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# 6 International Portfolio Diversification: Short-Term Financial Assets and Gold

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## 6.1 Introduction

During the last decade, there has been a relaxation of international capital controls, a dramatic expansion in the volume of trading in international financial markets, and, more generally, an increase in international financial integration. The period of generalized floating has also been associated with a significant increase in uncertainty about exchange rates, interest rates, and prices. In this context, our paper discusses international portfolio selection by individuals, firms, and government agencies. Specifically, we present a model of optimal portfolio diversification by risk-averse agents who consume goods produced in various countries. They are able to continuously reshuffle the composition of their wealth, which is held in assets with known nominal interest rates denominated in different currencies. Given risk and consumption preferences and uncertainty about the prices of goods, prices of assets, and exchange rates, this model indicates the optimal combination of assets chosen by an agent who wishes to maximize returns and minimize fluctuations in the purchasing power of his portfolio. The theory of international portfolio diversification thus explains how risk-averse investors may reduce uncertainty about real rates of return and provides a way of understanding the portfolio behavior of a given agent with an international horizon.

This analysis is an extension of the classic mean-variance framework of Tobin (1965). When continuous trading is possible, Merton (1971) spelled out the conditions under which intertemporal maximization of expected util-

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ity would allow the separation of the portfolio rule from the consumption rule. In particular, he showed that, if asset prices are generated by stationary and lognormally distributed continuous-time stochastic processes (geometric Brownian motion) and if the instantaneous utility function of the agent is homothetic with constant relative risk aversion, a time-invariant portfolio rule could be derived. Furthermore, this rule would be the same as the one obtained if the agent was maximizing period by period a linear function of mean real return and the variance of return.

There have been several applications of the Tobin-Merton framework to international finance. Most have been surveyed out by Branson and Henderson (1984). In section 6.2, we present a version of the model developed in Meerscham (1983) which allows the international investor to hold assets with uncertain prices, such as gold. This generalizes the currency diversification rules derived by Kouri and Macedo (1978) and Macedo (1979, 1983). Section 6.3 adds gold to the optimal portfolios of short-term financial assets analyzed by Goldstein (1983). Using monthly data and quarterly holding periods from April 1973 to March 1981, the evolution of optimal portfolios is discussed. The conclusion summarizes the main results.

## **6.2 Optimal Portfolio Rules**

In this section, we present the optimal diversification rule for an agent who consumes fixed proportions of  $N$  composite goods produced in  $N$  countries and who holds a portfolio (that can be continuously reshuffled) of  $M$  assets with known nominal returns in domestic currency. The prices of the  $N$  goods, the prices of the  $M$  assets, and the  $N - 1$  exchange rates are uncertain and are specified as continuous stochastic processes. As a result, real wealth accumulation, equal to the difference between the real rate of return on the portfolio and the rate of real consumption, is described by a stochastic differential equation. Given this flow budget constraint, the agent chooses at each moment in time a portfolio of assets and a consumption bundle. The optimal portfolio rule is thus one of the outcomes of the intertemporal constrained maximization of the expected utility of consumption from time 0 to time  $T$ .<sup>1</sup> Since we are interested in the problem of an individual agent rather than in the determination of goods and assets prices and exchange rates in general equilibrium, we assume that prices are exogenous.<sup>2</sup> We specify prices in terms of the numeraire (arbitrarily defined as

1. A constant discount rate could easily be introduced. For a variable discount rate and infinite time horizon, see Nairay (1981).

2. For an endogenous determination of these processes, see Nairay (1981). Simplified applications to international finance are in Bortz (1982) and Stulz (1982).

the currency of country  $N$ ) as stationary and lognormally distributed stochastic processes.<sup>3</sup> Then, for  $M = N$  and  $i = 1, \dots, N$ , we have

$$(1) \quad \frac{dG_i}{G_i} = \pi_i dt + \sigma_i dz_i$$

$$\frac{dP_i}{P_i} = \mu_i dt + \delta_i du_i,$$

where  $G_i$  is the price of the asset  $i$  expressed in terms of the numeraire, so that  $G_i = G_i^d/S_i$  and  $G_N = G_N^d$ ,  $G_i^d$  being the domestic currency price of asset  $i$  and  $S_i$  the price of currency  $i$  in terms of the numeraire;  $P_i$  is the price of the good produced in country  $i$  expressed in terms of the numeraire, so that  $P_i = P_i^d/S_i$  and  $P_N = P_N^d$ ,  $P_i^d$  being the domestic currency price of the good;  $\pi_i(\mu_i)$  is the instantaneous conditional mean proportional change per unit of time of  $G_i(P_i)$ ;  $\sigma_i^2(\delta_i^2)$  is the instantaneous conditional variance per unit of time of  $G_i(P_i)$ ,  $\sigma_{ij}$ ,  $\delta_{ij}$ ,  $\theta_{ij}$  being the instantaneous conditional covariances per unit of time between  $G_i$  and  $G_j$ ,  $P_i$  and  $P_j$ , and  $G_i$  and  $P_j$ , respectively; and  $dz_i$  and  $du_i$  are Wiener processes with zero mean and unit variance, and instantaneous correlation coefficients  $\rho_{ij}$  (between  $dz_i$  and  $dz_j$ ) and  $\bar{\rho}_{ii}$  (between  $dz_i$  and  $du_i$ ).

It is convenient to measure (positive or negative) asset holdings as a proportion of real wealth,  $W$ . The share of wealth held in asset  $i$  is defined as

$$(2) \quad x_i = \frac{N_i Q_i}{W}, \quad i = 1, \dots, N;$$

where  $N_i$  are the holdings of asset  $i$  and  $Q_i = G_i \prod_j^N P_j^{\alpha_j}$  is the purchasing power of asset  $i$  over the  $N$  goods,<sup>4</sup>  $\alpha_j$  being the share of good  $j$  in total expenditure.

Utility is a strictly concave function of the instantaneous rate of consumption  $X_j$  of the  $N$  goods, constant expenditure share  $\alpha_j$ , and constant relative risk aversion  $1 - \gamma$ . Given the state of the system, described by real wealth, we use the method of dynamic stochastic programming in order to find the optimal paths of the control variables  $x_i$  and  $X_j$ . Hence, we define the value function,

$$(3) \quad J(W) = \max E_t \int_t^T \frac{1}{\gamma} \prod_j^N X_j(\tau)^{\alpha_j \gamma} d\tau,$$

3. More general exogenous processes are used in Macedo (1983), Macedo, Golstein, and Meerscham (1982), henceforth MGM, and Meerscham (1983).

4. The purchasing power of a currency is the optimal price index when the indirect utility functions are separable. See more on the concept in Kouri and Macedo (1978) and Macedo (1982). Work with more general utility functions has been done by Stulz (1980).

where  $E_t$  denotes the expectation conditional upon information available at time  $t$ . From intertemporal utility maximization subject to the wealth accumulation constraint and the unity constraint on asset shares, we obtain first-order conditions from which the consumption and portfolio rules can be derived.<sup>5</sup> Stacking the  $M$  first-order conditions on portfolio shares, we obtain

$$(4) \quad r + (1 - \gamma)\Theta\alpha - (1 - \gamma)Gx - (\lambda/J_w)e = \underline{0},$$

where  $r$  is the vector of real returns;  $1 - \gamma = -(\delta^2 J/\delta W^2)(W\delta J/\delta W)$ ;  $\alpha$  is the vector of expenditure shares;  $x$  is the vector of portfolio shares;  $\lambda$  is the Lagrange multiplier;  $J_w = W\delta J/\delta W$ ;  $e$  is an  $N$  column vector of ones;  $\underline{0}$  is an  $N$  column vector of zeros;  $G = \{\sigma_{ij}\}$  is the  $N$ -by- $N$  variance-covariance matrix of changes in asset prices expressed in terms of the numeraire; and  $\Theta = \{\theta_{ij}\}$  is the  $N$ -by- $N$  covariance matrix of changes in asset prices and changes in goods prices both expressed in terms of the numeraire. Note that the expected real return on each asset is obtained by adding the expected proportional change in the purchasing power of the asset to its known nominal return in domestic currency:  $r_i = R_i + dQ_i/Q_i$ ,  $i = 1, \dots, N$ .

Using the unity constraint on the portfolio shares (multiplied by  $\gamma - 1$ ), we augment (4) by another row, to get

$$(5) \quad \begin{bmatrix} x \\ \lambda/J_w(1 - \gamma) \end{bmatrix} = \begin{bmatrix} G & e \\ e' & 0 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \Theta & \underline{0} \\ \underline{0}' & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ 0 \end{bmatrix} + \frac{1}{1 - \gamma} \begin{bmatrix} r \\ 1 - \gamma \end{bmatrix} \right\}.$$

Now we invert the augmented  $G$  matrix in (5):

$$(5') \quad \begin{bmatrix} G & e \\ e' & 0 \end{bmatrix}^{-1} = \begin{bmatrix} G^{-1}K & y \\ y' & -1/e'G^{-1}e \end{bmatrix},$$

where  $y = G^{-1}e/e'G^{-1}e$  and  $K = I - ey'$ ,  $I$  being the identity matrix of order  $N$ . Omitting the  $N + 1$  row (which is the definition of  $\lambda$ ) and substituting (5') into (5), we obtain an expression for the vector of  $N$  optimal portfolio shares:

$$(6) \quad x = y + G^{-1}K\Theta\alpha + \frac{1}{1 - \gamma} G^{-1}Kr.$$

The optimal portfolio decomposes into a capital position  $y$ , such that  $e'y = 1$ , and two zero-net-worth portfolios. The latter are constructed by comparing the mean and variance of the real return on the particular asset

5. The derivations are in MGM.

(respectively involving  $r$  and  $\Theta\alpha$ ) with the mean and variance of the real return on the capital position. This is done through the ‘‘comparison matrix’’  $K$ , such that  $e'G^{-1}K = 0'$ . We refer to  $y + G^{-1}K\Theta\alpha$  as the minimum variance portfolio,  $x^m$ , and to  $G^{-1}Kr/1 - \gamma$  as the speculative portfolio,  $x^s$ .<sup>6</sup>

To interpret (6) further, it is convenient to decompose the  $N$ -by- $N$  variance-covariance matrix of changes in numeraire prices of assets ( $G$ ) and the  $N$ -by- $N$  covariance matrix of changes in numeraire prices of assets and goods ( $\Theta$ ), namely,

$$(7) \quad \begin{aligned} G &= G^d + \hat{S} - \hat{E} - \hat{E}' \\ \Theta &= H + \hat{S} - \hat{E} - \hat{\Psi}, \end{aligned}$$

where  $G^d = \{g_{ij}\}$  is the  $N$ -by- $N$  variance covariance matrix of changes in the domestic currency price of assets;  $\hat{S}$  is  $S = \{\hat{\sigma}_{ij}\}$ , the  $N-1$ -by- $N-1$  variance-covariance matrix of exchange rate changes, bordered by zeros;  $\hat{E}$  is  $E = \{\epsilon_{ij}\}$ , the  $N$ -by- $N - 1$  covariance matrix between changes in domestic currency prices of assets and bilateral exchange rates, augmented by a column vector of zeros;  $H = \{\eta_{ij}\}$  is the  $N$ -by- $N$  covariance matrix of changes in domestic currency prices of assets and goods, and  $\hat{\Psi}$  is  $\psi = \{\psi_{ij}\}$  the  $N-1$ -by- $N$  covariance matrix between changes in bilateral exchange rates and domestic goods prices, augmented by a row vector of zeros.

