Cen

[redacted]

Discussion paper

Tilburg University

Center
for
Economic Research

No. 9925

INTERTEMPORAL EXPANSION OF BACKSTOP CAPACITIES

By Franz Wirl and Cees Withagen

March 1999

ISSN 0924-7815

Intertemporal Expansion of Backstop Capacities

Franz Wirl

Otto-von-Guericke University of Magdeburg, Germany

Cees Withagen

Free University of Amsterdam and Tilburg University, The Netherlands

Mailing address:

Franz Wirl

Faculty of Economics and Management

Otto-von-Guericke University of Magdeburg

P. O. Box 4120

D-39016 Magdeburg, Germany

Telephone: +49-391-67-18811/18810

Fax: +49-391-67-11142

E-mail: franz.wirl@ww.uni-magdeburg.de

Abstract

This paper considers an economy using a product that adds to a stock of pollution. Examples that come to mind are SO_2 emissions from burning coal accumulating in the soil and CO_2 emissions from fossil energy use which are retained in the atmosphere. The stock of pollutants is subject to natural decay, albeit not necessarily of the simple often assumed linear type. In addition, a clean or so-called backstop technology is available that requires costly investments but is characterised by low variable costs (e.g., solar energy or wind power). The costly investments lead to a slow build-up of the capacity of the backstop. On the modelling side, this is an essential extension of most of the literature that considers the unrealistic case that a backstop is instantaneously available. The second extension the present paper makes is to consider not only the planning problem but also competitive outcomes. One of the interesting results is that stable limit cycles may characterise the socially optimal long run outcome as well as the competitive equilibrium. In a competitive equilibrium pollution control policy is not necessarily optimal in the sense of corresponding with the social optimum. Although cycling can occur in a competitive equilibrium, just as in the social optimum, relaxation of the control increases the set of parameter values for which complex and unstable behaviour arises.

1. Introduction

This paper considers an economy using a product that causes pollution emissions that accumulate (e.g., SO_2 accumulates in the soil, CO_2 in the atmosphere). The pollution stock is subject to natural decay, albeit not necessarily of the simple linear type. In addition, a clean or so-called backstop technology is available that requires costly investment but can be exploited at low variable costs (say solar energy, wind power, etc.). The costly investments lead to slow build-up of the capacity of the backstop. On the modelling side, this offers an essential extension of most of the literature that considers the unrealistic case that a backstop is instantaneously available. Indeed, the obvious fact that all conceivable backstop technologies (say nuclear, solar, renewables) can impossibly overtake from one day to another such large markets as the world energy market, is somewhat overlooked in the literature on backstops (for an exception see Wirl (1991)). This paper provides a theoretical analysis of the introduction of such a backstop including a complete stability analysis for the social optimum and a competitive equilibrium. One of the interesting results of this framework is that stable limit cycles may characterise the socially optimal long run outcome. In addition, the paper investigates a competitive equilibrium if the externalities of the dirty good are not or not optimally internalised.

2. The model.

The following model is a straightforward amendment of the model studied by Withagen and Toman (1996) to allow for a sluggish build-up of backstop capacities. Environmental pollution (P) or degradation of nature increases by the amount of pollution emitted to the environment, which is assumed to be proportional to the consumption of the dirty product x, minus natural decay, $A(P)$:

(1) $\dot{P}(t) = x(t) - A(P(t)), P(0) = P_0$, given

The stock of pollution is modelled as in Withagen and Toman (1996), except for choosing a decay function that covers both cases, decay increasing or decreasing with increasing stock ($A' > 0$ and $A' < 0$) In fact we assume that decay is inverted U-shaped and concave, $A'' < 0$. See Figure 1 for a graphical illustration of the decay function. The familiar logistic function provides an arithmetical example. A familiar and plausible motivation of this specification is that the pollutant has to be 'digested' by the natural environment. Natural decay is low at low levels of pollution because the low levels of pollution rarely interact with a corresponding, say, bacteria, as well as at high levels of pollution because of the extensive load of pollution hindering decay.

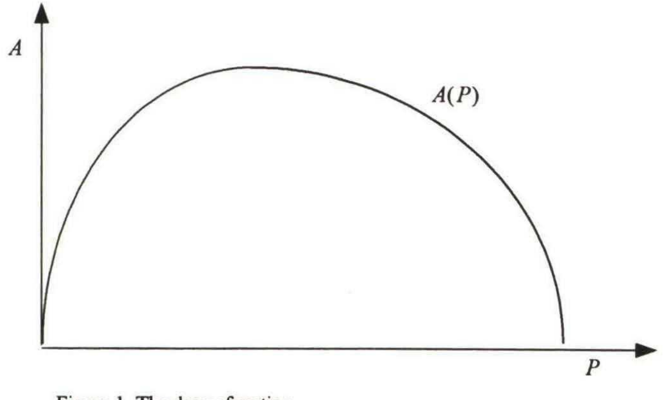

Figure 1. The decay function

We consider a partial equilibrium framework where consumers have utility U from consumption and are indifferent with respect to the origin of the consumer good: the dirty good x or the backstop product B. It is assumed that the instantaneous utility function U is strictly concave and strictly increasing. The Inada conditions, $U'(0) = \infty$ and $U'(\infty) = 0$, are assumed to hold. They will ensure an interior solution. The marginal willingness to pay, U', describes the inverse demand function, i.e., the market clearing price given aggregate supplies. Existing pollution has social cost D that is strictly increasing and convex. We do not treat the environment as providing raw materials or anything like that. The costs of producing the dirty good, $C(x,P)$, are assumed to be increasing and convex in output x, $C_x > 0, C_{xx} \geq 0$. The cost function may include pollution as an argument. Then costs are increasing and convex in pollution, $C_P > 0, C_{PP} > 0$, and the cost function is convex in (x,P), so that $C_{xx}\, C_{PP} - C_{xP}^2 > 0$. In that case we could also assume that pollution increases the marginal costs of producing the dirty commodity, $C_{xP} > 0$. In the sequel we assume that the costs depend on production only. This does not exclude the (likely) possibility that production costs increase with pollution (e.g., requiring dams, filters and other largely fixed costs elements), because these (additive) costs can be integrated in D, but it excludes that pollution increases the marginal costs of producing the dirty good. The reason for this simplification is purely arithmetical because retaining the mixed derivative complicates matters considerably, without adding further insights.

