Discussion Papers in Economics

Separability and Aggregation of Equivalence Relations

Dinko Dimitrov

Thierry Marchant

Debasis Mishra

August 2009

Discussion Paper 09-06

Indian Statistical Institute, Delhi
Planning Unit
7, S. J. S. Sansanwal Marg, New Delhi 110016, India

Separability and Aggregation of Equivalence Relations *

Dinko Dimitrov †, Thierry Marchant ‡, and Debasis Mishra §

August 26, 2009

Abstract

We provide axiomatic characterizations of two natural families of rules for aggregating equivalence relations: the family of join aggregators and the family of meet aggregators. The central conditions in these characterizations are two separability axioms. *Disjunctive separability*, *neutrality*, and *unanimity* characterize the family of join aggregators. On the other hand, *conjunctive separability* and *unanimity* characterize the family of meet aggregators. We show another characterization of the family of meet aggregators using conjunctive separability and two Pareto axioms, *Pareto*$^+$ and *Pareto*$^-$. If we drop Pareto$^-$, then conjunctive separability and Pareto$^+$ characterize the family of meet aggregators along with a trivial aggregator.

*Authors are listed alphabetically. The authors thank Arunava Sen for useful discussions.
†University of Munich, Email: dinko.dimitrov@lrz.uni-muenchen.de
‡Ghent University, Email: thierry.marchant@ugent.be
§Corresponding author, Indian Statistical Institute, Email: dmishra@isid.ac.in

1 Introduction

The theory of aggregating individual preferences into a social preference relation was initiated by the seminal work of Arrow (1951). Arrow's impossibility theorem inspired many scholars to apply the social choice approach to other environments as well. For instance, in many situations we are often required to classify a finite set of objects into "equivalence classes", where two objects belong to the same class if and only if they are assumed to be equivalent. Such a classification of objects into disjoint equivalence classes is called an *equivalence relation* and it is often based on the various *attributes* the objects may have. The objective is then to aggregate these equivalence relations of different attributes and form a holistic equivalence relation (Mirkin, 1975; Wilson, 1978; Fishburn and Rubinstein, 1986; Rubinstein and Fishburn, 1986; Barthélemy and Montjardet, 1986; Barthélemy, 1988). The critical difference of this problem from the Arrovian framework of individual preference aggregation is that an equivalence relation does not rank the equivalence classes it contains.

In this paper we focus on the aggregation of equivalence relations, consider two natural families of aggregation rules, and axiomatically characterize them. To start with, let us consider an example with three objects, say a, b, and c, and two attributes, say gender and nationality. Objects a and b are of the same gender and c is of a different gender. Objects a and c are of the same nationality and b is of a different nationality. We will now describe the two families of aggregators using this example. For both families, we first identify a non-empty subset of attributes, called *decisive* attributes. For each possible choice of decisive attributes, we then define two aggregators. Thus, each family of such aggregators has $2^n - 1$ members, where n is the number of attributes.

The first aggregator puts two objects in the same equivalence class if and only if each decisive attribute puts them in the same equivalence class. Thus, it reflects the consensus view of the decisive attributes and is a member of the family of *meet aggregators*. In the example above, if we take both the attributes as decisive, then the equivalence relation given by this meet aggregator puts each object in a separate equivalence class.

Each aggregator in the second family of rules we discuss has an entirely different approach in aggregation. It takes the *union* (or *join*) of the equivalence classes of every decisive attribute, and then builds the closure of this union[1]. In some sense, it tries to "satisfy" each decisive attribute, and belongs to the family of *join aggregators*. In the example above, if we take both attributes as decisive, the join aggregator puts a and b together due to the gender attribute. Also, it puts a and c together due to the nationality attribute, and hence, the three objects form an equivalence class.

[1] Unlike the meet operation, the join operation over equivalence relations may lead to intransitive relations. Hence, we need to take the closure. In the example above, if we take the union of equivalence relations of all attributes, then it requires a and b to be together due to attribute gender, and a and c to be together due to attribute nationality. As for transitivity, one needs to put b and c together as well.

1.1 Our Contribution

We provide characterizations for the family of meet aggregators and for the family of join aggregators. The central theme of our characterizations is separability. As a motivating example, let us consider a city which wants to classify its citizens. It considers different attributes for classification, say family size, place of birth, and eating habits, and hires different consultants to give classifications of individuals on each attribute. Each consultant comes up with a profile (over attributes) of equivalence relations. Now, the city has two ways to aggregate the information of the consultants. First, it can apply an aggregator on the profile of equivalence relations of each consultant, come up with an equivalence relation for each consultant, and then combine these equivalence relations into one equivalence relation. Second, it can first combine the profile of equivalence relations into a single profile of equivalence relations, and then use the aggregator on the combined data. We use two axioms which reflect consistency in these two approaches, albeit two completely different notions of consistency.

Conjunctive separability requires that if we combine the profile of equivalence relations of each consultant using the meet operation, then the aggregator must output the same equivalence relation on this combined data as the equivalence relation obtained by taking the meet of the equivalence relations produced by the aggregator for each consultant's profile of equivalence relations. Analogously, *disjunctive separability* requires that if the combination of equivalence relations according to both approaches is done using the join operation (and taking the closure of it), then it must produce the same equivalence relation. These two axioms reflect two consistent ways of decentralizing the process of aggregation.

Besides these two separability axioms, we use some standard axioms from the aggregation theory literature. We start with three different Pareto-type axioms. The first one, *Pareto*$^+$, says that if two objects are in the same equivalence class according to every attribute, then the aggregator must put them in the same equivalence class. Analogously, we define the *Pareto*$^-$ axiom, which requires that if two objects are in different equivalence classes according to every attribute, then the aggregator must put them in different equivalence classes. Pareto$^+$ is satisfied by both types of aggregators, while Pareto$^-$ is satisfied by the meet aggregators but not by the join aggregators. Finally, we consider also a third Pareto-type axiom, *unanimity*, which is satisfied by the aggregators in both families. Unanimity requires that if the equivalence relation according to every attribute is the same, the aggregator must output this very equivalence relation. The combination of Pareto$^+$ and Pareto$^-$ implies unanimity.

The next standard axiom we use is *neutrality*. It requires that if we construct a new profile of equivalence relations from a given profile of equivalence relations by permuting the set of objects, then the outcome of the aggregator at the new profile must be the same permutation applied to the outcome of the old profile. Hence, the names of the objects should not matter to the aggregator.

Finally, we use an axiom called *non-triviality*. An aggregator is called trivial if it outputs the equivalence relation where all the objects are put in the same equivalence class for every profile of equivalence relations. An aggregator satisfies non-triviality if it is not a trivial aggregator.

We show that an aggregator satisfies unanimity, neutrality, and disjunctive separability if and only if it is one of the join aggregators. We obtain an almost dual characterization of the family of meet aggregators. An aggregator satisfies unanimity and conjunctive separability if and only if it is one of the meet aggregators. Neutrality is implied by conjunctive separability and unanimity (or Pareto$^+$). We show that one can replace unanimity by Pareto$^+$ and Pareto$^-$ in the characterization of the meet aggregators.

Pareto$^-$ is not satisfied by the join aggregators when $m \geq 3$, where m is the number of objects. A natural task is then to explore the possibility of dropping the Pareto$^-$ axiom in the characterization of the family of meet aggregators. We show that we do not get a significantly larger set of aggregators if we drop this axiom. In particular, an aggregator satisfies Pareto$^+$ and conjunctive separability if and only if it is one of the meet aggregators or the trivial aggregator. Hence, using non-triviality in place of Pareto$^-$ along with Pareto$^+$ and conjunctive separability characterizes the family of meet aggregators. No such characterization is possible for the family of join aggregators.

Using our characterizations, we can conclude that the families of join and meet aggregators essentially differ by two different notions of separability. These families contain (almost) dual aggregators (almost because, we need to take the closure of the join operation, while the relation produced by the meet operation is transitive) and our characterizations reflect this duality.