Next consider the case where the  $N$ th asset has a known domestic currency price, so that it is essentially a short bond or deposit denominated in the numeraire currency. The  $G$  and  $\Theta$  matrix can then be rewritten as

$$(8) \quad G = \left[ \begin{array}{c|c} G & 0 \\ \hline 0' & 0 \end{array} \right], \quad \Theta = \left[ \begin{array}{c} \Theta \\ \hline 0' \end{array} \right],$$

where  $0$  is a  $N - 1$  column vector of zeros.

Substituting (8) into (4), the last row becomes

$$(9) \quad r_N + \lambda/J_w = 0.$$

Using (9) to eliminate  $\lambda/J_w$  from (5), we now solve for  $x$ , the  $N - 1$  column vector of portfolio shares:

6. Kouri (1975) referred to the ‘‘hedging demand for forward exchange which is proportional to the value of imported goods consumed’’ and to the ‘‘speculative demand’’ in a two-country model where national investors have different preferences. The decomposition between minimum variance and speculative portfolios for the international investor holding  $N$  currencies when prices and exchange rates are lognormally distributed is in Kouri and Macedo (1978). Equation (6) is written out in full in the Appendix.

$$(10) \quad \underline{x} = \underline{G}^{-1}\Theta\alpha + \frac{1}{1-\gamma}\underline{G}^{-1}(\underline{r} - \underline{e}r_N),$$

where  $\underline{r} = (r_1 \dots r_{N-1})'$  and  $\underline{e}$  is a  $N - 1$  column vector of ones. To obtain  $x_N$ , we use the unity constraint:

$$(10') \quad x_N = 1 - \underline{e}'\underline{x}.$$

Denoting the identity matrix of order  $N - 1$  by  $\underline{I}$ , the rule for the  $N$  assets is then written as

$$(11) \quad x = \Gamma\alpha + \frac{1}{1-\gamma}\Sigma r,$$

$$\text{where } \Gamma = \begin{bmatrix} \underline{G}^{-1}\Theta \\ \underline{e}'(\underline{I} - \underline{G}^{-1}\Theta) \end{bmatrix}$$

is such that  $\underline{e}'\Gamma = \underline{e}'$ , and

$$\Sigma = \left[ \begin{array}{c|c} \underline{G}^{-1} & -\underline{G}^{-1}\underline{e} \\ \hline -\underline{e}'\underline{G}^{-1} & \underline{e}'\underline{G}^{-1}\underline{e} \end{array} \right]$$

is such that  $\underline{e}'\Sigma = \underline{0}'$  and  $\Sigma e = \underline{0}$ .

Comparing (6) to (11), it is clear that when one asset has a known price in terms of the numeraire the structure of the minimum variance portfolio changes. In this case,  $\Gamma\alpha$  cannot be decomposed into a capital position depending on asset price uncertainty and a zero-net-worth hedge portfolio determined by the covariance of changes in assets and goods prices in terms of the numeraire, weighted by preferences ( $\underline{G}^{-1}K\Theta\alpha$ ). Also, the zero-net-worth speculative portfolio is computed in terms of real returns relative to the  $N$ th asset ( $\Sigma r / (1 - \gamma)$ ) rather than relative to the capital position ( $\underline{G}^{-1}Kr / (1 - \gamma)$ ).<sup>7</sup>

When all asset prices are known,  $G^d$ ,  $E$ , and  $H$  in (7) vanish and the  $\underline{G}$  and  $\Theta$  matrices can be written as

$$(7') \quad \begin{aligned} \underline{G} &= S \\ \Theta &= \tilde{S} - \Psi \end{aligned}$$

where

$$\tilde{S} = [S \mid \underline{0}].$$

7. Note that, by Itô's lemma, mean real return differentials depend on the variance of the exchange rate as well as on the covariance of prices and exchange rates, weighted by  $\alpha$ . This implies that  $\partial x_i / \partial \alpha_i > 0$  if  $\gamma < 0$ , that is to say the individual is more risk averse than the Bernouilli investor. See references in Macedo (1982).

The  $\Sigma$  matrix used to weight real returns in (11) now becomes the augmented inverse of the variance-covariance matrix of exchange rate changes. The  $\Gamma$  matrix used to weight consumption preferences in (11) decomposes further, so that the minimum variance portfolio for the  $N - 1$  assets can be written as

$$(12) \quad x^m = (\hat{I} - S^{-1}\Psi)\alpha,$$

where  $\hat{I} = S^{-1}\hat{S}$  is the  $(N - 1$  by  $N)$  matrix obtained by augmenting  $I$  by an  $N - 1$  column vector of zeros.

Using the unity constraint to obtain  $x_N$  we can express the total portfolio and its components as

$$(13) \quad x = (I - \Phi)\alpha + \frac{1}{1 - \gamma} \Sigma r$$

$$(13a) \quad x^m = (I - \Phi)\alpha$$

$$\text{where } \Phi = \begin{bmatrix} S^{-1}\Psi \\ -e'S^{-1}\Psi \end{bmatrix}$$

is such that  $e'\Phi = \underline{0}'$ ;

$$(13b) \quad x^s = \frac{1}{1 - \gamma} \Sigma r.$$

It is clear from (13a) that the capital position is given by the expenditure shares so that the minimum variance portfolio reduces to  $\alpha$  when goods prices are known.<sup>8</sup> Also, we again have the two zero-net-worth portfolios of (6), one hedging against changes in domestic currency prices of goods and in exchange rates ( $-\Phi\alpha$ ), the other,  $x^s$ , based on real returns relative to the  $N$ th currency.

Consider now the special case of purchasing power parity. In that case there are no relative price changes, so that there is only one random domestic currency good price, say in the  $N$ th currency, and  $P_i = P_N$  for all  $i$  in (1) above. Then the  $\Theta$  matrix in (7') can be expressed as

$$(7'') \quad \Theta = -\Psi_N e',$$

where  $\Psi_N$  is the  $N$ th column of  $\Psi$ . Using (7'') in the minimum variance portfolio, we see that preferences drop out and that the capital position is all in the  $N$ th asset:<sup>9</sup>

$$(14) \quad x^m = \underline{1}_N - \Phi_N$$

8. This result is emphasized by Adler and Dumas (1982).

9. In Kouri (1977), the assumption of purchasing power parity and no inflation in the  $N$ th country eliminates hedging so that the minimum variance portfolio is all in the  $N$ th currency,  $x^m = \underline{1}_N$ . A similar result holds in the model of Solnik (1973).



where  $\underline{1}_N$  is an  $N$  column vector with zeros in the first  $N - 1$  rows and one in the  $N$ th row, and  $\Phi_N = \begin{bmatrix} S^{-1} \\ -e'S^{-1} \end{bmatrix} \Psi_N$ .

The rule in (14) is applicable to the case where  $P_j^d = P_i S_j$  is the only random price and  $\Theta = -\Psi_j e'$  and also to an investor who only consumes the  $j$ th good because then  $I - \Phi$  reduces to  $\underline{1}_j - \Phi_j$ .<sup>10</sup>

Finally, consider the problem of the investor who holds currencies and one asset with an uncertain price in terms of the numeraire. In this case, a rule in the form of (13) still applies. This is shown in the Appendix. The reason for this equivalence is that the asset with an uncertain price has the same effect on the portfolio rule as the currency of a country whose good is not consumed by the investor. Henceforth, we will interpret the portfolio rule in (13) as an  $N + 1$  rule, where the first element in  $x$  is the gold share and  $\alpha$  has a zero in the first row.

### 6.3 Optimal Portfolios of Short-Term Financial Assets and Gold Computed

#### 6.3.1 Overview

In this section, we apply the  $N + 1$  time-invariant portfolio rule derived in section 6.2 to investors holding gold and short-term financial assets (with 3-month maturities) denominated in eight major currencies; the United States dollar (\$), used as the numeraire currency, the Canadian dollar (C\$), the French franc (FF), the German mark (DM), the Italian lira (IL), the Japanese yen (¥), the Swiss franc (SF), and the pound sterling (£). Interest rates and the domestic currency prices of these short-term financial assets are assumed known. Gold (GO), in turn, is a non-interest-bearing asset with an uncertain domestic price,  $G_0$ , which is expressed in terms of the numeraire. The  $N - 1$  bilateral exchange rates,  $S_i$ , are defined in (1) as units of domestic currency per dollar. It is convenient to express the price of gold in ounces per dollar or as  $1/G_0$ . As defined above, real returns are equal to the known interest rate plus the proportional rate of change of the purchasing power of the currency (or of gold) over the previous 3 months. Investors are assumed to have static expectations about the rate of change of exchange rates, the price of gold, and numeraire prices of the goods in their consumption basket.

In section 6.2, we assumed that the investor consumes a basket composed

10. If exchange rate changes are typically not passed on to prices, (14) is the relevant rule, making  $\phi_N = \phi_{CPI} \beta$  where  $\phi_{CPI}$  captures the covariance between exchange rates and the components of the  $N$ th country's CPI and  $\beta$  are the CPI weights as in Macedo (1982). See Branson and Henderson (1984).

of goods produced in the various countries with weights given by constant expenditure shares  $\alpha_j$ . We refer to these goods by the country name: Canada (CA), France (FR), Germany (GE), Italy (IT), Japan (JA), Switzerland (SZ), the United Kingdom (UK), and the United States (US). For empirical purposes, however, we identify each one of these national goods with the consumer price index of the country in question. As a consequence of this simplification, we refer to an investor consuming only the goods included in the consumer price index of, say, Germany as the "German investor" even though the German CPI includes imported goods. In terms of the utility function in (5) above, the "national investor" of country  $j$  is defined as having  $\alpha_j = 1$  and  $\alpha_i = 0$  for  $i \neq j$ . This contrasts with the "international consumer-investor" who weights national consumer price indexes by the share of each country in total trade and can thus be thought of as a weighted average of national investors.<sup>11</sup> The role of preferences in optimal portfolios is shown by comparing different national investors to the international investor.

In section 6.3.2, we focus on optimal portfolios and their determinants for the 3-month holding period, April 1–June 30, 1981. We refer to these as March 1981 portfolios given that, with the exception of known interest rates, real returns and their distribution are computed on the basis of data available prior to the holding period, that is, April 1973–March 1981. The use of all available data since April 1973 in the computation of the matrix of covariances between exchange rates and price changes and the variance-covariance matrix of exchange rate changes is motivated in part by the results of tests for the stationarity of these variance-covariance structures.<sup>12</sup>

The analysis of the joint distribution of the uncertain component of real returns which determines the composition of optimal portfolios is interesting because it offers a convenient summary of assets' risk and return characteristics and indicates the scope for risk-reducing diversification. In addition, these determinants serve to depict explicitly the substitutability and complementarity relationships between assets. We emphasize the total portfolios of United States and international investors, but the total portfolios of other

11. These weights are given as the simple average of the dollar value of imports and exports of the eight countries. The United States dollar share is 25%, which makes the comparison of the United States investor (with a share of 100% in the United States consumer price index) to the international investor particularly instructive in attempts at bracketing the dollar share in optimal portfolios. See a discussion of weighting schemes in Macedo (1982).

