuous-time model. In doing so they are able to delimit the scope of the applicability of Campbell's results. In light of the work of Fisher and Gilles, continuous-time modeling is seen to provide powerful analytical tools for the models considered here.

Even though continuous-time modeling offers great flexibility to transform expressions, in some ways it is more restrictive. In particular, not all discrete-time normalizations can be implemented directly in continuous time. Discrete-time modelers can rely on the "today-tomorrow" distinction of discrete time and need not necessarily consider what happens in the limit as the size of the time step goes to zero. On the other hand, it is often straightforward to translate a normalization that works in continuous time into discrete time. It turns out that many interrelations are more transparent in the continuous-time representations than in those of discrete-time. Fortunately, these continuous-time representations can be brought to bear on discrete-time models by, for example, treating the continuous-time representations as approximations to the discrete-time "truth."

Yet even if one grants the analytical advantages of continuous-time modeling, it is not clear what advantages, if any, it offers for numerical solutions. On this front, continuous-time models seem to be disadvantaged. For example, consider the solution technique of value-function iteration for dynamic programming problems. Judd (1998, p. 440) states the apparent difficulties with continuous-time models in this regard:

> Value function iteration is an important method for solving discrete-time dynamic programming problems. Unfortunately, the structure of value function iteration is itself tied to the discrete nature of time in those models, and cannot be used for continuous-time problems, since there is no today-tomorrow distinction in continuous time.
>
> One approach is to replace the continuous-time structure with a discrete-time structure with short periods and then use discrete-time methods. Using short periods will make the discount factor, β, close to unity, and imply very slow convergence.

For concreteness, consider a case where value function iteration can be reduced to solving for a fixed point by iteratively solving a system of difference equations.

Judd's point is that, given a prespecified convergence criterion, decreasing the size of the time step has the effect of increasing the number of steps required to reach convergence.

Yet a different moral can be drawn from the observation that there is no today-tomorrow distinction in continuous time: The absence of this distinction liberates one from the tyranny of a uniform step size. In the limit, a system of difference equations becomes a system of ordinary differential equations (ODEs). Modern ODE solvers choose the step size adaptively, according to various error criteria computed as part of the solution process. As Press, Teukolsky, Vettering, and Flannery (1992) state in their chapter on the integration of ODEs,[1]

> A good ODE integrator should exert some adaptive control over its own progress, making frequent changes in its stepsize. Usually the purpose of this adaptive stepsize control is to achieve some predetermined accuracy in the solution with minimum computational effort. Many small steps should tiptoe through treacherous terrain, while a few great strides should speed through smooth uninteresting countryside. The resulting gains in efficiency are not mere tens of percents or factors of two; they can be factors of ten, a hundred, or more.

As a result, the amount of computational effort required to reach convergence with a given accuracy can be far less for a differential equation than for a difference equation. When this problem is seen in this light, efficient numerical solutions based on continuous-time limits emerge.

This paper provides a discrete-time presentation of the approach of Fisher and Gilles (1998) to modeling and solving general-equilibrium asset-pricing problems. It compares their approach to existing approaches for solving models with both recursive and standard time-separable preferences. In order to highlight the advantages of their approach, their model is presented as an approximation to discrete-time models. Many presentations of discrete-time models implicitly take the size of the time step to be unity and adopt normalizations that rely on the today-tomorrow distinction with the result that the connection to continuous-time models is not transparent. In this paper the size of the time step, Δ, is explicit. The continuous-time results are all seen to be first-order approximations around $\Delta = 0$.

I adopt the Euler approximation to continuous-time Ito processes as the true discrete-time process. The Euler approximation converges to an Ito process both pathwise and in distribution and consequently it can be used to solve stochastic differential equations (SDEs).[2] Nevertheless, for some purposes the most natural discrete-time model is the binomial model. Not only does it converge on an Ito process in the limit, but it also maintains completeness of markets: A stock and a bond are sufficient to span markets in either a binomial model or a continuous-time

[1] Judd (1998) does not discuss variable step sizes.

[2] Kloeden and Platen (1991) provide a thorough discussion of Monte Carlo solutions to SDEs using the Euler approximation as well as more sophisticated approximations that converge more rapidly. For a brief introduction to these issues, see their section titled "A brief survey of stochastic numerical methods" (pp. *xxiii*–*xxxv*).

model with a single Brownian motion. By contrast, for the Euler approximations adopted here, the market is incomplete for any finite stepsize and any finite number of securities. Since the main concern here is with equilibrium asset pricing, this does not pose a problem. Having provided this note of caution, I will nevertheless refer to various "absence-of-arbitrage conditions" that are not strictly such in the discrete-time setting.

For technical issues regarding the continuous-time derivations of the continuous-time formulas in this paper, see Fisher and Gilles (1998) and the references therein, especially Schroder and Skiadas (1999). The derivations of the continuous-time formulas in this paper are intended to be heuristic. Continuous-time restrictions are derived by finding a first-order approximation in the stepsize that allows one to cancel out the stepsize.[3]

The absence of the today-tomorrow distinction in the limit as the stepsize goes to zero requires the adoption of normalizations that are not typical of discrete-time modeling. First, explicit reference to next period's asset price is replaced by the growth-rate dynamics of the asset price. Second, cum-dividend asset prices that include the current dividend flow as used, rather than ex-dividend prices that exclude it. As a result, the dividend that appears in the pricing equation is known at the beginning of the period and zero-coupon bond prices have a value of one at maturity rather than zero. The third necessary modification relates to the way in which state variables enter the model. In discrete-time models, state variables are often linked to backward-looking realized growth rates. But in stochastic continuous-time models based on Brownian motion, instantaneous realized growth rates do not exist. Instead, state variables can be linked with forward-looking expected growth rates, which do exist in the limit. On purely economic grounds, expected growth rates are more appropriate as hitching posts for state variables than realized growth rates: As is well-known, variation in realized growth rates that is unrelated to expected future growth rates is irrelevant. As a result, when actual growth rates are not perfectly correlated with expected growth rates, it is more parsimonious to use expected growth rates as state variables.

The final issue has to do with the size of the time step and the quality of the continuous-time approximations. Obviously, continuous-time approximations will well approximate a discrete-time model when the size of the time step is small. In other words, if the size of the time step is three months or one year, continuous-time approximations may do poorly. However, for the asset pricing models considered in this paper, I believe time steps of the magnitude of a day or a week are more appropriate. It is certainly true that at this frequency, the time step will not match the sampling frequency of the data. As the stepsize decreases, the parameters in the processes for the state variables will need to be adjusted in order to match the moments of the less-frequently sampled data.

Conceptual overview. A perpetuity is the canonical infinitely-lived asset. The absence-of-arbitrage condition for the price of a perpetuity is a functional equation for which the perpetuity price is the fixed point. Zero-coupon bond prices

[3]A version of Ito's lemma for the discrete-time processes in this paper is presented in Appendix A.

form the natural basis for the value of a perpetuity, and the positivity of the asymptotic forward rate is necessary and sufficient for the convergence of annuity prices to a perpetuity price. The functional operator in the perpetuity equation can be decomposed into a "bond-shift" operator plus a dividend accumulator. Any infinitely-lived asset with a strictly positive dividend flow can be transformed into a dividend-denominated perpetuity which can be analyzed in terms of its associated dividend-denominated zero-coupon bonds and dividend-denominated asymptotic forward rate.

In an equilibrium setting with an infinitely-lived representative agent, wealth is the asset for which consumption as the strictly positive dividend flow, making the wealth–consumption ratio a dividend-denominated perpetuity. If the agent has time-separable preferences, the dividend-denominated intertemporal marginal rate of substitution can be expressed in terms of equilibrium consumption growth. This allows one to identify the dividend-denominated interest rate (and the price of risk) and apply the solution techniques based on the perpetuity approach. Moreover, by structuring the model so that the dividend-denominated bond prices are exponential-affine, the curse of dimensionality can be eliminated since the functional form of the solution is known. The value of the wealth–consumption ratio at any point in the multidimensional state space can be evaluated by a univariate sum of dividend-denominated bond prices.

With recursive preferences, however, the dividend-denominated intertemporal marginal rate of substitution is not determined solely by consumption growth in general, and so the direct application of the approach outlined above is not possible. For recursive preferences, the first problem is simply to compute the utility of a given consumption process. Homotheticity guarantees that utility depends on two state variables, consumption (which delimits current opportunities) and an "information" variable that summarizes what is known about future opportunities. The key to computing utility is to find an expression for the information variable in terms of the growth rate of consumption (through which the state dependence of utility enters). The second problem is to solve for optimal consumption given a stochastic return process. This problem involves maximization, which was absent from the previous recursive utility problem. Marigal utility provides the link to optimization.

Outline of the rest of the paper. The paper is divided into four main parts. In Section 2, asset pricing is introduced in a generic setting. I interpret the price–dividend ratio as the value of a dividend-denominated perpetuity and derive exact closed-form solutions as infinite sums of dividend-denominated zero-coupon bond prices. Absence-of-arbitrage conditions produce systems of difference equations for bond prices. The infinite sums can be thought of as generated by repeated application of a contraction operator related to Bellman's equation. We show how the asymptotic dividend-denominated forward rate can be used to determine whether the iterative procedure will converge, even when Blackwell's sufficient conditions are not satisfied.

In Section 3, we present solutions based on continuous-time approximations for models with standard preferences. The solution method proposed for these models amounts to the integral of bond prices and can be thought of as the result of a continuous-time contraction. Bond prices are characterized by a system of ordinary differential equations (ODEs). As noted above, one of the advantages of the continuous time formulation is that the numerical solution to differential equations is very flexible: ODE solvers are free to adapt the step size to the features of the solution, which allows continuous-time formulations to converge much more rapidly than corresponding discrete-time formulations.

In Section 4, recursive preferences are introduced. A method of solving for utility (given a consumption process) is presented that parallels the solution method for asset pricing presented earlier. The supporting price system and returns process are derived, and the role of the information variable as a weighted forecast is explained.

In Section 5, the solution method of undetermined coefficients is applied to the recursive preference problem. Regions in the preference-parameter space where no solution to the infinite-horizon problem are identified.

2. Asset pricing

The *cum-dividend* value of an asset is the present value of the future dividends plus the value of the current dividend:

$$\begin{aligned} p(t) &= d(t)\,\Delta + E_t\left[\sum_{i=1}^{\infty}\left(\frac{m(t+i\,\Delta)}{m(t)}\right) d(t+i\,\Delta)\,\Delta\right] \\ &= d(t)\,\Delta + E_t\left[\left(\frac{m(t+\Delta)}{m(t)}\right) p(t+\Delta)\right]. \end{aligned} \tag{2.1}$$

$E_t[\,\cdot\,]$ is the conditional expectation operator, Δ is the size of the time step (the length of the period), $d(t+i\,\Delta)$ is the rate of dividend flow at time $t+i\,\Delta$, $m(t+i\,\Delta)/m(t)$ is the discount factor, and $p(t)$ is the cum-dividend asset price. The ex-dividend price is simply $p(t)-d(t)\,\Delta$. The existence of a well-defined price in (2.1) depends on the convergence of an infinite sum. If dividends grow too fast asymptotically relative to the discount factor, the sum will not converge. For the time being, let us assume the infinite-horizon problem does have a solution. In the second line of (2.1), the infinite-horizon problem has been converted to a one-period problem where $p(t+\Delta)$ captures the value of all of the future dividends. Since absolute time plays no role in (2.1), the notation can be streamlined:

$$p = d\,\Delta + E_t\left[(m'/m)\,p'\right], \tag{2.2}$$

where unprimed variables represent beginning-of-period values and primed variables represent end-of-period values.

Bond prices, the interest rate, and the state–price deflator. The simplest asset is a zero-coupon bond that pays one unit when it matures at time $t+\tau$, where $\tau \geq 0$ is the bond's term to maturity measured in units of time.[4] (It matures in

[4]The derivation of discrete-time bond prices parallels that of Backus and Zin (1994).

τ/Δ periods.) Its cum-dividend value at time t is

$$B(t, t+\tau) = E_t\left[\frac{m(t+\tau)}{m(t)}\right]. \tag{2.3}$$

Comparing (2.3) with (2.1), we see that the dividend flows for a τ-maturity zero-coupon bond are

$$d(t + i\,\Delta) = \begin{cases} 0 & \text{for } i \neq \tau/\Delta \\ 1/\Delta & \text{for } i = \tau/\Delta. \end{cases}$$

To streamline bond-price notation, let B_τ be the current price of a τ-maturity zero-coupon bond. Next period the τ-maturity bond will have become a $(\tau-\Delta)$-maturity bond, so its value today can be expressed as

$$B_\tau = E_t[(m'/m)\,B'_{\tau-\Delta}]. \tag{2.4}$$

Upon maturity we have $B_0 = 1$, regardless of the state of the world. Thus the value of the shortest (nonzero) maturity bond is

$$B_\Delta = E_t[m'/m].$$

The short-term interest rate can be defined in terms of B_Δ:

$$r := -\log(B_\Delta)/\Delta = -\log(E_t[m'/m])/\Delta.$$

The conditional distribution of the state–price deflator. The discount factor, m'/m, is the growth rate of the state–price deflator.[5] In equilibrium, m'/m is the intertemporal marginal rate of substitution measured in the chosen numeriare. We will assume that the state–price deflator is conditionally log-normally distributed. In particular, we will decompose the growth rate of the state–price deflator into expected and unexpected components as follows:

$$\log(m'/m) = \widetilde{\mu}_m\,\Delta + \sigma_m \cdot \varepsilon\,\sqrt{\Delta}, \tag{2.5}$$

where $\widetilde{\mu}_m$ is a (possibly state-dependent) scalar, σ_m is a (possibly state-dependent) vector, ε is a vector of independent standard normal shocks, and $y \cdot z$ is the inner (dot) product of two vectors y and z. We refer to $\widetilde{\mu}_m$ as the drift and σ_m as the diffusion or the volatility. Equation (2.5) implies

$$\log(m') \sim \mathcal{N}(\log(m) + \widetilde{\mu}_m\,\Delta, \|\sigma_m\|\,\sqrt{\Delta}), \tag{2.6}$$

where $\|z\| := \sqrt{z \cdot z}$ and $z \sim \mathcal{N}(\mu, \sigma)$ signifies that z is a random variable that is normally distributed with mean μ and standard deviation σ. Given (2.6) we see that $\widetilde{\mu}_m$ and σ_m have been normalized to measure the mean and the variance of the growth rate of the state–price deflator per unit of time.

Given (2.6) and using the fact that $E[e^z] = \exp(\mu + \sigma^2/2)$ for $z \sim \mathcal{N}(\mu, \sigma)$, we have $B_\Delta = \exp\left(\left\{\widetilde{\mu}_m + \frac{1}{2}\,\|\sigma_m\|^2\right\}\Delta\right)$, and therefore

$$r = -\widetilde{\mu}_m - \frac{1}{2}\,\|\sigma_m\|^2. \tag{2.7a}$$

[5]See Duffie (1996) for an extensive discussion of the state–price deflator.