1.2 Relation to Prior Work

The literature on aggregating equivalence relations started with the works of Mirkin (1975), Barthélemy and Montjardet (1986), and Fishburn and Rubinstein (1986). In their work, Rubinstein and Fishburn (1986) discuss a general model of preference aggregation which covers the aggregation of equivalence relations, while Fishburn and Rubinstein (1986) consider explicitly the family of meet aggregators. To our knowledge, there is no work which discusses the family of join aggregators.

Fishburn and Rubinstein (1986) use Pareto$^+$, Pareto$^-$, and *binary independence* to characterize the family of meet aggregators. Binary independence is an axiom in the spirit of Arrow's independence of irrelevant alternatives. It requires that the aggregated equivalence relation between any two objects must depend only on the equivalence relations between these two objects in every attribute. Our characterization of the family of meet aggregators replaces binary independence by conjunctive separability for $m \geq 3$. Is binary independence weaker than conjunctive separability? We give examples to illustrate that neither of them imply the other. However, we show that conjunctive separability and binary independence are equivalent axioms (for $m \geq 3$) in the presence of Pareto$^+$.

Our results for the family of meet aggregators are slightly tighter than the results in Fishburn and Rubinstein (1986). First, the binary independence axiom used in Fishburn and Rubinstein (1986) has no bite when $m = 2$. This is not the case for conjunctive separability, and as a result, we get a characterization which works for all m. Second, when $m \geq 3$, we show that Pareto^{-} in the characterization of Fishburn and Rubinstein (1986) can be weakened to non-triviality.

Another related strand of the aggregation literature considers environments in which every individual has a view about how a society he is a member of should be partitioned into classes (see for example, Houy (2007); Dimitrov and Puppe (2009)). A group identity function assigns then to each profile of views a societal decomposition into classes. Hence, this aggregation problem is formally equivalent to the aggregation of equivalence relations and it extends environments in which the number of social groups is assumed to be fixed and their names matter (see for example, Çengelci and Sanver (2008); Dimitrov et al. (2007); Houy (2007); Kasher and Rubinstein (1997); Miller (2008); Samet and Schmeidler (2003), among others). It is worth noting that the specific features of the group identification problem allows one to introduce liberalism-type axioms which have no meaning in the more general framework we consider in the present paper; for instance, Houy (2007) uses such kind of axioms, along with binary independence, to characterize the grand meet aggregator.

The study of the impact of appropriately defined meet and join separability axioms on group identification rules was undertaken by Miller (2008) for a specific context, where the number of social groups is fixed. This author then shows that the two requirements basically define a class of one-vote rules, in which one opinion determines whether an individual is considered to be a member of a group.

Another type of aggregation problems in which individuals submit a menu of options are considered by Ahn and Chambers (2008). These authors formulate a disjoint additivity axiom which can be seen as a weaker version of disjunctive separability for the corresponding context, and show that disjoint additivity, anonymity, unanimity, and a monotonicity axiom characterize the grand join aggregator in their model. In the characterizations we provide in this paper, the two separability ideas are formulated as to fit into the more general setting of aggregating equivalence relations and to clearly stress the duality between the aggregators in the considered families.

The rest of the paper is organized as follows. Section 2 discusses the model, the notations, and the general framework, while Section 3 introduces the axioms we use in the paper. We give our characterization of the family of join aggregators in Section 4 and present different characterizations of the family of meet aggregators in Section 5. Section 6 is devoted then to the duality in the characterizations of the two families. We conclude in Section 7 with discussions on the effect of (a) adding anonymity to our characterizations and (b) imposing both forms of separability on an aggregator.

2 Framework

Let $M = \{a, b, c, \ldots\}$ be a finite set of $m \geq 2$ objects and $N = \{1, 2, \ldots, n\}$ be a finite set of $n \geq 2$ attributes. An equivalence relation on M is a reflexive, symmetric, and transitive binary relation. Formally, $\sim$ is an **equivalence relation** on M if for all $a, b \in M$, we have

- (Reflexivity) $a \sim a$,
- (Symmetry) $a \sim b \Rightarrow b \sim a$,
- (Transitivity) $a \sim b$ and $b \sim c \Rightarrow a \sim c$.

Equivalently, one can think of an equivalence relation to be a partitioning of set of objects into **equivalence classes**, where objects a and b belong to the same equivalence class in equivalence relation $\sim$ if and only if $a \sim b$.

We will denote the equivalence relation of attribute $i \in N$ as $\sim_i$. Thus, $(\sim_1, \ldots, \sim_n)$ will denote a profile of equivalence relations. Sometimes, we will refer to a profile $(\sim_1, \ldots, \sim_n)$ as $(\sim_i)_N$.

One can think of an equivalence relation as being an undirected graph which is transitive. Such a graph will have a node for every object, i.e., the set of nodes is M. There is an edge between the different objects a and b if and only if $a \sim b$. We denote the (undirected) edge between any two different objects a and b as (a, b) [2]. Hence, there are three types of equivalence classes: a single node which is not part of any edge (equivalence class with a single object), a pair of nodes which are joined by an edge but not part of any cycle (equivalence class with two objects), and a set of nodes forming a cycle (equivalence classes with more than two objects).

Let $\mathbb{E}$ be the set of all equivalence relations on M. An **aggregator** is a mapping $F : \mathbb{E}^n \to \mathbb{E}$. So, the aggregator outputs a holistic equivalence relation for every profile of equivalence relations. To simplify notations, sometimes we will write $F(\sim_1, \ldots, \sim_n)$ as $\sim$ and $F(\sim'_1, \ldots, \sim'_n)$ as $\sim'$, etc.

The following specific types of equivalence relations will be useful in what follows. We will say that an equivalence relation $\sim$ is

- *empty*, if every equivalence class in $\sim$ contains a single object (i.e., the graph corresponding to $\sim$ has no edges),
- *single-edged*, if one equivalence class in $\sim$ contains two objects, while every other equivalence class in $\sim$ contains a single object (i.e., the graph corresponding to $\sim$ has exactly one edge),

[2] It is appropriate to denote an undirected edge between a and b by $\{a, b\}$. For convenience, we abuse notation here to denote it as (a, b).

- *complete*, if $\sim$ contains a single equivalence class consisting of all objects (i.e., the graph corresponding to $\sim$ is complete).

We define two operations on equivalence relations. The *meet* of two equivalence relations $\sim$ and $\sim'$ is denoted as $\sim \wedge \sim'$, and defined as follows. Any two objects a and b are equivalent in $\sim \wedge \sim'$ if and only if a and b are equivalent in $\sim$ and $\sim'$. Clearly, $\sim \wedge \sim'$ is an equivalence relation. The family of *meet agggregators* is defined as follows. For every non-empty $S \subseteq N$,

$$F_{\wedge}^{S}(\sim_1, \ldots, \sim_n) = \wedge_{i \in S} \sim_i \qquad \forall\ (\sim_1, \ldots, \sim_n) \in \mathbb{E}^n.$$

We call $F_{\wedge}^{N}$ the *grand meet* aggregator.

The *join* of two equivalence relations $\sim$ and $\sim'$ is denoted as $\sim \vee \sim'$, and defined as follows. Any two objects a and b are equivalent in $\sim \vee \sim'$ if and only if a and b are equivalent in $\sim$ or $\sim'$. Unlike the meet operation, the join operation is not closed, i.e., $\sim \vee \sim'$ need not be an equivalence relation. To make $\sim \vee \sim'$ an equivalence relation, we take the closure of $\sim \vee \sim'$. In graph terms, if there is an edge between a and b and between b and c, then we put an edge between a and c as well. The closure of $\sim \vee \sim'$ is denoted as $\sim \vee^c \sim'$. The family of *join aggregators* is defined as follows. For every non-empty $S \subseteq N$,

$$F_{\vee}^{S}(\sim_1, \ldots, \sim_n) = \vee_{i \in S}^{c} \sim_i \qquad \forall\ (\sim_1, \ldots, \sim_n) \in \mathbb{E}^n.$$

We call $F_{\vee}^{N}$ the *grand join* aggregator.

