CESifo Working Papers

www.cesifo.org/wp

Dark Clouds or Silver Linings? Knightian Uncertainty and Climate Change

Yu-Fu Chen
Michael Funke
Nicole Glanemann

CESIFO WORKING PAPER NO. 3516
CATEGORY 9: RESOURCE AND ENVIRONMENT ECONOMICS
JULY 2011

CESifo *Working Paper No. 3516*

Dark Clouds or Silver Linings? Knightian Uncertainty and Climate Change

Abstract

This paper examines the impact of Knightian uncertainty upon optimal climate policy through the prism of a continuous-time real option modelling framework. We analytically determine optimal intertemporal climate policies under ambiguity. Additionally, numerical simulations are provided to illustrate the properties of the model. The results indicate that increasing Knightian uncertainty accelerates climate policy, i.e. policymakers become more reluctant to postpone the timing of climate policies into the future.

JEL-Code: C610, D810, Q540.

Keywords: climate change, Knightian uncertainty, κ ambiguity, real options.

Yu-Fu Chen
Economic Studies
University of Dundee
Dundee / UK
y.f.chen@dundee.ac.uk

Michael Funke
Department of Economics
Hamburg University
Hamburg / Germany
funke@econ.uni-hamburg.de

Nicole Glanemann
Department of Economics
Hamburg University
Hamburg / Germany
nicole.glanemann@wiso.uni-hamburg.de

June 2011
The research was supported through the Cluster of Excellence "CliSAP", University of Hamburg, funded by the German Science Foundation (DFG).

1 Introduction

The future dynamics of greenhouse gas emissions, and their implications for global climate conditions in the future, will be shaped by the way in which policymakers respond to climate information, react to model uncertainty, and derive resultant mitigation decisions. When governments make climate policy decisions, they do not have complete confidence in the probability measure they utilise as a description of future climate uncertainty. Given the enormous complexity of the nonlinear physical system, they may think other probability measures divergent from their own measure are also possible. Such uncertainty, characterized not by a single probability measure but a set of probability measures, is called Knightian uncertainty. In contrast, uncertainty that is reducible to a single probability measure with known parameters is usually referred to as risk. Given the deep and irreducible uncertainties in the processes and implications of climate change, along with the many economic complexities that climate adaptation and mitigation decisions entail, standard tools of policy analysis are often not up to the task. The evolution of the IPCC guidelines on risk and uncertainties from the 3rd to the 4th report can be read as a move away from a purely probabilistic view of risk, to include more complex aspects of uncertainty.[1] Continuing along the same line, we formally develop mathematical tools for situations in which probabilities are not well defined, but not totally unknown either. In other words, we contribute to the climate change literature by developing continuous-time models with irreversibilities, Knightian uncertainty, and imprecise probabilities which appear on the informational radar screen of policymakers. We will then illustrate how the conceptualization of Knightian uncertainty alters optimal behaviour.[2]

Recent theoretical analyses of decisions under uncertainty have highlighted the effects of irreversibility in generating "real options". In these models, the interaction of time-varying uncertainty and irreversibility leads to a range of inaction where policymakers refer to "wait and see" rather than undertaking a costly action with uncertain consequences. We employ this recent literature and interpret climate policies as consisting of a portfolio of options. The general idea underpinning the view that climate policies are option-rights is that climate policy can be seen as analogous in its nature to the purchase of a financial call option, where the investor pays a premium price in order to get the right to buy an asset for some time at a predetermined price (exercise price), and eventually different from the spot market price of the asset. In this analogy, the policymaker, through his/her climate policy decision, pays

[1] A number of methods have been employed to provide information about future climate dynamics. Golub et al. (2011) have recently provided a non-technical summary of alternative approaches modelling uncertainty in the economics of climate change.