In addition, it is convenient to adopt the following renormalization of the volatility of m:

$$\lambda = -\sigma_m, \tag{2.7b}$$

where λ is called the price of risk. Given these expressions for r and λ, we can write the conditional dynamics of the state–price deflator as

$$\log(m'/m) = -\left(r + \frac{1}{2}\,\|\lambda\|^2\right)\Delta - \lambda \cdot \varepsilon\,\sqrt{\Delta}. \tag{2.8}$$

The term structure of interest rates. Let us consider longer-term bond prices. We can rewrite (2.4) as

$$E_t\left[(m'/m)\,(B'_{\tau-\Delta}/B_\tau)\right] = 1. \tag{2.9}$$

We now assume that the conditional growth rate of a bond's price is also log-normally distributed:

$$\log(B'_{\tau-\Delta}/B_\tau) = \widetilde{\mu}_{B_\tau}\,\Delta + \sigma_{B_\tau} \cdot \varepsilon\,\sqrt{\Delta}.$$

Inserting the expressions for m'/m and $B'_{\tau-\Delta}/B_\tau$ into (2.9), we have

$$\exp\left(\left\{-r - \frac{1}{2}\,\|\lambda\|^2 + \widetilde{\mu}_{B_\tau} + \frac{1}{2}\,\|\sigma_{B_\tau} - \lambda\|^2\right\}\Delta\right) = 1.$$

Taking logs, dividing by Δ, and rearranging produces

$$\mu_{B_\tau} = r + \lambda \cdot \sigma_{B_\tau}, \tag{2.10}$$

where $\mu_{B_\tau} := \widetilde{\mu}_{B_\tau} + \frac{1}{2}\,\|\sigma_{B_\tau}\|^2$. Equation (2.10) expresses the well-known relation between risk and return: The expected return μ_{B_τ} equals the risk-free rate r rate plus the covariance-based risk premium $\lambda \cdot \sigma_{B_\tau}$.

In order to derive explicit expressions for bond prices, we introduce Markovian state variables. This allows us to turn (2.10) into a system of difference equations by the matching undetermined coefficients. Our goal is to find a Markovian bond pricing function $\mathcal{B}(x,\tau)$ such that $B_\tau = \mathcal{B}(X,\tau)$ for some Markovian state variables X. In order to acheive this goal, how must the state variables enter the marginal rate of substitution? Examining (2.10), we see that r and λ must be Markovian. In other words, we will need functions $R(x)$ and $\Lambda(x)$ that determine how the interest rate and the price of risk depend on the state of the world: $r = R(X)$ and $\lambda = \Lambda(X)$. Note however that m'/m itself need not be Markovian. Markovianizing m'/m is at best unnecessary and at worst restictive.[6]

Let x be a scalar state variable that is conditionally normally distributed:

$$x' - x = \mu_X\,\Delta + \sigma_X \cdot \varepsilon\,\sqrt{\Delta}, \tag{2.11}$$

where μ_X and σ_X can themselves be state-dependent. For the moment, however, we will leave unspecified the way in which μ_X and σ_X depend on x. Suppose the solution for bond prices can be written as follows:

$$\mathcal{B}(x,\tau) = \exp(a_\tau + b_\tau\,x). \tag{2.12}$$

[6]It is even more restrictive to Markovianize the state–price deflator m. See Fisher and Gilles (1999).

Since $B_0 = 1$, we must have $a_0 = b_0 = 0$. The coefficients $\{a_\tau\}$ and $\{b_\tau\}$ for $\tau = 1, 2, 3, \ldots$ are as yet undetermined.

It is convenient to introduce forward rates here. Forward rates are defined as

$$f_\tau := -\log(B_\tau / B_{\tau+\Delta})/\Delta, \tag{2.13}$$

where the indices have been chosen so that $f_0 = r$. In light of (2.12), forward rates can be expressed as

$$f_\tau = \left(\frac{a_{\tau+\Delta} - a_\tau}{\Delta}\right) + \left(\frac{b_{\tau+\Delta} - b_\tau}{\Delta}\right) x. \tag{2.14}$$

Since $f_0 = r$, it follows that $r = (a_\Delta/\Delta) + (b_\Delta/\Delta)\, x$.

To recover all of the coefficients $\{a_\tau\}$ and $\{b_\tau\}$, we proceed as follows. Given (2.12) and (2.11), we can write

$$\begin{aligned} \log\left(\frac{B'_{\tau-\Delta}}{B_\tau}\right) &= (a_{\tau-\Delta} + b_{\tau-\Delta}\, x') - (a_\tau + b_\tau\, x) \\ &= (-f_{\tau-\Delta} + b_{\tau-\Delta}\, \mu_X)\, \Delta + b_{\tau-\Delta}\, \sigma_X \cdot \varepsilon \sqrt{\Delta}, \end{aligned}$$

and therefore

$$\mu_{B_\tau} = -f_{\tau-\Delta} + b_{\tau-\Delta}\, \mu_X + b^2_{\tau-\Delta}\, \frac{1}{2}\, \|\sigma_X\|^2 \tag{2.15a}$$

$$\sigma_{B_\tau} = b_{\tau-\Delta}\, \sigma_X. \tag{2.15b}$$

In (2.15a) we see that the expected return on a bond is composed of three parts, induced by (*i*) the forward rate (solely from the passage of time), (*ii*) the expected change in the state variable, and (*iii*) the nonlinear relationship between bond prices and the state variable (the Jensen's inequality part). Using (2.15) we can write (2.10) as

$$-f_{\tau-\Delta} + b_{\tau-\Delta}\, \hat{\mu}_X + b^2_{\tau-\Delta}\, \frac{1}{2}\, \|\sigma_X\|^2 - r = 0, \tag{2.16}$$

where

$$\hat{\mu}_X := \mu_X - \lambda \cdot \sigma_X$$

is the so-called risk-adjusted drift of x.

We now make specific assumptions about how the state variable enters the system. We specify how each of r, λ, $\hat{\mu}_X$, and σ_X depend on x. In order to maintain consistency with our previous assumptions, r, $\hat{\mu}_X$, and $\|\sigma_X\|^2$ must be affine in x.[7] For this example, assume $r = x$, $\mu_X = \kappa\,(\bar{x} - x)$, and $\sigma_X = s_X$ and λ are scalar constants. In this case we have $\hat{\mu}_X = \kappa\,(\bar{X} - x)$, where $\bar{X} := \bar{x} - \lambda\,\sigma_X/\kappa$. Given this specification (2.10) becomes

$$\left(\frac{a_{\tau-\Delta} - a_\tau}{\Delta}\right) + \left(\frac{b_{\tau-\Delta} - b_\tau}{\Delta}\right) x + b_{\tau-\Delta}\, \big(\kappa\,(\bar{X} - x)\big) + b^2_{\tau-\Delta}\, \frac{1}{2}\, \|s_X\|^2 - x = 0.$$

[7]The conditions are similar for continuous-time models. See Duffie and Kan (1996).

By matching coefficients, a pair of difference equations is produced:

$$\left(\frac{a_\tau - a_{\tau-\Delta}}{\Delta}\right) = b_{\tau-\Delta}\,\kappa\,\bar{X} + b_{\tau-\Delta}^2\,\frac{1}{2}\,\|s_X\|^2 \tag{2.17a}$$

$$\left(\frac{b_\tau - b_{\tau-\Delta}}{\Delta}\right) = -1 - \kappa\,b_{\tau-\Delta}, \tag{2.17b}$$

subject to $a_0 = b_0 = 0$. The solution to (2.17) is

$$\begin{aligned} a_\tau = {} & \left(\frac{s_X^2}{2\,\kappa^2} - \bar{X}\right)\tau + \frac{\left(1 - (1-\kappa\,\Delta)^{\tau/\Delta}\right)\bar{X}}{\kappa} - \\ & \frac{\left(2\left(1 - (1-\kappa\,\Delta)^{\tau/\Delta}\right) - \frac{1-(1-\kappa\,\Delta)^{2\,\tau/\Delta}}{2-\kappa\,\Delta}\right)s_X^2}{2\,\kappa^3} \end{aligned} \tag{2.18a}$$

$$b_\tau = \frac{(1-\kappa\,\Delta)^{\tau/\Delta} - 1}{\kappa}. \tag{2.18b}$$

The bond-price solution is obtained by substituting (2.18) into (2.12).

The value of a perpetuity. A perpetuity is the canonical infinitely-lived asset. It pays dividends at the constant rate of one: $d(t + i\,\Delta) = 1$ for all i in (2.1). For a perpetuity, (2.2) becomes

$$p = \Delta + E_t\left[(m'/m)\,p'\right]. \tag{2.19}$$

The value of a perpetuity is the limiting value, if the limit exists, of the value of an annuity as the horizon goes to infinity. The value of an annuity is the sum of zero-coupon bond prices:

$$p_\tau := \sum_{i=0}^{\tau/\Delta} B_{i\,\Delta}\,\Delta, \tag{2.20}$$

where p_τ denote the value of a annuity that matures in τ units of time, so that $p = \lim_{\tau\to\infty}\; p_\tau$ if the limit exists. We address the question of existence in terms of the asymptotic forward rate.

A necessary condition for convergence is that bond prices go to zero as the maturity goes to infinity: $\lim_{\tau\to\infty}\; B_\tau = 0$. From (2.18b) we have $\lim_{\tau\to\infty}\; b_\tau = -1/\kappa$. Therefore, we need $\lim_{\tau\to\infty}\; a_\tau = -\infty$, or equivalently $\lim_{\tau\to\infty}\; a_\tau - a_{\tau-\Delta} < 0$. From (2.17a) we see that

$$\lim_{\tau\to\infty}\; \frac{a_\tau - a_{\tau-\Delta}}{\Delta} = \lim_{\tau\to\infty}\; b_{\tau-\Delta}\,\kappa\,\bar{X} + b_{\tau-\Delta}^2\,\|s_X\|^2 = -\bar{X} + \frac{\|s_X\|^2}{2\,\kappa^2}.$$

Therefore, bond prices will go to zero if and only if $\bar{X} - \frac{1}{2}\,(\|s_X\|/\kappa)^2 > 0$. At the same time, inserting (2.17) into (2.14), we have

$$f_\infty := \lim_{\tau\to\infty}\; f_\tau = \bar{X} - \frac{1}{2}\left(\frac{\|s_X\|}{\kappa}\right)^2.$$

Thus bond prices will go to zero if and only if $f_\infty > 0$. Assuming bond prices do go to zero, we can apply the ratio test for convergence. The ratio of successive terms is $B_{\tau+\Delta}/B_\tau = e^{-f_\tau \Delta}$. Thus we have

$$\lim_{\tau\to\infty} B_{\tau+\Delta}/B_\tau = \lim_{\tau\to\infty} e^{-f_\tau \Delta} = e^{-f_\infty \Delta} < 1.$$

For this example we conclude that the positivity of the asymptotic forward rate is both necessary and sufficient for the convergence of annuity prices to the price of a perpetuity as the horizon goes to infinity.[8]

Functional iteration and fixed points. We can think of (2.19) as expressing a functional relation: $p = T(p')$, where $T(\,\cdot\,) = \Delta + T_B(\,\cdot\,)$, and T_B is the "bond-shift" operator: $T_B(B_\tau)$ returns $B_{\tau+\Delta}$. We are looking for the fixed point p such that $p = T(p)$. We can confirm our solution in terms of bond prices:

$$T\left(\sum_{i=0}^{\infty} B_{i\,\Delta}\,\Delta\right) = \Delta + T_B\left(\sum_{i=0}^{\infty} B_{i\,\Delta}\,\Delta\right) = \Delta + \sum_{i=0}^{\infty} T_B(B_{i\,\Delta})\,\Delta = \sum_{i=0}^{\infty} B_{i\,\Delta}\,\Delta.$$

Moreover, this suggests that we can solve for the fixed point by starting with $p_0 = \Delta$ and iterating according to $p_{\tau+\Delta} = T(p_\tau)$.[9] Bond prices provide closed-form expressions for the increments, $B_{\tau+\Delta} = (p_{\tau+\Delta} - p_\tau)/\Delta$.

Blackwell's sufficient conditions for T to be a contraction are (i) monotonicity and (ii) discounting. Let f and g be two functions in the space T operates on, and let $f - g \geq 0$. T satisfies the monotonicity condition if $T(f) - T(g) \geq 0$. In our case $T(f) - T(g) = T_B(f - g) \geq 0$. Let $c \geq 0$ be a scalar constant. T satisfies the discounting condition if $T(f + c) - T(f) \leq c\,\delta$, where $\delta \in (0, 1)$. To see that this condition is not satisfied in general, let $f = 0$ and $c = 1$. Then $T(f + c) - T(f) = T_B(1) - T_B(0) = e^{-r\,\Delta}$. Note that $e^{-r\,\Delta} \geq 1$ where $r \leq 0$, which violates the discounting condition. We have seen, however, that it is possible to derive necessary and sufficient conditions for convergence in terms of the asymptotic forward rate.

A first continuous-time approximation. Equation (2.10) exemplifies the restrictions we will derive as continuous-time approximations. In this case, of course, no approximations were required. The restriction involves in the parameters of forward-looking growth rate processes. The introduction of Markovian state variables turns the restriction into a system of difference equations.

It is straightforward to construct a continuous-time approximation to the solution for the value of a perpetuity given the Markovian structure of the example above. Assume that bond prices maintain their exponential-affine form as in (2.12). The pair of difference equations that characterize the bond-price solution, (2.17), become a pair of ordinary differential equations in the limit as $\Delta \to 0$. The solution to the

[8] This analysis extends to the class of exponential-affine and exponential-quadratic models as long as the state variables are stationary.

[9] We could just as well start with $p_{-\Delta} = 0$.

differential equations is the limit of the solution to the difference equations:[10]

$$a_\tau = \left(\frac{s_X^2}{2\,\kappa^2} - \bar{X}\right)\tau + \frac{\left(1 - e^{-\kappa\,\tau}\right)\,\bar{X}}{\kappa} - \frac{\left(3 + e^{-2\,\kappa\,\tau} - 4\,e^{-\kappa\,\tau}\right)\,s_X^2}{4\,\kappa^3} \tag{2.21a}$$

$$b_\tau = \frac{e^{-\kappa\,\tau} - 1}{\kappa}. \tag{2.21b}$$

Taking the limit as the step size goes to zero, we have $p_\tau = \int_{s=0}^{\tau} B_s\,ds$, and $p = \lim_{\tau\to\infty}\ p_\tau$ if the limit exists. The condition for the positivity of the asymptotic forward rate is the same in the continuous-time limit as in discrete-time; the integral converges if and only if $\bar{X} - (\|s_X\|/\kappa)^2/2 > 0$. The advantage of the continuous-time approximation lies in the efficiency gains from evaluating integrals using adaptive quadrature instead of evaluating sums.[11]

We can easily determine the magnitude of the approximation error when the interest rate is constant. The value of the continuous-time perpetuity is simply $p = 1/r$, whereas the value of the discrete-time perpetuity is $p = \Delta/(1 - e^{-r\,\Delta})$. The continuous-time approximation understates the discrete-time solution by about $50\,r\,\Delta$ percent. For $\Delta = 1/12$ and $r = 0.05$, the understatement is about 0.21 percent.

Dividend-denominated perpetuities. We now consider assets with more general dividend processes, where the dividend flow is always positive. We observe that for assets with such dividend streams, the price–dividend ratio is itself the value of a perpetuity measured in terms of the dividend process. In particular, (2.2) is homogeneous in prices and dividend: One is free to multiply each of p, d, and p' by a constant $\alpha > 0$. We can remove this degree of freedom by normalizing (2.2) by the dividend d, assuming it is not zero, to produce

$$\pi = \Delta + E_t\left[(m_d'/m_d)\,\pi'\right], \tag{2.22}$$

where $\pi = p/d$ is the price–dividend ratio and $m_d'/m_d = (m'\,d')/(m\,d)$ is the discount factor measured in terms of the dividend process. Equation (2.22) shows that the price–dividend ratio is itself the value of an asset when measured in terms of the dividend process: Its dividend is identically one (*i.e.*, one unit of the dividend process per unit of time); the asset is a dividend-denominated perpetuity.

Let the growth-rate for dividends be given by

$$\log(d'/d) = \widetilde{\mu}_d\,\Delta + \sigma_d \cdot \varepsilon\,\sqrt{\Delta},$$

[10]This is essentially the Vasicek (1977) model of the term structure of interest rates. See Duffie and Kan (1996) for the general exponential-affine case.