Note that the binary operations $\vee^c$ and $\wedge$ are associative. Hence, for any $S \subseteq N$, the operations $\vee_{i \in S}^{c}$ and $\wedge_{i \in S}$ are well-defined. Clearly then, based on the choice of S, we will have different meet aggregators and different join aggregators.

3 Axioms

In this section, we define the axioms we use later. We start with some well-known axioms in the preference aggregation literature.

Axiom **1** *Let σ be a permutation of M and for any binary relation $\sim$ define $\sim^{\sigma}$ as $a \sim^{\sigma} b$ if and only if $\sigma(a) \sim \sigma(b)$ for all $a, b \in M$. We say an aggregator F satisfies* **neutrality** *if and only if $F((\sim_i^{\sigma})_N) = F((\sim_i)_N)^{\sigma}$ for all profiles of equivalence relations $(\sim_1, \ldots, \sim_n)$.*

Neutrality requires that the aggregator should not distinguish between objects based on their names. Similarly, if we do not distinguish between attributes based on their names, then we have the following axiom.

Axiom **2** *An aggregator F satisfies* **anonymity** *if for any permutation ρ of N and any profile of equivalence relations $(\sim_1, \ldots, \sim_n)$ we have $F((\sim_i)_N) = F((\sim_{\rho(i)})_N)$.*

The next two axioms are the equivalent of Pareto axiom in the Arrovian framework.

Axiom **3** *An aggregator F satisfies* **Pareto$^+$** *if and only if for all $a, b \in M$ and for all profiles of equivalence relations $(\sim_1, \ldots, \sim_n)$ such that $a \sim_i b$ for all $i \in N$, we have $a \sim b$, where $\sim \equiv F(\sim_1, \ldots, \sim_n)$.*

Axiom **4** *An aggregator F satisfies* **Pareto$^-$** *if and only if for all $a, b \in M$ and for all profiles of equivalence relations $(\sim_1, \ldots, \sim_n)$ such that $a \nsim_i b$ for all $i \in N$, we have $a \nsim b$, where $\sim \equiv F(\sim_1, \ldots, \sim_n)$.*

It is easy to see that Pareto$^+$ is satisfied by the family of meet and join aggregators. On the other hand, Pareto$^-$ is satisfied by the family of meet aggregators but not by the family of join aggregators. To see this, consider an example with two attributes 1 and 2 and three objects a, b, and c. Consider the following profile of equivalence relations: $a \sim_1 b$ but $a \nsim_1 c$ and $b \nsim_1 c$ and $a \sim_2 c$ but $a \nsim_2 b$ and $c \nsim_2 b$. Note that $b \nsim_i c$ for all $i \in \{1, 2\}$. Hence, by Pareto$^-$, b and c should belong to different equivalence classes in the aggregated equivalence relation. Now, consider the grand join aggregator on this profile. It will put a, b, and c in one equivalence class.

The following axiom is a weakening of the combination of Pareto$^+$ and Pareto$^-$, and is satisfied by the families of join and meet aggregators.

Axiom **5** *An aggregator F satisfies* **unanimity** *if and only if for all profiles of equivalence relations $(\sim_1, \ldots, \sim_n)$ such that $\sim_1 = \ldots = \sim_n$, we have $F(\sim_1, \ldots, \sim_n) = \sim_1$.*

Clearly, the combination of Pareto$^+$ and Pareto$^-$ imply unanimity.

Next, we define two axioms related to separability.

Axiom **6** *An aggregator F satisfies* **conjunctive separability** *if and only if for all pairs of profiles $(\sim_i)_N$ and $(\sim'_i)_N$, we have $F((\sim_i \wedge \sim'_i)_N) = F((\sim_i)_N) \wedge F((\sim'_i)_N)$.*

Axiom **7** *An aggregator F satisfies* **disjunctive separability** *if and only if for all pairs of profiles $(\sim_i)_N$ and $(\sim'_i)_N$, we have $F((\sim_i \vee^c \sim'_i)_N) = F((\sim_i)_N) \vee^c F((\sim'_i)_N)$.*

The following independence axiom was used in Fishburn and Rubinstein (1986).

Axiom **8** *An aggregator F satisfies* **binary independence** *if and only if for every $a, b \in M$ and for all pairs of profiles $(\sim_i)_N$ and $(\sim'_i)_N$ such that $a \sim_i b$ if and only if $a \sim'_i b$ for all $i \in N$, we have $a \sim b$ if and only $a \sim' b$, where $\sim \equiv F((\sim_i)_N)$ and $\sim' \equiv F((\sim'_i)_N)$.*

Finally, we say an aggregator F is **trivial** if for all profiles $(\sim_i)_N$, $F((\sim_i)_N)$ is the complete equivalence relation.

Axiom **9** *An aggregator satisfies* **non-triviality** *if it is not a trivial aggregator.*

4 Join Aggregators

In this section, we give a characterization of the family of join aggregators. Our characterization uses the neutrality, unanimity, and disjunctive separability axioms. To give an idea why the characterization works, consider an aggregator which satisfies neutrality, unanimity, and disjunctive separability. Now, consider an example with three attributes ($N = \{1, 2, 3\}$) and four objects ($M = \{a, b, c, d\}$). Call a profile of equivalence relations a *single-edged profile* if there is exactly one attribute with a single-edged equivalence relation and all other attributes have empty equivalence relations. The first step in the proof is to show that the aggregator is

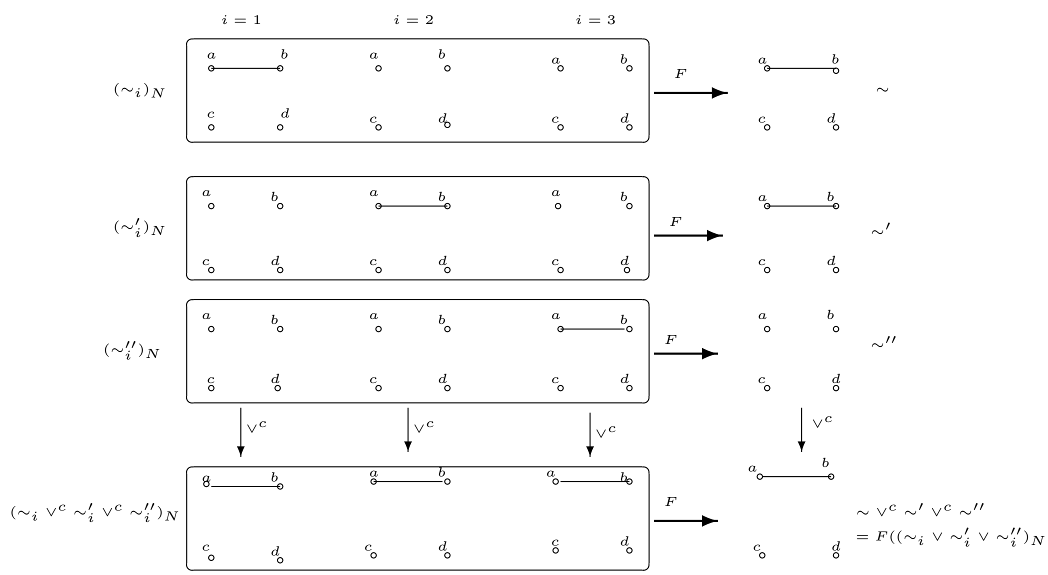

Figure 1: Single-edged profiles

a join aggregator for any single-edged profile. To do so, consider the single-edged profiles in Figure 1. Each triplet of relations in a rounded rectangle represents a profile. Each column corresponds to an attribute. The join of these single-edged profiles is a profile (last row), where unanimity axiom can be applied. Hence, the aggregator (applied to the last row) must output the equivalence relation shown in Figure 1. By disjunctive separability, there is some non-empty set of single-edged profiles ($(\sim_i)_N$, $(\sim'_i)_N$ or $(\sim''_i)_N$), such that the corresponding aggregated equivalence relations (resp. $\sim$, $\sim'$ or $\sim''$) have edge (a, b). Moreover, no other edges must be present in $\sim$, $\sim'$ or $\sim''$. Suppose the aggregator outputs a non-empty equivalence relation whenever the single-edged profile has edge (a, b) in attributes 1 or 2. Denote these attributes as S^{ab}. By neutrality, $S^{ab} = S^{cd} = S$ for all (a, b) and (c, d). Thus, the set S collects all decisive attributes and we call it a **decisive set**. In Figure 1, we have $S = \{1, 2\}$ as the decisive set.