The extension we propose is to incorporate a backstop capacity that provides a clean substitute free of variable costs and that is characterised by a sluggish build-up. The assumption of zero variable costs ensures that the installed capacity of the backstop technology is always utilised and thus consumed. Consumption from the backstop can therefore be identified with the backstop capacity B. Of course, zero variable costs are not essential. They just have to be lower than the costs of the dirty product and the technology itself must be economically viable. The sluggish build-up is due to the fact that the backstop is produced from a capital stock, say fusion or wind power plants, photovoltaic cells, etc., which cannot be implemented at once for various reasons such as adjustment costs. Indeed, the obvious fact that all conceivable backstop technologies can impossibly overtake from one day to another such large markets as the world energy market is somewhat overlooked in the literature on backstops. The accumulation of the backstop capacity is described as follows:

(2) $\dot{B}(t) = y(t) - \delta B(t), B(0) = 0$

Here y is the rate of investment in the backstop and δ is the constant rate of depreciation. The investment y in the backstop technology has costs I that are strictly increasing and convex.

In the following we will study the intertemporal evolution of both states, backstop technology and pollution, under two different institutional arrangements: in a social optimum and in a competitive equilibrium.

3. Social optimum

We start with an analysis of the social optimum. The objective is to determine optimal production of the dirty commodity and investments in the backstop in order to maximise the aggregate net present value of social welfare:

$$\max_{\{x(t),y(t)\}} \int_0^\infty e^{-rt}\left[U(x(t)+B(t)) - D(P(t)) - C(x(t)) - I(y(t))\right]dt$$

subject to (1) and (2). The discount rate is denoted by r.

Current social welfare includes the consumer surplus U, the external costs D due to existing pollution, the investment costs I, and the costs of producing the dirty good C.

In order to solve the optimal control problem we define the current value Hamiltonian (omitting the time argument t)

$$H(x,y,P,B) = U(x+B) - D(P) - C(x) - I(y) + \lambda[x - A(P)] + \mu[y - \delta B]$$

Note that the Hamiltonian is jointly concave in states and controls, due to the concave objective and given that the state P carries a negative shadow price λ. Therefore, the first order conditions together with the transversality conditions are sufficient for an optimal program. The transversality conditions are satisfied if the states and co-states converge to a finite steady state or remain bounded, as in the case of limit cycles.

The first order conditions for an interior solution are the Hamiltonian maximising conditions,

$$H_x = 0: \; U' - C' + \lambda = 0$$

$$H_y = 0: \; -I' + \mu = 0$$

and the intertemporal evolution of the shadow prices of the stock of pollution λ and of the backstop capacity μ, given by:

$$\dot{\lambda} = (r + A')\lambda + D'$$

$$\dot{\mu} = (r + \delta)\mu - U'$$

If we assume that interior controls exist that are implicitly determined with the derivatives shown above, then we can write $x = X(B,\lambda)$ and $y = Y(\mu)$, with $X_B = U''/(C''-U'')$, $X_\lambda = 1/(C''-U''), Y_\mu = 1/I''$. In the sequel it is assumed that there exist steady states where the controls are indeed interior, e.g., due to the above mentioned Inada conditions. With regard to the points we wish to make this is not restrictive. We shall also provide some examples where this is straightforward to establish. Substitution of the optimal controls into state and co-state equations yields the following canonical equation system:

$$\dot{P} = X(B,\lambda) - A(P)$$

$$\dot{B} = Y(\mu) - \delta B$$

$$\dot{\lambda} = (r + A'(P))\lambda + D'(P)$$

$$\dot{\mu} = (r + \delta)\mu - U'(X(B,\lambda) + B)$$

From this system it follows that in the steady state $r + A' > 0$, because the co-state of pollution is negative.

Proposition 1: *Suppose that in the stationary state* $A' > 0$. *Then the optimal steady state is asymptotically locally stable.*

The result follows from applying standard sufficiency criteria (such as developed by Brock, Scheinkmann and others). The formal proof is relegated to Appendix A, where however other more direct techniques are employed. The economic consequence of the result is that sufficient environmental concern, implying that pollution is below what 'nature' could digest, P such that $A'(P) > 0$, does not only lower stationary pollution but ensures stability as well. However, other cases might occur as well.

Proposition 2:*If stationary pollution is 'large' so that $A' < 0$, then this steady state may be asymptotically locally stable but it may also be unstable and, in particular, there may exist stable limit cycles.*

The existence of limit cycles means that the build-up of the backstop capacity in order to reduce pollution and thus to lower the pressure on the environment, is followed by an increased consumption of the dirty product as the backstop capacity depreciates. Yet at higher pollution levels, the backstop is again pushed back into the market, and so on forever. The reason for the existence of limit cycles is that the framework described by the differential equations (1) - (2) includes one of the routes to limit cycles in strictly concave models addressed in Wirl (1996). More precisely, Wirl (1996) shows that growth, in the case at hand amounting to $r > \partial \dot{P} / \partial P = -A'(P) > 0$, is a necessary condition for Hopf bifurcation, which ensures the existence of stable (and generic) limit cycles. However, the restriction to steady states satisfying $r + A' > 0$ is not necessary, because accumulating pollution beyond the point where $r + A' = 0$, which defines the optimal long run stock absent any pollution externality, is suboptimal given the damage. Furthermore it is worth noting that while 'growth' is destabilising, the second dynamics $r > \partial \dot{B} / \partial B = -\delta < 0$ is stabilising. Hence $\delta > 0$ restricts the domain of complexities even for $A' < 0$ up to the point of ensuring overall stability if the rate of depreciation is sufficiently large. Or, the other way around, long lasting backstop (i.e., small δ) capacities are suitable to yield complex solutions including limit cycles. However, this restriction imposed by δ is implicit rather than explicit for the social optimum. It is discussed in detail in appendix B.

The Hopf bifurcation theorem requires that three properties hold:

i) there exists a pair of purely imaginary eigenvalues for a proper choice of the parameter that is varied (called the critical value or bifurcation point).

ii) the derivative of the real part with respect to the parameter is different from zero so that the critical value of the parameter separates the domains where the linearised system is stable (possibly restricted to a 'stable' manifold) from the domain of locally unstable spirals.

iii) a negative coefficient of a quadratic term of the so-called normal form (i.e., the system remains stable at the critical value).