[11]Press, Teukolsky, Vettering, and Flannery (1992) do not discuss adaptive quadrature directly in their chapter on the integration of functions. Instead they refer the reader to their chapter on the integration of ODEs for adaptive stepsize algorithms. Judd (1998) briefly discusses adaptive quadrature. He seems to suggest that every subinterval gets subdivided, rather than just those that need it, which would indeed make the method quite costly to use in general. An elementary introduction to adaptive quadrature can be found in Skeel and Keiper (1993), where the authors present "an algorithm that *automatically* determines an *efficient* partition that is fine enough for the desired accuracy." [Emphasis in the original.] For a textbook discussion of adaptive Gaussian quadrature, see Buchan and Turner (1992).

so that

$$\widetilde{\mu}_{m_d} = \widetilde{\mu}_m + \widetilde{\mu}_d \quad \text{and} \quad \sigma_{m_d} = \sigma_m + \sigma_d.$$

The expressions for the interest rate and the price of risk in (2.7a) and (2.7b) in terms of the drift and diffusion of the state–price deflator are not numeraire-specific. In particular, they apply to m'_d/m_d:

$$r_d = -\widetilde{\mu}_{m_d} - \frac{1}{2}\,\|\sigma_{m_d}\|^2 \quad \text{and} \quad \lambda_d = -\sigma_{m_d}. \tag{2.23}$$

We refer to r_d and λ_d as the dividend-denominated interest rate and price of risk. Given the dynamics of m and d, we can solve for dividend-denominated bond prices, B_d^τ, and thereby solve for value of the asset, the price–dividend ratio,

$$\pi = \sum_{i=0}^{\infty} B_d^{i\,\Delta}\,\Delta.$$

Equilibrium asset pricing: Standard preferences. Here we apply some of the ideas already developed to equilibrium asset pricing.[12] With standard time-separable preferences, utility has the following recursive structure:

$$V = u(c)\,\Delta + e^{-\beta\,\Delta}\,E_t\left[V'\right], \tag{2.24}$$

where β is the rate of time-preference and $u(x) = (x^\rho - 1)/\rho$. As is well-known, the marginal rate of substitution is $e^{-\beta\,\Delta}\,(c'/c)^{\rho-1}$. For optimal consumption, the intertemporal marginal rate of substitution equals with the one-period discount factor as given by the growth rate of the state–price deflator: $m'/m = e^{-\beta\,\Delta}\,(c'/c)^{\rho-1}$. In this setting, the dividend is consumption itself, the value of the asset is wealth, the price–dividend ratio is the wealth–consumption ratio, and the dividend-denominated intertemporal marginal rate of substitution is $m'_d/m_d = e^{-\beta\,\Delta}\,(c'/c)^{\rho}$.

Let the conditional growth rate of consumption be given by

$$\log(c'/c) = \widetilde{\mu}_c\,\Delta + \sigma_c \cdot \varepsilon\,\sqrt{\Delta},$$

so that

$$r_d = \beta + \rho\,\widetilde{\mu}_c - \rho^2\,\frac{1}{2}\,\|\sigma_c\|^2 \quad \text{and} \quad \lambda_d = \rho\,\sigma_c.$$

Any state-dependence in π must enter through either $\widetilde{\mu}_c$ or σ_c. Therefore, to complete the model, let $\widetilde{\mu}_c = x$ where the evolution of the scalar state variable x is determined by

$$x' - x = \kappa\,(\bar{x} - x)\,\Delta + (s_X, 0) \cdot \varepsilon\,\sqrt{\Delta}, \tag{2.25}$$

[12] Below we will equilibrium asset pricing in the more general setting of recursive preferences that includes standard preferences as a special case.

where κ, $\bar{x}$, and s_X are constant. Also let $\sigma_c = (s_1, s_2)$, where s_1 and s_2 are constant. With these dynamics,

$$r_d = \beta + \rho\, x + \rho^2\, \frac{1}{2}\left(s_1^2 + s_2^2\right) \quad \text{and} \quad \lambda_d = \begin{pmatrix} \rho\, s_1 \\ \rho\, s_2 \end{pmatrix}. \tag{2.26}$$

As noted in the introduction, the state variable is related to be the expected growth rate of consumption, rather than the realized growth rate as is typical in discrete-time models. In the typical formulation, $\log(c'/c)$ follows and AR(1). As formulated here, $\log(c'/c)$ follows an ARMA(1,1), since $x' - x$ is not perfectly correlated with $\log(c'/c)$.

The solution to this equilibrium asset pricing models can follow the same steps as the solution to the perpetuity problem: Solve for (dividend-denominated) bond prices and add them up. If $\rho = 0$, then $r_d = \beta$, the value of a dividend-denominated zero coupon bond is $B_d^\tau = e^{-\beta\,\tau}$, and the wealth–consumption ratio is $\Delta/(1 - e^{-\beta\,\Delta}) \approx 1/\beta$. For $\rho \neq 0$, the (2.26) constitute a Gaussian model of the term structure that is essentially the same as the bond example solved above. The aymptotic dividend-denominated forward rate is given by

$$\beta + \rho\,\bar{x} + \rho^2\,\frac{1}{2}\left(s_1^2 + s_2^2 - \frac{2\,s_1\,s_X}{\kappa} - \frac{s_X^2}{\kappa^2}\right).$$

3. Continuous-time approximations

In the previous section, the connection between the canonical perpetuity model and general equilibrium asset pricing models with standard preferences was established.[13] In this section, continuous-time approximations to the discrete-time models are developed. The continuous-time approximations are based on first-order approximations around $\Delta = 0$, where Δ is the size of the time step. This allows us to treat, for example, both z and z^2 as normally distributed.[14] This result provides continuous time (and discrete-time approximations based on it) great analytical flexibility.

The approach to characterizing asset prices described here, which is borrowed from the continuous-time setting, eliminates explicit reference to π' in (2.22) since in continuous time there is no "next period." In its place are the dynamics of the growth rate $\log(\pi'/\pi)$. We proceed by dividing (2.22) by π and rearranging:

$$\Delta/\pi + E_t[\zeta] - 1 = 0, \qquad \text{where } \zeta := (m_d'/m_d)\,(\pi'/\pi). \tag{3.1}$$

Let $\log(\zeta) \sim \mathcal{N}(\widetilde{\mu}_\zeta\,\Delta, \sigma_\zeta\,\sqrt{\Delta})$, so that $E_t[\zeta] = e^{\mu_\zeta\,\Delta}$, where $\mu_\zeta := \widetilde{\mu}_\zeta + \frac{1}{2}\,\|\sigma_\zeta\|^2$. Since

$$\Delta/\pi + e^{\mu_\zeta\,\Delta} - 1 = (1/\pi + \mu_\zeta)\,\Delta + \mathcal{O}(\Delta^2),$$

---

[13]Only the case where the forcing variable is consumption, as in an endowment economy, was treated. The case where the forcing variable is a stochastic return process is treated below as a special case of optimization with recursive preferences.

[14]If $z' \sim \mathcal{N}(z + \mu\,\Delta, \sigma\,\sqrt{\Delta})$, then, to a first-order approximation in Δ, $(z')^2 \sim \mathcal{N}(z^2 + (2\,z\,\mu + \sigma^2)\,\Delta, 2\,z\,\sigma\,\sqrt{\Delta})$. See Appendix A.

we can write (3.1) as

$$1/\pi + \mu_\zeta = 0 \tag{3.2}$$

to a first-order approximation, where we have divided through by $\Delta \neq 0$.

We now assume the growth rates of m_d and π is log-normally distributed:

$$\log(\pi'/\pi) = \widetilde{\mu}_\pi \, \Delta + \sigma_\pi \cdot \varepsilon \sqrt{\Delta} \tag{3.3a}$$

$$\log(m_d'/m_d) = \widetilde{\mu}_{m_d} \, \Delta + \sigma_{m_d} \cdot \varepsilon \sqrt{\Delta}. \tag{3.3b}$$

This assumption involves an implicit approximation, since as we saw in the example above, π is not exactly log-normally distributed for finite Δ. Given (3.3), we have

$$\log(\zeta) = (\widetilde{\mu}_\pi + \widetilde{\mu}_{m_d}) \, \Delta + (\sigma_\pi + \sigma_{m_d}) \cdot \varepsilon \sqrt{\Delta},$$

and thus

$$\widetilde{\mu}_\zeta = \widetilde{\mu}_\pi + \widetilde{\mu}_{m_d} \quad \text{and} \quad \sigma_\zeta = \|\sigma_\pi + \sigma_{m_d}\|. \tag{3.4}$$

Therefore, $\mu_\zeta = \widetilde{\mu}_\pi + \widetilde{\mu}_{m_d} + \frac{1}{2} \|\sigma_\pi + \sigma_{m_d}\|^2$, and we can write (3.2) as

$$1/\pi + \widetilde{\mu}_\pi + \widetilde{\mu}_{m_d} + \frac{1}{2} \|\sigma_\pi + \sigma_{m_d}\|^2 = 0. \tag{3.5}$$

In order to solve (3.5) for π, we need to specify state variables along with their dynamics and how $\widetilde{\mu}_{m_d}$ and σ_{m_d} depend on those state variables.

Given (2.23), we can rewrite (3.5) as

$$1/\pi + \mu_\pi = r_d + \lambda_d \cdot \sigma_\pi, \tag{3.6}$$

where $\mu_\pi := \widetilde{\mu}_\pi + \frac{1}{2} \|\sigma_\pi\|^2$ is the expected capital gain. Equation (3.6) expresses the risk-return condition for a perpetuity: The expected return (relative dividend plus relative capital gain) equals the risk free rate plus a risk premium. Since π is the value of a perpetuity, we see immediately that if r_d is constant then $\pi = 1/r_d$, regardless of the behavior of λ_d. The exact solution to the discrete-time problem when r_d is constant is $\pi = \Delta/(1 - e^{-r_d \, \Delta})$. The magnitude of the approximation error can be determined by comparison.

Markovian asset price functions. To put some Markovian meat on the bones on (3.5), typically one assumes there is an unknown function $\mathit{\Pi}$ such that $\pi = \mathit{\Pi}(x)$ and $\pi' = \mathit{\Pi}(x')$ for some vector of state variables x that evolve according to specified laws of motion:

$$x' - x = \mu_X(x) \, \Delta + \sigma_X(x) \, \varepsilon \sqrt{\Delta}, \tag{3.7}$$

where $\mu_X(x)$ and $\sigma_X(x)$ are (respectively) vector and matrix functions of x. The dependence of the solution $\mathit{\Pi}$ on the state variables x arises solely through the dependence of μ_{m_d} and σ_{m_d} on the state variables, as illustrated by the example at the end of Section 2.

For our purpose here, it is convenient to write (3.6) as

$$1 + \bar{\mu}_\pi = r_d \, \pi + \lambda_d \cdot \bar{\sigma}_\pi, \tag{3.8}$$

where $\bar{\mu}_\pi = \mu_\pi \, \pi$ and $\bar{\sigma}_\pi = \sigma_\pi \, \pi$, as in $\pi' - \pi = \bar{\mu}_\pi \, \Delta + \bar{\sigma}_\pi \cdot \varepsilon \sqrt{\Delta}$. The discrete-time version of Ito's lemma (see Appendix A) delivers

$$\begin{aligned} \bar{\mu}_\pi &= \mu_X(x) \, \mathit{\Pi}_x(x) + \frac{1}{2} \, \mathit{\Pi}_{xx}(x) \, \|\sigma_X(x)\|^2 \\ \bar{\sigma}_\pi &= \sigma_X(x) \, \mathit{\Pi}_x(x) \end{aligned}$$

to a first order approximation in Δ. Thus we can write (3.8) as

$$1 + \hat{\mu}_X \, \mathit{\Pi}_x + \frac{1}{2} \, \mathit{\Pi}_{xx} \, \|\sigma_X\|^2 - R_d \, \mathit{\Pi} = 0, \tag{3.9}$$

where $\hat{\mu}_X := \mu_X - \Lambda_d \cdot \sigma_X$.

Solution methods. Let there be a single state variable x with the following risk-adjusted dynamics: $\hat{\mu}_X = \kappa \, (\bar{X} - x)$ and $\sigma_X = s_X$, where κ, $\bar{X}$, and s_X are constants, and let $R_d(x) = x$. In this example, dividend-denominated bond prices are given by $B_d(x, \tau) = e^{a(\tau) + b(\tau) \, x}$, where $a(\tau) = a_\tau$ and $b(\tau) = b_\tau$ are given in (2.21). Therefore the solution is given by $\mathit{\Pi}(x) = \int_{\tau=0}^\infty e^{a(\tau)+b(\tau) \, x} \, d\tau$, if the integral converges.

In order to introduce a solution method that will prove quite useful for solving models with recursive preferences, we will apply the semi-analytic method of undetermined coefficients to (3.9). First, insert the power-series representation $\mathit{\Pi}(x) = \sum_{n=0}^\infty \delta_n \, x^n$ into (3.9):

$$1 + \kappa \, (\bar{X} - x) \sum_{n=1}^\infty n \, \delta_n \, x^{n-1} + \frac{1}{2} \, s_X^2 \sum_{n=2}^\infty n \, (n-1) \, \delta_n \, x^{n-2} - x \sum_{n=0}^\infty \delta_n \, x^n = 0.$$

Next, collect the coefficients of powers of x to produce a system of linear equations in the unknown δ coefficients:

$$-\delta_n - (n+1) \, \kappa \, \delta_{n+1} + (n+2) \, \kappa \, \bar{X} \, \delta_{n+2} + (n+2) \, (n+3) \, \frac{1}{2} \, s_X^2 \, \delta_{n+3} = 0,$$

for $n \geq -1$ where $\delta_{-1} := -1$. The first N equations (for $N \geq 1$) contain $N + 2$ coefficients: δ_0 through δ_{N+1}. To solve this under-determined system, we replace the true coefficients δ_n with approximations δ_n^N and set $\delta_N^N = \delta_{N+1}^N = 0$, producing a system of N equations in N unknowns: δ_0^N through δ_{N-1}^N. By increasing N, one can examine the convergence of δ_n^N for specific n. Of course in this example we can compute the coefficients in the power series directly from bond prices. The Taylor series for bond prices is

$$B_d(x, \tau) = \sum_{n=0}^\infty \frac{e^{a(\tau)} \, b(\tau)^n}{n!} \, x^n,$$

so that $\delta_n = \frac{1}{n!} \int_{\tau=0}^\infty e^{a(\tau)} \, b(\tau)^n \, d\tau$.