The proof is then concluded by observing that any profile of equivalence relations can be decomposed into single-edged profiles whose join is the original profile of equivalence relations. Figure 2 illustrates this using a profile of equivalence relations $(\sim_i)_N$ which is the

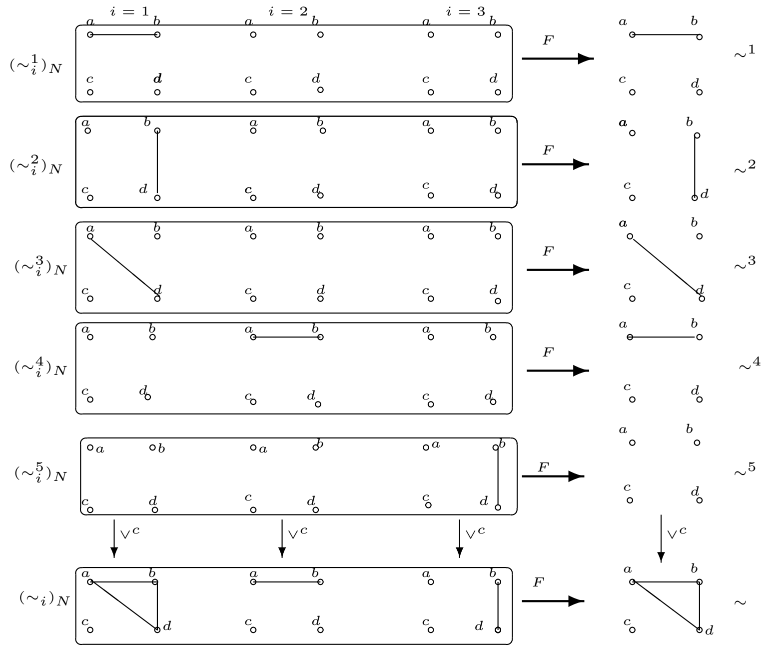

Figure 2: Decomposing a profile into single-edged profiles

join of 5 (total number of edges in this profile of equivalence relations) single-edged profiles of equivalence relations denoted by $(\sim_i^1)_N$, $(\sim_i^2)_N$, $(\sim_i^3)_N$, $(\sim_i^4)_N$ and $(\sim_i^5)_N$. Using the first step of the proof, we know $\sim^1, \ldots, \sim^5$. The join of these five relations is the cycle with edges a, b, c. By disjunctive separability, $F((\sim_i)_N) = \sim$ must be equal to this cycle, that is, the aggregator must take the join (and closure) of the equivalence relations in the decisive set.

We now state the theorem and prove it formally.

Theorem **1** *An aggregator satisfies neutrality, unanimity, and disjunctive separability if and only if it is belongs to the family of join aggregators.*

Proof: It is not difficult to see that each join aggregator satisfies neutrality, unanimity, and disjunctive separability. Suppose F is an aggregator which satisfies these three axioms.

Let $(\sim_i^{k,ab})_N$ be a single-edged profile of equivalence relations where edge (a, b) is present in attribute k only, and no other edges are present in any attribute. By construction, the profile $\left(\vee_{k\in N}^c \sim_i^{k,ab}\right)_N$ is a profile of equivalence relations as in the statement of the unanimity axiom. Hence, $F(\left(\vee_{k\in N}^c \sim_i^{k,ab}\right)_N)$ is a single-edged equivalence relation with edge (a, b). By disjunctive separability, the edge (a, b) is present in $F((\sim_i^{k,ab})_N)$ for at least one k. Define the set S^{ab} as follows.

$$S^{ab} = \{k \in N : \text{ edge } (a, b) \text{ belongs to } F((\sim_i^{k,ab})_N)\}.$$

We can also define S^{cd} for all $c, d \in M$ using profiles $(\sim_i^{k,cd})_N$. By neutrality, $S^{ab} = S^{cd} = S$ for all $a, b, c, d \in M$. Call S the set of decisive attributes (the decisive set). Hence, for every single-edged profile, the aggregator is the join over the decisive set.

Now, consider any arbitrary profile of equivalence relations $(\sim_i)_N$. We can decompose it into a finite number of single-edged profiles $(\sim_i^j)_N$, where $j \in \{1, \dots, l\}$ and l is the number of edges in the profile $(\sim_i)_N$. Thus, $(\sim_i)_N = (\vee_{j=1,\dots,l}^c \sim_i^j)_N$. By disjunctive separability, $F((\sim_i)_N) = F((\vee_{j=1,\dots,l}^c \sim_i^j)_N) = \vee_{j=1,\dots,l}^c F((\sim_i^j)_N)$. Since the aggregator for the single-edged profiles is the join over the corresponding decisive set, it follows that the aggregator over any profile is also the join over that decisive set. The result then follows because the decisive set is non-empty. ■

To complete the characterization in Theorem 1, we show that the axioms neutrality, unanimity, and disjunctive separability are independent. The trivial aggregator satisfies neutrality and disjunctive separability, but it fails unanimity. The grand meet aggregator satisfies neutrality and unanimity but fails disjunctive separability.

To show that neutrality is not implied by unanimity and disjunctive separability, we construct the following aggregator.

Example **1** *Fix two objects $a, b \in M$. For a given profile, $(\sim_i)_N$, define a binary relation I as follows. We have aIb if and only if $a \sim_1 b$. For any $(c, d) \neq (a, b)$, we have cId if and only if $c \sim_i d$ for some $i \in N$. Define $F((\sim_i)_N)$ as the symmetric and transitive closure of I.*

The aggregator in Example 1 satisfies unanimity and disjunctive separability, but fails neutrality. The fact that it fails neutrality is clear (it distinguishes between pairs (a, b) and other pairs). Also, it is easy to see that it satisfies unanimity. To show that it satisfies disjunctive separability requires some effort. We do this in the Appendix.

Also, note that if $m = 2$, then every aggregator satisfies neutrality. So, Example 1 works only when $m \geq 3$.

5 Meet Aggregators

In this section, we set out to give a characterization of the family of meet aggregators using conjunctive separability. Our aim is to give a characterization which is analogous to the characterization of the family of join aggregators in Theorem 1. Fishburn and Rubinstein (1986) provided the following characterization.

Theorem **2** (**Fishburn and Rubinstein (1986)**) *Suppose $m \geq 3$. An aggregator satisfies Pareto$^+$, Pareto$^-$, and binary independence if and only if it belongs to the family of meet aggregators.*

Our first characterization of the meet aggregators replaces binary independence in Theorem 2 by conjunctive separability. Unlike Theorem 2, this characterization works also for $m = 2$.

THEOREM **3** *An aggregator satisfies Pareto$^+$, Pareto$^-$, and conjunctive separability if and only if it belongs to the family of meet aggregators.*

Proof: It is easy to see that each meet aggregator satisfies Pareto$^+$, Pareto$^-$, and conjunctive separability. Consider an aggregator F which satisfies these three axioms. We how that it must be a meet aggregator. We do the proof in two cases.

CASE 1: Suppose $m \geq 3$. We use the following lemma for this case.

LEMMA **1** *If an aggregator F satisfies Pareto$^+$ and conjunctive separability, then it satisfies binary independence.*

Proof: Fix $a, b \in M$. Consider two profiles of equivalence relations $(\sim_i)_N$ and $(\sim'_i)_N$ which satisfy the premises of binary independence axiom for a and b. Assume for contradiction F violates binary independence in this case. This implies that edge (a, b) belongs to either $F((\sim_i)_N)$ or $F((\sim'_i)_N)$ but not both. Hence, the edge (a, b) does not belong to $F((\sim_i)_N) \wedge F((\sim'_i)_N)$. By conjunctive separability, the edge (a, b) does not belong to $F((\sim_i \wedge \sim'_i)_N)$.