The first two conditions are sufficient for the existence of a limit cycle. Yet if condition (iii) is violated the cycle is unstable. That is, such a cycle repels all motions starting arbitrarily close to the cycle (and within the stable manifold): motions starting inside the cycle converge to the steady states, those starting outside the cycle either diverge or converge to another steady state (if existing). The geometric intuition of the theorem is straightforward: at the point of the bifurcation, the steady state of the linearised system becomes a centre, but the non-linear system remains stable because of the negative quadratic terms. Now for parameter values slightly beyond the critical value, there are two opposing forces. First the negative quadratic term addressed in (iii) dominates the linear terms, at least sufficiently off the equilibrium. Second, close to the equilibrium this quadratic term is irrelevant so that the linear terms, with positive real parts, lead to locally 'exploding' spirals near the equilibrium. The limit cycle arises from balancing these two forces and constitutes the attractor of this system. If on the other hand the quadratic term is positive, a cycle requires that the linear terms provide the stabilising elements so that the steady state is locally stable and the cycle becomes repelling. For further details see Guckenheimer-Holmes (1983). In the following we will concentrate on condition (i). We will verify the other conditions numerically.

In the context of our model the growth condition mentioned requires that $A' < 0$ meaning a relatively large socially optimal pollution stock. Given this condition, very simple examples, with high discount rates and/or highly convex investment costs, allow for a Hopf bifurcation and hence for stable limit cycles. We consider an example with linear external costs $D(P) = dP$, linear production costs $C(x) = cx$, quadratic consumer surplus $U(z) = 1 - \frac{1}{2}z^2$, so that demand is linear with a maximal willingness to pay of 1$ and maximum demand of 1 unit, linear-quadratic investment costs $I(y) = ay + \frac{1}{2}by^2$, and logistic decay $A(P) = P(1-P)$. For the parameter values we choose $a = 0.1$, $c = 0.2$, $d = 0.3$, $\delta = 0.05$ and $r = 1.8$. Thus the average lifetime of a backstop plant is 20 years. We use the parameter b, which determines the convexity of the investment costs, as our bifurcation parameter. This approach leads to a pair of purely imaginary eigenvalues of the Jacobian at $b = b^{crit} = 6.56596...$, where the derivative of the real part does not vanish and the bifurcation is supercritical, i.e. the quadratic term of the normal form is negative so that the cycle is stable and not repelling. This can be verified numerically using LOCBIF (see Khibnik et al. (1992)). The corresponding Lyapunov number turns out to be -2.02042, so that limit cycles characterise the optimal policy in a local, one-sided surrounding of $b > 6.56596...$ In Figure 2, we vary the adjustment cost parameter b as indicated for the reported Hopf bifurcation that leads to the interesting result, that 'intermediate' and high values induce limit cycles or instability, but low and very high values for b imply stability. Of course, this instability results only upon entering the domain $P > \frac{1}{2} = \arg\max A(P)$ so that $A' < 0$.

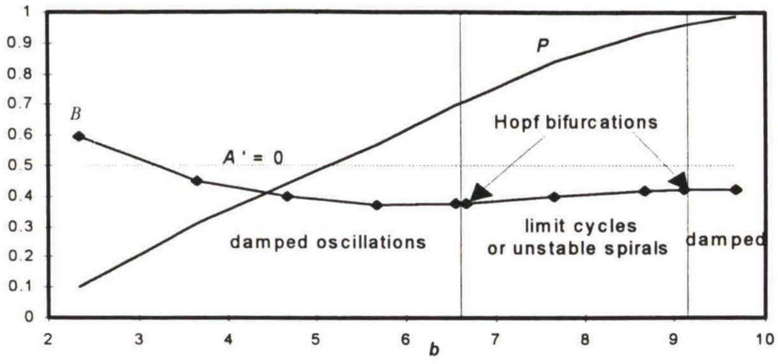

Figure 2. Steady states versus the convexity of investment costs, stability properties and Hopf bifurcations

Figure 3 uses the discount rate as the bifurcation parameter, all other parameters as above and b at the critical value for $r = 1.8$. This leads to stable Hopf bifurcation too and it highlights at the same time the non-monotonic dependence of the steady states on the parameter and the associated stability properties.

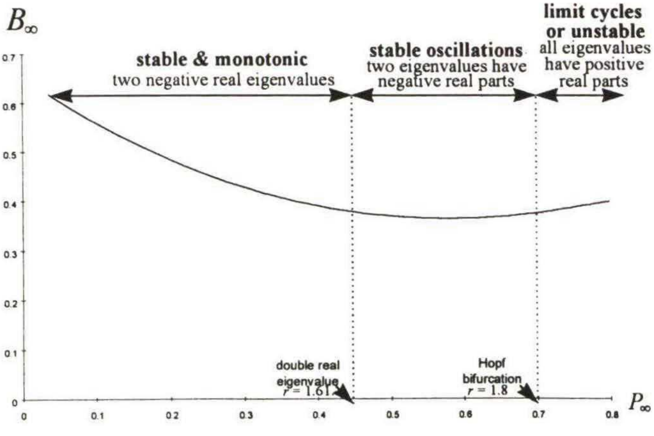

Figure 3. Stationary solutions and stability properties for different values of r.

Remark: Setting $\delta = 0$ and assuming $I(0) = 0$ implies the same steady state as reported in Withagen and Toman (1996). The inclusion of adjustment costs can fundamentally alter the stability properties. More precisely, the adjustment costs can destabilise an otherwise stable equilibrium, but cannot stabilize an otherwise unstable equilibrium, see Feichtinger, Novak and Wirl (1994).

4. Competitive equilibria

In this section, we study the perfect foresight competitive equilibrium where the externality is not at all or not optimally internalised by governmental environmental policy. The representative firm faces the following problem:

$$\max_{\{x(t),y(t)\}} \int_0^\infty e^{-rt}\left[p(t)[x(t)+B(t))]-C(x(t))-\tau(t)x(t)-I(y(t))\right]dt$$

subject to

$$\dot{B} = y - \delta B,\ B(0) = 0.$$

That is, the firm decides at each point of time how much to pollute and how much to expand the backstop capacities, taking the market price p and the pollution stock as given. In the objective function τ denotes the tax rate on the production of the dirty commodity, which is exogenous to the firm.. Each individual firm neglects the damage function D and pollution does not enter the private cost function either. Firm-specific feedback of pollution by means of the tax rate is now essential in contrast to the social optimum where such feedbacks were included in D. Absent such a feedback, the tragedy of the commons arises, ultimately destroying the environment's inherent abatement capabilities. The consequences of such irreversibilities are analysed by Tahvonen and Withagen (1996) in the context of a planning problem. The irreversibility creates a non-concavity for the planning problem that would be irrelevant in our competitive setting, because we are working in a decentralised economy.

