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CHAPTER VI

THE WHITTAKER-HENDERSON METHOD OF GRADUATION.

In smoothing data some departure from observed facts is, of course, necessary. In any particular case, the question naturally arises as to how far it is desirable to depart from the observed facts in order to obtain smoothness. If departure from the observed facts be measured by means of the sum of the squares of the deviations of the data from the graduated curve, and if roughness or lack of smoothness be measured by the sum of the squares of the third differences of the graduated curve, the problem is to decide at what stage it is undesirable to increase the sum of the squares of the deviations of the data from the graduated curve in order to decrease the sum of the squares of the third differences of the graduated curve itself.

Such a problem might be formulated as follows: How shall k times the sum of the squares of the deviations of the data from the graduated curve plus the sum of the squares of the third differences of that graduated curve be made a minimum? The larger the value chosen for k the more importance would be attached to closeness of fit as compared with smoothness. As the value of k approached infinity, the resulting graduated curve would approach the data points. As the value of k approached zero, the resulting graduated curve would approach a second degree parabola fitted to the observations by the method of least squares.[1]

In 1923, Professor E. T. Whittaker drew attention to the fact that if data be smoothed in such a manner that k times the sum of the squares of the deviations of the data from the smooth curve plus the sum of the squares of the third differences of the smooth curve itself be made a minimum, the smooth curve will be such that each of its ordinates will equal a corresponding ordinate of the data plus $\frac{1}{k}$ times a sixth difference of the smooth curve.[2] He developed a method for computing such a smooth curve.[3]

In 1924, Robert Henderson published an article in which he showed that if

[1] The form of the smooth curve in this limiting case is the result of making the sum of the squares of the *third* differences vanish and the sum of the squares of the deviations a minimum. If the sum of the squares of the *second* differences be made to vanish, the limiting case will give a straight line as the graduated curve.

[2] A complete set of sixth differences is obtained by extending the graduated curve by means of second-degree parabolas put through the first three and last three graduated points.

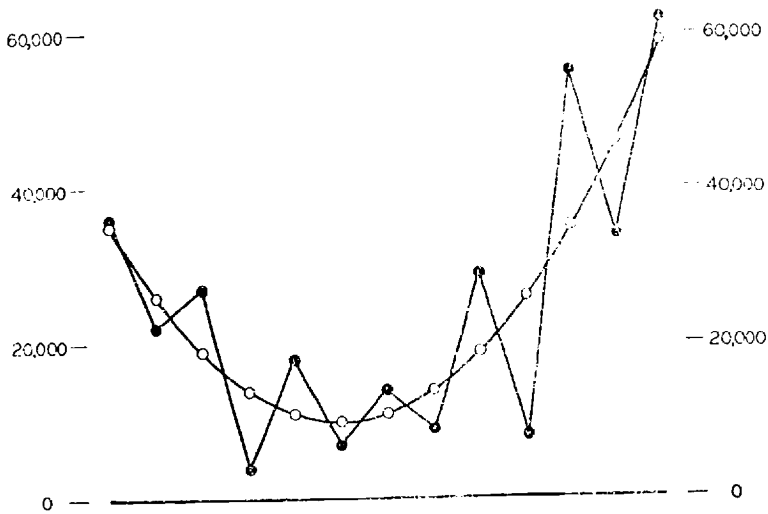
[3] See E. T. Whittaker and G. Robinson, *The Calculus of Observations*, pp. 303 to 315, inclusive. For an even earlier suggestion of the method, see Henderson and Sheppard, *Graduation of Mortality and Other Tables* (1919), page 6, eleven lines from bottom of page and onwards.

$$k = \frac{16\ (2n + 3)^2}{n\ (n + 1)^3\ (n + 2)^3\ (n + 3)},$$

the numerical calculation of the ordinates of the smooth curve can be arranged in a simple form for all values of k corresponding to integral values

CHART III

ILLUSTRATION OF
ROBERT HENDERSON'S
METHOD OF GRADUATION



of n.[1] While Henderson's calculation scheme is quite simple, the reader will probably understand it more easily if it is illustrated than he would if

[1] For the theory back of Henderson's method of fitting, see his article, *A New Method of Graduation*, Transactions of The Actuarial Society of America, May 1924, and Robert Henderson, *Some Points in the General Theory of Graduation*, Proceedings of the International Mathematical Congress—Toronto 1924—Volume II, pages 815-820, inclusive.

it were merely described. Appendix VI contains a numerical paradigm with complete instructions for computation. Chart III illustrates the fit obtained in Appendix VI.