Now, without loss of generality, let the edge (a, b) belong to $F((\sim_i)_N)$. Then, we modify the profile of equivalence relations $(\sim'_i)_N$ to construct $(\sim''_i)_N$ in the following manner. The edge (a, b) is present in $\sim''_i$ for all $i \in N$. For any $(c, d) \neq (a, b)$ and for all $i \in N$, the edge (c, d) is present in $\sim''_i$ if and only if the edge (c, d) is present in $\sim'_i$. Thus, for all $i \in N$, $\sim''_i$ is $\sim_i$ along with the edge (a, b). Note that by definition, for all $i \in N$, $\sim_i \wedge \sim'_i = \sim_i \wedge \sim''_i$. Hence, $F((\sim_i \wedge \sim'_i)_N) = F((\sim_i \wedge \sim''_i)_N)$. This implies that the edge (a, b) does not belong to $F((\sim_i \wedge \sim''_i)_N)$. By Pareto$^+$, the edge (a, b) belongs to $F((\sim''_i)_N)$. By assumption, the edge (a, b) belongs to $F((\sim_i)_N)$. Hence, the edge (a, b) belongs to $F((\sim_i)_N) \wedge F((\sim''_i)_N)$. This violates conjunctive separability, and gives us a contradiction. ■

Using Theorem 2 along with Lemma 1, the result now follows.

CASE 2: Suppose $m = 2$. Let $M = \{a, b\}$. Call a set of attributes $S \subseteq N$ **decisive** if

- $a \sim b$ for all profiles $(\sim_i)_N$ such that $a \sim_i b$ for all $i \in S$
- and $a \nsim b$ for all profiles $(\sim_i)_N$ such that $a \nsim_i b$ for some $i \in S$,

where $\sim \equiv F((\sim_i)_N)$. By Pareto$^+$, a decisive set exists. By Pareto$^-$, the decisive set is non-empty. We will be done if we can show that the decisive set is unique. Suppose S and T are two different decisive sets. Consider a profile $(\sim_i)_N$ in which edge (a, b) is present in $\sim_i$ if and only if $i \in S$. Similarly, consider a profile $(\sim'_i)_N$ in which edge (a, b) is present in $\sim'_i$ if and only if $i \in T$. By definition, $(\sim_i)_N \neq (\sim'_i)_N$ and $S \cap T \neq S$ or $S \cap T \neq T$. Since S and T are decisive sets, both $F((\sim_i)_N)$ and $F((\sim'_i)_N)$ are complete equivalence relations (i.e., contain edge (a, b)). Hence, $F((\sim_i)_N) \wedge F((\sim'_i)_N)$ is a complete equivalence relation. By definition, the profile $(\sim_i)_N \wedge (\sim'_i)_N$ has edge (a, b) only in attributes in $S \cap T$. By conjunctive separability $F((\sim_i)_N \wedge (\sim'_i)_N) = F((\sim_i)_N) \wedge F((\sim'_i)_N)$. But $(S \cap T) \subsetneq S$. Since S is decisive, this is a contradiction. ■

Is Theorem 3 a tighter characterization of the family of meet aggregators than the one in Theorem 2? We try to answer this question. We give two examples to show that conjunctive separability and binary independence do not imply each other.

EXAMPLE **2** *Fix $a, b \in M$. The aggregator F is defined as follows. For all $c, d \in M$ where $(c, d) \neq (a, b)$ and for all profiles $(\sim_i)_N$, we have*

- *$c \nsim d$,*
- *and $a \sim b$ if and only if $a \sim_1 b$ and $a \nsim_i b$ for all $i \neq 1$,*

where $\sim \equiv F((\sim_i)_N)$.

The aggregator F in Example 2 satisfies binary independence. To see this, fix $x, y \in M$ and consider two profiles $(\sim_i)_N$ and $(\sim'_i)_N$ as in the premises of the binary independence axiom. If $(x, y) \neq (a, b)$, then the edge (x, y) is absent in both $F((\sim_i)_N)$ and $F((\sim'_i)_N)$. If $(x, y) = (a, b)$, then clearly, the edge $(x, y) = (a, b)$ is present in $F((\sim_i)_N)$ if and only if it is present in $F((\sim'_i)_N)$.

However, the aggregator F in Example 2 fails conjunctive separability. An example is shown in Figure 3. We have only shown objects a and b in Figure 3. The rest of the objects can be put in any arbitrary equivalence class in each attribute. Figure 3 shows the output of the aggregator in three profiles, where the third profile is the meet of the first two profiles. It is easy to see that the aggregator violates conjunctive separability.

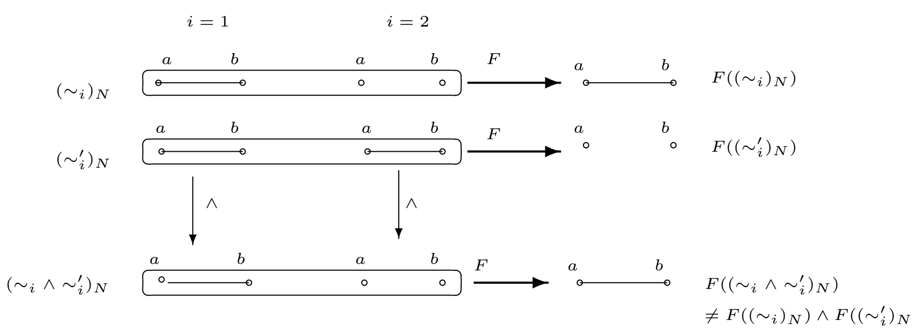

Figure 3: Binary independence does not imply conjunctive separability

The following example illustrates that conjunctive separability does not imply binary independence when $m \geq 3$.

EXAMPLE **3** *Suppose $m \geq 3$. Define the aggregator F as follows. For any profile $(\sim_i)_N$,*

- *if $(\sim_i)_N$ is a profile of complete equivalence relations, then $F((\sim_i)_N)$ is a complete equivalence relation,*

- *else, it is the empty equivalence relation.*

Clearly, the aggregator in Example 3 satisfies conjunctive separability. Figure 4 shows two profiles where this aggregator fails binary independence.

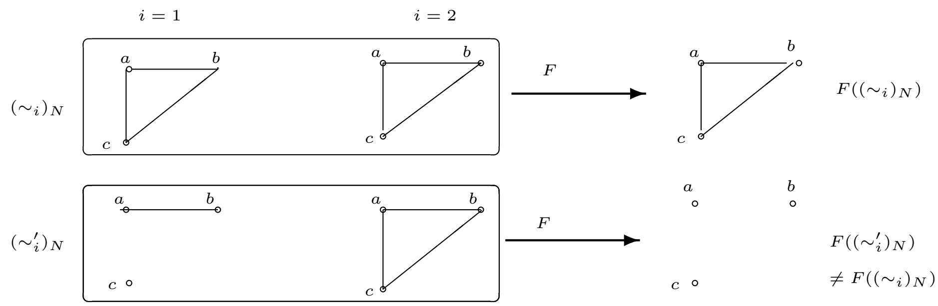

Figure 4: Conjunctive separability does not imply binary independence

Notice that the aggregators in Examples 2 and 3 do not satisfy Pareto$^+$. This is no coincidence as the following proposition proves that conjunctive separability and Pareto$^+$ are equivalent to binary independence and Pareto$^+$ if $m \geq 3$.

Proposition **1** *Suppose $m \geq 3$ and let F be an aggregator which satisfies Pareto$^+$. The aggregator F satisfies conjunctive separability if and only if it satisfies binary independence.*

Proof: We need the following lemma. The lemma is due to Fishburn and Rubinstein (1986), but we give a proof in the Appendix for completeness.