[2] The tools developed in this paper have a clear importance for the potential impact of decadal variability, predictability and prediction. They provide a better understanding of how a change in Knightian uncertainty on the basis of decadal predictions will affect decision-making. For example, increased information about the unfolding of climate dynamics may change the course of investment in adaptation and mitigation technologies as well as the willingness to join in various versions of bi-, multi- or global climate agreements.

a price which gives him/her the right to use a mitigation strategy, now or in the future, in return for lower damages. Taking into account this options-based approach, the calculus of suitability cannot be done simply applying the net present value rule, but rather has to consider the following three salient characteristics of the environmental policy decision: (i) there is uncertainty about future payoffs from climate policies; (ii) waiting allows policymakers to gather new information on the uncertain future; and (iii) climate policies are at least partially irreversible. These characteristics are encapsulated in the concept of real option models.[3] This strand of literature now constitutes a significant branch of the climate economics literature.

A limited, but growing, strand of literature – particularly in mathematical economics - has extended the real options approach to analyse the interplay of irreversibility and uncertainty under Knightian uncertainty. A first axiomatic foundation of Knightian uncertainty or ambiguity was given by Gilboa and Schmeidler (1989). The impact of Knightian uncertainty on optimal timing decisions was further investigated by Nishimura and Ozaki (2007) and Trojanowska and Kort (2010) in continuous-time models. Recently, Asano (2010) and Vardas and Xepapadeas (2010) have transferred these theoretical advances into environmental economic issues. In this paper, we expand the paper by Pindyck (2009) on uncertain outcomes and climate change policy focussing on the impact of tail effects on Knightian uncertainty in a continuous-time setting. In particular, we shall investigate the impact of Knightian uncertainty on the optimal climate threshold policies and their values. This anchors our approach in the existing literature.

The remainder of the paper is organized as follows. In Section 2, the comprehensive modelling set-up is presented. The framework incorporates cross-discipline interactions in order to derive dynamically optimal policy responses to Knightian uncertainty. Subsequently, in Section 3 we illustrate the working of the model through numerical exercises and examine the sensitivity of the main results with respect to key parameters. The paper concludes in Section 4 with a brief summary and suggestions for further research. Omitted details of several derivations are provided in appendices.

2 The Model

Over the last decades, climate models have been developed to an impressive level of complexity. Over a similar period, there has been growing interest in the uncertainty of future climate scenarios. Future climate predictions are uncertain because both the initial conditions and the computational representation of the known equations of motion of the natural system are uncertain. To aid future climate policy decisions, accurate quantitative descriptions of the

[3] Concise surveys of the real options literature are provided by Bertola (2010), Dixit and Pindyck (1994) and Stokey (2009).

uncertainty in climate outcomes under various possible policies and scenarios are needed. Of course, the multidisciplinary nature of the field presents a challenge. This requires integrating different natural and social sciences modelling paradigms traditions in a unified decision tool. Here, we have decided to extend the modelling framework of Pindyck (2009, 2010) that embodies, in a simplified way, all essential ingredients by allowing for real options under Knightian uncertainty. The stochastic dynamic programming framework quantifies scientific uncertainties to the extent possible, and explains the potential implications of Knightian uncertainty for the outcomes of concern to the policymakers. It should be noted that the most obvious challenge along the way is to minimise complexity so that the model setup under massive uncertainty is still tractable.[4]

Let's first focus our attention on the expected temperature change $\Delta\mathbb{T}$. What will the world be like in 50 or 100 years when climate change may become acute? Most climate change scenarios project that greenhouse gas concentrations G_t will increase through 2100 with a continued increase in average global temperatures. How much and how quickly the temperature will increase remains unknown given the uncertainty of future greenhouse gas, aerosol emissions and the Earth's response to changing conditions. The Intergovernmental Panel on Climate Change (IPCC, 2007) has estimated that the Earth's temperature is likely to increase within the interval $1.1°C < \Delta\mathbb{T} < 6.4°\text{C}$ by the end of the 21st century, relative to 1980-1990 level, with a best estimate of $1.8°C < \Delta\mathbb{T} < 4.0°\text{C}$, with a long tail of small but finite probabilities of large temperature increases.[5] So although the basics of global warming are not in scientific dispute, the uncertainties about the future state of nature are immense and the range of uncertainty still is a factor of 2-5. This illustrates the scale of the problem. How should policymakers respond to that kind of large-scale uncertainty?