The Hamiltonian of the system reads

$$H(x,y,B,v) = p[x+B]-C(x)-\tau x - I(y) + v[y-\delta B]$$

Assuming an interior solution we find as a first set of necessary conditions, again the Hamiltonian maximising conditions

$$H_x = p - C_x - \tau = 0$$
$$H_y = -I' + \nu = 0$$

Another necessary condition describes the evolution of the single adjoint variable of the representative firm's backstop capacity, denoted by ν, to differentiate from the socially optimal solution where we used μ:

$$\dot{\nu} = (r+\delta)\nu - p$$

Market clearing at any instant of time requires:

$$p(t) = U'(x(t) + B(t))$$

The optimal controls, i.e. the production of the dirty product and the investment in the backstop interior controls are now implicitly defined by $x^c = X^c(B;\tau)$ *and* $y^c = Y^c(\nu)$, with $X_B^c = -U''/(U''-C'')$, $X_\tau^c = 1/(U''-C'')$ and $Y_\nu^c = 1/I''$. Moreover, equation 1 describing the evolution of the stock of pollution must hold. Although the competitive firms have no control over the stock of pollution, they anticipate the evolution of pollution perfectly (rational expectations). Therefore, the competitive equilibrium is described by the following system of differential equations:

$$\dot{B} = Y^c(\nu) - \delta B$$

$$\dot{\nu} = (r+\delta)\nu - U'(X^c(B;\tau) + B)$$

$$\dot{P} = X^c(B;\tau) - A(P)$$

Proposition 3: *Suppose the tax rate is set such that it solves* $\tau = -\lambda$, *where* λ *is the co-state of the pollution stock in the social optimum, and satisfies* $\dot{\lambda} = (r + A')\lambda + D'$ *and a transversality condition. Then the competitive equilibrium is identical to the social optimum.*

Proof: Addition of this differential equation for $\dot{\lambda}$ to the competitive equilibrium and replacing τ by $-\lambda$ yields the same system as in section 3 except for a different labelling of ν instead of μ.

This proposition identifies the interrelationship between a competitive equilibrium and the social optimum. It leads immediately to the corollary that competitive equilibria allow for limit cycles too. Nevertheless we study in the following also competitive equilibrium where internalisation is possibly sub-optimal, so that the tax rate is not equal to the negative of the shadow price of pollution.

Proposition 4: *Suppose that the steady state is in the domain of* $A' > 0$. *Then the steady state is asymptotically locally stable.*

This proposition is completely analogous to proposition 1. The next two propositions investigate the domain $A' < 0$.

Proposition 5: *Suppose that the tax rate* τ *is constant over time. Then a steady state in the domain of* $A' < 0$ *is (generically) unstable, i.e., only one eigenvalue is negative so that the stability is restricted to a one-dimensional manifold of the initial conditions in the* (P, B)*-plane.*

The instability addressed in proposition 5 is usually associated with multiple equilibria with the consequence that applying even the long run optimal tax is insufficient, if initial pollution is large. Note that the instability addressed in proposition 5 can occur in all other cases considered in this paper. This may be surprising to some of the readers given the strict and joint concavity of our model, since in the literature most of these kinds of instabilities are associated with (local) convexities.

An immediate consequence of proposition 5 is that we have to extend the analysis so as to allow for a tax that is increasing in pollution, to get complex solutions. The reason is that the kind of instability addressed in proposition 5 excludes local instabilities of the kind required for a Hopf bifurcation, where two eigenvalues must have negative real parts. The relation $\tau = \tau(P)$ may cover to some extent not only the tax but also other adversities to the firm related to pollution, such as the increase of marginal costs of producing the dirty good. Substitution of this modification into the above differential equation system characterises the competitive equilibrium facing a state contingent, instead of a constant or just time-dependent, tax rate.

Proposition 6: *Instabilities and limit cycles can characterise a competitive equilibrium with a stationary pollution such that* $A' < 0$. *However, the domain for complexities in competitive equilibria is not restricted by the discount rate of the (representative) firm, i.e.,* $r + A' < 0$ *is a possible long run outcome under competition. Yet depreciation restricts the domain of possible complexities of limit cycles to sufficiently long lasting capacities,* $\delta + A' < 0$, *so that for* $\delta = 0$ *the entire domain* $A' < 0$ *permits complexities.*

The reason for this restriction is the stability inherent to the accumulation of the backstop capacities. In contrast to the social optimum an explicit bound can be given concerning the domain of complex solutions. An example establishing the claim of limit cycles in Proposition 6 run as follows. It borrows from the social planning example the specifications of the surplus, $U(z) = z - \frac{1}{2}z^2$, and of the investment costs, $I(y) = ay + \frac{1}{2}by^2$, but assumes quadratic production costs $C(x) = \frac{1}{2}cx^2$, decay slightly different from the logistic one, $A(P) = P(1 - \sqrt[3]{P})$, and firm specific taxes/costs linear in pollution, $\tau(P) = sP$. A theoretical discussion and details concerning this example are given in the appendix A. Setting the parameters $a = 0.1, c = 0.05, r = 0.3, \delta = 0.05, s = 0.1$, yields a Hopf bifurcation for variations in b at $b = 3.870334$ (i.e., a pair of purely imaginary eigenvalues satisfying the conditions about a non-zero crossing velocity and having a negative Lyapunov number -0.0332 according to the calculations performed with LOCBIF). The resulting steady state of large pollution exceeds pollution feasible for a socially optimal programme and thus highlights that the domain of complex policies is enlarged for competitive outcomes.

Figure 4 shows on the left hand side the evolution of the steady states and the corresponding stability properties for the intertemporal, competitive equilibrium associated with the above mentioned bifurcation but considering variations in the discount rate r instead of b. For the purpose of comparison, the right-hand side shows the social optimum, which leads to comparable pollution. This is achieved by setting $d = .005$, so that the tax leads to similar steady states with rather large pollution due to the low damage parameter.

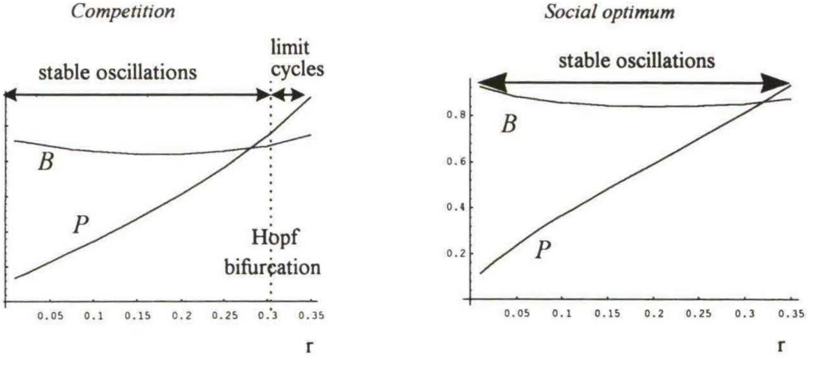

Figure 4. Steady states as a function of the discount rate and the associated stability properties.