12. Results in Goldstein (1983, chap. 3) show that the structure of the  $S^{-1}\Psi$  matrix has remained significantly the same since the widespread introduction of floating exchange rates. This implies that the best forecast of the determinants of the inflation hedge portfolio is based on all available data since April 1973 to the period immediately prior to the chosen holding period. Tests for the stationarity of the variance-covariance matrix of exchange rate changes used in the calculation of the speculative portfolio are less conclusive. This may imply that shorter sample periods should be used in the estimation of the joint distribution of returns in order to avoid the bias which would result if there has been structural change in the parameters of the stochastic processes generating exchange rate changes. These results exclude consideration of gold.

national investors can easily be calculated. This is done by adding the computed speculative portfolio which does not depend on consumption preferences to the national investor's minimum variance portfolio. Section 6.3.3 investigates the evolution of these portfolios since September 1974 as investors revise their estimates of variances and covariances at the end of every quarter by including the new observations on the risk and return characteristics of each asset.

### 6.3.2 Optimal Portfolios and Their Determinants, March 1981

In table 6.1, we present the pattern of correlations and covariances between exchange rate (and gold price) changes which underlies the computation of the speculative portfolio as well as the computation of the minimum variance portfolios of different investors. The upper triangular matrix reports estimates of the  $S$  matrix (including the price of gold). Since mean changes in exchange rates are expressed in number per quarter, we multiply their variances and covariances by 100 and refer to the units as percentages. Since variances and covariances are not directly comparable (because the variables have different means), correlation coefficients are reported in the lower triangle. It is clear from the table that the correlation coefficients between "Ecu area" currencies—including the Swiss franc but excluding the pound sterling—are uniformly higher than all other correlation coefficients. The lowest of the Ecu area correlations, between the lira and the Swiss franc, is 0.5. The table also shows that the correlation coefficients between the Canadian dollar and the other currencies are the lowest (and negative). Between these two extremes, we find the correlation coefficients of gold, the yen, and the pound with the other currencies. The highest variance is the variance of the price of gold. On the other hand, the Canadian dollar ranks lowest in variance of dollar exchange rate changes. The two "hard currencies" of Europe (DM and SF) exhibit a somewhat higher variance than the other currencies.

As was mentioned in section 6.2, the speculative portfolio is based on the inverse of  $S$ , each element of which shows the effect of change in the return differential relative to the United States dollar on the speculative demand of all investors for a particular currency or gold. Therefore, the elements of  $S^{-1}$  provide estimates on the degree of substitutability (negative entries) and complementarity (positive entries) between assets. For an investor with unitary risk aversion ( $\gamma = 0$ ), the own and cross effects of an increase in the real return of a given asset on speculative shares are obtained by augmenting  $S^{-1}$  by a row (column) equal to minus the sum of the elements of all other columns (rows). The resulting matrix, which we denoted above by  $\Sigma$ , is reported in table 6.2 using an ordering of the assets which emphasizes the strength of the substitutability (–) and complementarity (+) relationships between assets.

It is clear from table 6.2 that, in addition to the strong substitutability

**Table 6.1 Exchange Rates and Gold: Covariances and Correlations (April 1973–March 1981)**

	GO (ounces/\$)	C\$ (Canadian dollars/\$)	FF (French francs/\$)	DM (DM/\$)	IL (Lira/\$)	¥ (Yen/\$)	SF (Swiss francs/\$)	£ (pounds/\$)
GO	2.281	.032	.326	.461	.245	.042	.411	.308
C\$	.1	.044	-.016	*	.025	.020	*	-.001
FF	.4	-.1	.287	.268	.205	.190	.267	.140
DM	.5	*	.8	.377	.190	.123	.346	.149
IL	.3	-.2	.7	.6	.287	.084	.188	.143
¥	*	-.2	.4	.4	.3	.300	.184	.107
SF	.4	*	.7	.8	.5	.5	.461	.156
£	.4	-.1	.5	.5	.5	.4	.5	.257

Note: Upper-triangular matrix is  $G_o = \{\sigma_i \sigma_j \rho_{ij}\}$ , defined in equation (A.8) of the Appendix (in number per quarter squared times 100). Lower-triangular matrix reports  $\rho_{ij}$ .

\*Less than .05 in absolute value.

**Table 6.2** Own and Cross Effects (% , April 1973–March 1981)

	SF	DM	FF	IL	C\$	\$	¥	£	GO
SF	<u>2.0</u>	-1.5	-.4	*	-.3	.6	-.7	.2	-.1
DM	-1.5	<u>3.2</u>	-1.6	*	-.5	.3	.2	-.1	-.1
FF	-.4	-1.6	<u>3.7</u>	-1.2	.7	-1.0	*	-.2	*
IL	*	*	-1.2	<u>2.0</u>	.7	-1.1	.1	-.5	*
C\$	-.3	-.5	.7	.7	<u>6.7</u>	-7.4	.5	-.2	-.1
\$	.6	.3	-1.0	-1.1	-7.4	<u>9.7</u>	-1.1	-.3	.1
¥	-.7	.2	*	.1	.5	-1.1	<u>1.4</u>	-.5	.2
£	.2	-.1	-.2	-.5	-.2	-.3	-.5	<u>1.8</u>	-.2
GO	-.1	-.1	*	*	-.1	.1	.2	-.2	<u>.2</u>

Notes:  $\Sigma_0$  matrix defined by equation (A12) in the Appendix. Columns and rows may not add to zero due to rounding.

\*Less than .05%.

between the United States and the Canadian dollar and, to a lesser degree, between the Deutschmark and the Swiss franc, there are two partly overlapping currency blocs: the Ecu bloc and the dollar bloc. The criterion for a bloc is a cross effect of at least 1%. While the French franc and the lira belong to both blocs, the pound does not belong to either one, as all its cross effects are less than or equal to 0.5% in absolute value. Table 6.2 also shows that gold's own and cross effects are quite small.<sup>13</sup>

The Canadian and United States dollars' own effects far exceed those of other currencies. In the Canadian dollar's case, this is partly the result of the fact that, as noted, it exhibits the lowest variance of exchange rate changes. The high value of the United States own effect is observed here because it equals the sum of all elements of the  $S^{-1}$  matrix. In general, the own effects are much greater than the absolute value of the cross effects. One notable exception is the cross effect between United States and Canadian dollar assets which exhibits, by far, the highest degree of substitutability. A 1% increase in the real rate of return on one asset decreases the other's share in the speculative portfolio by 7.4% of the initial share (when  $\gamma = 0$ ). Contrary to the presumption in two-country models, we find that the United States dollar and German mark as well as United States dollar and Swiss franc are complements in the speculative portfolio.<sup>14</sup> Also, with the exception of the observed complementarity between the pound and the Swiss franc, the cross effects between all other European currencies are negative.

For given consumption preferences, the inflation hedge portfolio is determined by the estimates of the degree of substitutability and complementarity between assets shown in the  $\Sigma$  matrix (table 6.2), together with the covariances between changes in exchange rates and domestic currency prices of national goods. In table 6.3, we report the correlation coefficients between changes in dollar exchange rates (and in the price of gold) and national inflation rates, which we denoted in (1) above by  $\tilde{\rho}_{ij}$ . It is evident that these correlations are generally small. Note that the negative correlations in the Canadian row imply that the Canadian dollar appreciates relative to the United States dollar not only when foreign consumer prices rise but also when Canadian prices increase. Similarly, a rise in United States prices is associated with a depreciation of the French franc, the German mark, the yen, and the Swiss franc vis-à-vis the dollar. While the low values of the elements of table 6.3 (particularly the underlined ones) indicate little correlation between domestic price and exchange rate movements, they do not, by themselves, imply the rejection of the relative purchasing power parity hypothesis. We can derive the correlation coefficients which would obtain if

13. As can be seen in equation (A8) in the Appendix, if the price of gold is uncorrelated with exchange rate changes,  $G_0^{-1}$  becomes block diagonal. Further details on this case can be found in MGM.

14. See, e.g., Dornbusch (1980).

**Table 6.3** The Correlation Matrix of Exchange Rates and National Consumer Price Indices (1973:4-1981:3)

Asset	Good							
	CA	FR	GE	IT	JA	SZ	UK	US
GO	.1	-.1	-.1	-.4	-.3	*	*	-.2
C\$	-.1	-.3	-.3	-.1	-.3	-.2	*	-.2
FF	.1	.1	.2	.1	.1	.2	-.1	.1
DM	.3	.2	.2	*	*	.3	*	.2
IL	*	.1	.3	.3	.2	.2	-.1	-.1
¥	*	.2	.3	.3	.2	.4	.1	.2
SF	.2	.1	.3	.1	*	.3	.1	.2
£	*	*	*	.1	.2	.1	.1	-.2

Note:  $\psi_0$  matrix defined in equation (A7') of the Appendix.  
 \*Less than .05 in absolute value.

purchasing power parity (PPP) prevailed. In all cases, they are vastly different from those reported in table 6.3.

Note further that each vector  $-S^{-1}\Psi_i$  has a simple interpretation: it gives the shares of the  $N - 1$  currencies in the inflation hedge portfolio of the national investor of country  $i$ .<sup>15</sup> The dollar share of the inflation hedge portfolio is then obtained residually. Adding this portfolio to the expenditure share of the national investor of country  $i$  (given by a vector with one in row  $i$  and zeros elsewhere) we obtain the minimum variance portfolio of the national investor of country  $i$ . These portfolios are reported in table 6.4. Together they form what we denoted in section 6.2 as the  $I - \Phi$  matrix (expressed in percent). For example, the minimum variance portfolio of the German investor (table 6.4, col. 3) would include long positions in marks (98%), Canadian dollars (8%), French francs (5%), pounds (2%), and gold (1%), and short positions in lire (5%), United States dollars (5%), Swiss francs (3%), and yen (1%). We find that inflation risk is minimized for most national investors by holding gold, pound, French franc, and Canadian dollar assets, while borrowing in United States dollars, Swiss francs, yen, and marks.