Though Robert Henderson's method of computing the Whittaker-Henderson graduation is a highly elegant contribution to mathematical technique, and appears extremely simple, the actual computation is not without some serious pitfalls.

In the first place, the reader must remember that the three points from which the computation in the paradigm in Appendix VI began, were selected after the exact graduation was already known. They were selected in such a manner as to give immediately accurate results. In practice *the preliminary* operation (see Appendix VI) which is concerned with obtaining three points which shall closely approximate three points at one end of the true smooth curve, is often lengthy—particularly if the three points from which the preliminary operation commences are taken too close to the end or are at all badly chosen.[1]

Though each operation is extremely simple, being merely a multiplication by a small number or an addition or a subtraction, the operations have to be performed seriatim. In actual computation, this leads to errors being made with great ease. In any summation formula, each operation is com-

[1] See Note 1, page 152.

pleted before the next one is begun. For example, the first operation connected with the 43-term approximately fifth-degree parabolic formula is the taking of a 12-months moving total of *all* the data. The next step is the taking of an 8-months moving total of the first moving total—for *all* the data. Each type of operation is completed before the next is begun. Computers are much less likely to make mistakes if they do not have to change the nature of their operations momentarily.

Furthermore, while there are checks on the accuracy of the Henderson computation, such checks cannot be used until after the entire graduation has been calculated—unless we except the important check of repeating the preliminary operation backwards and forwards the entire length of the data until three identical, or practically identical, end figures have been arrived at. From the nature of the Henderson computation scheme errors are not easy to spot on a chart, as they tend to be distributed smoothly over a considerable range. This is not so with errors in some of the other graduations which have already been discussed. For example, errors in the computation of the parabolic summation formulas nearly always occur in the weight multiplications or in the final additions and subtractions of such results.[1] Now, such errors generally stand out as cusps on a chart. Moreover,

[1] The moving totals are automatically checked.

when discovered they can easily be corrected, whereas an error in the Henderson computation involves, at least theoretically, a re-computation of the whole *second half*.

If n equals 3, as in the paradigm in Appendix VI, the value of

$$\frac{4\ (2n+3)}{(n+1)\ (n+2)^2\ (n+3)}$$

is exactly .06. However, if n equals 2 or 4 or 5, for example, the value of the above function is no longer a simple decimal of few terms. The third figure of each column in the computation will then have to be obtained from a calculating machine.

It may well be that, for any particular set of observations, no integral value of n will give the exact graduation desired. If a fractional value of n be required, the computation becomes considerably lengthened.[1]

A Whittaker-Henderson graduation needs no extrapolation; it covers the entire range of the

[1] In 1925 Mr. Henderson published another article in which he described a method of applying the same graduation to a series of *weighted* observations. This second method he designated "Method B." Though "Method B" is primarily designed for use where the observations are weighted, it can be applied to unweighted observations by assuming a uniform weight of unity. However, when applied to unweighted observations, it has the disadvantage of being considerably more laborious in computation than Mr. Henderson's earlier method. On the other hand, it is more self-checking, and the difficulties connected with obtaining the three end points are taken care of in the method itself without involving any guessing as in the

data. This is a distinct element of mathematical elegance and sometimes an important practical consideration.[1] However, graduation of the ends of almost any series is necessarily extremely hypothetical unless facts outside the range covered by the graduation are used in obtaining the graduation. This is as true of the Whittaker-Henderson graduation as of any other type. If it were not so, we should have the philosopher's stone which turns all things to gold. A graduation could be applied to the course of stock or commodity prices, and successful speculation be based on the slope of the curve at its end point. Though mathematically inelegant, the most desirable procedure in a majority of the cases of graduation is to graduate not only the actual data, but extrapolated data which sometimes may be extremely crude estimates. However, we usually know *something* about the general nature of possible data outside of the range for which we have definite figures. It is highly undesirable not to make some use of such knowledge,

earlier method. Perhaps its outstanding superiority to the earlier method is that it is not restricted to any specific values of k. As in the earlier method, the graduation cannot be extended to include new data without re-calculation.

See Robert Henderson, *Further Remarks on Graduation*, Transactions of The Actuarial Society of America, 1925, pages 52 to 57 inclusive.