4. Recursive preferences

The first problem one faces when dealing with recursive preferences is simply evaluating the utility of a given consumption process. Recursive utility in discrete time has the following form:[15]

$$V = \mathfrak{A}\big(c,\, \mathfrak{C}(V')\big), \tag{4.1}$$

where $\mathfrak{C}(\,\cdot\,)$ is the certainty equivalent operator and $\mathfrak{A}(\,\cdot\,,\,\cdot\,)$ is the intertemporal aggregator. In particular,

$$\mathfrak{A}(c,z) = \begin{cases} \big((1-e^{-\beta\,\Delta})\,c^\rho + e^{-\beta\,\Delta}\,z^\rho\big)^{1/\rho}, & 0 \neq \rho \leq 1, \\ c^{1-e^{-\beta\,\Delta}}\,z^{e^{-\beta\,\Delta}}, & \rho = 0 \end{cases} \tag{4.2a}$$

and

$$\mathfrak{C}(x) = \begin{cases} E_t\left[x^\alpha\right]^{1/\alpha}, & 0 \neq \alpha \leq 1, \\ \exp\big\{E_t\left[\log(x)\right]\big\}, & \alpha = 0, \end{cases} \tag{4.2b}$$

where $\beta > 0$ is the rate of time preference, $\eta = (1-\rho)^{-1}$ is the elasticity of intertemporal substitution, and $\gamma = 1-\alpha$ is the coefficient of relative risk aversion for static wealth gambles. The certainty equivalent for the lognormal distribution has a simple expression: $E\left[x^\alpha\right]^{1/\alpha} = \exp(\mu + \alpha\,\sigma^2/2)$ for $\log(x) \sim \mathcal{N}(\mu,\sigma)$. Inserting (4.2) into (4.1) produces (for $\rho \neq 0$ and $\alpha \neq 0$)

$$V = \left(\left(1-e^{-\beta\,\Delta}\right)c^\rho + e^{-\beta\,\Delta}\,E_t\left[\left(V'\right)^\alpha\right]^{\rho/\alpha}\right)^{1/\rho}, \quad \text{subject to } V_T = \zeta\,c_T, \tag{4.3}$$

where $\zeta\,c_T$ is the *terminal reward* and $\zeta > 0$.[16] Equation (4.3) has the recursive structure of the Bellman equation, but without optimization. Since V is homogeneous in consumption of degree one, we can write $V = c\,\psi$, $V' = c'\,\psi'$, and $V_T = c_T\,\psi_T$ for some process ψ, and we can replace c, c', and c_T with $\ell\,c$, $\ell\,c'$, and $\ell\,c_T$ for $\ell > 0$. By choosing $\ell = (c\,\psi)^{-1}$ we can write (4.3) as

$$\left(\left(1-e^{-\beta\,\Delta}\right)\psi^{-\rho} + e^{-\beta\,\Delta}\,E_t\left[(c'/c)^\alpha(\psi'/\psi)^\alpha\right]^{\rho/\alpha}\right)^{1/\rho} - 1 = 0, \qquad \text{subject to } \psi_T = \zeta. \tag{4.4}$$

Equation (4.4) is scale-free: It involves consumption only through its growth rate. Note that the similarity between (4.4) and (3.1).

We model the conditional growth rates of c and ψ as log-normally distributed,

$$\log(c'/c) = \widetilde{\mu}_c\,\Delta + \sigma_c \cdot \varepsilon\,\sqrt{\Delta} \tag{4.5a}$$

$$\log(\psi'/\psi) = \widetilde{\mu}_\psi\,\Delta + \sigma_\psi \cdot \varepsilon\,\sqrt{\Delta}. \tag{4.5b}$$

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[15] See Epstein and Zin (1991).

[16] Fisher and Gilles (1998) treat the case where $\zeta = 0$.

Given (4.5), we can write (4.4) as

$$\left(\left(1 - e^{-\beta\,\Delta}\right)\psi^{-\rho} + e^{-\beta\,\Delta}\,\exp\left\{\rho\left(\widetilde{\mu}_c + \widetilde{\mu}_\psi + \alpha\,\frac{1}{2}\,\|\sigma_c + \sigma_\psi\|^2\right)\Delta\right\}\right)^{1/\rho} - 1 = 0. \tag{4.6}$$

Taking the first-order approximation to (4.6) around $\Delta = 0$ and canceling Δ produces

$$\widetilde{\mu}_c + \widetilde{\mu}_\psi + \alpha\,\frac{1}{2}\,\|\sigma_c + \sigma_\psi\|^2 + \beta\,u(1/\psi) = 0, \quad \text{subject to } \psi_T = \zeta, \tag{4.7}$$

where

$$u(x) = \frac{x^\rho - 1}{\rho}.$$

Equation (4.7) is valid for all parameter values including $\alpha = 0$ and $\rho = 0$, where $u(x) = \log(x)$ in the latter case. Given the dynamics of consumption, the process ψ that solves (4.7) provides the solution for utilty. We discuss how to solve (4.7) below.

Marginal utility and the supporting price system. Marginal utility provides the link to optimality.

We can use (4.3) to express how utility is expected to change over time. Define

$$\Phi(z) = \frac{z^\alpha - 1}{\alpha},$$

and let $\bar{V}_t = \Phi(V_t)$. We can write (4.3) as

$$\bar{V} = \frac{\left(\left(1 - e^{-\beta\,\Delta}\right)c^\rho + e^{-\beta\,\Delta}\,E_t\left[1 + \alpha\bar{V}'\right]^{\rho/\alpha}\right)^{\alpha/\rho} - 1}{\alpha}.$$

The first-order approximation around $\Delta = 0$ produces

$$\begin{aligned}\bar{V} &\doteq \bar{f}(c, E_t[\bar{V}'])\,\Delta + E_t[\bar{V}'] \\ &\doteq \bar{f}(c, \bar{V})\,\Delta + E_t[\bar{V}'], \qquad \text{subject to } V_T = \Phi(\zeta\,c_T),\end{aligned} \tag{4.8}$$

where

$$\bar{f}(c, v) = z\,\beta\,u\left(c\,z^{-1/\alpha}\right), \qquad \text{where } z = 1 + \alpha\,v.$$

Note that discounting, intertemporal substitution, and risk aversion are all in impounded in $\bar{f}$. We can write (4.8) as

$$\bar{V}_t \doteq E_t\left[\sum_{s=t}^{T-\Delta} \bar{f}(c_s, \bar{V}_s)\,\Delta + \Phi(\zeta\,c_T)\right].$$

To get an expression for marginal utility, take the total derivative of (4.8):[17]

$$\delta\bar{V} = \bar{f}_c(c, \bar{V})\,\Delta\,\delta c + \bar{f}_V(c, \bar{V})\,\Delta\,\delta\bar{V} + E_t[\delta\bar{V}'], \tag{4.9}$$

[17] Note that δV is the current value of marginal utility, not the change in utility as time changes, which is $\bar{V}' - \bar{V}$. The change in marginal utility as time changes is $\delta\bar{V}' - \delta\bar{V}$.

where $\bar{f}_c$ and $\bar{f}_V$ are the partial derivatives of $\bar{f}$. Solving (4.9) for $\delta\bar{V}$, we have:

$$\delta\bar{V} = \frac{\bar{f}_c(c,\bar{V})\,\Delta\,\delta c + E_t[\delta\bar{V}']}{1 - f_V(c,\bar{V})\,\Delta} \doteq \bar{f}_c(c,\bar{V})\,\Delta\,\delta c + e^{\bar{f}_V(c,\bar{V})\,\Delta}\,E_t[\delta\bar{V}'], \tag{4.10}$$

where the approximation is first-order in Δ. From (4.10), we see the marginal utility of current consumption per unit of time is $\bar{f}_c(c,\bar{V})$ and the rate at which utility is discounted is $-\bar{f}_V(c,\bar{V})$. Since (4.10) holds for marginal utility next period as well, we can write

$$\delta\bar{V} = \bar{f}_c(c,\bar{V})\,\Delta\,\delta c + e^{\bar{f}_V(c,\bar{V})\,\Delta}\,E_t\Bigg[\overbrace{\bar{f}_c(c',\bar{V}')\,\Delta\,\delta c' + e^{\bar{f}_V(c',\bar{V}')\,\Delta}\,E_{t+\Delta}\big[\delta\bar{V}''\big]}^{\delta\bar{V}'}\Bigg]. \tag{4.11}$$

Setting $\delta\bar{V} = \delta\bar{V}'' = 0$ in (4.11), rearranging, and canceling Δ, we have

$$-\delta c = E_t\left[\left(e^{\bar{f}_V(c,\bar{V})\,\Delta}\,\frac{\bar{f}_c(c',\bar{V}')}{\bar{f}_c(c,\bar{V})}\right)\delta c'\right]$$

for any feasible intertemporal rearrangement of consumption between c and c'. Therefore, we can express the one-period intertemporal marginal rate of substitution $\mathcal{G}'/\mathcal{G}$ as $\mathcal{G}'/\mathcal{G} = e^{\bar{f}_V(c,\bar{V})\,\Delta}\,\bar{f}_c(c',\bar{V}')/\bar{f}_c(c,\bar{V})$.[18] Given $\bar{V} = \Phi(V) = \Phi(c\,\psi) = \left((c\,\psi)^\alpha - 1\right)/\alpha$ and (4.7), we can write

$$\bar{f}_c(c,\bar{V}) = \beta\,c^{\alpha-1}\,\psi^{\alpha-\rho} \tag{4.12a}$$

$$\begin{aligned}\bar{f}_V(c,\bar{V}) &= -\beta + (\alpha-\rho)\,\beta\,u(1/\psi)\\ &= -\beta - (\alpha-\rho)\left(\widetilde{\mu}_c + \widetilde{\mu}_\psi + \alpha\,\frac{1}{2}\,\|\sigma_c + \sigma_\psi\|^2\right).\end{aligned} \tag{4.12b}$$

Given (4.12) we can write

$$\begin{aligned}\mathcal{G}'/\mathcal{G} = \exp\Bigg\{ &-\left(\beta + (1-\rho)\,\widetilde{\mu}_c - \alpha\,(\rho-\alpha)\,\frac{1}{2}\,\|\sigma_c + \sigma_\psi\|^2\right)\Delta\\ &- \big((1-\alpha)\,\sigma_c + (\rho-\alpha)\,\sigma_\psi\big)\cdot\varepsilon\,\sqrt{\Delta}\Bigg\}.\end{aligned} \tag{4.13}$$

Given the dynamics of consumption, the only unknown in the marginal rate of substitution is the volatility of the information variable.

The supporting price system satisfies $m'/m = \mathcal{G}'/\mathcal{G}$. Using (2.7), which expresses the interest rate and the price of risk in terms of the dynamics of the log of the state–price deflator, we have

$$r = \beta + (1-\rho)\,\widetilde{\mu}_c - \alpha\,(\rho-\alpha)\,\frac{1}{2}\,\|\sigma_c + \sigma_\psi\|^2 - \frac{1}{2}\,\|\lambda\|^2 \tag{4.14a}$$

$$\lambda = (1-\alpha)\,\sigma_c + (\rho-\alpha)\,\sigma_\psi. \tag{4.14b}$$

Before proceeding we can use (4.14a) to confirm the interpretations of β and η. Let $\sigma_c = 0$ and $\widetilde{\mu}_c$ be constant. Since there is no state variation in this case, $\sigma_\psi = 0$ and

[18] Duffie and Skiadas (1994) derive this result in a semi-martingale setting that includes both discrete time and continuous time as special cases.

(4.14a) becomes $r = \beta + (1-\rho)\,\widetilde{\mu}_c$. For constant consumption, $r = \beta$, the rate of time preference. In addition, $d\widetilde{\mu}_c/dr = (1-\rho)^{-1} = \eta$, confirming that η measures the elasticity of intertemporal substitution.

Wealth and the supporting returns process. Wealth k is the present value of consumption:

$$k = c\,\Delta + E_t[(m'/m)\,k']. \tag{4.15}$$

The dynamics of wealth are given by

$$k' = (k - c\,\Delta)\,(\phi'/\phi), \tag{4.16}$$

where ϕ'/ϕ is the return on wealth. We can interprete ϕ as the value of an asset that earns the return on wealth with no drawdowns for consumption. As such, we refer to ϕ as the value of the *capital account*. Eliminating k' from (4.15) and (4.16) produces the absence-of-arbitrage condition for ϕ:

$$E_t[(m'/m)\,(\phi'/\phi)] = 1. \tag{4.17}$$

Equation (4.17) provides a link from the capital account to the price system and hence to preferences. Next we establish a second link from the capital account to preferences.

As Fisher and Gilles (1998) show, the wealth–consumption ratio can be expressed as the marginal utility of a permanent increase in consumption normalized by the marginal utility of current consumption. The marginal utility of a permanent increase in utility can be expressed as

$$\frac{\partial \bar{V}}{\partial c} = \frac{\partial}{\partial c}\,\frac{(c\,\psi)^\alpha - 1}{\alpha} = c^{\alpha-1}\,\psi^\alpha,$$

and the marginal utility of current consumption is given in (4.12a). Therefore,

$$\pi = \psi^\rho/\beta, \tag{4.18}$$

where π is the wealth–consumption ratio in this setting. Together (4.16) and (4.18) provide the second link:

$$\phi'/\phi = \frac{(c'/c)\,(\psi'/\psi)^\rho}{1 - c/k\,\Delta}. \tag{4.19}$$

Given the dynamics of the stochastic return,

$$\log(\phi'/\phi) = \widetilde{\mu}_\phi\,\Delta + \sigma_\phi \cdot \varepsilon\,\sqrt{\Delta}, \tag{4.20}$$

Equations (4.17) and (4.19) imply

$$\widetilde{\mu}_\phi = r + \lambda\cdot\sigma_\phi - \frac{1}{2}\,\|\sigma_\phi\|^2 \tag{4.21a}$$

$$\sigma_\phi = \sigma_c + \rho\,\sigma_\psi. \tag{4.21b}$$

Note that the returns process is independent of wealth. Together (4.14) and (4.21) establish the relations among the dynamics of consumption, the supporting price system, and the supporting returns process. For example, we can use (4.14b) to eliminate σ_c from (4.21b) and express the price of risk as

$$\lambda = \gamma\,\sigma_\phi + (1-\gamma)\left(\frac{\sigma_\psi}{\eta}\right). \tag{4.22}$$

Evidently risk premia depend on the covariance with the returns process and with the information variable. Campbell (1993) derives an equivalent expression for risk premia (his Equation (25)), which he refers to as the "cross-sectional asset pricing formula that makes no reference to consumption." We can obtain Campbell's parameterization with a change of variables. Define $\omega := \psi^{1/\eta}$. Then $\sigma_\psi = \eta\,\sigma_\omega$, and (4.22) becomes $\lambda = \gamma\,\sigma_\phi + (1-\gamma)\,\sigma_\omega$.

The information variable as a weighted forecast. Given (4.8), we have

$$\hat{V}' - \hat{V} = -\bar{f}(c,\bar{V})\,\Delta + \sigma_{\bar{V}}\cdot\varepsilon\,\sqrt{\Delta},$$

for some $\sigma_{\bar{V}}$. Let $\hat{V} = g(\bar{V})$. Using the discrete-time version of Itô's lemma, we can compute

$$\hat{V}' - \hat{V} = \mu_{\hat{V}}\,\Delta + \sigma_{\hat{V}}\cdot\varepsilon\,\sqrt{\Delta},$$

where

$$\mu_{\hat{V}} = -g'(\bar{V})\,\bar{f}(c,\bar{V}) + g''(\bar{V})\,\frac{1}{2}\,\|\sigma_{\bar{V}}\|^2 \qquad\text{and}\qquad \sigma_{\hat{V}} = g'(\bar{V})\,\sigma_{\bar{V}}. \tag{4.23}$$

We can use the expression for $\sigma_{\hat{V}}$ and the inverse function $g^{-1}(\hat{V})$ to eliminate $\sigma_{\bar{V}}$ and $\bar{V}$ from $\mu_{\hat{V}}$:

$$\mu_{\hat{V}} = -g'\left(g^{-1}(\hat{V})\right)\,\bar{f}\left(c, g^{-1}(\hat{V})\right) + \frac{g''\left(g^{-1}(\hat{V})\right)}{g'\left(g^{-1}(\hat{V})\right)^2}\,\frac{1}{2}\,\|\sigma_{\hat{V}}\|^2$$

By choosing $g(x) = ((1+\alpha\,x)^{\rho/\alpha} - 1)/\rho$, we can make the first term in $\mu_{\hat{V}}$ linear in $\hat{V}$:

$$\mu_{\hat{V}} = -\beta\,(u(c) - \hat{V}) - \frac{\alpha-\rho}{1+\rho\,\hat{V}}\,\frac{1}{2}\,\|\sigma_{\hat{V}}\|^2.$$

Thus we can write

$$\hat{V} = \left(\beta\,(u(c) - \hat{V}) + \frac{\alpha-\rho}{1+\rho\,\hat{V}}\,\frac{1}{2}\,\|\sigma_{\hat{V}}\|^2\right)\Delta + E_t[\hat{V}'],$$

which is equivalent to

$$\begin{aligned}\hat{V} &= \frac{\left(\beta\,u(c) + \frac{\alpha-\rho}{1+\rho\,\hat{V}}\,\frac{1}{2}\,\|\sigma_{\hat{V}}\|^2\right)\Delta + E_t[\hat{V}']}{1+\beta\,\Delta} \\ &\doteq \left(\beta\,u(c) + \frac{\alpha-\rho}{1+\rho\,\hat{V}}\,\frac{1}{2}\,\|\sigma_{\hat{V}}\|^2\right)\Delta + e^{-\beta\,\Delta}\,E_t[\hat{V}'].\end{aligned} \tag{4.24}$$

Equation (4.24) shows that the current flow of utility is composed of two parts. The first part is the standard "felicity" of consumption, $\beta\,u(c)$. The second part depends on the conditional variance of utility itself. The sign of the second part is determined by the sign of $\alpha - \rho$. If $\alpha - \rho < 0$, then an increase in the conditional variance of utility reduces current utility. This can be interpreted as a preference for early resolution of uncertainty. Similarly, $\alpha - \rho > 0$ indicates a preference for late resolution of uncertainty. Standard preferences ($\alpha - \rho = 0$) display indifference to the timing of the resolution of uncertainty.