In the following we give a formal exposition of the model. We first focus on future temperature changes. What can be expected from some specified increase in the concentration of greenhouse gases like carbon dioxide? In order to model future warming, we adopt the commonly used climate sensitivity function in Weitzman (2009a) and Pindyck (2009, 2010).

$$d\Delta\mathbb{T}_t = m_1 \left(\frac{\ln (G_t/G_0)}{\ln 2} - m_2 \Delta\mathbb{T}_t \right) dt = m_1 \left(1 - m_2 \Delta\mathbb{T}_t\right) dt, \tag{1}$$

where G_0 is the inherited pre-industrial baseline level of greenhouse gas, and m_1 and m_2 are positive parameters; with a further assumption that G_t initially doubles to $2G_0$. Equation (1) shows that the changes in temperature are a mean-reverting process with a moving target, if

[4]The plethora of potentially significant contributions to overall atmospheric heat balance that are not treated in the simple model used here includes changes in other well-mixed greenhouse gases, ozone, snow albedo, cloud cover, solar irradiance, and aerosols. From this list, it should be clear that the objectives of the present paper are limited ones. A more complete assessment of outcome probabilities would include detailed models of the past and future of each of these effects.

[5]The gradual and continuous temperature increase reflects the strong inertia in the climate system which will expose the earth to some degree of warming irrespective of what policymakers do to curb emissions in the future.

the green house concentration keeps rising. Let H be the time horizon for which we assume that $\Delta\mathbb{T}_t = \Delta\mathbb{T}_H$ at $t = H$ and $\Delta\mathbb{T}_t \rightarrow 2\Delta\mathbb{T}_H$ as $t \rightarrow \infty$, which means that $2\Delta\mathbb{T}_H = 1/m_2$.[6] This implies that the change in temperature is given by

$$\Delta\mathbb{T}_t = e^{-m_1 m_2 t}\Delta\mathbb{T}_0 - \left(\frac{e^{-m_1 m_2 t}}{m_2} - \frac{1}{m_2}\right) = 2\Delta\mathbb{T}_H\left(1 - e^{-at}\right), \tag{2}$$

where the initial value for $\Delta\mathbb{T}_0$ is set to zero, and $a = m_1 m_2$ is equivalent to the adjustment speed of the mean-reverting process of $\Delta\mathbb{T}_t$ approaching the eventual temperature change $2\Delta\mathbb{T}_H$. Using the assumption that $\Delta\mathbb{T}_t = \Delta\mathbb{T}_H$ at $t = H$, it follows that

$$\Delta\mathbb{T}_t = 2\Delta\mathbb{T}_H\left(1 - (1/2)^{t/H}\right), \tag{3}$$

which is the same as in Pindyck (2009, 2010) with $\ln(1/2)/H = -\alpha$.[7] It is easy to verify that due to $2\Delta\mathbb{T}_H = 1/m_2$ and $\ln(1/2)/H = -a = -m_1 m_2$, it follows that

$$d\Delta\mathbb{T}_t = \frac{\ln(2)}{H}\left(2\Delta\mathbb{T}_H - \Delta\mathbb{T}_t\right)dt, \tag{4}$$

and

$$\Delta\mathbb{T}_t = 2\Delta\mathbb{T}_H\left(1 - e^{-\frac{\ln 2}{H}t}\right), \tag{5}$$

where $\ln(2)/H$ denotes the adjustment speed of changes in temperature to the eventual changes in temperature $2\Delta\mathbb{T}_H$.[8] Equation (4) is convenient to use in the real options setting, because the differential equation allows one to derive the corresponding partial differential equation related to the real options terms and thus to solve the optimal stopping problem in a straightforward way. For analytical convenience, we assume that the policymaker solves the following isoelastic objective function under climate uncertainty:

$$W = E\left[\int_{t=0}^{\infty} \frac{C_s^{1-\delta}}{1-\delta}e^{-rs}ds\right], \tag{6}$$

where $E[.]$ is the expectation operator, C is the aggregate consumption over time with the initial value normalised to 1, $\delta \geq 0$ is the inverse of the intertemporal elasticity of substitution,

[6] Equation (1) indicates that the eventual target temperature is $1/m_2$ and the adjustment speed is $m_1 m_2$, since equations (1) can be written as $d\Delta\mathbb{T}_t = m_1 m_2 \left(1/m_2 - \Delta\mathbb{T}_t\right)dt = (\text{adjustment speed})\,(\text{mean} - \Delta\mathbb{T}_t)\,dt$.