5. Concluding remarks

It has been shown that with sluggish build-up of a backstop technology complex behaviour of pollution and backstop capacity can arise, in a social optimum as well as in a competitive equilibrium. A remarkable result is that in a competitive equilibrium with non-optimal taxation the scope for such behaviour is larger. This fact has been established by means of a numerical example. Figure 5 summarises these general findings.

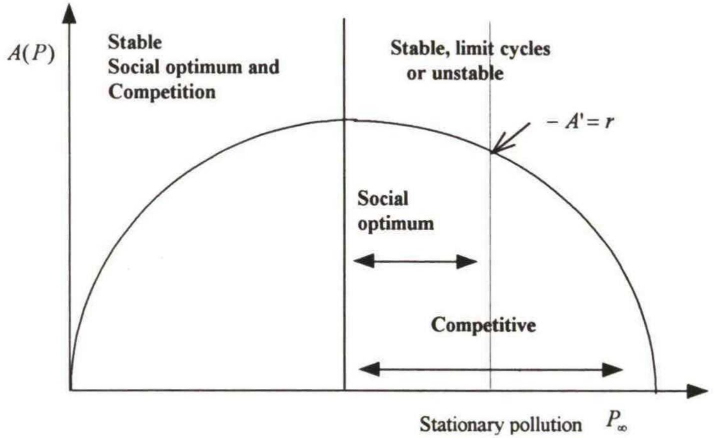

Figure 5. Comparison of the stability properties of social planning and competition

The basic reason for this phenomenon of limit cycles is that the consideration of negative marginal decay introduces 'growth', which is a pathway for Hopf bifurcation and thus for limit cycles in strictly concave dynamic optimisation models. Local convexities substantially simplify the derivation of limit cycles from the first order conditions, but these conditions are not sufficient anymore. Wirl (1996) shows this for planning problems and Wirl (1997) for competitive equilibria. The reason for the differences between the social optimum and the competitive equilibrium is that the accumulation of pollution beyond the stock where $r + A' = 0$ is always socially suboptimal, but it is a feasible outcome under competition if the externality is insufficiently internalised.

There are two extensions of this research that are worth pursuing. First it would be interesting to investigate the effect of introducing variable costs associated with the production from the backstop technology. Second, and more importantly, it is worthwhile to see if set-up costs for the backstop technology would drastically alter the main results.

Acknowledgement. The authors wish to thank Jeroen van den Bergh and Peter Kort for their comments on an earlier version of the paper. All remaining errors are ours.

References:

Brock, W. and Malliaris, A. (1989), Differential Equations, Stability and Chaos in Dynamic Economics, North-Holland, Amsterdam.

Dockner, E. (1985): "Local stability analysis in optimal control problems with two state variables", in G. Feichtinger (ed.), Optimal Control Theory and Economic Analysis 2, 89-103, North-Holland, Amsterdam.

Feichtinger, G., Novak, A. and Wirl, F. (1994): "Limit cycles in intertemporal adjustment models", Journal of Economic Dynamics and Control 18, pp. 353-380.

Guckenheimer, J. and Holmes, P. (1983), Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields, Springer Verlag, New York.

Khibnik, A., Kuznetsov, Y., Levitin, V. and Nikolaev, E. (1992), Interactive LOCal BIFurcation Analyzer (Manual), Expertise Centre for Computer Algebra in the Netherlands (CAN), Amsterdam.

Tahvonen, O. and Withagen, C. (1996): "Optimality of irreversible pollution accumulation, Journal of Economic Dynamics and Control 20, pp. 1775 - 1795.

Wirl, F. (1991): "Monopolistic resource extraction and limit pricing: the market penetration of competitively produced synfuels", Environmental and Resource Economics 1, pp. 157-178.

Wirl, F. (1996): "Pathways to Hopf bifurcations in dynamic, continuous time optimization models", Journal of Optimization Theory and Applications 91, pp. 299 - 320.

Wirl, F. (1997): "Stability and limit cycles in one-dimensional dynamic optimizations of competitive agents with a market externality", Journal of Evolutionary Economics 7, pp. 73- 89.

Withagen and Toman (1996): "Cumulative pollution with a backstop", mimeo Tilburg University.

Appendix A: Stability analysis in the social optimum

The Jacobian of the system of differential equations discussed in section 3 is:

$$J = \begin{pmatrix} \frac{\partial \dot{P}}{\partial P} & \frac{\partial \dot{P}}{\partial B} & \frac{\partial \dot{P}}{\partial \lambda} & \frac{\partial \dot{P}}{\partial \mu} \\ \frac{\partial \dot{B}}{\partial P} & \frac{\partial \dot{B}}{\partial B} & \frac{\partial \dot{B}}{\partial \lambda} & \frac{\partial \dot{B}}{\partial \mu} \\ \frac{\partial \dot{\lambda}}{\partial P} & \frac{\partial \dot{\lambda}}{\partial B} & \frac{\partial \dot{\lambda}}{\partial \lambda} & \frac{\partial \dot{\lambda}}{\partial \mu} \\ \frac{\partial \dot{\mu}}{\partial P} & \frac{\partial \dot{\mu}}{\partial B} & \frac{\partial \dot{\mu}}{\partial \lambda} & \frac{\partial \dot{\mu}}{\partial \mu} \end{pmatrix}$$

All the elements this matrix are evaluated at the steady state. The stability properties of this canonical equations system may be studied applying the global asymptotic stability criteria developed in the seventies by Brock, Malliaris, Rockefeller, Scheinkman and others, which are well summarised in the book of Brock and Malliaris (1989). However, we opt for a local stability analysis following Dockner (1985) who derives an explicit formula for the eigenvalues, E_i (i = 1,2,3,4)

$$E_i = \frac{r}{2} \pm \sqrt{\left(\frac{r}{2}\right)^2 - \frac{K}{2} \pm \frac{1}{2}\sqrt{K^2 - 4\det(J)}}\ ,\ i = 1,2,3,4$$

Here K is defined as follows (see Dockner (1985), p.96).