15. Notice that each element  $ij$  of the  $S^{-1}\Psi$  matrix involves the ratio of the standard deviation of the change in the price of good  $j$  to the standard deviation of the change in the dollar exchange rate of currency  $i$ . These ratios are in the 20%-40% range for Italy, Japan, and the United Kingdom, countries with a relatively high variance of inflation, and in the 10%-20% range for the other countries. Thus, for example, when  $N = 3$  the 1, 2 element of  $S^{-1}\Psi$  would be  $\Phi_{12} = (\zeta_2/\sigma_1)R_{12}$  where  $R_{12} = \bar{\rho}_{12} - \rho_{12}\rho_{22}/1 - \rho_{12}^2$ . When gold is included, we have instead

$$R_{12} = \frac{(1 - \rho_1^2)(\bar{\rho}_{12} - \rho_1\bar{\rho}_2) - (\rho_{12} - \rho_1\rho_2)(\bar{\rho}_{22} - \rho_2\bar{\rho}_2)}{1 - \rho_1^2 - \rho_2^2 - \rho_{12}^2 + 2\rho_{12}\rho_1\rho_2}$$

where  $\rho_i(\bar{\rho}_i)$  refers to the correlation of the price of gold with exchange rate  $i$  (price of good  $j$ ).

The underlined element in each column of table 6.4 may also be interpreted as the extent to which a long position in the domestic currency of a given national investor is chosen in the construction of the inflation hedge portfolio. This is consistent with the domestic currency being a "preferred monetary habitat" and is thus only supported for those currencies whose "diagonal" element in table 6.4 is greater than 100, that is, Canada, France, and Switzerland.<sup>16</sup> Hence, a "preferred local currency habitat" may be observed as a result of the inflation-hedging portfolio provided by one's domestic money, even in the absence of transaction or information costs.

The last column of table 6.4 is of particular interest because, as noted at the end of section 6.2, if relative prices between national goods do not change, the minimum variance portfolio is invariant to consumption preferences. In this context, relative purchasing power parity would imply that uncertainty with respect to the  $N$  national goods prices collapses into uncertainty about the price of a single national good, for example, the good pro-

**Table 6.4** The Minimum Variance Portfolio of National Investors (% , April 1973–March 1981)

Currency of Holding	Home Country of Investor							
	CA	FR	GE	IT	JA	SZ	UK	US
GO	-.1	.7	.8	4.4	4.8	.1	.5	1.2
C\$	<u>105.1</u>	7.6	8.1	-4.9	21.0	6.7	8.4	8.5
				(-2.5)				
FF	3.6	<u>100.6</u>	4.6	4.0	-5.8	3.3	19.7	1.6
DM	-6.5	-3.3	<u>98.4</u>	1.6	-4.0	-10.4	-6.6	-6.0
				(2.9)	(-2.5)			
IL	2.2	.3	-5.3	<u>86.8</u>	-3.3	-.5	-.3	5.0
¥	.9	-.8	-1.3	-4.3	<u>98.5</u>	-7.2	-2.9	-.9
				(-8.4)	(94.0)			(-2.1)
SF	-.1	.3	-3.0	-1.3	8.3	<u>96.6</u>	-5.8	-3.0
					(11.8)			
£	.4	.8	2.3	-.8	-9.6	3.6	<u>95.7</u>	3.9
		(1.6)		(4.2)	(-4.1)			(5.3)
\$	-5.5	-6.2	-4.5	14.6	-9.8	1.1	-8.7	<u>89.7</u>
				(11.6)	(-13.1)			

*Notes:* Numbers in parentheses refer to the corresponding element in the  $I - \Phi$  matrix without gold (noted only when significantly different). Columns may not add to 100 due to rounding.  $\Phi_0$  matrix defined by equation (A11) in the Appendix subtracted from the identity matrix times 100.

16. This correspond to a negative "diagonal" element in the  $S^{-1}\Psi$  matrix. Using the expression in the previous footnote, we see that the "own" inflation hedge in table 6.4 of -3.4% for Switzerland corresponds to  $(\zeta/\sigma)_{SZ} = 14\%$  and  $R_{SZ} = -0.24$  (while the underlined element in table 6.3 was  $\hat{\rho}_{SZ} = 0.3$ ) and that the value of 13.2% for Italy corresponds to  $(\zeta/\sigma)_{IT} = 28\%$  and  $R_{IT} = 0.47$  ( $\hat{\rho}_{IT} = 0.3$ ).



duced in the country of the numeraire currency.<sup>17</sup> With the United States dollar chosen as the numeraire, the minimum variance portfolio of the United States investor would also be the “universal” minimum variance portfolio under purchasing power parity. The portfolio is dominated by a long position in United States dollars (90%). The United States (cum universal PPP) investor holds less than his consumption share in dollars in order to maintain an 8.5% long position in Canadian dollar assets, while mark, yen, and Swiss franc-denominated liabilities finance short-term investments in gold, French franc, lira, and pound assets.

Contrasting the last column of table 6.4 with the other columns reveals that relative price changes were important, particularly in the cases of Italy and Japan. Specifically, we find that the Japanese investor’s minimum variance portfolio differs significantly from the universal PPP portfolio. Of particular note are the sign and magnitude of positions in Canadian dollar, French franc, lira, Swiss franc, and pound sterling assets. The last row of table 6.4, which reports the residually determined shares of the United States dollar, also reflects the significance of relative price changes. Note that the 89.7% dollar share in the “universal” minimum variance portfolio stems from  $-10.3\%$  dollar share in the “universal” inflation-hedge portfolio. It is thus smaller than the dollar share in the minimum variance/inflation-hedge portfolios of all national investors, especially those of the Italian and Swiss investors. In sum, this analysis shows that, since national inflation rates are not fully anticipated and relative prices change, not even investors who consume only domestic goods (and are infinitely risk averse) will hold a portfolio consisting only of home-currency-denominated claims. Rather, national investors exploit inflation risk-minimizing gains to diversification as provided by the variance-covariance structure of exchange rate changes relative to the covariance of exchange rate and domestic price changes.

Having presented and interpreted the  $\Sigma$  and  $I - \Phi$  matrices, we are now in a position to report the components of the total portfolio computed under alternative assumptions about consumption preferences and risk aversion. This is done in table 6.5 for the United States investor (left panel) and an international investor (right panel). The speculative portfolio of the Bernoulli investor ( $\gamma = 0$ ) is reported in the center column. It is common to both investors because there is no significant difference in the speculative portfolio when computed with real rates of return relevant to the international investor compared with real returns relevant to national investors. This is a consequence of the fact that own and cross effects in the  $\Sigma$  matrix

17. If price indices in different countries were constructed using identical goods and weights, the composition of the universal PPP minimum variance portfolio would be independent of the choice of the numeraire. However, when goods and weights and hence price indexes vary by country, the universal minimum variance portfolio is determined according to the choice of the numeraire. See 10 above.

**Table 6.5 Optimal Portfolio Shares and Their Components (% , April 1973–March 1981)**

Asset	U.S. Investor				International Investor			
	Total Portfolio (1)	Minimum Variance Portfolio		Speculative Portfolio (R.A. = 1) (3)	Minimum Variance Portfolio		Total Portfolio (5)	
		Capital Position (2a)	Inflation Hedge (2b)		Inflation Hedge (4b)	Capital Position (4a)		
GO	4	0	1	3	2	0	5	
C\$	0(3)	0	8	-8(-6)	8	8	8(11)	
FF	1(1)	0	2	-1(-1)	4	12	15(15)	
DM	-5(-4)	0	-6	1(2)	-4	19	16(17)	
IL	0(-1)	0	5	-5(-6)	-1	8	2(2)	
¥	4(0)	0	-1	5(2)	-2	14	17(12)	
SF	-3(-1)	0	-3	0(2)	-1	3	2(6)	
£	3(9)	0	4	-1(3)	0	11	10(15)	
\$	96(94)	100	-10	6(4)	-6	25	25(22)	
Total	100(100)	100	0	0(0)	0	100	100(100)	

Notes: Col. 1 = col. 2a + col. 2b + col. 3. Col 5 = col. 4a + col. 4b + col. 3. Numbers in parentheses in cols. 1, 3, and 5 refer to the optimal shares when gold is excluded. Risk-aversion (R.A.) is unity (Bernouilli investor).

are far greater in magnitude than differences in national versus international investor's real rates of return. In fact, the composition of the speculative portfolio is invariant to the choice of real returns versus nominal interest rates adjusted for exchange rates changes.

As expected, the United States and international investor's minimum variance portfolios are significantly different (col. 2a + 2b vs. col. 4a + 4b). With the exception of the lira, however, we find that the sign of the difference between expenditure shares and minimum variance portfolio shares is independent of consumption preferences. For example, both United States and international investors have greater holdings of gold, Canadian dollars, and French francs than is implied by their respective capital position (i.e., inflation hedge portfolio shares greater than zero). On the other hand, the zero-net-worth inflation hedge portfolio decreases the share of mark, yen, Swiss franc, and United States dollar assets in the minimum variance portfolio.

The relationship between the minimum variance portfolio and consumption preferences can be illustrated by multiplying each element  $ij$  of the  $I - \Phi$  matrix by the ratio of the expenditure share  $j$  (column) to the minimum variance portfolio share  $i$  (row). We then obtain a matrix of elasticities of the shares of the international investor's minimum variance portfolio with respect to shares in expenditure. For example, the own elasticity for the United States dollar is 1.16. A 10% increase in the international investor's share of expenditure on United States goods would increase the dollar component of the minimum variance portfolio from 19% to 22% ( $= 19 \times 1.16$ ). Other countries with own elasticities greater than one are Germany, Italy, Japan, Switzerland, and the United Kingdom. Sizable cross elasticities with respect to an increase in the United States expenditure share are on holdings of Canadian dollars and lire.

The speculative portfolio, dependent on own and cross effects between assets and real return differentials with the United States dollar, includes long positions in United States dollars (6%), yen (5%), and gold (3%) and short positions in Canadian dollars (-8%) and lire (-5%). The relatively large positive share of the United States dollar is attributable less to its mean real return (-1.9%) than to its substitutability with Canadian dollar, French franc, and lira assets and to its complementarity with the Swiss franc. Return differentials with the dollar largely explain the attractiveness of yen assets (4% return differential) and gold (20% differential) and the short position in lire (-1.5% differential). While the return differential for the Swiss franc was the same as for the yen, its share is zero rather than 5%. The primary reason is the interaction between these assets' risk and return characteristics and gold. If gold is excluded from the portfolio the shares are the same. The high degree of substitutability between the Canadian and United States dollars is reflected by the fact that a relatively small difference

in mean real returns results in a long position in United States dollar assets financed by Canadian dollar liabilities.

The total portfolios of the international and United States investors are computed under the assumption of unitary risk aversion. Of course, the higher the degree of risk aversion, the smaller the contribution of the speculative to the total portfolio. At the limit, when risk aversion is infinite, the speculative portfolio disappears so that the minimum variance and total portfolios are the same and optimal shares are independent of returns. It is clear from table 6.5, column 5, that the total portfolio of the international investor is dominated by the minimum variance portfolio. The long positions of gold, marks, yen, and United States dollars in the latter are reinforced by the speculative portfolio.

We now analyze the effect of excluding gold from the available menu of assets, reported in parentheses in tables 6.4 and 6.5. The elements of the  $\Sigma$  matrix are not sensitive to the exclusion of gold, as expected from the low own effect in table 6.2. We note from table 6.3 that the price of gold has the largest correlation with the Italian and Japanese consumer price indexes (respectively,  $-0.4$  and  $-0.3$ ). Accordingly, the exclusion of gold results in significant changes in the minimum variance portfolio of the Italian and Japanese investors (col. 4 and 5 of table 6.4). These differences do not affect the international investor, however, as can be seen in column 4b of table 6.5, while the last column of table 6.4 suggested little change in the United States (cum universal PPP) investor minimum variance portfolio.