Lemma **2 (Fishburn and Rubinstein (1986))** *Suppose $m \geq 3$, and let F be an aggregator which satisfies binary independence and Pareto$^+$. Consider $a, b, c, d \in M$ and two profiles $(\sim_i)_N$ and $(\sim'_i)_N$ such that for every $i \in N$, the edge (a, b) belongs to $\sim_i$ if and only if the edge (c, d) belongs to $\sim'_i$. Then the edge (a, b) belongs to $F((\sim_i)_N)$ if and only if the edge (c, d) belongs to $F((\sim'_i)_N)$.*

Due to Lemma 1, we only have to prove that Pareto$^+$ and binary independence imply conjunctive separability. Assume for contradiction that conjunctive separability does not hold. Then, there are two profiles $(\sim_i)_N$ and $(\sim'_i)_N$ and $a, b \in N$ such that either

a) the edge (a, b) belongs to $F((\sim_i)_N) \wedge F((\sim'_i)_N)$ but it does not belong to $F((\sim_i \wedge \sim'_i)_N)$,

b) the edge (a, b) does not belong to $F((\sim_i)_N) \wedge F((\sim'_i)_N)$ but it belongs to $F((\sim_i \wedge \sim'_i)_N)$.

Suppose (a) holds. Denote $F((\sim_i)_N)$ as $\sim$ and $F((\sim'_i)_N)$ as $\sim'$. Since the edge (a, b) belongs to $\sim \wedge \sim'$, it belongs to both $\sim$ and $\sim'$. On the other hand the edge (a, b) does not belong to $F((\sim_i \wedge \sim'_i)_N)$. By Pareto$^+$, there must exist some $i \in N$ such that the edge (a,b) does not belong to $\sim_i \wedge \sim'_i$. Define the following sets:

$$S = \{i \in N : \text{edge } (a,b) \text{ belongs to } \sim_i\} \quad (1)$$
$$S' = \{i \in N : \text{edge } (a,b) \text{ belongs to } \sim'_i\} \quad (2)$$
$$S'' = \{i \in N : \text{edge } (a,b) \text{ belongs to } \sim_i \wedge \sim'_i\}. \quad (3)$$

By definition $S'' \subseteq S$ and $S'' \subseteq S'$. We first show that S and S' are distinct. Assume for contradiction $S = S'$. Then, $S'' = S = S'$, and by binary independence, the edge (a,b) must belong to $F((\sim_i \wedge \sim'_i)_N)$, which is a contradiction. Similarly, by binary independence again, S and S' are strict supersets of S'', i.e., $S'' \subsetneq S$ and $S'' \subsetneq S'$.

Consider a profile $(\dot\sim_i)_N$ and $c \in M \setminus \{a,b\}$ such that

$$\{i \in N : \text{edge } (a,b) \text{ belongs to } \dot\sim_i\} = S$$
$$\{i \in N : \text{edge } (b,c) \text{ belongs to } \dot\sim_i\} = S'$$
$$\{i \in N : \text{edge } (a,c) \text{ belongs to } \dot\sim_i\} = S''.$$

Note that $(\dot\sim_i)_N$ is well defined. Denote $F((\dot\sim_i)_N)$ as $\dot\sim$. By binary independence, edge (a,b) belongs to $\dot\sim$ since it belongs to $\sim$. By Lemma 2, edge (b,c) belongs to $\dot\sim$ since the edge (a,b) belongs to $\sim'$. By transitivity, edge (a,c) belongs to $\dot\sim$. By Lemma 2, the edge (a,b) must then belong to $F((\sim_i \wedge \sim'_i)_N)$. This is a contradiction.

Assume now (b) holds. Since the edge (a,b) does not belong to $(\sim \wedge \sim')$, it does not belong to $\sim$ or $\sim'$. Without loss of generality, suppose (a,b) does not belong to $\sim$. Define S and S'' as in Equations 1 and 3 respectively. Since the edge (a,b) belongs to $F((\sim_i \wedge \sim'_i)_N)$, by binary independence, $S'' \subsetneq S$.

Consider a profile $(\dot\sim_i)_N$ and $c \in M \setminus \{a,b\}$ such that

$$\{i \in N : \text{edge } (a,b) \text{ belongs to } \dot\sim_i\} = S$$
$$\{i \in N : \text{edge } (b,c) \text{ belongs to } \dot\sim_i\} = \{i \in N : \text{edge } (a,c) \text{ belongs to } \dot\sim_i\} = S''.$$

Note that $(\dot\sim_i)_N$ is well defined. Denote $F((\dot\sim_i)_N)$ as $\dot\sim$. By binary independence, the edge (a,b) does not belong to $\dot\sim$ since it does not belong to $\sim$. But by Lemma 2, the edges (b,c) and (a,c) belongs to $\dot\sim$ since (a,b) belongs to $F((\sim_i \wedge \sim'_i)_N$. By transitivity, (a,b) belongs to $\dot\sim$. This is a contradiction. ■

Though conjunctive separability and binary independence are equivalent in the presence of Pareto$^+$, conjunctive separability and binary independence are not equivalent in the presence of Pareto$^-$. This is easily verified from Examples 2 and 3 (the aggregators in Examples 2 and 3 satisfy Pareto$^-$).

Proposition 1 is not true when $m = 2$. In this case, we can easily construct rules which satisfies Pareto$^+$ but does not satisfy conjunctive separability (and binary independence is satisfied vacuously). For example, the grand join aggregator satisfies Pareto$^+$ but fails conjunctive separability.

5.1 Weakening Pareto$^-$

The characterizations in Theorems 2 and 3 use Pareto$^-$, a form of Pareto axiom which is not satisfied by the join aggregators. We wish to replace it with a weaker axiom which is satisfied by those aggregators. The following theorem shows that any aggregator which satisfies Pareto$^+$ and conjunctive separability must either be a meet aggregator or the trivial aggregator.

Theorem **4** *Consider the following statements.*

1. *An aggregator satisfies conjunctive separability, Pareto$^+$, and non-triviality.*
2. *An aggregator satisfies binary independence, Pareto$^+$, and non-triviality.*
3. *An aggregator belongs to the family of meet aggregators.*

If $m \geq 3$, then (1), (2), and (3) are equivalent. If $m = 2$ then (1) and (3) are equivalent.

Proof: Consider the case when $m \geq 3$. The fact that (1) $\Leftrightarrow$ (2) follows from Proposition 1. Clearly, (3) implies (1) and (2). We show that (1) implies (3). We need the following lemma.

Lemma **3** *Let F be an aggregator which satisfies binary independence and Pareto$^+$. Then, F satisfies neutrality.*

Proof: For $m = 2$, every aggregator satisfies neutrality, and hence the claim holds. For $m \geq 3$, the proof is almost immediate from Lemma 2. To see this, consider a permutation π of the set of objects and a profile $(\sim_i)_N$. Let $(\sim_i^\pi)_N$ be the permuted profile. Suppose F does not satisfy neutrality. Then, some edge (a, b) belongs to $F((\sim_i)_N)$ but the edge $(\pi(a), \pi(b))$ does not belong to $F((\sim_i^\pi)_N)$. Note that the profiles $(\sim_i)_N$ and $(\sim_i^\pi)_N$ satisfy the premises of Lemma 2 with edge $(\pi(a), \pi(b))$ taking the role of (c, d). Hence, by Lemma 2, the edge (a, b) belongs to $F((\sim_i)_N)$ if and only if the edge $(\pi(a), \pi(b))$ belongs to $F((\sim_i^\pi)_N)$. This is a contradiction. ■

By Proposition 1 and Lemma 3, if an aggregator satisfies conjunctive separability and Pareto$^+$, then it satisfies neutrality. Now, consider an aggregator F which satisfies conjunctive separability, Pareto$^+$, and non-triviality. We now do the proof in two steps.

Step 1: In this step, we show that if $(\sim_i)_N$ is a profile of empty equivalence relations then $F((\sim_i)_N)$ is the empty equivalence relation. By neutrality, $F((\sim_i)_N)$ is either the empty equivalence relation or the complete equivalence relation. Suppose it is the complete equivalence relation. Consider another profile $(\sim_i')_N$ such that $F((\sim_i')_N)$ is not the complete equivalence relation. Such a profile exists because of non-triviality. But for every $i \in N$, $\sim_i \wedge \sim_i' = \sim_i$. Hence, $F((\sim_i \wedge \sim_i')_N)$ is a complete equivalence relation. By conjunctive separability, $F((\sim'_i)_N)$ is a complete equivalence relation. This is a contradiction.