$$K = \begin{vmatrix} \frac{\partial \dot{P}}{\partial P} & \frac{\partial \dot{P}}{\partial \lambda} \\ \frac{\partial \dot{\lambda}}{\partial P} & \frac{\partial \dot{\lambda}}{\partial \lambda} \end{vmatrix} + \begin{vmatrix} \frac{\partial \dot{B}}{\partial B} & \frac{\partial \dot{B}}{\partial \mu} \\ \frac{\partial \dot{\mu}}{\partial B} & \frac{\partial \dot{\mu}}{\partial \mu} \end{vmatrix} + 2\begin{vmatrix} \frac{\partial \dot{P}}{\partial B} & \frac{\partial \dot{P}}{\partial \mu} \\ \frac{\partial \dot{\lambda}}{\partial B} & \frac{\partial \dot{\lambda}}{\partial \mu} \end{vmatrix}$$

The formula allows for a complete characterisation of the local dynamics of the linearised system and provides the ideal test recommended in Brock-Malliaris (1989, bottom of p. 148). For the general case of section 3 we have:

$$J = \begin{pmatrix} -A' & \frac{U''}{C''-U''} & \frac{1}{C''-U''} & 0 \\ 0 & -\delta' & 0 & \frac{1}{I''} \\ A''\lambda + D'' & 0 & r + A' & 0 \\ 0 & -\frac{U''C''}{C''-U''} & -\frac{U''}{C''-U''} & r+\delta \end{pmatrix}$$

Calculation of the coefficients in Dockner's formula yields for K:

$$K = -A'[r + A'] - \delta[r + \delta] - \frac{A''\lambda + D''}{C''-U''} + \frac{C''U''}{I''[C''-U'']}$$

All terms except for the first term $-A'[r + A']$ are definitely negative due to the assumed second order derivatives and due to the fact the stationary shadow price of pollution, $\lambda = -D'/(r + A')$, is negative. The determinant equals:

$$\det(J) = \delta A'[r + \delta][r + A'] + \frac{\delta[r + \delta][[\lambda A''+D'']}{C''-U''} - \frac{U''[\lambda A''+D''+A'C''[r + A']]}{I''[C''-U'']}$$

According to the above calculations, low pollution i.e., $A' > 0$, implies $K < 0$ and $\det(J) > 0$. In view of Dockner's formula these inequalities are sufficient for J to have two eigenvalues, which are either negative or have negative real parts. That implies saddlepoint stability, albeit that damped oscillation may be optimal, since the eigenvalues of the stable manifold can be complex (as shown in the example in the main text). This proves proposition 1.

We show next that proposition 2 holds. First, $A' < 0$ still allows for stability. This follows directly from the above calculations because $A' < 0$ is compatible with $K < 0$ and $\det(J) > 0$, which in turn are sufficient for saddlepoint stability. The example in Figure 2 shows that even local monotonicity is possible for $A' < 0$. The existence of limit cycles according to the Hopf bifurcation theorem requires inter alia the existence of a pair of purely imaginary eigenvalues. In Dockner's formula this implies

$$\det(J) = (K / 2)^2 + r^2 K / 2, \ K > 0$$

which in turn requires that the determinant is positive too.

According to the above calculation, $K > 0$ requires $A' < 0$. Ironically enough, this negative derivative introduces 'growth', which according to Wirl (1996) is a pathway for limit cycles. As mentioned earlier, Wirl (1996) shows that growth, i.e. $r > \partial\dot{P} / \partial P = -A' > 0$, is a necessary condition for Hopf bifurcation in strictly concave dynamic optimisation models and of the possible pathways only this one is present. Although his theorem is, strictly speaking, not applicable because of its restriction to a single control, the consequence on $K > 0$ is tied to $A' < 0$. However, depreciation of the backstop capacities adds a stabilising (i.e. negative) factor in K so that $A'[r + A']$ must at least outweigh $\delta[\delta + r]$. Hence, long lasting investments are helpful for $K > 0$, a prerequisite for complex solutions such as limit cycles.

For the example in Section 3 we obtain the optimal controls, $x^* = -B - c + \lambda, y^* = [\mu - a]/b$. The canonical equations (retaining $A = P[1-P]$ and $A' = 1 - 2P$ as short hand) are:

$$\dot{P} = (-B - c + \lambda) - A(P)$$
$$\dot{B} = [\mu - a]/b - \delta B$$
$$\dot{\lambda} = [r + A']\lambda + d$$
$$\dot{\mu} = [r + \delta]\mu + \lambda - c$$

which yields

$$J = \begin{pmatrix} -A' & -1 & 1 & 0 \\ 0 & -\delta & 0 & 1/b \\ -2\lambda & 0 & r + A' & 0 \\ 0 & 0 & 1 & r + \delta \end{pmatrix}$$

This system allows for a closed form analytical solution, of the steady states and even of the critical value of the bifurcation parameter, at least for the parameter b. However, all these expressions are extremely cumbersome so that we report here only the crucial coefficient K.

$$K = -A'[r + A'] + 2\lambda - \delta[r + \delta].$$

Fairly similar are the results for the example in Section 4 (albeit we could not obtain a closed form solution anymore given the quadratic production costs):

$$\dot{P} = \frac{1 - B + \lambda}{1 + c} - A(P)$$

$$\dot{B} = \frac{\mu - a}{b} - \delta B$$

$$\dot{\lambda} = [r + A']\lambda + d$$

$$\dot{\mu} = [r + \delta]\mu - \frac{c[1 - B] - \lambda}{1 + c}$$

This system yields the following Jacobian:

$$J = \begin{pmatrix} -A' & \frac{-1}{1+c} & \frac{1}{1+c} & 0 \\ 0 & -\delta & 0 & 1/b \\ \lambda A'' & 0 & r + A' & 0 \\ 0 & \frac{c}{1+c} & \frac{1}{1+c} & r + \delta \end{pmatrix}$$

The corresponding K is

$$K = -A'[r + A'] - \frac{A''\lambda}{1+c} - \frac{c}{b[1+c]} - \delta[r + \delta]$$

Appendix B. Stability in the competitive equilibrium

Although the following analysis allows for taxes depending on pollution we start with the case where the tax rate is independent of pollution. Then the corresponding Jacobian equals

$$J = \begin{pmatrix} -\delta & \frac{1}{I''} & 0 \\ \frac{U''C''}{[U''-C'']} & r+\delta & 0 \\ \frac{-U''}{U''-C''} & 0 & -A' \end{pmatrix}$$

Again the stability properties can be obtained from calculating the eigenvalues of the Jacobian. The eigenvalues are the roots of the characteristic polynomial:

$$p(e) = e^3 - tr(J)e^2 + k\,e - \det(J)$$

where k is the sum of the principal minors of dimension 2 of the Jacobian. It plays a role similar to K in appendix A. The calculation of the coefficients of the characteristic polynomial proceeds in several steps.