In fact, larger effects can be seen in the speculative portfolio. Excluding gold, the asset with the highest mean return, leads to an increase in the share of the mark, the Swiss franc, the pound, the Canadian dollar, and the French franc totaling 13% (to 49%) and a decline of the share of the yen and the dollar totaling 8% (to 39%), the difference being the (5%) share in gold. These shifts illustrate the interaction of the change in the variance-covariance structure and of the change in return differentials on the speculative portfolio, a topic to which we return at the end of the next subsection.

### 6.3.3 The Evolution of Optimal Portfolios over Time

Table 6.5 reported March 1981 minimum variance speculative and total optimal portfolios calculated with data from the full sample period. It must be emphasized that these portfolios are period specific, that is, optimal for a single holding period. If, as we have assumed, the variance-covariance structure generating  $\Sigma$  and  $\Phi$  were stationary and, in addition, investors had perfect knowledge of these true underlying structures, the inflation hedge portfolio would not change over time and speculative portfolios would change only as a consequence of changes in real returns. In the absence of such knowledge, investors must compute sample moments from observed exchange rate and price data and utilize these statistics to infer their true

values. That is, investors' expectations regarding the determinants of optimal portfolios change over time as their information set is enlarged by the availability of new data. Thus, optimal inflation hedge and speculative portfolios may be reshuffled as investors improve their estimates of both variance-covariance structures and real returns.

In the preceding subsection, optimal portfolios and their determinants were calculated with data from April 1973 to March 1981 for the April 1–June 30, 1981 holding period. In this subsection, we study the evolution of these optimal portfolios for interim periods and assess whether changes in these portfolios were attributable to changes in expected real returns differentials or to changes in observed variance-covariance structures. It should be emphasized that in computing these portfolios we avoid the use of *ex post* data as if the information were known *ex ante*.<sup>18</sup> In other words, the information set available to an investor choosing an internationally diversified portfolio is limited to data available prior to the holding period in which optimal shares are calculated.

In table 6.6, we report the United States dollar share in the inflation hedge portfolios of the different investors as well as of the international investor. The inflation hedge portfolio share of the dollar is the minimum variance portfolio share for all but the United States and international investors. In the case of the United States investor (international investor), the minimum variance portfolio share of the dollar is obtained by adding the capital position of 100 (25) to the inflation hedge portfolio share. Movements in the United States dollar shares are implied by changes in the sum of all other inflation hedge portfolio shares since dollar shares are determined residually. It is clear from table 6.6 that the dollar shares of all investors change substantially from year to year. Some patterns, however, do emerge. Since 1978, the short positions in United States dollars of both the international (col. 9) and the German investor decline. The reduction in the Japanese investor's short position in dollars begins in 1976. The decline in the long position in dollars held by the Italian investor begins in 1977 but is reversed in 1981. This strengthening in the inflation hedge demand for the dollar (smaller short positions and larger long positions) in 1981 is evidenced in all minimum variance portfolios except those of the Swiss and the United States investors. Over the entire period, we observe like movements in the minimum variance dollar shares of the United States and international investors. Although it is only roughly reflected in table 6.6, we also found that the change over time in the share of many of the assets in the minimum variance portfolio is similar regardless of the choice of expenditure weights.

Next, we turn to table 6.7, which summarizes the evolution of the own and cross effects of changes in the rate of return on the United States dollar.

18. This procedure, developed in Goldstein (1983), is motivated by critiques of earlier finance-theoretic work on international portfolio selection along the lines of Tobin (1982).

**Table 6.6 The U.S. Dollar Share in the Inflation Hedge Portfolio of Different National Investors (%)**

From April 1973 to March of:	Investor Consuming Only the Good of								Inter- national Investor <sup>a</sup>
	Canada	France	Germany	Italy	Japan	Switzerland	U.K.	U.S.	
1975	-2	-33	-4	-28	-43	20	-70	-7	-22
1976	21	-6	-2	-6	-50	29	-33	6	-8
1977	-2	-7	-11	30	-47	-2	-9	-13	-11
1978	-1	-6	-13	23	-45	2	-20	-15	-13
1979	-9	-11	-10	15	-33	-4	-8	-17	-12
1980	-7	-8	-6	10	-13	3	-7	-14	-8
1981	-6	-6	-5	15	-10	1	-9	-10	-6

<sup>a</sup>Weighted sum of national investor's inflation hedge portfolio where weights are given by the capital position in table 6.3, col. 4a.

**Table 6.7** Cross and Own Effects with the U.S. Dollar (%) and the U.S. Dollar Share in the Speculative Portfolio

1973: April to March of	GO	CA	FR	GE	IT	JA	SZ	UK	US	Speculative Share of the U.S. Dollar
1975	.6	-33.5	-8.2	4.2	-2.5	2.7	-.9	4.5	33.0	33.0
1976	-.3	-27.8	-7.0	4.4	-4.3	.8	.9	4.0	29.4	-14.6
1977	-.4	-10.5	-2.6	1.3	-1.7	-1.5	1.9	.5	13.1	7.1
1978	-.5	-10.8	-2.5	2.2	-2.0	-2.7	1.1	.3	14.8	7.6
1979	-.2	-9.1	-1.4	.7	-1.8	-1.0	.9	-.3	12.2	10.0
1980	.0	-8.0	-1.3	.5	-1.4	-1.3	.7	.1	10.9	3.8
1981	.1	-7.4	-1.0	.3	-1.1	-1.1	.6	-.3	9.7	6.4

Specifically, this table reports the last row of  $\Sigma$ . It is determined residually so that each element of this row is minus the sum of the column elements of the inverse of the variance-covariance matrix of exchange rate (and gold price) changes. The sum of all the elements of this matrix is equal to the element in the United States column (own effect) of table 6.7. In the last column of this table we report the United States dollar share in the speculative portfolio.

Except for a slight increase in 1978, there has been a steady and substantial decline in the own effect of an increase in the real return on the United States dollar-denominated asset on its speculative share. Similarly, the magnitude of the cross effects of changes in dollar asset returns on the speculative shares of other assets has generally declined over the sample period. This pattern is most apparent in the Canadian and French columns. The reduction in the size of own and cross effects of changes in United States real returns on speculative portfolios shares is associated with the observed pattern of increased variances and covariances of exchange rate and gold price changes. Between December 1975 and March 1981, the observed variance of exchange rate changes increased for all currencies except the German mark and French franc. We also found that the own and cross effects of changes in other assets' real returns have generally declined over the sample period. The cross effects between the European currencies have exhibited the greatest stability over time, with respect to both sign and magnitude.

As noted in the preceding subsection, the elements of  $\Sigma$  indicate the degree of substitutability and complementarity between assets. We thus interpret the first eight columns of table 6.7 as reporting the evolution of the substitutability/complementarity relationships of all assets with the United States dollar. The consistently strong substitution effects between the Canadian and United States dollars, noted above, are evident in their negative signs and high absolute values. For example, in the late 1970s, they were close to 10%, showing that a 10% increase in the return on United States dollar assets decreases the speculative demand for Canadian dollars by 1%. For the pound sterling, the strong complementarity before the dramatic mid-1976 depreciation is followed by a very weak and erratic relationship. The degree of dollar-mark complementarity has significantly diminished over time. The increasing weakness in this relationship became particularly pronounced following the decline in the value of the United States dollar in late 1978.

In figure 6.1, we show the evolution of the optimal United States dollar share in the total portfolio for the Bernouilli ( $\gamma = 0$ ) United States and international investors. These shares correspond to the sum of the appropriate column of table 6.6 plus the last column of table 6.7, to which we add the capital position (100 for the United States investor and 25 for the international investor). The similarity of the evolution in these shares is apparent.



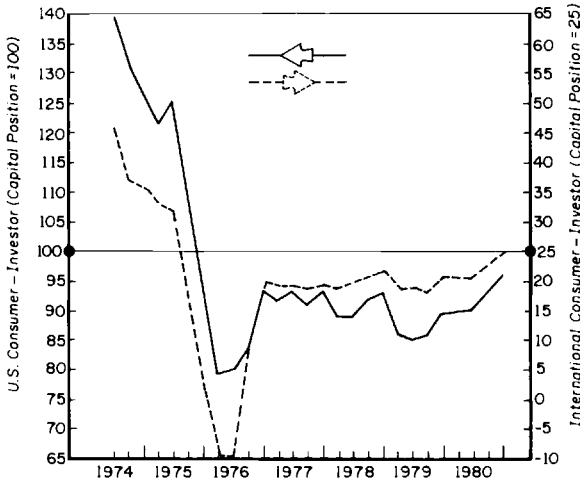


Fig. 6.1 Total optimal U.S. dollar shares (%) April 1974–March 1981

We noted above that the choice of expenditure weights did not greatly affect the sign of inflation hedge portfolio shares. This is also evident when comparing movements in the United States and international investors' minimum variance portfolios. Further, the speculative portfolio is, for empirical purposes, common to all investors regardless of their expenditure patterns.

Figure 6.1 reveals that the sharp decline in the attractiveness of the dollar between 1974 and mid-1976 was partly reversed in 1976, and that since 1977 rather stable shares obtained. Over the late 1975 to early 1977 period, both the precipitous decline and the subsequent increase in the total optimal share of the United States dollar were the result of similar movements in the speculative portfolios. In the period prior to September 1975, we found that the United States dollar held the dominant share in the speculative portfolio. After that time, no asset clearly dominated this portfolio. Finally, it should be noted that the increase in the total dollar share for both the United States and the international investor in the 1980–81 period was caused by like movements of the dollar share in both the inflation hedge portfolio (becoming less negative) and the speculative portfolio.

Table 6.8 reports expected mean real returns on both United States dollar assets and optimal portfolios computed with different degrees of relative risk aversion. It is evident that the expected mean real return on the United States dollar is consistently negative and less than the expected return on the minimum variance portfolio (and a fortiori less than the expected return on

**Table 6.8** Expected Mean Real Return on the U.S. Dollar and on the Optimal Portfolios<sup>a</sup> (% p.a.)

From April 1973 to March of:	Return on U.S. Dollar (1)	Minimum Variance Portfolio (2)	Speculative Portfolio (3)	Total Portfolio		
				$\gamma = 0$ (4a)	$\gamma = -1$ (4b)	$\gamma = 1/2$ (4c)
1975	-6.4	-2.1	3.2	1.1	-.5	4.3
1976	-2.9	-1.9	3.1	1.2	-.2	4.3
1977	-2.5	-1.4	1.3	-.1	-.7	1.2
1978	-3.7	-1.5	2.1	.6	.2	2.7
1979	-3.8	-1.0	1.3	.3	-.3	1.6
1980	-3.1	-.5	1.1	.6	.1	1.6
1981	-1.9	-.3	.9	.6	.3	1.5

Notes: Col. 4a = col. 2 + col. 3. Col. 4b = col. 2 + (1/2 × col. 3). Col. 4c = col. 2 + (2 × col. 3).