Step 2: In this step, we show that F satisfies Pareto$^-$, and by Theorem 3, F is an aggregator in the family of meet aggregators. To show that F satisfies Pareto$^-$, fix $a, b \in M$, and consider a profile $(\sim_i)_N$ such that the edge (a, b) is not present in $\sim_i$ for all $i \in N$. Let $(\sim'_i)_N$ be a profile of empty equivalence relations. These two profiles satisfy the premises in the binary independence axiom. Since by Step 1, $F((\sim'_i)_N)$ is an empty equivalence relation, we get that the edge (a, b) is not present in $F((\sim_i)_N)$ because of binary independence (F satisfies binary independence by Proposition 1). Hence, Pareto$^-$ holds.

Now, when $m = 2$, we can go back to Case 2 in the proof of Theorem 3. In the absence of Pareto$^-$, we will either have a non-empty unique decisive set or an empty decisive set. An empty decisive set implies a trivial aggregator. Hence, by non-triviality, we must have a unique non-empty decisive set. The rest of the proof follows as in Case 2 in the proof of Theorem 3. Hence, (1) and (3) are equivalent for any $m \geq 2$. ■

The equivalence of (2) and (3) in Theorem 4 is a tighter characterization of the family of meet aggregators than the characterization of Fishburn and Rubinstein (1986) (Theorem 2). This is because non-triviality is weaker than Pareto$^-$. For the same reason, the equivalence of (1) and (3) in Theorem 4 is a tighter characterization than Theorem 3.

6 Duality in the Characterizations

It is interesting to investigate if similar or "dual" characterizations are possible for the families of meet and join aggregators. The two main axioms, conjunctive and disjunctive separability, used in the characterizations in Theorems 1, 3, and 4 are dual to each other. However, we seem to require neutrality in Theorem 1, but not in Theorems 3 and 4 because it is implied by conjunctive separability and Pareto$^+$ (Lemma 3). One wonders if unanimity can be relaxed in the characterization of the join aggregators. For instance, does non-triviality, Pareto$^+$, and disjunctive separability characterize the family of join aggregators? The answer is no. The aggregator in Example 1 satisfies all these axioms but it is not a join aggregator. Does neutrality, non-triviality, Pareto$^+$, and disjunctive separability characterize this family? The answer is again no. The aggregator in the following example satisfies all these axioms but it is not a join aggregator.

Example 4 *The aggregator outputs the empty equivalence relation if the profile of equivalence relations include only empty equivalence relations, else it outputs the complete equivalence relation.*

Another approach is to modify the characterizations of the family of meet aggregators to make it look analogous to Theorem 1. The following theorem attempts to do that.

Theorem 5 *An aggregator satisfies unanimity and conjunctive separability if and only if it belongs to the family of meet aggregators.*

Proof: Clearly, any meet aggregator satisfies unanimity and conjunctive separability. Now, let F be an aggregator which satisfies unanimity and conjunctive separability. We show that F satisfies Pareto$^+$.

Consider a profile $(\sim_i)_N$ such that the edge (a,b) belongs to $\sim_i$ for all $i \in N$. Consider another profile $(\sim_i')_N$ such that for all $i \in N$, the only edge in equivalence relation $\sim_i'$ is the edge (a,b). By unanimity, the edge (a,b) belongs to $F((\sim_i')_N)$. Assume for contradiction that edge (a,b) does not belong to $F((\sim_i)_N)$. By conjunctive separability, the edge (a,b) does not belong to $F((\sim_i \wedge \sim_i')_N)$. Since for all $i \in N$, $\sim_i \wedge \sim_i' = \sim_i'$ and the edge belongs to $F((\sim_i')_N)$, the edge must also belong to $F((\sim_i \wedge \sim_i')_N)$. This is a contradiction.

Since F satisfies unanimity, it satisfies non-triviality. Hence, F satisfies conjunctive separability, Pareto$^+$, and non-triviality, and by Theorem 4, it must belong to the family of meet aggregators. ■

Clearly, unanimity is weaker than the combination of Pareto$^+$ and Pareto$^-$. Hence, Theorem 5 is a tighter characterization than Theorem 3. Moreover, Theorem 5 gives us an almost dual characterization of the meet aggregators to the characterization in Theorem 1 of the join aggregators.

7 Conclusion

We conclude by giving two remarks on our characterizations.

Anonymity. It is not difficult to see that adding anonymity to the list of axioms in Theorems 3, 4, and 5 give us the grand meet aggregator. It can be shown that anonymity is independent of the axioms used in these theorems. On the other hand, anonymity, unanimity, and disjunctive separability characterize the grand join aggregator. This can be seen from the proof of Theorem 3. We use neutrality in the proof of Theorem 3 to conclude $S^{ab} = S^{cd}$. But we can use anonymity instead of neutrality to conclude $S^{ab} = S^{cd} = N$. It is not difficult to argue that anonymity, unanimity, and disjunctive separability are independent axioms.

Dictatorial Aggregators. Call an aggregator F **dictatorial** if there exists an attribute (dictator) $j \in N$ such that for every profile of equivalence relations $(\sim_i)_N$ we have $F((\sim_i)_N) = \sim_j$. Note that an aggregator F is a dictatorial aggregator if and only if $F = F^S_{\vee^c} = F^S_{\wedge}$ for some $S \subseteq N$ and $|S| = 1$. Thus, dictatorial aggregators are the only aggregators which belong to the families of meet and join aggregators. Using this fact and our results we can give various characterizations of the dictatorial aggregators. First, an aggregator is a dictatorial aggregator if and only if it satisfies conjunctive separability, disjunctive separability, Pareto$^+$, and Pareto$^-$. This follows from Theorems 1 and 3. Second, an aggregator is a dictatorial aggregator if and only if it satisfies conjunctive separability, disjunctive separability, Pareto$^+$, and non-triviality. This follows from Theorems 1 and 4. Finally, an aggregator is a dictatorial aggregator if and only if it satisfies conjunctive separability, disjunctive separability, and unanimity. This follows from Theorems 1 and 5.

References

Ahn, D. S. and C. P. Chambers (2008): "What's on the Menu? Deciding What is Available to the Group?" Working Paper, California Institute of Technology.

Arrow, K. J. (1951): *Social Choice and Individual Values*, New York: Wiley, second Edition: 1963.

Barthélemy, J. P. (1988): "Comments on: Aggregation of Equivalence Relations by P.C. Fishburn and A. Rubinstein," *Journal of Classification*, 5, 85–87.

Barthélemy, J. P. and B. Montjardet (1986): "On the Use of Ordered Sets in Problems of Comparison and Consensus of Classifications," *Journal of Classification*, 3, 187–224.

Çengelci, M. and R. Sanver (2008): "Simple Collective Identity Functions," *Theory and Decision*, forthcoming.

Dimitrov, D. and C. Puppe (2009): "Non-bossy Social Classification," Working Paper, University of Karlsruhe.

Dimitrov, D., S.-C. Sung, and Y. Xu (2007): "Procedural Group Identification," *Mathematical Social Sciences*, 54, 137–146.

Fishburn, P. C. and A. Rubinstein (1986): "Aggregation of Equivalence Relations," *Journal of Classification*, 3, 61–65.

Houy, N. (2007): ""I want to be a J!": Liberalism in Group Identification Problems," *Mathematical Social Sciences*, 54, 59–70.

Kasher, A. and A. Rubinstein (1997): "On the Question "Who is a J?" A social Choice Approach," *Logique et Analyse*, 160, 385–395.

Miller, A. (2008): "Group Identification," *Games and Economic Behavior*, 63, 188–202.