$$tr(J) = r - A'$$

$$\det(J) = A'\delta(r+\delta) + \frac{U''A'C''}{I''(U''-C'')}$$

and

$$k = -\delta[r+\delta] - rA' - \frac{C''U''}{I''[U''-C'']}$$

Since $A' < 0$ implies det(J) < 0, an instability arises whenever $A' < 0$ and thus the impossibility of limit cycles. This verifies proposition 5. For $A' > 0$ we have det(J) >0 as well as $k < 0$, which are sufficient for saddlepoint stability (in fact sufficient for real eigenvalues and thus for local monotonicity). Therefore, the existence of complex competitive equilibria requires a state contingent effect, either as a tax or through a feedback on costs.Taking into account $\tau = \tau(P)$ in the system derived in section 4 and applying the chain rule yields the following Jacobian:

$$J = \begin{pmatrix} -\delta & \frac{1}{I''} & 0 \\ \frac{U''C''}{U''-C''} & r+\delta & \frac{-U''\tau'}{U''-C''} \\ \frac{-U''}{U''-C''} & 0 & -A' + \frac{\tau'}{U''-C''} \end{pmatrix}$$

We now have

$$tr(J) = r - A' + \frac{\tau'}{U''-C''}$$

$$\det(J) = A'\delta(r+\delta) + \frac{U''(\tau' + A'C'') - \delta\tau' I''(r+\delta)}{I''(U''-C'')}$$

and:

$$k = -\delta[r+\delta] - rA' - \frac{C''U''}{I''[U''-C'']} + \frac{r\tau'}{U''-C''}$$

From det$(J) > 0$ it follows that either one eigenvalue, say e_1, is positive and the other two (say e_2 and e_3) are negative, or all three are positive. Since $k = e_3[e_1 + e_2] + e_1 e_2$, $k < 0$ is only possible if both e_2 and e_3 are negative (because they cannot have opposite sign due to the fact that $\det(J) > 0$). Therefore, the properties $k < 0$ and $\det(J) > 0$ are sufficient for saddlepoint stability. Similarly, $\det(J) > 0$ combined with $tr(J) < 0$ is sufficient for saddlepoint stability.

The existence of a pair of purely imaginary eigenvalues as a necessary condition for a Hopf bifurcation requires that $\det(J) = tr(J)k$ and that all elements of this equation are positive. Therefore, again the sum the principal minors of dimension 2 must be positive, $k > 0$. Again $A' < 0$ is helpful for a positive trace and a positive k, but can lead to a negative determinant, which implies an instability, yet simultaneously excluding limit cycles. However, $A' < 0$ is not sufficient since $k > 0$ demands at the minimum $\delta + A' < 0$. The reason is similar to the social optimum: $A' < 0$ introduces growth into the externality and thus helps to destabilise the system while $\partial \dot{B} / \partial B = -\delta$ destabilises the system. Hence instabilities of any kind require that the destabilising element must outweigh the stabilising element $\delta + A' < 0$. However, in contrast to the social optimum, low discount rates do not constrain the domain of feasible equilibria so that $r + A' < 0$ is feasible.

No.	Author(s)	Title
9876	J. Timmer, P. Borm and J. Suijs	Linear transformation of products: games and economies
9877	T. Lensberg and E. van der Heijden	A cross-cultural study of reciprocity, trust and altruism in a gift exchange experiment
9878	S.R. Mohan and A.J.J. Talman	Refinement of solutions to the linear complementarity problem
9879	J.J. Inman and M. Zeelenberg	"Wow, I could've had a V8!": The role of regret in consumer choice
9880	A. Konovalov	Core equivalence in economies with satiation
9881	R.M.W.J. Beetsma and A.L. Bovenberg	The optimality of a monetary union without a fiscal union
9882	A. de Jong and R. van Dijk	Determinants of leverage and agency problems
9883	A. de Jong and C. Veld	An empirical analysis of incremental capital structure decisions under managerial entrenchment
9884	S. Schalk	A model distinguishing production and consumption bundles
9885	S. Eijffinger, E. Schaling and W. Verhagen	The term structure of interest rates and inflation forecast targeting
9886	E. Droste and J. Tuinstra	Evolutionary selection of behavioral rules in a cournot model: A local bifurcation analysis
9887	U. Glunk and C.P.M. Wilderom	High performance on multiple domains: Operationalizing the stakeholder approach to evaluate organizations
9888	B. van der Genugten	A weakened form of fictituous play in two-person zero-sum games
9889	A.S. Kalwij	Household wealth, female labor force participation and fertility decisions
9890	T. Leers, L. Meijdam and H. Verbon	Ageing and Pension reform in a small open economy: The role of savings incentives
9891	R.T. Frambach, J. Prabhu and T.M.M. Verhallen	The influence of business strategy on market orientation and new product activity
9892	H. Houba and G. van Lomwel	Counter intuitive results in a simple model of wage negotiations
9893	T.H.A. Bijmolt and R.G.M. Pieters	Generalizations in marketing using meta-analysis with Multiple measurements
9894	E. van Damme and J.W. Weibull	Evolution with mutations driven by control costs
9895	A. Prat and A. Rustichini	Sequential common agency

No.	Author(s)	Title
9896	J.H. Abbring, G.J. van den Berg P.A. Gautier, A.G.C. van Lomwel and J.C. van Ours	Displaced workers in the United States and the Netherlands
9897	G.J. van den Berg, A.G.C. van Lomwel and J.C. van Ours	Unemployment dynamics and age
9898	J. Fidrmuc	Political support for reforms: economics of voting in transition countries
9899	R. Pieters, H. Baumgartner, J. Vermunt and T. Bijmolt	Importance, cohesion, and structural equivalence in the evolving citation network of the international journal of research in marketing
98100	A.L. Bovenberg and B.J. Heijdra	Environmental abatement and intergenerational distribution
98101	F. Verboven	Gasoline or diesel? Inferring implicit interest rates from aggregate automobile purchasing data
98102	O.J. Boxma, J.W. Cohen and Q. Deng	Heavy-traffic analysis of the *M/G/*1 queue with priority classes
98103	S.C.W. Eijffinger, M. Hoeberichts and E. Schaling	A theory of central bank accountability
98104	G.J. van den Berg, P.A. Gautier, J.C. van Ours and G. Ridder	Worker turnover at the firm level and crowding out of lower educated workers
98105	Th. ten Raa and P. Mohnen	Sources of productivity growth: technology, terms of trade, and preference shifts
98106	M.P. Montero Garcia	A bargaining game with coalition formation
98107	F. Palomino and A. Prat	Dynamic incentives in the money management tournament
98108	F. Palomino and A. Prat	Risk taking and optimal contracts for money managers
98109	M. Wedel and T.H.A. Bijmolt	Mixed tree and spatial representation of dissimilarity judgments
98110	A. Rustichini	Sophisticated Players and Sophisticated Agents
98111	E. Droste, M. Kosfeld and M. Voorneveld	A Myopic adjustment process leading to best-reply matching
98112	J.C. Engwerda	On the scalar feedback Nash equilibria in the infinite horizon LQ-game
98113	J.C. Engwerda, B. van Aarle and J.E.J. Plasmans	Fiscal policy interaction in the EMU
98114	K.J.M. Huisman and P.M. Kort	Strategic investment in technological innovations