Repeated recursive substitution for expected future utility in (4.24) produces

$$\begin{aligned}\hat{V}_t &= E_t\left[\sum_{s=t}^{T-\Delta} e^{-\beta\,(s-t)}\left\{\beta\,u(c_s) + \frac{\alpha-\rho}{1+\rho\,\hat{V}_s}\,\frac{1}{2}\,\|\sigma_{\hat{V}_s}\|^2\right\}\Delta + e^{-\beta\,(T-t)}\,u(\zeta\,c_T)\right] \\ &\doteq E_t\left[\int_{s=t}^{T} e^{-\beta\,(s-t)}\left\{\beta\,u(c_s) + \frac{\alpha-\rho}{1+\rho\,\hat{V}_s}\,\frac{1}{2}\,\|\sigma_{\hat{V}_s}\|^2\right\}ds + e^{-\beta\,(T-t)}\,u(\zeta\,c_T)\right],\end{aligned} \tag{4.25}$$

where we have approximated the sum with an integral and we have used the fact that $\hat{V} = u(V)$ in writing the terminal reward. Using $\hat{V} = u(c\,\psi)$, $\|\sigma_{\hat{V}}\|^2/(1+\rho\,\hat{V}) = V^\rho\,\|\sigma_V/V\|^2$, and $u(x)/y^\rho + u(1/y) = u(x/y)$, we can write (4.25) as

$$\begin{aligned}u(\psi) = \int_{s=t}^{T} \beta\,e^{-\beta\,(s-t)}\,E_t\left[u\left(\frac{c_s}{c_t}\right)\right]ds + e^{-\beta\,(T-t)}\,E_t\left[u\left(\zeta\,\frac{c_T}{c_t}\right)\right] + \\ (\alpha-\rho)\int_{s=t}^{T} e^{-\beta\,(s-t)}\,E_t\left[\psi_s^\rho\,\frac{1}{2}\,\|\sigma_{c_s}+\sigma_{\psi_s}\|^2\right]ds.\end{aligned} \tag{4.26}$$

For $\rho = 0$ (4.26) becomes

$$\begin{aligned}\log(\psi) = \int_{s=t}^{T} \beta\,e^{-\beta\,(s-t)}\,E_t\left[\log\left(\frac{c_s}{c_t}\right)\right]ds + e^{-\beta\,(T-t)}\,E_t\left[\log\left(\zeta\,\frac{c_T}{c_t}\right)\right] + \\ \alpha\int_{s=t}^{T} e^{-\beta\,(s-t)}\,E_t\left[\frac{1}{2}\,\|\sigma_{c_s}+\sigma_{\psi_s}\|^2\right]ds.\end{aligned} \tag{4.27}$$

Equation (4.27) expresses the information variable as a weighted forecast of consumption growth rates plus a weighted forecast of conditional variances. If the conditional variances are constant, then variation in the information variable will come solely from variation in the weighted forecasts of consumption growth.

We can easily convert (4.27) into a weighted forecast of growth rates of the capital account. For $\rho = 0$, $\widetilde{\mu}_c = \widetilde{\mu}_\phi - \beta$ and $\sigma_c = \sigma_\phi$, so we can write (4.27) as

$$\begin{aligned}\log(\psi) = \int_{s=t}^{T} \beta\,e^{-\beta\,(s-t)}\,E_t\left[\log\left(\frac{\phi_s}{\phi_t}\right)\right]ds + E_t\left[\log\left(\zeta\,\frac{\phi_T}{\phi_t}\right)\right] - \\ \left(1 - e^{-\beta\,(T-t)}\right) + \alpha\int_{s=t}^{T} e^{-\beta\,(s-t)}\,E_t\left[\frac{1}{2}\,\|\sigma_{\phi_s}+\sigma_{\psi_s}\|^2\right]ds\end{aligned} \tag{4.28}$$

The approximations in Campbell (1993) are implicitly based on (4.28).

Optimal consumption. Up to this point, we have treated the consumption process as given: Current opportunities have been given by current consumption and future opportunities have been determined by the dynamics of consumption. Now we change perspective. Consumption is no longer given exogenously. Instead, current opportunities are given by current wealth and future opportunities are determined by stochastic investment returns—either directly via the investment technology or indirectly via the price system.

The dynamics of wealth embody the consumption–investment trade-off. We can interpret $d\phi/\phi$ as the return on optimally invested wealth and σ_ϕ as the volatility of the optimal portfolio. We refer to ϕ as the value of the *capital account*. The capital account tracts the value of a marginal investment. The source of returns could be a portfolio of securities, or it could be a single stochastic investment technology. In either case, the value of the capital account represents the outcome of the following investment strategy: invest one unit of the consumption good in the returns process at time zero and thereafter continuously reinvest the proceeds.

In the current setting, the information variable will summarize all relevant information about future opportunities as reflected in the dynamics of either the state–price deflator or the capital account. In other words, the information variable must conform to the dynamics of the *forcing variable*. Previously, in the endowment setting, the forcing variable was consumption. We now allow the forcing variable to be the state–price deflator or the capital account. The restriction ψ must satisfy when the forcing variable is the state–price deflator is obtained by eliminating $\widetilde{\mu}_c$ and σ_c from (4.7) using (4.14). Similarly, the restriction ψ must satisfy when the forcing variable is the capital account is obtained by eliminating $\widetilde{\mu}_c$ and σ_c from (4.7) using (4.14) and (4.21).

TABLE 1. Coefficients for Equation (4.29).

y	a_0	a_1	a_2
c	0	1	$1-\gamma$
ϕ	$-\beta\,\eta$	η	$(1-\gamma)/\eta$
$1/m$	$-\beta\,\eta$	η	$(1-\gamma)/(\eta\,\gamma)$

Thus there are three versions of (4.7), the central restriction on the information variable, each depending on a different choice for the forcing variable. It is convenient to formally unify all three restrictions. To that end, we denote the generic forcing variable y and its dynamics $d\log(y(t)) = \widetilde{\mu}_y(t)\,dt + \sigma_y(t)^\top dW(t)$, where y is either consumption (c), the capital account (ϕ), or the inverse of the state–price deflator ($1/m$). We can write all three restrictions as

$$a_0 + a_1\,\widetilde{\mu}_y + \widetilde{\mu}_\psi + a_2\,\frac{1}{2}\,\|\sigma_\psi + a_1\,\sigma_y\|^2 + \beta\,u(1/\psi) = 0, \quad \text{subject to } \psi(T) = \zeta, \tag{4.29}$$

where the coefficients a_i are given in Table 1. Given the dynamics of the forcing variable and the solution to (4.29), we can use (4.14) and (4.21) to compute the dynamics of the remaining variables.

Wealth–consumption ratio, again. For $\eta \neq 1$, a change of variable from ψ to $\pi = k/c$ allows us to express (4.29) as (3.6). Using $\pi = \psi^\rho/\beta$, $\rho\,\widetilde{\mu}_\psi = \widetilde{\mu}_\pi$, $\rho\,\sigma_\psi = \sigma_\pi$, and $\widetilde{\mu}_\pi = \mu_\pi - \frac{1}{2}\,\|\sigma_\pi\|^2$, we can rewrite (4.29) as (3.6), where

$$r_d = d_0 + d_1\,\widetilde{\mu}_y + d_1\,d_2\,\frac{1}{2}\,\|\sigma_y\|^2 - (\varepsilon/d_1)\frac{1}{2}\,\|\sigma_\pi\|^2 \tag{4.30a}$$

$$\lambda_d = -d_2\,\sigma_y + (\varepsilon/d_1)\,\sigma_\pi. \tag{4.30b}$$

The coefficients d_i are given in Table 1. As long as $\eta \neq 1$, Equations (4.29) and (3.6) are equivalent in the sense that the relation between ψ and π can also be inverted. However, when $\eta = 1$, (3.6) devolves to $1/\pi = \beta$, and ψ cannot be recovered from π. In this case, one must attack (4.29).

Table 2. The coefficients of Equation (4.30) in terms of the preference parameters (columns 2–5).

y	d_0	d_1	d_2	$\varepsilon = d_1 + d_2$
c	β	$1/\eta - 1$	$1-\gamma$	$1/\eta - \gamma$
ϕ	$\eta\,\beta$	$1-\eta$	$1-\gamma$	$2-\eta-\gamma$
$1/m$	$\eta\,\beta$	$1-\eta$	$1/\gamma - 1$	$1/\gamma - \eta$

In (4.30), the dividend-denominated interest rate and price of risk are not exogenous unless either $\sigma_\pi = 0$ (in which case r_d must be deterministic) or $\varepsilon = 0$. The latter case depends on the parameter values and the forcing process. For example, if the forcing variable is consumption, then standard preferences imply $\varepsilon = 0$. By contrast, if the capital account is the forcing variable, then standard preferences imply $\varepsilon \neq 0$.

5. Numerical solutions for recursive utility

Solving the model amounts to solving for the information variable given the dynamics of a forcing variable. In this section we solve (4.29) for the information variable in a continuous-time Markovian setting.

Markovian structure. We suppose there are d Markovian state variables X driving the forcing process y, where $y = c$, or $1/m$, or ϕ. The joint dynamics of X and y are given by

$$\begin{pmatrix} dX(t) \\ d\log(y(t)) \end{pmatrix} = \begin{pmatrix} \mu_X(X(t)) \\ \widetilde{\mu}_Y(X(t)) \end{pmatrix} dt + \begin{pmatrix} \sigma_X(X(t))^\top \\ \sigma_Y(X(t))^\top \end{pmatrix} dW(t),$$

where $W^\top = \left(W_x^\top, W_y\right)$ is a l-dimensional vector of orthonormal Brownian motions, with W_x $(l-1)$-dimensional and W_y scalar. The dimensions of $\sigma_X(x)$ and $\sigma_Y(x)$ are respectively $l \times d$ and $l \times 1$. We assume that the last column of $\sigma_X(x)^\top$ is a vector of zeros, so that the state variables are not affected by W_y: $\sigma_X(x)^\top = \big(\Sigma_X(x)^\top \; 0\big)$, where $\Sigma_X(x)$ is $(l-1) \times d$.[19] Note that even if the state variables are deterministic ($\sigma_X(x) \equiv 0$), y can be stochastic; a completely deterministic economy would require $\sigma_Y(x) \equiv 0$ as well. The following functions of the state variables are called collectively *the data*:

$$\mu_X(x), \quad \widetilde{\mu}_Y(x), \quad \sigma_X(x)^\top \sigma_X(x), \quad \sigma_X(x)\,\sigma_Y(x), \quad \text{and} \quad \sigma_Y(x)^\top \sigma_Y(x). \tag{5.1}$$

For the most part, we will take $\zeta > 0$ so that the information variable is strictly positive and finite. In addition, since we have assumed the data do not depend on time, the solution will be time-homogeneous. As a consequence, it is convenient to model $\log(\psi(t)) = \mathit{\Omega}(X(t), T-t)$. Given the function $\mathit{\Omega}(x,\tau)$, we can compute the functions for the drift and diffusion $\widetilde{\mu}_\psi(t) = \mu_\mathit{\Omega}(X(t),t)$ and $\sigma_\psi(t) = \sigma_\mathit{\Omega}(X(t),t)$, in terms of the partial derivatives of $\mathit{\Omega}$:

$$\mu_\mathit{\Omega}(x,\tau) = \mathit{\Omega}_x(x,t)^\top \mu_X(x) + \frac{1}{2}\,\mathrm{tr}\left[\sigma_X(x)^\top \sigma_X(x)\,\mathit{\Omega}_{xx}(x,\tau)\right] - \mathit{\Omega}_\tau(x,\tau) \tag{5.2a}$$

$$\sigma_\mathit{\Omega}(x,\tau) = \mathit{\Omega}_x(x,\tau)\,\sigma_X(x). \tag{5.2b}$$

The data turn (4.7) into a quasi-linear partial differential equation (PDE) in terms of the unknown function $\mathit{\Omega}$,

$$\widetilde{\mu}_Y(x) + \mu_\mathit{\Omega}(x,\tau) + \alpha\,\|\sigma_Y(x) + \sigma_\mathit{\Omega}(x,\tau)\|^2 + \beta\,u\left(e^{-\mathit{\Omega}(x,\tau)}\right) = 0, \tag{5.3}$$

subject to the boundary condition $\mathit{\Omega}(x,0) = \log(\zeta)$.[20] It can be shown that if the data are real analytic, a unique real analytic function $\mathit{\Omega}(x,\tau)$ exists in the neighborhood of $\tau = 0$.[21] The theorem does not guarantee the existence of a solution for an arbitrary finite horizon. In Section 5 we will encounter an example that fails to have such a solution. However, if solutions exist for all finite horizons, then they converge to an infinite-horizon solution if and only if $\lim_{\tau\to\infty} \; \mathit{\Omega}_\tau(x,\tau) = 0$.

Solution method. We present a method for numerically solving (5.3). Our numerical solution technique can be thought of as the method of undetermined coefficient functions. It is based on the exact solutions described in the previous section for $\rho = 0$. To illustrate our method and to reduce the notational burden, we suppose there is a single state variable. Given real analytic data, $\mathit{\Omega}(x,\tau)$ has the power-series representation expanding around $x = x_0$ and treating τ as a parameter: $\mathit{\Omega}(x,\tau) = \sum_{n=0}^{\infty} \delta_n(\tau)\,(x-x_0)^n$. The condition for convergence to an

[19]This assumption is without loss of generality. It simply allows for the possibility that there exists a shock, W_y, that affects y but not X.
[20]Duffie and Lions (1992) address the existence and uniqueness of V in a similar setting.
[21]This is due to the main existence theorem for PDEs, the Cauchy–Kowaleskaya theorem. See Rauch (1991, Chapter 1), for example.

infinite-horizon solution is $\Omega_\tau(x,\tau) = 0$. The partial derivatives are given by

$$\begin{aligned}
\Omega_\tau(x,\tau) &= \sum_{n=0}^{\infty} \delta'_n(\tau)\,(x-x_0)^n \\
\Omega_x(x,\tau) &= \sum_{n=1}^{\infty} n\,\delta_n(\tau)\,(x-x_0)^{n-1} \\
\Omega_{xx}(x,\tau) &= \sum_{n=0}^{\infty} n\,(n-1)\,\delta_n(\tau)\,(x-x_0)^{n-2}.
\end{aligned}$$

The boundary condition, $\Omega(x,0) = \log(\zeta)$, implies $\delta_0(0) = \log(\zeta)$ and $\delta_i(0) = 0$ for $i \geq 1$. The solution method becomes operational by approximating $\Omega(x,\tau)$ as

$$\Omega^N(x,\tau) := \sum_{n=0}^{N} \delta_n^N(\tau)\,(x-x_0)^n,$$

which is inserted into (5.2) and the result is inserted into (5.3), upon which the N-th order Taylor approximation is computed. The result can be separated into a system of nonlinear ordinary differential equations. In the previous section with $\rho = 0$, we saw three examples where this representation provided exact solutions with finite N.