Mirkin, B. (1975): "On the Problem of Reconciling Partitions," in *Quantitative Sociology, International Perspectives on Mathematical and Statistical Modelling*, ed. by H. M. Blalock, A. Aganbegian, F. Borodkin, R. Boudon, and V. Capecchi, Academic Press, New York, 441–449.

Rubinstein, A. and P. C. Fishburn (1986): "Algebraic Aggregation Theory," *Journal of Economic Theory*, 38, 63–77.

Samet, D. and D. Schmeidler (2003): "Between Liberalism and Democracy," *Journal of Economic Theory*, 110, 213–233.

Wilson, R. (1978): "On the Theory of Aggregation," *Journal of Economic Theory*, 10, 89–99.

Appendix

Aggregator in Example 1 Satisfies Disjunctive Separability

Consider two profiles $(\sim_i)_N$ and $(\sim'_i)_N$. We have to show that

$$F((\sim_i \vee^c \sim'_i)_N) = F((\sim_i)_N) \vee^c F((\sim'_i)_N).$$

Suppose (a,b) is an edge in $F((\sim_i \vee^c \sim'_i)_N)$. First, we show that (a,b) in also an edge in $F((\sim_i)_N) \vee^c F((\sim'_i)_N)$. We consider two cases.

Case 1: Suppose (a,b) is an edge in $\sim_1 \vee^c \sim'_1$. This in turn induces two subcases.

- Case 1a: The edge (a,b) is in $\sim_1$ or in $\sim'_1$. In such a case, the edge (a,b) will either be in $F((\sim_i)_N)$ or $F((\sim'_i)_N)$. Hence, it will be in $F((\sim_i)_N) \vee^c F((\sim'_i)_N)$.
- Case 1b: There is a chain of edges $(a,a_1),(a_1,a_2),\ldots,(a_r,b)$ which belongs to $\sim_1 \vee \sim'_1$. Hence, these chain of edges are also present in $F((\sim_i)_N) \vee F((\sim'_i)_N)$. Consequently, the edge (a,b) is present in $F((\sim_i)_N) \vee^c F((\sim'_i)_N)$.

Case 2: Suppose (a,b) is not an edge in $\sim_1 \vee^c \sim'_1$. Since (a,b) is an edge in $F((\sim_i \vee^c \sim'_i)_N)$, there must exist $a_0,a_1,\ldots,a_r$ with $a_0 = a$ and $b_0 = b$ and $j_1,\ldots,j_r \in N$ such that for $k \in \{1,\ldots,r\}$, each edge (a_{k-1},a_k) belongs to $(\sim_{j_k} \vee \sim'_{j_k})$ or there is chain of edges $(a_{k-1},c_1),(c_1,c_2),\ldots,(c_q,a_k)$ each of which belongs to $(\sim_{j_k} \vee \sim'_{j_k})$. By definition of the aggregator, each edge (a_{k-1},a_k) is in $F((\sim_i)_N) \vee^c F((\sim'_i)_N)$. This in turn implies that (a,b) is in $F((\sim_i)_N) \vee^c F((\sim'_i)_N)$.

Suppose now $(c,d) \neq (a,b)$ is an edge in $F((\sim_i \vee^c \sim'_i)_N)$. Using a similar reasoning as above, we can conclude that (c,d) is an edge in $F((\sim_i)_N) \vee^c F((\sim'_i)_N)$.

Now, we show that if (a,b) is an edge in $F((\sim_i)_N) \vee^c F((\sim'_i)_N)$, then it is also an edge in $F((\sim_i \vee^c \sim'_i)_N)$. Again, there are multiple cases to consider.

Case 1: The edge (a, b) is in $F((\sim_i)_N)$. So, either (a, b) is an edge in $\sim_1$ or there is a chain of edges $(a, a_1), (a_1, a_2), \ldots, (a_r, b)$ which belongs to $(\sim_i)_N$. In the first case, the edge (a, b) also belongs to $(\sim_1 \vee^c \sim'_1)$. Hence, it belongs to $F((\sim_i \vee^c \sim'_i)_N)$. In the second case, the same chain of edges belongs to $(\sim_i \vee^c \sim'_i)_N$. Hence, (a, b) belongs to $F((\sim_i \vee^c \sim'_i)_N)$.

Case 2: The edge (a, b) is in $F((\sim'_i)_N)$. Again, as in Case 1, we can argue that (a, b) belongs to $F((\sim_i \vee^c \sim'_i)_N)$.

Case 3: There is a chain of edges $(a, a_1), (a_1, a_2), \ldots, (a_r, b)$ which belongs to $F((\sim_i)_N) \vee F((\sim'_i)_N)$. Each of these edges can be supposed to be different from edge (a, b). Consider any arbitrary edge (c, d) in this chain. Edge (c, d) is either in $(\sim_i)_N$ or in $(\sim'_i)_N$ or there is a chain of edges $(c, c_1), (c_1, c_2), \ldots, (c_q, d)$ in $(\sim_i \vee \sim'_i)_N$. In each of these cases, edge (c, d) belongs to $(\sim_i \vee^c \sim'_i)_N$. Since $(c, d) \neq (a, b)$, we conclude that (c, d) belongs to $F((\sim_i \vee^c \sim'_i)_N)$. This is true for any edge in the chain $(a, a_1), (a_1, a_2), \ldots, (a_r, b)$. Hence, the edge (a, b) belongs to $F((\sim_i \vee^c \sim'_i)_N)$.

Finally, suppose $(c, d) \neq (a, b)$ is an edge in $F((\sim_i)_N) \vee^c F((\sim'_i)_N)$. Using a similar reasoning as above, we can conclude that (c, d) is an edge in $F((\sim_i \vee^c \sim'_i)_N)$.

Proof of Lemma 2

Proof: If $(a, b) = (c, d)$, then the claim is trivial. Assume a, b, c are distinct. Consider two profiles $(\sim_i)_N$ and $(\sim'_i)_N$ such that the edge (a, b) is in $\sim_i$ if and only if the edge (b, c) is in $\sim'_i$. Consider another profile of equivalence relations $(\sim''_i)_N$ which satisfies the following for every $i \in N$:

- the edge (a, b) belongs to $\sim''_i$ if and only if it belongs to $\sim_i$,
- the edge (b, c) belongs to $\sim''_i$ if and only if it belongs to $\sim'_i$ (if and only if the edge (a, b) belongs to $\sim_i$),
- and the edge (a, c) belongs to $\sim''_i$.

Note that such a profile of equivalence relations $(\sim''_i)_N$ exists. By Pareto$^+$, the edge (a, c) belongs to $F((\sim''_i)_N)$. By transitivity, the edge (a, b) belongs to $F((\sim''_i)_N)$ if and only if the edge (b, c) belongs to $F((\sim''_i)_N)$. By binary independence, edge (a, b) belongs to $F((\sim_i)_N)$ if and only if it belongs to $F((\sim''_i)_N)$. Similarly, by binary independence again, edge (b, c) belongs to $F((\sim_i)_N)$ if and only if it belongs to $F((\sim''_i)_N)$. Hence, edge (a, b) belongs to $F((\sim_i)_N)$ if and only if edge (b, c) belongs to $F((\sim'_i)_N)$.

Now, consider a pair of profiles $(\sim_i)_N$ and $(\sim'_i)_N$ such that for every $i \in N$, the edge (a, b) belongs to $\sim_i$ if and only if the edge (c, d) belongs to $\sim'_i$. Now, construct $(\sim''_i)_N$ such that it satisfies the following: for every $i \in N$, the edge (a, b) belongs to $\sim''_i$ if and only if the edge (b, c) belongs to $\sim_i''$ if and only if the edge (c, d) belongs to $\sim_i''$ if and only if the edge (a, b) belongs to $\sim_i$ if and only if the edge (c, d) belongs to $\sim_i''$. Applying the result in the previous paragraph multiple times, we get that the edge (a, b) belongs to $F((\sim_i)_N)$ if and only if the edge (b, c) belongs to $F((\sim_i'')_N)$ if and only if the edge (c, d) belongs to $F((\sim_i')_N)$. This proves the lemma. ■