No.	Author(s)	Title
98115	A. Cukierman and Y. Spiegel	When do representative and direct democracies lead to similar policy choices?
98116	A. Cukierman and F. Lippi	Central bank independence, centralization of wage bargaining, inflation and unemployment -theory and some evidence
98117	E.G.A. Gaury, J.P.C. Kleijnen and H. Pierreval	Customized pull systems for single-product flow lines
98118	P.J.J. Herings, G. van der Laan and D. Talman	Price-quantity adjustment in a Keynesian economy
98119	R. Nahuis	The dynamics of a general purpose technology in a research and assimilation model
98120	C. Dustmann and A. van Soest	Language fluency and earnings: estimation with misclassified language indicators
98121	C.P.M. Wilderom and P.T. van den Berg	A test of the leadership-culture-performance model within a large, Dutch financial organization
98122	M. Koster	Multi-service serial cost sharing: an incompatibility with smoothness
98123	A. Prat	Campaign spending with office-seeking politicians, rational voters, and multiple lobbies
98124	G. González-Rivera and F.C. Drost	Efficiency comparisons of maximum likelihood-based Estimators in GARCH models
98125	H.L.F. de Groot	The determination and development of sectoral structure
98126	S. Huck and M. Kosfeld	Local control: An educational model of private enforcement of public rules
98127	M. Lubyova and J.C. van Ours	Effects of active labor market programs on the transition rate from unemployment into regular jobs in the Slovak Republic
98128	L. Rigotti	Imprecise beliefs in a principal agent model
98129	F. Palomino, L. Rigotti and A. Rustichini	Skill, strategy and passion: an empirical analysis of soccer
98130	J. Franks, C. Mayer and L. Renneboog	Who disciplines bad management?
98131	M. Goergen and L. Renneboog	Strong managers and passive institutional investors in the UK: stylized facts
98132	F.A. de Roon and Th.E. Nijman	Testing for mean-variance spanning: A survey
98133	A.C. Meijdam	Taxes, growth and welfare in an endogenous growth model with overlapping generations

No.	Author(s)	Title
98134	A. Scott and H. Uhlig	Fickle investors: An impediment to growth?
98135	L.W.G. Strijbosch, R.M.J. Heuts and E.H.M. van der Schoot	Improved spare parts inventory management: A case study
98136	E. Schaling	The nonlinear Phillips curve and inflation forecast targeting - symmetric versus asymmetric monetary policy rules
98137	T. van Ypersele	Coordination of capital taxation among a large number of asymmetric countries
98138	H. Gruber and F. Verboven	The diffusion of mobile telecommunications services in the European Union
98139	F. Verboven	Price discrimination and tax incidence - Evidence from gasoline and diesel cars
98140	H.G. Bloemen	A model of labour supply with job offer restrictions
98141	S.J. Koopman, N. Shephard and J.A. Doornik	Statistical algorithms for models in state space using SsfPack 2.2
98142	J. Durbin and S.J. Koopman	Time series analysis of non-gaussian observations based on state space models from both classical and Bayesian perspectives
9901	H. Pan and T. ten Raa	Competitive pressures on income distribution in China
9902	A. Possajennikov	Optimality of imitative behavior in Cournot oligopoly
9903	R.G.M. Pieters and M. Zeelenberg	Wasting a window of opportunity: Anticipated and experiences regret in intention-behavior consistency
9904	L.C. Koutsougeras	A remark on the number of trading posts in strategic market games
9905	B. Roorda, J. Engwerda and H. Schumacher	Performance of delta-hedging strategies in interval models - a robustness study
9906	M. Slikker	Link monotonic allocation schemes
9907	W.A. van den Broek	Moving horizon control in dynamic games
9908	F.J.G.M. Klaassen	Long swings in exchange rates: Are they really in the data?
9909	F.J.G.M. Klaassen	Purchasing power parity: Evidence from a new test
9910	F.J.G.M. Klaassen	Have exchange rates become more closely tied? Evidence from a new multivariate GARCH model
9911	J.P.C. Blanc	On the stability of communication systems with timed token protocols

No.	Author(s)	Title
9912	B. Donkers, B. Melenberg and A. Van Soest	Estimating risk attitudes using lotteries; a large sample approach
9913	P. Francois and Shouyong Shi	Innovation, growth and welfare-improving cycles
9914	Pinelopi Koujianou Goldberg and Frank Verboven	The evolution of price discrimination in the European car market
9915	Bas Jacobs, Richard Nahuis and Paul J.G. Tang	Sectoral productivity growth and R&D spillovers in the Netherland
9916	R.C.H. Cheng, J.P.C. Kleijnen and V.B. Melas	Optimal design of experiments with simulation models of nearly saturated queues
9917	Pieter A. Gautier, Gerard J. van den Berg, Jan C. van Ours and Geert Ridder	Separations at the firm level
9918	Kuno J.M. Huisman and Peter M. Kort	Strategic technology investment under uncertainty
9919	Gerard J. van den Berg, Anders Holm and Jan C. van Ours	Does work experience help to become a medical specialist?
9920	Robert P. Gilles and Guillermo Owen	Cooperative games and disjunctive permission structures
9921	Flip Klijn, Stef Tijs and Herbert Hamers	Balancedness of permutation games and envy-free allocations in indivisible good economies
9922	Sharon Schalk	Proof of the existence theorem of a model distinguishing production and consumption bundles
9923	J.J.A. Moors	Double checking for two error types
9924	Chris Elbers and Cees Withagen	General equilibrium models of environmental regulation and international trade
9925	Franz Wirl and Cees Withagen	Intertemporal expansion of backstop capacities

CentE

Warandelaan 2
P.O. Box 90153
5000 LE Tilburg
The Netherlands

phone +31 13 4663050
fax +31 13 4663066
e mail center@kub.nl
www center.kub.nl

Tilburg University