For comparison with Campbell (1993) and Campbell and Koo (1997), we treat the case where the forcing variable is the return on optimally invested wealth. We adopt the following dynamics:

$$dx = \kappa\,(\bar{x} - x)\,dt + s_X\,dW_1 \tag{5.4a}$$

$$d\log(y) = (a_y + b_y\,x)\,dt + s_1\,dW_1 + s_2\,dW_2. \tag{5.4b}$$

In order to directly compare with their results, we adopt a change of variables. Define $\omega := \psi^{1/\eta}$. We can write (4.29) in terms of ω for $y = \phi$:

$$\beta + \eta\left(\widetilde{\mu}_\phi + \widetilde{\mu}_\omega + (1-\gamma)\,\frac{1}{2}\,\|\sigma_\omega + \sigma_\phi\|^2\right) + \beta\,u(1/\omega^\eta) = 0, \tag{5.5}$$

subject to $\omega(T) = \zeta^{1/\eta}$. Now we let $\log(\omega(t)) = \Omega(X(t), T-t)$. Given (5.4) we have

$$\mu_\Omega(x) = \kappa\,(\bar{x} - x)\,\Omega_x(x,\tau) + s_X^2\,\frac{1}{2}\,\Omega_{xx}(x,\tau) - \Omega_\tau(x,\tau) \tag{5.6a}$$

$$\sigma_\Omega(x) = \begin{pmatrix} s_X\,\Omega_x(x,\tau) \\ 0 \end{pmatrix}. \tag{5.6b}$$

We are now set to apply our truncated series representation to the Markovian version of (5.5). Let $\zeta = 1$ so that $\Omega(x,0) = 0$. For example, with $N = 1$ and

$x_0 = \bar{x}$, we have

$$\delta_0^{1\prime}(\tau) = \bar{x} - \beta + \frac{\beta\left(1 - e^{(1-\eta)\,\delta_0^1(\tau)}\right)}{1-\eta} + (1-\gamma)\,\frac{1}{2}\left(\left(s_1 + s_X\,\delta_1^1(\tau)\right)^2 + s_2^2\right) \quad (5.7a)$$

$$\delta_1^{1\prime}(\tau) = 1 - \left(\kappa + \beta\, e^{(1-\eta)\,\delta_0^1(\tau)}\right)\delta_1^1(\tau), \quad (5.7b)$$

subject to $\delta_0^1(0) = \delta_1^1(0) = 0$. For $N = 2$, we must add terms to the right-hand sides of (5.7) (changing δ_i^1 to δ_i^2): $s_X^2\,\delta_2^2(\tau)$ to (5.7a) and $s_X\,(1-\gamma)\,(s_1 + s_X\,\delta_1^2(\tau))\,\delta_2^2(\tau)$ to (5.7b), where

$$\delta_2^{2\prime}(\tau) = -\frac{1}{2}\,\beta\, e^{(1-\eta)\,\delta_0^2(\tau)}\,(1-\eta)\,\delta_1^2(\tau)^2 - \left(2\,\kappa + \beta\, e^{(1-\eta)\,\delta_0^2(\tau)}\right)\delta_2^2(\tau) + 2\,(1-\gamma)\,s_X^2\,\delta_2^2(\tau)^2, \quad (5.8)$$

subject to $\delta_2(0) = 0$.

Special cases. There are two cases where (5.7) provides an exact solution. Both cases involve a linearization of (5.7). These cases can be understood in terms of the dividend-denominated asymptotic forward rate, $f_d(\infty)$. Using the relations already established above, we can write the dividend-denominated interest rate and price of risk as[22]

$$r_d = \eta\,\beta + (1-\eta)\left\{\widetilde{\mu}_\phi + (1-\gamma)\,\frac{1}{2}\,\|\sigma_\phi\|^2 - (2-\eta-\gamma)\,\frac{1}{2}\,\|\sigma_\psi\|^2\right\} \quad (5.9a)$$

$$\lambda_d = (\gamma-1)\,\sigma_\phi - (2-\eta-\gamma)\,\sigma_\psi. \quad (5.9b)$$

A potential difficulty we face in using (5.9) is the presence of σ_ψ, which is part of the solution, in the expressions for both r_d and λ_d. However note that when $\eta + \gamma = 2$, the terms involving σ_ψ drop out of (5.9), leaving (in this example) an exponential-affine model of the dividend-denominated term structure.

For the first case, consider $\eta = 1$. In this case, $r_d = \beta$. Therefore $f_d(\infty) = \beta$. For $\beta > 0$, convergence to an infinite-horizon is guaranteed. In this case, (5.7) becomes

$$\delta_0^{1\prime}(\tau) = \bar{x} - \beta - \beta\,\delta_0^1(\tau) + (1-\gamma)\,\frac{1}{2}\left(\left(s_1 + s_X\,\delta_1^1(\tau)\right)^2 + s_2^2\right) \quad (5.10a)$$

$$\delta_1^{1\prime}(\tau) = 1 - (\kappa+\beta)\,\delta_1^1(\tau). \quad (5.10b)$$

Moreover, the right-hand side of (5.8) is proportional to $\delta_2(\tau)$ when $\eta = 1$, which makes it identically zero, given its starting value. All higher-order terms are similarly zero. Since convergence is guaranteed, $\delta_0^{1\prime}(\infty) = \delta_1^{1\prime}(\infty) = 0$. Therefore we can reexpress (5.7b) for the infinite-horizon problem as a system of algebraic equations,

$$0 = \bar{x} - \beta - \beta\,\delta_0(\infty) + (1-\gamma)\,\frac{1}{2}\left((s_1 + s_X\,\delta_1(\infty))^2 + s_2^2\right) \quad (5.11a)$$

$$0 = 1 - (\kappa+\beta)\,\delta_1(\infty), \quad (5.11b)$$

[22] See Fisher and Gilles (1998) for a unified treatment of all three forcing variables.

with the unique solution

$$\delta_0(\infty) = \frac{\bar{x}-\beta}{\beta} + \frac{(1-\gamma)}{\beta}\,\frac{1}{2}\left(\left(s_1 + \frac{s_X}{\kappa+\beta}\right)^2 + s_2^2\right) \tag{5.12a}$$

$$\delta_1(\infty) = \frac{1}{\beta+\kappa}. \tag{5.12b}$$

For the second case, let $f_d(\infty) \leq 0$. In this case there is no infinite-horizon solution: The wealth–consumption ratio, which is the value of an annuity, grows without bound, and its inverse, the consumption–wealth ratio, shrinks to zero as the horizon goes to infinity. Recall that $c/k = \beta\,\psi^{1-\eta}$. (Consequently $\psi \to 0$ for $\eta < 1$ and $\psi \to \infty$ for $\eta > 1$.) The term $\beta\, e^{(1-\eta)\,\delta_0^N(\tau)}$ captures the scale of c/k, and so it goes goes to zero, thereby linearizing the system of ODEs. For large τ, (5.7) becomes

$$\delta_0^{1\prime}(\tau) \doteq \bar{x} - \beta + \frac{\beta}{1-\eta} + (1-\gamma)\,\frac{1}{2}\left(\left(s_1 + s_X\,\delta_1^1(\tau)\right)^2 + s_2^2\right) \tag{5.13a}$$

$$\delta_1^{1\prime}(\tau) \doteq 1 - \kappa\,\delta_1^1(\tau). \tag{5.13b}$$

For $N > 1$, all higher-order coefficients are asymptotically zero, so that the model is asymptotically first-order in the region of nonconvergence. Using (5.13b), we can compute $\lim_{\tau\to\infty}\ \delta_1(\tau) = 1/\kappa$. Inserting this into (5.13a) produces

$$\lim_{\tau\to\infty}\ \delta_0{}'(\tau) = \bar{x} - \beta + \frac{\beta}{1-\eta} + (1-\gamma)\,\frac{1}{2}\left((s_1 + s_X/\kappa)^2 + s_2^2\right). \tag{5.14}$$

Because the model is asymptotically first-order in the region where it does not converge, we have $\lim_{\tau\to\infty}\ \sigma_\psi = (s_X/\kappa, 0)$.[23] As a result, we can use (5.9) to identify the regions of nonconvergence. For the regions of nonconvergence in this example, we can write $r_d = \alpha_0 + \alpha_1\,x$ and $\lambda_d = (\ell_1, \ell_2)$, where

$$\begin{aligned}
\alpha_0 &= \eta\,\beta + (1-\eta)\,\frac{1}{2}\left\{(1-\gamma)\,(s_1^2+s_2^2) - (2-\eta-\gamma)\,(s_X/\kappa)^2\right\} \\
\alpha_1 &= 1-\eta \\
\ell_1 &= (\gamma-1)\,s_1 - (2-\eta-\gamma)\,(s_X/\kappa) \\
\ell_2 &= (\gamma-1)\,s_2.
\end{aligned}$$

are scalar constants. The asymptotic forward rate is

$$f_d(\infty) = \alpha_0 + \alpha_1\left(\bar{x} - \frac{s_X\,\ell_1}{\kappa}\right) - \left(\frac{\alpha_1\,s_X}{\sqrt{2}\,\kappa}\right)^2. \tag{5.15}$$

Note that the condition $f_d(\infty) = 0$ is equivalent to the condition $\lim_{\tau\to\infty}\ \delta_0{}'(\tau) = 0$ in (5.14).

[23] In essence what happens is this. As the horizon increases, the annuity value comes to be dominated by limiting values of the underlying discount bonds. If the data are structured to deliver exponential-affine dividend-denominated discount bonds absent the σ_ψ terms in r_d and λ_d, the state-dependence of the annuity will inherit that structure in the nonconvergent case, guaranteeing that r_d and λ_d including the σ_ψ terms will deliver the same formal state-dependence asymptotically.

Let $h(\tau) \equiv (1-\eta)\,\delta_0^1(\tau)$ and $f := f_d(\infty)$. For $f \le 0$ and τ large, we write (5.7a) as

$$h'(\tau) = f - \beta\, e^{h(\tau)}. \tag{5.16}$$

The solution to (5.16) subject to $h(0) = h_0$ is

$$h(\tau) = \begin{cases} -\log\left(\beta/f + e^{-f\,\tau}\,\left(e^{-h_0} - \beta/f\right)\right) & \text{for } f < 0 \\ -\log\left(e^{-h_0} + \beta\,\tau\right) & \text{for } f = 0. \end{cases} \tag{5.17}$$

Therefore, for $f \le 0$, we have $\lim_{\tau\to\infty}\ e^{h(\tau)} = 0$ and $\lim_{\tau\to\infty}\ h'(\tau) = f$. Since $\lim_{\tau\to\infty}\ (1-\eta)\,\delta_0^1(\tau) = -\infty$ for $f \le 0$, we have in this case

$$\lim_{\tau\to\infty}\ \delta_0^1(\tau) = \begin{cases} -\infty & \text{for } \eta < 1 \\ +\infty & \text{for } \eta > 1. \end{cases}$$

Notwithstanding the ill behavior of the scale of the economy in terms of the wealth–consumption ratio, the dynamics of consumption growth and of the state–price deflator (*i.e.*, the interest rate and the price of risk) are well-behaved as the horizon goes to infinity in the nonconvergent case. Using the relations already established, we can write the process for the state–price deflator and for optimal consumption as follows:

$$r = \widetilde{\mu}_\phi + \left((1-\gamma) - \frac{1}{2}\right)\|\sigma_\phi\|^2 + (\gamma-1)\,\sigma_\psi^\top\sigma_\phi \tag{5.18a}$$

$$\lambda = \gamma\,\sigma_\phi + (\gamma-1)\,\sigma_\psi \tag{5.18b}$$

$$\widetilde{\mu}_c = \eta\left(\widetilde{\mu}_\phi - \beta\right) + (1-\eta)\,(\gamma-1)\,\frac{1}{2}\,\|\sigma_\phi + \sigma_\psi\|^2 \tag{5.18c}$$

$$\sigma_c = \sigma_\phi + (1-\eta)\,\sigma_\psi. \tag{5.18d}$$

The dynamics in (5.18) depend only on σ_ψ which in turn depends only on $\delta_1 = 1/\kappa$.

Numerical investigation. For a numerical investigation, let

$$\beta = 0.06,\ \kappa = 2.67,\ s_X = 0.126,\ \bar{x} = 0.065,\ s_1 = 0.16,\ \text{and } s_2 = 0.04.$$

The parameter values are all measured per annum, and have been chosen to (roughly) match the monthly moments in Campbell (1993). Numerical solutions for various combinations of η and γ are summarized in Tables 3–7. The first column indicates the order of the approximation, N, which runs from 1 to 9. In the second column, $\varXi_\tau^N(x_0,\tau)$ is computed as a measure of whether there has been convergence. In the tables, $x_0 = .065$ and $\tau = 10^5$ years.[24] Numbers less the 10^{-16} in absolute value are reported as zero.

Table 3 presents the results for $\eta = 1$ and $\gamma = 2$. The results in the table confirm our analysis in this case. The value of the time derivative in column 2 indicates that an infinite-horizon solution does indeed exist. The first-order solution appears to be exact, since higher-order terms contribute nothing. Even absent analytical proof, we can always insert the solution into the PDEs, and evaluate the absolute value of the residual as a function of the state variable at the horizon in question.

[24] The initial stepsize taken in solving the ODEs is 10^{-6} years (≈ 31.5 seconds).

N	$\Xi_\tau^N(x_0,\tau)$	$\delta_0^N(\tau)$	$\delta_1^N(\tau)$	$\delta_2^N(\tau)$	$\delta_3^N(\tau)$	$\delta_4^N(\tau)$	$\delta_5^N(\tau)$	$\delta_6^N(\tau)$
1	0	$-2.8416(-1)$	$3.6630(-1)$					
2	0	$-2.8416(-1)$	$3.6630(-1)$	0				
3	0	$-2.8416(-1)$	$3.6630(-1)$	0	0			
4	0	$-2.8416(-1)$	$3.6630(-1)$	0	0	0		
5	0	$-2.8416(-1)$	$3.6630(-1)$	0	0	0	0	
6	0	$-2.8416(-1)$	$3.6630(-1)$	0	0	0	0	0
7	0	$-2.8416(-1)$	$3.6630(-1)$	0	0	0	0	0
8	0	$-2.8416(-1)$	$3.6630(-1)$	0	0	0	0	0
9	0	$-2.8416(-1)$	$3.6630(-1)$	0	0	0	0	0

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

TABLE 3. $\eta = 1$, $\gamma = 2$, $\tau = 10^5$, and $x_0 = 0.065$.