<sup>a</sup>These returns are computed for the international investor.

the speculation and total portfolios). We also found that the return on the speculative portfolio is always lower than the mean real return on gold (the lowest return on gold ranged from 7.3% to 2.3% over this period). The yield on the speculative portfolio was also less than the return on the German mark asset in all reported periods except March 1976 and March 1981. As a result, the expected return on the total portfolio for the Bernoulli investor is relatively low. The expected return on the total portfolio is even lower when we increase the degree of relative risk aversion (e.g.,  $\gamma = -1$ ).

Changes in speculative shares were, in many periods, the dominant factor in the determination of movements in the total optimal portfolio. Clearly, observed changes in the speculative portfolio were a consequence of changes both in real returns and in the inverse of the augmented variance-covariance matrix of gold price and exchange rate changes,  $\Sigma$ . In table 6.9, we report

**Table 6.9** Real Return Differentials with the U.S. Dollar (% p.a.)

From April 1973 to March of:	GO	C\$	FF	DM	IL	¥	SF	£
1975	40.7	-.7	8.9	13.9	-.2	-.9	13.7	2.1
1976	15.3	.8	4.1	5.4	-5.5	.7	8.1	-3.3
1977	12.8	.3	2.0	5.6	-3.9	2.7	6.4	-4.3
1978	14.1	-.8	3.4	7.1	-1.1	5.8	10.5	-1.1
1979	15.6	-1.3	4.2	6.8	.0	6.5	10.1	.4
1980	26.4	-.7	3.6	5.6	.2	2.5	7.4	4.1
1981	20.2	-.7	1.3	2.8	-1.4	4.2	4.2	2.4

Note: Mean real return on asset in column minus mean real return on U.S. dollar (see table 6.8).

expected real return differentials with United States dollar assets observed in March of each year from 1975 to 1981. The importance of capital gains on gold, which bears no interest, is evident. The consistently positive expected yield differentials in favor of French franc-, mark-, yen-, and Swiss franc-denominated assets are also apparent. It is interesting to note that while the expected real return differential between Canadian and United States dollar assets is low, we have observed large movements in the speculative shares of these assets in response to small changes in their return differential. This is a consequence of the high degree of substitutability between these currencies.

In table 6.10, we record the percentage of the year-over-year change in speculative portfolio shares attributable to changes in real return differentials. That is, we decompose the relative effects of changes in expected real returns and in the observed variance-covariance matrix of exchange rate (and gold price) changes on movements over time in the speculative portfolio shares of all assets. It should be emphasized that under the assumption that the variance-covariance structure of exchange rate (and gold price) changes is stationary and known with certainty by the investor, movements in speculative portfolio shares would be entirely due to changes in real returns. This would imply that investors' estimates of the true stationary  $\Sigma$  matrix are not subject to sampling error. In this case, all of the elements in table 6.10 would be 100%, indicating that changes in speculative portfolios are fully attributable to real returns. In those cases where the reported percentage is 0%–100%, changes in the observed variance-covariance structure were found to reinforce the effect of changes in the real return differentials on (positive or negative) movements in speculative shares. Alternatively, elements greater than 100% imply that changes in the observed variance-covariance structure were a countervailing influence. A negative element in the table indicates that the movement in the speculative share was dominated by changes in the observed  $\Sigma$  matrix while the countervailing influence became the change in the real return vector.

Only in 1976 and 1978 were year-over-year changes in the speculative share of the United States dollar dominated by changes in real return differentials. For example, between March 1977 and March 1978 the optimal dollar share increased by 0.5%. If the observed  $\Sigma$  matrix had remained constant over this period, however, the share of United States assets in the speculative portfolio would have increased by 2.1%. Alternatively, the March 1979 dollar share increased by 2.4% over the previous year. If the  $\Sigma$  matrix observed in March 1978 had prevailed, the dollar speculative share would have fallen by 0.5% as a consequence of increased gold, French franc, yen, and pound assets' return differentials (see table 6.9). Thus, the increase in the share of the dollar over the year was entirely the consequence of favorable changes in its substitutability-complementarity relationships with other assets. Similarly, between March 1980 and March 1981, we ob-

**Table 6.10**      **Percentage Change in Speculative Portfolio Shares Due to Changes in Real Returns**

12-Month Change in Portfolio in March of:	GO	CA	FR	GE	IT	JA	SZ	UK	US
1976	67.3	79.2	76.3	99.6	-84.8	107.5	61.8	72.4	86.4
1977	-5.1	16.1	40.0	51.9	431.5	73.4	24.6	-8.0	16.6
1978	324.9	179.0	186.3	144.0	330.4	141.3	114.9	-304.1	445.8
1979	96.0	-90.3	131.4	-23.0	-20.0	-16.2	17.5	44.3	-22.4
1980	73.9	50.1	87.7	53.5	88.4	86.4	79.4	49.1	39.8
1981	70.8	-39.2	52.0	23.9	28.1	75.8	91.7	68.0	-39.0

*Note:*  $\sum(r - r_{-12})$  as a percentage of  $(\sum r - \sum_{-12} r_{-12})$ , the change in Bernoulli investor speculative portfolio changes in the previous 12-month period.

serve a 2.7% increase in the optimal dollar share. Changes in return differentials alone would have resulted in a 1% decline in the optimal share. This effect, however, was overwhelmed by a 3.7% increase in the dollar share attributable to changes in the observed variance-covariance structure (i.e., the optimal dollar share would have increased by 3.7% if real return differentials had remained constant at their March 1980 level).

In contrast to the case of the United States, changes in German mark speculative portfolio shares were, in most periods, principally due to changes in differentials. In 1976, for example, the 14.1% drop in the optimal German mark share was entirely the consequence of changes in real return differentials (e.g., between March 1975 and March 1976, the return differential in favor of German mark assets declined from 13.9% to 5.4%). In March 1978, the 304% decline in the optimal German mark share was fully attributable to changes in the vector of real returns. In this instance, however, changes in the observed variance-covariance structure served to reduce the magnitude of this effect.

In table 6.10, we also observe a similarity in the relative contribution of changes in real return differentials across assets in a given year. That is, in 1977 and 1978, changes in the observed  $\Sigma$  matrix played a significant role in the determination of changes in most speculative shares. In comparison, in March of 1976, 1980, and 1981, movements in real return differentials were of relatively greater importance in the reshuffling of the observed speculative portfolios.

## **6.4 Conclusion**

Using a continuous-time finance-theoretic framework, section 6.2 of this paper presented the optimal portfolio rule of an international investor who consumes  $N$  national composite goods and who holds  $M$  domestic-currency-denominated assets with known nominal interest rates in an environment where prices of goods, assets, and exchange rate follow geometric Brownian motion. It is shown that the optimal portfolio decomposes into a capital position and two zero-net-worth portfolios. The derivation presents a capital position which depends only on the relative variances and covariances of changes in asset prices in terms of the numeraire. The first zero-net-worth portfolio, scaled by risk aversion, depends on a comparison of mean real return to the return on the capital position.

When the portfolio is restricted to  $N$  short-term financial assets with known prices and one asset with a random price in terms of the numeraire (e.g., gold), both the general rule of Meerscham (1983) and the currency rule derived in Macedo (1983) are applicable, as shown in the Appendix. In this case, the capital position depends only on expenditure shares. The zero-net-worth inflation hedge portfolio is determined by the covariances between exchange rate and gold price changes and by their interactions with goods

price changes. The zero-net-worth speculative portfolio depends on the risk preferences of the consumer-investor, the returns on the assets, and exchange rate and gold price changes.

Optimal portfolios are presented in section 6.3. These portfolios are based on the inflation-hedging potential provided by short-term financial assets denominated in different currencies and gold as well as on the substitutability/complementarity relationships among these assets.

In general, optimal diversification involves departures from both the "preferred monetary habitat" hypothesis, according to which portfolio shares would match expenditure shares, and the "purchasing power parity" hypothesis, according to which preferences would not affect the minimum variance portfolio. Specifically we found that the optimal portfolio of an investor who consumed goods from all major industrialized countries (according to their weight in total trade) would be dominated in March 1981 by long positions in United States dollars (25%) yen (17%), German marks (16%), French francs (15%), and pounds sterling (10%). An investor who consumed only United States goods, by contrast, would hold 96% of his optimal portfolio in United States dollars. The inflation hedge portion of this portfolio reveals that inflation risk is minimized, for both the international and United States investor would hold lire and pounds, while the international investor would borrow lire.

In March 1981, the optimal speculative portfolio, maximizing mean real returns, would include long positions in United States, German, and Japanese assets and in gold and short positions in Canadian dollars, French francs, lire, and pounds. The exclusion of gold generates substantial reshuffling in the speculative portfolio. With the exception of the pound, however, there is no change in the sign of optimal positions. The analysis of the speculative portfolio also reveals strong substitutability between United States and Canadian dollars and, to a lesser extent, between United States dollars and French francs and Italian lire. Weak complementarity relationships are observed between United States dollar and German mark and Swiss franc assets.

The analysis of the evolution of portfolios over time showed that even if the optimal portfolio rule is time invariant, optimal portfolios are not. Shares changed as expectations about the joint distribution of returns were revised. Share movements were most dramatic at the beginning of the period, and optimal positions did not begin to approach their March 1981 levels until the end of 1976.<sup>19</sup> In the case of the yen and the pound there were oscillations throughout the period. With respect to the dollar share in the optimal portfolio of the United States and international investor, we found that, in the period between late 1974 and mid-1976, a period in which the dollar is

19. Optimal portfolio shares computed with constant (two-year) sample length but different base periods exhibit even greater variability. See Goldstein (1983, chap. 5).

considered to have been strong, a large decline in its optimal share took place. This shows the importance of the variability (and the associated uncertainty) of the changes in the value of the United States dollar, even when the currency itself is strong. After the lows reached in mid-1976, the share increased again and stabilized in mid-1977 at levels well below those of before the end of 1974.

In sum, the finance-theoretic framework presented in this paper is motivated by the need to analyze the microeconomic foundations of international portfolio demands. The theory of optimal international portfolio selection highlights both the importance of risk and return considerations in the determination of the composition of a multicurrency portfolio and the potential for risk-reducing gains from diversification. Together with consumption and risk preferences, the demand for a given asset by a utility-maximizing investor is determined by its relative return as well as by its inflation-hedging potential and substitutability and complementarity relationships with assets and liabilities denominated in other currencies.

Ultimately, the analysis of market portfolio behavior will be facilitated by the relaxation of assumptions regarding the stochastic processes generating goods prices, asset prices, and exchange rates in a general rather than partial equilibrium model. The analysis of macroeconomic policy in interdependent economies also requires knowledge about the portfolio behavior of major participants in international financial markets. The theoretical and empirical framework developed in this paper provides a useful foundation for studying actual portfolio diversification across currencies by individual investors. For example, regression results in Goldstein (1984) cast doubt on the correspondence between observed United States banks' portfolio behavior and optimal international diversification. This may be explained by the estimation of cross-spectral densities between rates of return and asset positions: his results suggest that there are significant differences in portfolio demands across currencies and maturities, as well as in the portfolio behavior of smaller versus larger banks. Pursuing the line of research of Healy (1981), reserve diversification by central banks can also be studied using the model presented here.