[1] Though there are simple mathematical methods of extrapolating any graduation in such a manner as to cover the entire range of the data, the labor involved in computation is, in most cases, large. See Chapter VIII.

indefinite though it may be.[1] While such a procedure may sometimes give a poorer fit, if tested by mathematical criteria based merely on the data covered by the graduation, it will tend to give a distinctly more rational fit. It will not allow any mere irregularity of the particular sample to determine the position of the graduation in the end regions.

Professor Whittaker stresses the fact that in obtaining the graduation all observations are used. The position of each datum point affects the position of every point on the smooth curve. While this is theoretically true, it is not of great practical importance. The alteration in the shape of the smooth curve at any point which would be caused by a change in a far distant datum point is negligible. The effect upon the smooth curve of changing a datum point is most pronounced in the immediate vicinity of the datum point. The results of such a change are less and less as the distance from the datum point increases. Moreover, it is highly desirable that such should be the case. It would be highly undesirable that a change in the position of a datum point should seriously affect the position of distant parts of the smooth

[1] One of the reasons why the Henderson graduations of the Call Money data, January 1886 to January 1894, look a little different from the other graduations at the ends is that, in computing the other graduations, data outside of the range, January 1886 to January 1894, were used. See Appendix VIII.

curve. For example, one of the great disadvantages of harmonic analysis is that the configuration of the data in one section may seriously affect the shape of the fitted curve in a far distant section. The Whittaker-Henderson graduation is free from this objection.

The Whittaker-Henderson graduation does not necessarily eliminate seasonal fluctuations. Only if n be taken sufficiently large will the percentage of seasonal fluctuation remaining be negligible. For example, if the seasonal fluctuation were a 12-months sine curve and if n equaled 5, over 95 per cent of the seasonal fluctuation would be eliminated. However, if n equaled 4, less than 89 per cent would be eliminated, and if n equaled 3, less than 69 per cent.

If k equal infinity (n equal zero) the graduation will coincide with the data no matter how irregular they may be. For all other values of k, the graduation "dampens" the data, no matter how smooth such data may be—with certain parabolic exceptions.[1]

If a Whittaker-Henderson graduation be applied to a regular and indefinitely extended sine series with complete period M the resulting smooth

[1] Strictly speaking a Whittaker-Henderson graduation in which n does not equal zero will not fall exactly on even a mathematical curve unless that curve be a second-degree parabola or a straight line.

curve will be a sine curve whose amplitude will be the amplitude of the original sine curve divided by

$$1 + \frac{4n\ (n+1)^3\ (n+2)^3\ (n+3)}{(2n+3)^2} \sin^6 \frac{\pi}{M}$$ [1]

The dampening effect on sine curves of various periods when n = 3, 4, or 5 may be seen from columns 25, 26 and 27 of the table in Appendix VII. It will be seen from that table that if we wish to eliminate the major portion of a 12-months sine seasonal we must make n as large as 5. Now if n = 5 all cycles whose period is less than 36 months are very inadequately fitted. A comparison of the figures for the 43-term graduation in column 24 of the same table will illustrate the point better than any lengthy discussion. The Whittaker-Henderson graduation is not particularly well adapted to saving the major portion of the amplitude of both short and long cycles, and at the same time eliminating seasonal fluctuations. This may be an important consideration in graduating such a series as monthly Call Money Rates.

Some of the purely practical disadvantages of the method have already been suggested. The computation is laborious—when account is taken of the difficulty of obtaining three accurate end points to begin *the first half* and the very considerable

[1] See Robert Henderson—*Discussion*—Transactions of The Actuarial Society of America, pages 306 and 307.

chance of error at all stages of the work. In using Method A, the computer does not have automatic checks as he proceeds with the work; if he use Method B, the advantage of checks is somewhat offset by the increased amount of computation involved. Upon the discovery of an error, the shortest procedure is usually to do the whole job over again. The graduation cannot be extended to include new data without re-calculating a large part of *the second half*—theoretically the whole of *the second half*. However, none of the above criticisms should be interpreted as in any way suggesting that the Henderson method is not a really superlative achievement in the theory and practice of graduation.

I wish to take this opportunity to thank Mr. Robert Henderson for his great kindness and courtesy in explaining his method of computation, answering numerous questions, and lending me illustrations of actual fittings.