N	$\Xi_\tau^N(x_0,\tau)$	$\delta_0^N(\tau)$	$\delta_1^N(\tau)$	$\delta_2^N(\tau)$	$\delta_3^N(\tau)$	$\delta_4^N(\tau)$	$\delta_5^N(\tau)$	$\delta_6^N(\tau)$
1	0	$-2.4934(-1)$	$3.6403(-1)$					
2	0	$-2.4914(-1)$	$3.6402(-1)$	$9.4148(-4)$				
3	0	$-2.4914(-1)$	$3.6402(-1)$	$9.4253(-4)$	$-7.3255(-5)$			
4	0	$-2.4914(-1)$	$3.6402(-1)$	$9.4261(-4)$	$-7.3314(-5)$	$4.6005(-6)$		
5	0	$-2.4914(-1)$	$3.6402(-1)$	$9.4261(-4)$	$-7.3318(-5)$	$4.6032(-6)$	$-2.2577(-7)$	
6	0	$-2.4914(-1)$	$3.6402(-1)$	$9.4261(-4)$	$-7.3318(-5)$	$4.6034(-6)$	$-2.2586(-7)$	$7.7784(-9)$
7	0	$-2.4914(-1)$	$3.6402(-1)$	$9.4261(-4)$	$-7.3318(-5)$	$4.6034(-6)$	$-2.2587(-7)$	$7.7793(-9)$
8	0	$-2.4914(-1)$	$3.6402(-1)$	$9.4261(-4)$	$-7.3318(-5)$	$4.6034(-6)$	$-2.2587(-7)$	$7.7790(-9)$
9	0	$-2.4914(-1)$	$3.6402(-1)$	$9.4261(-4)$	$-7.3318(-5)$	$4.6034(-6)$	$-2.2587(-7)$	$7.7790(-9)$

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

TABLE 4. $\eta = 2$, $\gamma = 2$, $\tau = 10^5$, and $x_0 = 0.065$.

For $\eta = 1$, the residual is indistinguishable from zero for all values of γ and at all horizons.

In Table 4 we move away from the analytically available solutions to $\eta = 2$ and $\gamma = 2$. In this case the solution has infinite order. Here we see that the coefficients converge quite rapidly of a function of N: δ_{N-2}^N has converged to five significant digits. Also note that the coefficients die off rapidly as a function of order. The same observations hold for the coefficients in Tables 5 and 6. Note that in Tables 4–6, δ_1^N remains close to $1/(\beta + \kappa)$.

N	$\Xi^N_\tau(x_0,\tau)$	$\delta^N_0(\tau)$	$\delta^N_1(\tau)$	$\delta^N_2(\tau)$	$\delta^N_3(\tau)$	$\delta^N_4(\tau)$	$\delta^N_5(\tau)$	$\delta^N_6(\tau)$
1	0	−3.3570(−1)	3.6861(−1)					
2	0	−3.3591(−1)	3.6862(−1)	−5.4124(−4)				
3	0	−3.3591(−1)	3.6862(−1)	−5.4061(−4)	−4.3393(−5)			
4	0	−3.3591(−1)	3.6862(−1)	−5.4066(−4)	−4.3356(−5)	−2.8664(−6)		
5	0	−3.3591(−1)	3.6862(−1)	−5.4066(−4)	−4.3359(−5)	−2.8645(−6)	−1.5459(−7)	
6	0	−3.3591(−1)	3.6862(−1)	−5.4066(−4)	−4.3359(−5)	−2.8647(−6)	−1.5451(−7)	−6.6105(−9)
7	0	−3.3591(−1)	3.6862(−1)	−5.4066(−4)	−4.3359(−5)	−2.8647(−6)	−1.5451(−7)	−6.6082(−9)
8	0	−3.3591(−1)	3.6862(−1)	−5.4066(−4)	−4.3359(−5)	−2.8647(−6)	−1.5451(−7)	−6.6082(−9)
9	0	−3.3591(−1)	3.6862(−1)	−5.4066(−4)	−4.3359(−5)	−2.8647(−6)	−1.5451(−7)	−6.6082(−9)

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

TABLE 5. $\eta = 0$, $\gamma = 2$, $\tau = 10^5$, and $x_0 = 0.065$.

N	$\Xi^N_\tau(x_0,\tau)$	$\delta^N_0(\tau)$	$\delta^N_1(\tau)$	$\delta^N_2(\tau)$	$\delta^N_3(\tau)$	$\delta^N_4(\tau)$	$\delta^N_5(\tau)$	$\delta^N_6(\tau)$
1	0	8.7011(−2)	3.6697(−1)					
2	0	8.7210(−2)	3.6697(−1)	6.8632(−4)				
3	0	8.7210(−2)	3.6697(−1)	6.8632(−4)	−5.4442(−5)			
4	0	8.7210(−2)	3.6697(−1)	6.8638(−4)	−5.4442(−5)	3.5329(−6)		
5	0	8.7210(−2)	3.6697(−1)	6.8638(−4)	−5.4446(−5)	3.5329(−6)	−1.8447(−7)	
6	0	8.7210(−2)	3.6697(−1)	6.8638(−4)	−5.4446(−5)	3.5331(−6)	−1.8447(−7)	7.3644(−9)
7	0	8.7210(−2)	3.6697(−1)	6.8638(−4)	−5.4446(−5)	3.5331(−6)	−1.8447(−7)	7.3644(−9)
8	0	8.7210(−2)	3.6697(−1)	6.8638(−4)	−5.4446(−5)	3.5331(−6)	−1.8447(−7)	7.3643(−9)
9	0	8.7210(−2)	3.6697(−1)	6.8638(−4)	−5.4446(−5)	3.5331(−6)	−1.8447(−7)	7.3643(−9)

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

TABLE 6. $\eta = 2$, $\gamma = 1$, $\tau = 10^5$, and $x_0 = 0.065$.

Even though the coefficients in Tables 4–6 have converged, both as functions of τ and N, there remains the question as to how well $\Xi^N(x,\infty)$ fits the defining restriction. Since the solution method is local in nature, the fit will be perfect at x_0 and decay as we move away.[25] There are two related issues here. First, what is the range over which we desire a good fit, and, second, how good a fit should the fit be? To help determine the appropriate range, we can compute the unconditional distribution of the state variable. In our example, $x \sim \mathcal{N}\left(\bar{x}, s_X/\sqrt{2\,\kappa}\right) = \mathcal{N}(0.065, 0.054526)$. A range centered on $\bar{x}$ that includes more than 99.9 percent of the PDF is $(-0.12, 0.25)$. Table 8 shows the errors at the endpoints for the parameters in Table 4. Even the first-order approximation is reasonably accurate over the region.

Tables 7 shows results for $\eta = 4$ and $\gamma = 0$. With these parameter values there is no convergence. The time derivative is clearly not zero, even at a horizon of 10^5 years. Moreover, all of the coefficients higher than first-order are effectively zero

[25]In general, the decay need not be monotonic in $|x - x_0|$, although it is for this example.

N	$\Xi_\tau^N(x_0,\tau)$	$\delta_0^N(\tau)$	$\delta_1^N(\tau)$	$\delta_2^N(\tau)$	$\delta_3^N(\tau)$	$\delta_4^N(\tau)$	$\delta_5^N(\tau)$	$\delta_6^N(\tau)$
1	7.2641(−3)	7.2684(2)	3.7453(−1)					
2	7.2641(−3)	7.2684(2)	3.7453(−1)	0				
3	7.2641(−3)	7.2684(2)	3.7453(−1)	0	0			
4	7.2641(−3)	7.2684(2)	3.7453(−1)	0	0	0		
5	7.2641(−3)	7.2684(2)	3.7453(−1)	0	0	0	0	
6	7.2641(−3)	7.2684(2)	3.7453(−1)	0	0	0	0	0
7	7.2641(−3)	7.2684(2)	3.7453(−1)	0	0	0	0	0
8	7.2641(−3)	7.2684(2)	3.7453(−1)	0	0	0	0	0
9	7.2641(−3)	7.2684(2)	3.7453(−1)	0	0	0	0	0

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

TABLE 7. $\eta = 4$, $\gamma = 0$, $\tau = 10^5$, and $x_0 = 0.065$.

N	$x = -0.12$	$x = 0.25$
1	1.7858(−4)	1.7074(−4)
2	3.8123(−6)	3.6916(−6)
3	5.8662(−8)	5.7292(−8)
4	6.6224(−10)	6.5169(−10)
5	5.0328(−12)	5.0061(−12)
6	1.1414(−14)	1.2122(−14)
7	3.5388(−16)	3.9205(−16)
8	0	0
9	0	0

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

TABLE 8. Errors at the end points for $\eta = 2$ and $\gamma = 2$.

at this horizon. The residual from the PDE is indistinguishable from zero at this horizon. Also note that $\delta_1^N = 1/\kappa = 0.37453$ as expected.

The regions of non-convergence can be identified by examining the dividend-denominated asymptotic forward rate. Using the expression for $f_d(\infty)$ in (5.15), we can map out the regions of nonconvergence. Panel (a) of Figure 1 shades the regions where $f_d(\infty) \leq 0$. Standard preferences are plotted as the rectangular hyperbola $\eta\gamma = 1$. Note that there is no infinite-horizon solution for standard preferences unless $0.26 < \gamma < 4.65$. This rules out the level of risk-aversion that has previously been found consistent with the moments of asset returns and consumption growth. Panel (b) of Figure 1 illustrates the effect of lowering the rate of time preference to $\beta = 0.02$ on the regions of nonconvergence. We see that a sizable fraction of the region Campbell studied is nonconvergent in this case. In the limit as $\beta \to 0$, the regions of nonconvergence form a checkerboard, approaching the point $(\eta, \gamma) = (1, 3.9195)$.

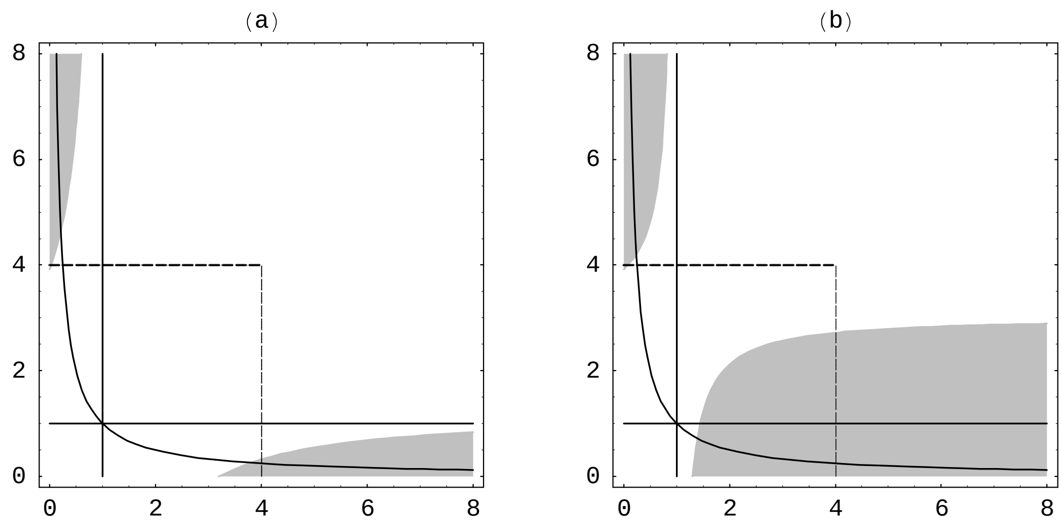

FIGURE 1. Preference parameter space: The elasticity of intertemporal substitution (η) on the horizontal axis and the coefficient of relative risk aversion (γ) on the vertical axis. Areas of nonconvergence are shaded. The dashed line delimits the region studied by Campbell (1993). The panels differ only in the rate of time preference (β). Panel (a) uses $\beta = 0.06$ from Campbell, while panel (b) uses $\beta = 0.02$.

Convergence and its absence can be viewed from the perspective of solving a system of nonlinear equations. In particular, setting the time derivatives to zero, (5.7) becomes a system of nonlinear algebraic equations in the unknown variables δ_0^1 and δ_1^1. For each of the parameter combinations in Tables 3–7, these equations are plotted in Figure 2, where the absence of a solution (at least in the neighborhood of the origin) is evident in the last frame. One advantage of the ODE approach it that it provides a solution method that finds the convergent solution if it exists, even in the presence of multiple solutions to the nonlinear equations.

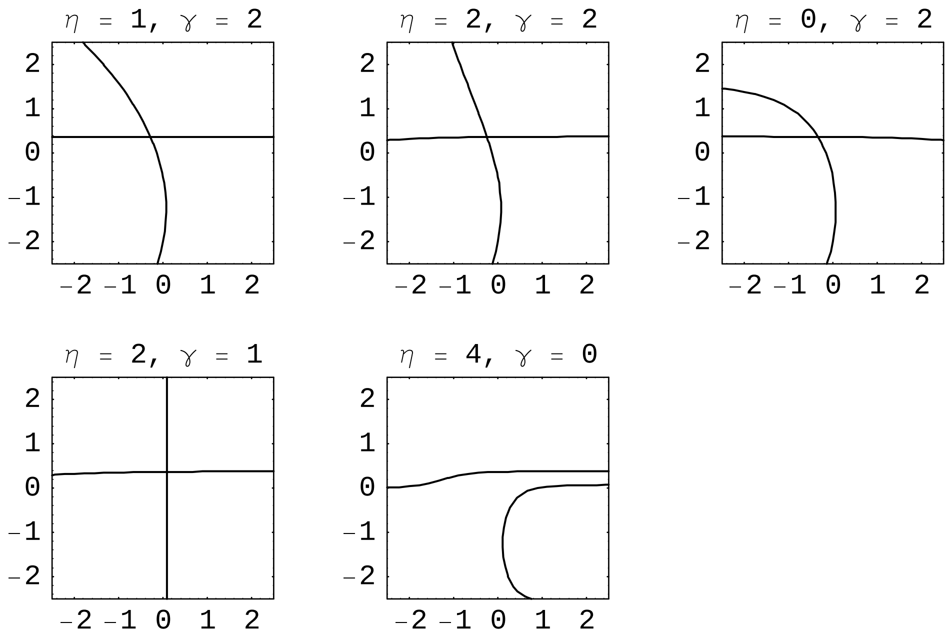

FIGURE 2. Contour plots associated with Tables 3–7. δ_0^1 is measured on the horizontal axis and δ_1^2 is measured on the vertical axis.

Now consider a model with two state variables. Augment the previous model with stochastic volatility. In particular, let

$$\begin{aligned} d\log(\phi) &= x\,dt + s_1\,dW_1 + \sqrt{y}\,dW_2 \\ dx &= \kappa\,(\bar{x} - x)\,dt + s_X\,dW_1 \\ dy &= \kappa_Y\,(\bar{y} - y)\,dt + s_Y\,\sqrt{y}\,dW_2. \end{aligned}$$

We keep the values for s_1, $\bar{x}$, κ, and s_X from the previous example, and let $\bar{y} = 0.04^2$ so that $\sqrt{\bar{y}} = .04$ ($= s_2$ from the previous example). Finally let $\kappa_Y = 1$ and $s_Y = .02$.[26] Table 9 shows some results for $\eta = 2$ and $\gamma = 2$. There are $(N+1)\,(N+2)/2$ coefficients $\delta_{ij}(\tau)$ for which $i + j \leq N$. The upper-left number in each block is the constant term $\delta_{00}(\tau)$. The remaining numbers in the first row of each block are the coefficients on powers of $x - \bar{x}$, while the remaining numbers in the first column of each block are the coefficients on powers of $y - \bar{y}$. The time derivatives are essentially zero, indicating the existence of an infinite-horizon solution for these parameter values. As in the one-factor example, the coefficients converge rapidly as a function of N. Note that the coefficients for x are little changed from the previous example, while at the same time y enters the solution with an impact of the same magnitude as x. In Table 10, the coefficient of relative risk aversion is set to one, which is the CAPM. In this case, the model reverts to a one-factor model: The volatility of the return on the market plays no role. In Table 11, the parameters

[26]These values have been chosen arbitrarily.

are $\eta = 5$ and $\gamma = 5$. In this case, the coefficients for y do not decay as rapidly as previously with respect to the order.

			x				
N	Ξ_τ^N	y	0	1	2	3	4
1	0	0	−2.4915(−1)	3.6404(−1)			
		1	−4.5584(−1)				
2	0	0	−2.4896(−1)	3.6402(−1)	9.4132(−4)		
		1	−4.5576(−1)	−3.3897(−3)			
		2	3.7764(−3)				
3	0	0	−2.4895(−1)	3.6402(−1)	9.4236(−4)	−7.3243(−5)	
		1	−4.5576(−1)	−3.3944(−3)	3.4113(−4)		
		2	3.7833(−3)	−5.6083(−4)			
		3	3.4483(−4)				
4	0	0	−2.4895(−1)	3.6402(−1)	9.4244(−4)	−7.3302(−5)	4.5999(−6)
		1	−4.5576(−1)	−3.3947(−3)	3.4145(−4)	−2.6436(−5)	
		2	3.7838(−3)	−5.6147(−4)	5.7989(−5)		
		3	3.4532(−4)	−5.8242(−5)			
		4	2.3228(−5)				

0 signifies less than 10^{-16} in absolute value.