By understanding agents' sensitivity to changes in real and nominal rates of return and to uncertainty deriving from exchange rate and price variability, one is better able to explore questions relating to the effects of monetary policy on portfolio shifts across currencies, the effectiveness of interventions policy, and the determination of exchange rates.

## Appendix

Equation (6) in the proper can be written in full as:

$$(A1) \quad x = \frac{G^{-1}e}{e'G^{-1}e} + G^{-1} \left[ I - \frac{e'e'G^{-1}}{e'G^{-1}e} \right] \Theta \alpha \\ + \frac{1}{1-\gamma} G^{-1} \left[ I - \frac{e'e'G^{-1}}{e'G^{-1}e} \right] r.$$

Rearranging terms, we see that

$$(A2) \quad x = G^{-1} \Theta \alpha + \frac{1}{1-\gamma} G^{-1} r \\ + \frac{G^{-1}e}{e'G^{-1}e} \left[ 1 - e'G^{-1} \Theta \alpha - \frac{1}{1-\gamma} e'G^{-1} r \right].$$

Since  $e'x = 1$ , a consistent solution for  $x$  is that

$$(A3) \quad e'G^{-1} \Theta \alpha + \frac{1}{1-\gamma} e'G^{-1} r = 1.$$

Using (A3) in (A2) we get the portfolio rule for the  $N - 1$  assets as

$$(A4) \quad x = G^{-1} \Theta \alpha + \frac{1}{1-\gamma} G^{-1} r.$$

Equation (28) in Meerscham (1983) shows a similar result for the case where there is one asset with an uncertain price and one asset with a certain price in each country, so that the vector of portfolio shares (excluding the numeraire currency) is  $2N - 1$ . Here we only have one asset with an uncertain price in the numeraire country, gold (subscript 0), and  $N - 1$  exchange rates (subscript 1 to  $N$ ), so that the vector of portfolio shares excluding the numeraire has dimension  $N$  and

$$(A5) \quad x_0 = G_0^{-1} \Theta_0 \alpha + \frac{1}{1-\gamma} G_0^{-1} r_0$$

where  $X_0(r_0)$  is an  $n \times 1$  column vector of portfolio shares (return differentials) obtained by adding row zero to the previous vector  $x(r)$ ;  $G_0$  is an  $N \times N$  variance-covariance matrix of exchange rates and gold price changes;  $\Theta_0$  is an  $N \times N$  covariance matrix of numeraire currency goods price changes with exchange rates and gold price changes.

Using the decomposition described in equation (7) in the text and preserving the notation introduced there, we get



$$\begin{aligned}
 \text{(A6)} \quad G_0 &= \begin{bmatrix} g^2 & \underline{0}' \\ \underline{0} & \underline{0} \end{bmatrix} + \begin{bmatrix} \underline{0} & \underline{0}' \\ \underline{0} & S \end{bmatrix} - \begin{bmatrix} \underline{0} & \underline{\epsilon}' \\ \underline{0} & \underline{0} \end{bmatrix} - \begin{bmatrix} \underline{0} & \underline{0}' \\ \underline{\epsilon} & \underline{0} \end{bmatrix} \\
 &= \begin{bmatrix} g^2 & -\underline{\epsilon}' \\ -\underline{\epsilon} & S \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{(A7)} \quad \Theta_0 &= \begin{bmatrix} \underline{\eta}' \\ \underline{0} & \underline{0} \end{bmatrix} + \begin{bmatrix} \underline{0}' & \underline{0} \\ S & \underline{0} \end{bmatrix} - \begin{bmatrix} \underline{0} & \underline{\epsilon}' \\ \underline{0} & \underline{0} \end{bmatrix} - \begin{bmatrix} \underline{0}' & \underline{0} \\ \psi & \end{bmatrix} \\
 &= \begin{bmatrix} -\underline{\epsilon}' & \underline{0} \\ S & \underline{0} \end{bmatrix} - \begin{bmatrix} -\underline{\eta}' \\ \psi \end{bmatrix} = \bar{S}_0 - \psi_0,
 \end{aligned}$$

where  $\underline{0}$  is an  $N - 1$ -by- $N - 1$  matrix of zeros,  $g$  is the variance of gold price changes,  $\underline{\epsilon}$  is an  $N - 1$  column vector of the covariance of gold price with exchange rate changes, and  $\underline{\eta}$  is an  $N$  column vector of the covariance of gold price with goods price changes. Since gold is priced in the numeraire currency it is useful to decompose the  $\psi_0$  matrix using the notation of (7'') in the text and the covariance  $\eta_N$  between the price of gold and the good of the numeraire currency:

$$\text{(A7')} \quad \psi_0 = \begin{bmatrix} -\bar{\eta}' & -\eta_N \\ \bar{\psi} & \psi_N \end{bmatrix}.$$

We now invert  $G_0$  and operate

$$\text{(A8)} \quad G_0^{-1} = \begin{bmatrix} \sigma_0 & \sigma_0 \underline{\epsilon}' S^{-1} \\ S^{-1} \underline{\epsilon} \sigma_0 & S^{-1} + \sigma_0 S^{-1} \underline{\epsilon} \underline{\epsilon}' S^{-1} \end{bmatrix},$$

where  $\sigma_0 = (g^2 - \underline{\epsilon}' S^{-1} \underline{\epsilon})^{-1}$ .

$$\text{(A9)} \quad G_0^{-1} \bar{S}_0 = \begin{bmatrix} \underline{0}' & \underline{0} \\ \underline{I} & \underline{0} \end{bmatrix} \alpha = \begin{bmatrix} \underline{0}' \\ \bar{I} \end{bmatrix} \alpha$$

$$\text{(A10)} \quad G_0^{-1} \psi_0 =$$

$$\begin{bmatrix} \sigma_0 \bar{\eta}' - \sigma_0 \underline{\epsilon}' S^{-1} \bar{\psi} & \sigma_0 \eta_N - \sigma_0 \underline{\epsilon}' S^{-1} \psi_N \\ \sigma_0 \bar{\eta}' S^{-1} \underline{\epsilon} - S^{-1} [\underline{I} + \sigma_0 \underline{\epsilon} \underline{\epsilon}' S^{-1}] \bar{\psi} & \sigma_0 \eta_N S^{-1} \underline{\epsilon} - S^{-1} [\underline{I} + \sigma_0 \underline{\epsilon} \underline{\epsilon}' S^{-1}] \psi_N \end{bmatrix}.$$

It is clear from (A9) and (A10) that these matrixes are the ones given in (12) in the text, with the first row referring to gold. The  $N + 1$  rule is therefore obtained by constructing the  $\phi_0$  and  $\Sigma_0$  matrixes which are the exact counterparts of the  $\phi$  and  $\Sigma$  matrixes defined in equations (13) and (11) of the text, respectively:

$$(A11) \quad \phi_0 = \begin{bmatrix} G_0^{-1} \psi_0 \\ -e' G_0^{-1} \psi_0 \end{bmatrix}$$

$$(A12) \quad \Sigma_0 = \left[ \begin{array}{c|c} G_0^{-1} & -G_0^{-1} e \\ \hline -e' G_0^{-1} & e' G_0^{-1} e \end{array} \right].$$

An alternative procedure, described in detail in MGM and alluded to in the text, interprets gold as the currency of a country whose good is not concerned by the international consumer inventory and hence has  $\alpha_0 = 0$  not generalize when there are several assets with uncertain prices in each country. In order to provide further insight, however, we do illustrate it for the case of gold and two currencies, modifying a similar presentation of the three-currency portfolio in Goldstein (1983, pp. 37–38). We chose currency 2 as the numeraire and denote the correlation coefficient between gold price and exchange rate changes by  $\rho$  and write

$$(A13) \quad \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_1 \end{pmatrix} - \begin{bmatrix} (\delta_1/g)l_{01} & (\delta_2/g)l_{02} \\ (\delta_1/\sigma)l_{11} & (\delta_2/\sigma)l_{12} \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \frac{1}{1 - \gamma} \frac{1}{g^2 \sigma^2 (1 - \rho^2)} \begin{bmatrix} \sigma^2 & -\rho g \\ -\rho g \sigma & g^2 \end{bmatrix} \begin{pmatrix} r_0 - r_2 \\ r_1 - r_2 \end{pmatrix},$$

where  $l_{0i} = \bar{p}_{0i} - \rho \bar{p}_{1i} / (1 - \rho^2)$

and

$$l_{1i} = \bar{p}_{1i} - \rho \bar{p}_{1i} / (1 - \rho^2),$$

for  $i = 1, 2$ . Using the unity constraint  $x_2 = 1 - x_0 - x_1$ , we get

$$(A14) \quad \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} - \begin{bmatrix} (\delta_1 l_{01}) & (\delta_2/g)l_{02} \\ (\delta_1/\sigma)l_{11} & (\delta_2/\sigma)l_{12} \\ (\delta_1/g l_{01} + \delta_1/\sigma l_{11}) & -(\delta_2/g l_{02} + \delta_2/\sigma l_{12}) \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$\begin{aligned}
 &+ \frac{1}{(1 - \gamma)(1 - \rho^2)} \\
 &+ \begin{bmatrix} 1/g^2 & -\rho/g\sigma & (\rho/\sigma - 1)g \\ -\rho/g\sigma & 1/\sigma^2 & (\rho/g - 1)\sigma \\ (\rho/\sigma - 1)g & (\rho/g - 1)\sigma & 1/g^2 + 1/\sigma^2 - 2\rho/g\sigma \end{bmatrix} \begin{pmatrix} r_0 \\ r_1 \\ r_2 \end{pmatrix}.
 \end{aligned}$$

**Comment**      Bernard Dumas

The recent work of Kouri, de Macedo, and others represents a welcome attempt by macroeconomists to use Markowitz portfolio theory as a way to determine asset demands. This attempt is long overdue; for years, macroeconomic analysis has postulated asset demand schedules without microeconomic foundations. However, it comes at a time when the Finance profession itself begins to seriously question the virtues of this approach. This reappraisal is motivated by theoretical and statistical considerations which I will review after some comments and clarifications pertaining to the specific contribution of Macedo, Goldstein, and Meerscham.