TABLE 9. $\eta = 2$, $\gamma = 2$, $\tau = 10^5$, $x_0 = 0.065$, $y_0 = 0.0016$.

APPENDIX A. ITO'S LEMMA IN DISCRETE TIME

In continuous time, Ito's lemma provides a rule for computing the Ito process for $f(x,t)$ given the Ito process for x. It requires f be twice continuously differentiable in x and one continuously differentiable in t. Here we derive a version of Ito's lemma for the discrete-time processes in this paper as a first-order approximation around $\Delta = 0$.[27] Let

$$x' - x = \mu_X \, \Delta + \sigma_X \cdot \varepsilon \sqrt{\Delta} \quad \text{and} \quad t' - t = \Delta. \tag{A.1}$$

We seek expressions for μ_f and σ_f such that

$$f(x', t') - f(x, t) = \mu_f \, \Delta + \sigma_f \cdot \varepsilon \sqrt{\Delta} \tag{A.2}$$

[27]See for example Hull (1993) for a similar derivation.

N	Ξ_τ^N	y	x = 0	1	2	3	4
1	0	0	8.7011(−2)	3.6697(−1)			
		1	0				
2	0	0	8.7210(−2)	3.6697(−1)	6.8632(−4)		
		1	0	0			
		2	0				
3	0	0	8.7210(−2)	3.6697(−1)	6.8632(−4)	−5.4442(−5)	
		1	0	0	0		
		2	0	0			
		3	0				
4	0	0	8.7210(−2)	3.6697(−1)	6.8638(−4)	−5.4442(−5)	3.5329(−6)
		1	0	0	0	0	
		2	0	0	0		
		3	0	0			
		4	0				

0 signifies less than 10^{-16} in absolute value.

Table 10. $\eta = 2$, $\gamma = 1$, $\tau = 10^5$, $x_0 = 0.065$, $y_0 = 0.0016$.

to a first-order approximation in Δ near zero. Given (A.1) we can write

$$\begin{aligned} f(x', t') - f(x, t) &= f(x + \mu_X\,\Delta + \sigma_X \cdot \varepsilon\,\sqrt{\Delta}, t + \Delta) - f(x, t) \\ &= \left(f_t + f_x\,\mu_X + \frac{1}{2}\,f_{xx}\,\|\sigma_X\|^2\,(\varepsilon)^2\right)\Delta + f_x\,\sigma_X\,\varepsilon\,\sqrt{\Delta} + \mathcal{O}(\Delta^{3/2}), \end{aligned} \tag{A.3}$$

where f_t, f_x, and f_{xx} are the obvious partial derivatives of f with respect to t and x. Note the presence of $(\varepsilon)^2\,\Delta$ in (A.3). Letting $z := f(x', t') - f(x, t)$ and using $E_t[(\varepsilon)^2] = 1$, we can compute from (A.3)

$$E_t\,[z] = \left(f_t + \mu_X\,f_x + \frac{1}{2}\,f_{xx}\,\sigma_X\right)\Delta + \mathcal{O}(\Delta^{3/2})$$

and

$$E_t\left[(z - E_t\,[z])^2\right] = f_x(x)^2\,\|\sigma_X\|^2\,\Delta + \mathcal{O}(\Delta^2).$$

Therefore, we have established, to a first-order approximation in Δ,

$$\mu_f = f_t + \mu_X\,f_x + \frac{1}{2}\,f_{xx}\,\|\sigma_X\|^2 \quad \text{and} \quad \sigma_f = \sigma_X\,f_x.$$

N	Ξ_τ^N	y	x: 0	1	2	3	4
1	0	0	−4.5912(−1)	3.2825(−1)			
		1	−1.3742(0)				
2	0	0	−4.5832(−1)	3.2745(−1)	1.4076(−2)		
		1	−1.3640(0)	−1.6253(−1)			
		2	5.5135(−1)				
3	0	0	−4.5829(−1)	3.2739(−1)	1.4247(−2)	−3.3503(−3)	
		1	−1.3633(0)	−1.6479(−1)	4.8315(−2)		
		2	5.6024(−1)	−2.3310(−1)			
		3	3.8396(−1)				
4	0	0	−4.5829(−1)	3.2739(−1)	1.4256(−2)	−3.3732(−3)	4.8814(−4)
		1	−1.3633(0)	−1.6489(−1)	4.8649(−2)	−7.6931(−3)	
		2	5.6054(−1)	−2.3458(−1)	3.9485(−2)		
		3	3.8537(−1)	−5.7982(−2)			
		4	−4.8659(−2)				

0 signifies less than 10^{-16} in absolute value.

TABLE 11. $\eta = 5$, $\gamma = 5$, $\tau = 10^5$, $x_0 = 0.065$, $y_0 = 0.0016$.

Generalizing to a vector of state variables produces, let

$$x' - x = \mu_X \, \Delta + \sigma_X \, \varepsilon \sqrt{\Delta},$$

where x and μ_X are vectors and σ_X is a matrix. In this case we have

$$\mu_f = f_t + \mu_X \cdot f_x + \frac{1}{2} \operatorname{tr} \left[f_{xx} \, \sigma_X \, \sigma_X^\top \right] \quad \text{and} \quad \sigma_f = \sigma_X \cdot f_x.$$

where f_x is the gradient vector, f_{xx} is the Hessian matrix, $\operatorname{tr}[z]$ is the trace of matrix z and $z^\top$ is the transpose of z.

Appendix B. Gaussian quadrature

Gaussian quadrature can be thought of as a grid-based method of solution. For a review of grid-based methods in general, see Rust (1996). The grid-based method pursued here is the Gaussian-quadrature approach of Tauchen and Hussey (1991). The quadrature-based approach persued here is that of Tauchen and Hussey (1991). They takes (2.2) as their starting point, using ex-dividend prices. In this appendix, the method is applied to the conditional expectation under the equivalent martingale measure where asset prices are measured cum-dividend.

The Feynman–Kac theorem suggests we can approximate the solution to (3.9) in terms of a discrete-time one-step ahead expectation:

$$\mathit{\Pi}(x) = \Delta + e^{-R_d(x)\,\Delta}\,\hat{E}[\mathit{\Pi}(x') \mid x], \tag{B.1}$$

where the change in x has been risk-adjusted:

$$x' - x = (\mu_X(x) - \sigma_X(x) \cdot \Lambda_d(x))\,\Delta + \sigma_X(x) \cdot \hat{\varepsilon}\,\sqrt{\Delta}.$$

The conditional expectation in (B.1) can be expressed as an integral

$$\hat{E}[\mathit{\Pi}(x') \mid x] = \int \mathit{\Pi}(x')\,\hat{f}(x' \mid x)\,dx = \int \mathit{\Pi}(x')\,\frac{\hat{f}(x' \mid x)}{\omega(x')}\,\omega(x')\,dx$$

where $\hat{f}(x' \mid x)$ is the risk-adjusted conditional PDF of x' given x, and $\omega(x')$ is some strictly positive weighting function (to be chosen). A popular choice for the weighting function is $\omega(x) = \hat{f}(x' \mid \bar{x})$, where $\bar{x}$ is the unconditional mean of x. A set of abscissas $\{x_1, x_2, \ldots, x_N\}$ and a corresponding set of weights $\{w_1, w_2, \ldots, w_N\}$ that are chosen for $\omega(x)$ by Gaussian quadrature. Then we can approximate the preceeding integral by a weighted sum:

$$\int \mathit{\Pi}(x')\,\frac{\hat{f}(x' \mid x)}{\omega(x')}\,\omega(x')\,dx \doteq \sum_{j=1}^{N} \mathit{\Pi}(x_j)\,\varpi_j(x),$$

where the weights are given by

$$\varpi_j(x) = \frac{\hat{f}(x_j \mid x)}{s(x)\,\omega(x_j)}\,w_j, \qquad \text{where } s(x) = \sum_{j=1}^{N} \frac{\hat{f}(x_j \mid x)}{\omega(x_j)}\,w_j.$$

The weights have been normalized so that $\varpi_j(x_i)$ can be interpreted as the transition probability from x_i to x_j in a discrete Markov chain. Taking each abscissa in turn as a conditioning variable we have

$$\mathit{\Pi}^N(x_i) = \Delta + e^{-R_d(x_i)\,\Delta} \sum_{j=1}^{N} \mathit{\Pi}^N(x_j)\,\varpi_j(x_i), \qquad i = 1, \ldots, N, \tag{B.2}$$

where $\mathit{\Pi}^N(x_i)$ is an approximation to $\mathit{\Pi}(x)$ at $x = x_i$. Equations (B.2) comprise a system of N linear equations in the N unknowns $\mathit{\Pi}^N(x_i)$. The solution can be extended to other values of x via

$$\mathit{\Pi}^N(x) = \Delta + e^{-R_d(x)\,\Delta} \sum_{j=1}^{N} \mathit{\Pi}^N(x_j)\,\varpi_j(x).$$

Greater accuracy can be achieved by increasing N.

Equivalent martingale measure. Given our representation for m'/m we can use (2.7a) and (2.7b) to write

$$m'/m = e^{-r\,\Delta}\,(\omega'/\omega), \tag{B.3}$$

where ω is the Radon–Nikodym derivative for the equivalent martingale measure[28] and

$$\log(\omega'/\omega) = -\frac{1}{2}\,\|\lambda\|^2\,\Delta - \lambda\cdot\varepsilon\,\sqrt{\Delta}. \tag{B.4}$$

Note that ω is an exponential martingale: $E_t[\omega'/\omega] = 1$.

Let z be a positive process where $\log(z'/z) = \widetilde{\mu}_z\,\Delta + \sigma_z\cdot\varepsilon\,\sqrt{\Delta}$. Note that

$$\left(\frac{\omega'}{\omega}\right)\left(\frac{z'}{z}\right) = \exp\left\{\left(\widetilde{\mu}_z - \frac{1}{2}\,\|\lambda\|^2\right)\Delta + (\sigma_z - \lambda)\cdot\varepsilon\,\sqrt{\Delta}\right\}.$$

Therefore

$$\begin{aligned} E_t\left[\left(\frac{\omega'}{\omega}\right)\left(\frac{z'}{z}\right)\right] &= \exp\left\{\left(\widetilde{\mu}_z - \lambda\cdot\sigma_z + \frac{1}{2}\,\|\sigma_z\|^2\right)\Delta\right\} \\ &= \hat{E}_t\left[\exp\left\{(\widetilde{\mu}_z - \lambda\cdot\sigma_z)\,\Delta + \sigma_z\cdot\hat{\varepsilon}\,\sqrt{\Delta}\right\}\right], \end{aligned} \tag{B.5}$$

where the standard normal shocks $\hat{\varepsilon}$ are conceptually distinct from ε and where $\hat{E}_t[\,\cdot\,]$ is the conditional expectation operator associated with those shocks. Equation (B.5) shows that we can compute the conditional expectation of $(\omega'/\omega)\,(z'/z)$ by instead computing the conditional expectation of z'/z where the expected change in z has been artifically modified. From a purely formal standpoint, we have changed the measure from the physical measure to the equivalent martingale measure and computed the expectation of the "risk-adjusted" process for z: $\log(z'/z) = (\widetilde{\mu}_z - \lambda\cdot\sigma_z)\,\Delta + \sigma_z\cdot\hat{\varepsilon}\,\sqrt{\Delta}$. We have established the result that

$$E_t[(\omega'/\omega)\,(z'/z)] = \hat{E}_t[(z'/z)], \tag{B.6}$$

for a lognormal process z. Given (B.3) and (B.6), we can reexpress (2.9):

$$e^{-r\,\Delta}\,\hat{E}_t[B'_{\tau-\Delta}/B_\tau] = 1,$$

where the risk-adjusted process for bond prices is given by $\log(B'_{\tau-\Delta}/B_\tau) = (\widetilde{\mu}_{B_\tau} - \lambda\cdot\sigma_{B_\tau})\,\Delta + \sigma_{B_\tau}\cdot\hat{\varepsilon}\,\sqrt{\Delta}$. In terms of the Markovian bond function, we can write

$$\mathcal{B}(x,\tau) = e^{-x\,\Delta}\,\hat{E}\left[B(x',\tau-\Delta)\mid x\right], \tag{B.7}$$

where the drift of x has been "risk-adjusted":

$$x' - x = \big(\mu_X - \lambda\cdot\sigma_X\big)\,\Delta + \sigma_X\cdot\hat{\varepsilon}\,\sqrt{\Delta}. \tag{B.8}$$

We can apply Gaussian quadrature to (B.7) because we have removed the explicit dependence on the shocks.

[28] See Duffie (1996) for discussions of the Radon–Nikodym derivative and the equivalent martingale measure

Quadrature under the physical measure. If we want to apply Gaussian quadrature under the physical measure, we must remove the explicit dependence on the shocks. We can decompose the marginal rate of substitution into two factors, one which is independent of the state variable and has unit conditional expectation and the other from which the shocks can be eliminated. Decompose $\lambda = \lambda_1 + \lambda_2$, where $\lambda_1 = C\,\sigma_X$ and $\lambda_2 = \lambda - C\,\sigma_X$ and where C is chosen so that $\lambda_2 \cdot \sigma_X = 0$; *i.e.*, $C = \lambda \cdot \sigma_X / \|\sigma_X\|^2$. In this case we can write

$$\begin{aligned} E_t\left[(m'/m)\,B_{\tau-\Delta}\right] &= e^{-x\,\Delta}\,E\left[(\omega_1'/\omega_1)\,(\omega_2'/\omega_2)\,\mathcal{B}(x',\tau-\Delta)\,\middle|\,x\right] \\ &= e^{-x\,\Delta}\,E\left[(\omega_1'/\omega_1)\,\mathcal{B}(x',\tau-\Delta)\,\middle|\,x\right], \end{aligned} \tag{B.9}$$

where

$$\omega_i'/\omega_i := \exp\left\{-\frac{1}{2}\,\|\lambda_i\|^2\,\Delta - \lambda_i \cdot \varepsilon\,\sqrt{\Delta}\right\}.$$

The second line in (B.9) follows from the fact that (i) ω_2 is conditionally independent of the other two factors in the expectation and (ii) that $E[\omega_2'/\omega_2 \mid x] = 1$. We can solve (2.11) for[29]

$$\sigma_X \cdot \varepsilon\,\sqrt{\Delta} = x' - x - \mu_X\,\Delta,$$

which allows us to write

$$\log(\omega_1'/\omega_1) = -\frac{1}{2}\,\|C\,\sigma_X\|^2\,\Delta - C\,(x' - x - \mu_X\,\Delta).$$

Inserting this expression into (B.9) provides the representation sought.

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[29]This step has no analog in continuous time.

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