The basic separation result of mean-variance portfolio theory, formulated in a way which makes it relevant for international finance, can be stated as follows: "Every investor in the world holds a combination (with weights based on his risk tolerance) of two portfolios: the portfolio of minimum variance in real terms (weight = one minus the risk tolerance); and the portfolio which would be held by a logarithmic investor (weight = risk tolerance). Under purchasing power parity the minimum variance portfolio is the same for all investors; otherwise it is specific to each one of them,<sup>1</sup> and it depends on the commodities price index used by the investor. The logarithmic portfolio, however, is universal and is independent of the price index used."<sup>2</sup>

Against this background, I would make two observations on the paper of de Macedo et al. that appear to call their results into question. One is mostly a matter of presentation, but the other is more fundamental. First, recall that Macedo et al. present the optimal portfolio composition in terms of *three* component portfolios: the capital position, the (zero net worth) inflation

1. Solnik (1974) and Sercu (1980) examine the case where each investor takes his anticipated home inflation to be zero. In that case, the minimum variance portfolio is entirely made up of the home currency short-term asset.

2. This observation appeared first in Hakansson (1969).

hedge, and the (zero net worth) speculative portfolio. As a means of describing the optimal choices, this approach seems inferior to the separation theorem I stated above: it uses three components when only two are required by the mean-variance framework, and two of these components (the inflation hedge and the speculative portfolio) are specific to the particular national investor being considered. As a consequence of this added complexity, the picture which emerges from the numerical results being presented is not as clear as it might have been.<sup>3</sup>

The second, more fundamental observation pertains to the authors' choice of the commodities price indexes. The results are quite sensitive to this choice. De Macedo et al. compute their own brand of price index based on home and foreign consumer price indexes and the shares of the imports from the various countries, in home consumption. An alternative procedure would be simply to use the home consumer price index of the investors. Quite obviously the two types of indexes behave very differently; the computed index reflects immediately the strong impact of exchange rates on foreign goods whereas the CPI does so very little. The consequences of the choice of the index on portfolio choice are momentous. In de Macedo et al. the minimum variance portfolio composition reflects mostly the geographic origin of the goods consumed by the investor, whereas using the CPI would let practically 100% of this portfolio fall on the home short-term nominal asset,<sup>4</sup> which appears minimally risky even in real terms (for countries of residence with "reasonable" inflationary processes). A strong "home currency preference" pattern comes out of the second specification.

At least two interpretations may be offered for the de Macedo et al. procedure. The first one is the one they themselves apparently advocate: they intend to refer to an international "average" investor. This reference to the aggregate seems to be the reason for computing a weighted average of national CPIs. This seems questionable, as aggregation of national portfolio compositions should not be performed by means of trade weights, but by using weights related to national wealths. (Which could perhaps be proxied by GNPs?)

A second (and to my mind more satisfactory) interpretation would consider the de Macedo price index as a proxy for home and foreign *output prices* weighted by the shares of import in the home consumption of one particular nation. In that case, the choice between the de Macedo index and the straight CPI is not simple to make. In principle, investors are confronted with consumption prices and not with import prices directly so that the CPI

3. Numerical results in the two-fund format are given in Adler and Dumas (1982), table 5.

4. See Adler and Dumas (1982), table 5.

should be used. Yet the impact of exchange rates on import prices must somehow have an impact on asset selection.<sup>5</sup>

In short, de Macedo et al. do not provide sufficient justification for the particular price index which they use in portfolio choice as against using the straight CPI and the results of the empirical analysis are markedly affected by this choice of an index. One cannot therefore consider them as definitive.

I now come to some general remarks on the validity and testability of portfolio theory, in its specific version used here. First, let us examine the validity of the analytical derivations as they are presented in the paper. Some Ito stochastic processes are postulated for prices and exchange rates, but for empirical purposes these processes are restricted to being stationary Brownian motions, and portfolio optimization based on some time-additive von Neuman–Morgenstern objective function is then performed.<sup>6</sup> If the equilibrium resulting from these asset demands were computed, some restriction (called the capital asset pricing model) on the parameters of the various Brownian motions would typically be reached. To be rigorous, however, one should then verify that prices do follow the postulated processes. That is rarely done. But we can be more negative: we know from Lucas (1978) that the processes governing asset prices cannot be stationary Brownian motions, as risk aversion implies that rates of return must be serially correlated. Hence the specific formulation presented here is contradictory as it is not compatible with general equilibrium. What we need, therefore, is a complete theory of asset prices based on more general processes for prices, and we must await this theory before optimal portfolios are presented.<sup>7</sup> This is apparently forthcoming, since the conclusion of the paper announces that one of the authors (Meerscham) is working on such a theory and another (Goldstein) is attempting empirically to measure the nonstationarity of the postulated processes.

I now come to some remarks on the testability of portfolio theory. They will throw some light on the validity of the numerical calculations produced by de Macedo et al. At the root of most of the statistical problems is the fact that *the sample distribution of optimal portfolio compositions is not*

5. But, contra de Macedo et al., the proper vehicle for this influence is not the minimum variance fund. If exchange rate changes are not immediately reflected in the CPI, it must be because firms engaged in import trading keep a stable sales price which they do not adjust very quickly to the import price. In the ideal case, where the shares of these firms would be traded on a stock exchange and included as separate investment lines into an optimal portfolio based, as it should be, on the CPI, investors would hold these shares in their logarithmic fund and would tend to hedge them by holding some foreign currency deposits. But note that this portfolio modification takes place via the logarithmic fund which is universal and not via the inflation hedge.

6. There may be an additional assumption that the invested wealth (a variable combination of asset prices) of each investor also follows an Ito process.

7. This is not being unduly cautious. There is considerable evidence of instability in the drift and standard deviation parameters of the postulated Brownian motions. Investors would therefore hold a number of hedge funds designed to anticipate on the shifts of the various state variables. The portfolio choice picture is likely to be *qualitatively* modified by these considerations.

*known*. Consider, for instance, the logarithmic portfolio. Its optimal composition is given by the inverse of the covariance matrix of returns pre-multiplying the vector of expected returns in excess of the reference currency interest rate. There is no difficulty in obtaining efficient unbiased (in the expected value sense) estimates of the covariance matrix and the excess returns, but we have no reason to think that these estimates would yield efficient (or even unbiased) estimates of portfolio compositions. If returns are normally distributed, the inverted covariance matrix estimate is distributed according to the inverse Wishart distribution and the expected return estimates are normal, but we know nothing of their product, especially because all these estimates are generally not independent. Casual reasoning suggests that the estimated optimal portfolio compositions have a very high variance:<sup>8</sup> the optimization procedure presumably rushes headlong on the securities whose sample risk happens to be low and whose sample return happens to be large, assigning very large positive weights to them and very large negative weights to other securities, while another sample might do just the reverse. And, indeed, every study based on actual rates-of-return data which I have seen produces such “unreasonable” optimal portfolio choices (in the absence of constraints such as short-selling limitations).

Simulation studies can clarify this issue to some extent. Consider the choice between two assets: one riskless with an interest rate of 10% and one risky with an average return of 11% and a standard deviation of return of 10% so that the true optimal weight (for a logarithmic investor) to be placed on the risky asset is equal to one. Drawing six times a sample of thirty-two observations,<sup>9</sup> the average estimated optimal weight has been found to be 0.8576 with a standard deviation of this estimate equal to 1.2988. Hence for any given sample one may easily obtain an optimal weight of +2 or -1 when the true optimal weight is +1. Estimates of optimal portfolio composition are not reliable.

Perhaps even more crucial is the restricted menu of assets considered here: only gold and short-term nominal assets. The diversification motive is central to portfolio choice; as a result, adding just one asset to a prior list of eight assets is quite easily capable of drastically changing the relative weights assigned to the eight assets themselves. For so long as we do not look at a reasonably complete list of individual assets available across the world, sample calculations of optimal portfolios yield almost arbitrary results. On the other hand, of course, a reasonably complete list would entail impossible data and computation requirements.

Assuming the above two problems solved, there would remain the actual task of testing the theory (an issue which de Macedo et al. do not touch upon) before we venture to provide advice to decision makers. One way to proceed would be to aggregate the optimal portfolios of all the economic

8. This is quite apart from nonstationarity questions.

9. This is the same sample size *relative* to the number of assets as in de Macedo et al. (128 observations for eight assets).

entities and compare the result with the observed world market capitalizations of the various assets (the so-called world market portfolio). As Roll (1977) has pointed out, identifying this portfolio with sufficient precision is quite difficult. Another way to proceed, which, to my knowledge, has not been attempted, would be to confront optimal portfolio revisions computed as above with balance of payments data on international capital flows. This is a possible, if hazardous, avenue for future research.

## Comment Jeffrey A. Frankel

I would like to follow up on Bernard Dumas's call for going beyond econometric estimation of optimal portfolios, to econometric testing of the hypothesis that they in fact correspond to the actual portfolios held by investors. That this hypothesis is in doubt should be clear from the fact that the optimal portfolios estimated by Macedo, Goldstein, and Meerscham, as in the earlier study by Kouri and Macedo, prescribe negative holdings of some countries' assets. These assets could be "shorted" by individual investors, but it is not possible that all investors hold negative amounts, as the net supplies of these assets to the market, that is, the net liabilities of governments, are known to be positive.

The hypothesis of portfolio optimization can be tested by nesting it within framework in which asset demand functions are of the portfolio balance variety but are not necessarily optimizing. Let  $x_t$  be a vector of demands for various assets as proportions of wealth, let  $r_t^e$  be the vector of expected returns, and let  $\beta$  be the matrix of coefficients in the demand function:

$$(1) \quad x_t = \alpha + \beta r_t^e.$$

If we invert the portfolio balance function and assume rational expectations, then we can use equation-by-equation OLS to estimate the system because the regression error  $\epsilon_{t+1}$  is the expectational error  $r_{t+1} - r_t^e$  that is independent of all variables known at time  $t$ :

$$(2) \quad r_{t+1} = -\beta^{-1}\alpha + \beta^{-1}x_t + \epsilon_{t+1}.$$

Under assumptions like those made by Macedo et al., it can be seen that if investors optimize with respect to the mean and variance of end-of-period wealth, the coefficient matrix will be given by

$$(3) \quad \beta^{-1} = \rho\Sigma,$$

where  $\rho$  is defined as the coefficient of relative risk aversion and  $\Sigma$  is defined as the return variance-covariance matrix:  $\Sigma \equiv E\epsilon_{t+1}\epsilon'_{t+1}$ . It is an unusual problem in econometrics to estimate an equation like (2) subject to a constraint like (3), a constraint not within the coefficient matrix but rather between the coefficient matrix and the variance-covariance matrix of the error term. But it can be done by maximum likelihood estimation. Then the like-

likelihood from estimating equation (2) unconstrained can be compared to the likelihood from estimating it constrained. If the fit does not worsen significantly, as decided by the likelihood ratio test, then we have failed to reject the null hypothesis.

If one accepts the optimizing hypothesis, either on the grounds of such a test or a priori, then the constrained estimates can be taken as efficient estimates of the parameters. The Macedo et al. paper is an important step forward over most previous work in the respect that it allows expected returns to vary over time, as they surely do both in the standard macroeconomic models and in the recent history of the world economy. However, it is still limited by the assumption that investors form expectations solely on the basis of past history. One might wish for coefficients in the portfolio balance function estimated under the assumption that investors form expectations rationally given all contemporaneous data. This can be done by using data on actual portfolios to infer expectations, as in the estimation framework suggested here; the expected returns are simply the fitted values of equation (2).

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