[7] Substituting $\Delta\mathbb{T}_t = \Delta\mathbb{T}_H$ at $t = H$ back into equation (2) gives $\ln(1/2)/H = -\alpha$ and hence equation (3) is obtained.

[8] There is considerable a priori uncertainty in the probability and scale of climate change, but at least there are historical time series data available to calibrate probability distributions for parameters important in modelling climate sensitivity. On the other hand, based on current knowledge there is a large a priori uncertainty concerning when dramatic technological breakthroughs might occur and how much impact they will have, so allowing for such possibilities should increase the spread of outcomes for global carbon emissions and their consequences.

and r is the risk-free social utility discount rate. In the simplest form, the level of consumption is equivalent to the level of GDP.[9]

There are countless estimates regarding the impact of climate change. Instead of trying to model climate impacts in any detail, we keep the problem analytically simple by assuming that damages depend only on the temperature change, which is chosen as a measure of climate change. To be precise, following Pindyck (2009, 2010) we assume that the damage from warming and the associated physical impacts of climate change as a fraction of GDP is implied by the exponential loss function

$$L\left(\Delta \mathbb{T}_t\right) = e^{-X_t\left(\Delta \mathbb{T}_t\right)^2}, \tag{7}$$

where X_t is a (positive) stochastic damage function parameter determining the sensitivity of losses to changes in temperature, $0 < L\left(\Delta \mathbb{T}_t\right) \leq 1$ and $\partial L / \partial \left(\Delta \mathbb{T}\right) < 0$. This yields GDP at time t net of damage from warming in the order of $L(\Delta \mathbb{T}_t) GDP_t$, i.e. climate-induced damages result in less GDP, and hence less consumption.[10] It follows that:

$$W = E\left[\int\limits_{t=0}^{\infty} \frac{\left(e^{-X_s\left(\Delta \mathbb{T}_s\right)^2} C_s\right)^{1-\delta}}{1-\delta} e^{-rs} d\mathrm{s}\right]. \tag{8}$$

The standard real options approach emphasizes the importance of uncertainty in determining option value and timing of option exercise. However, the standard real options approach rules out the situation where policymakers are unsure about the likelihoods of future events. It typically adopts strong assumptions about policymakers' beliefs and no distinction between risk and uncertainty is made. The usual prescription for decision making under risk then is to select an action that maximises expected utility. This is assumed although the knowledge of climate dynamics is still far from conclusive.[11] New modelling techniques in natural science and greater computing power provide more details and finer distinctions, but not necessarily more accurate predictions. In the more realistic Knightian uncertainty scenario,

[9] In our model framework we treat the world as a single entity in the interest of brevity. The world climate policy equilibrium can be constructed as a symmetric Nash equilibrium in mitigation strategies. The equilibrium can be determined by simply looking at the single country policy which is defined ignoring the other countries' mitigation policy decisions [Leahy (1993)].

[10] Due the scarcity of empirical information about the magnitude of the damages in question, the shape of the damage function is somewhat arbitrary. Pindyck (2009) has assumed the exponential function $L(\Delta \mathbb{T}) = \exp[-\beta(\Delta \mathbb{T}2)]$, where β follows a gamma distribution. This implies that future damages are fully captured by the probabilistic outcomes of a given distribution. This concept can be understood as risk. However, the present uncertainty about β also comprises the choice of the probability distribution, which will be tackled in this paper.

[11] One has to admit that despite more observations, more sophisticated coupled climate models and substantial increases in computing power, climate projections have not narrowed appreciably over the last two decades. Indeed, it has been speculated that foreseeable improvements in the understanding of the underlying physical processes will probably not lead to large reductions in climate sensitivity uncertainty. See Roe and Baker (2007).

policies therefore become more complex, as now the policymakers carry a set of probability measures for future climate change and consequently every policy measure is associated with an interval of expected costs. This implies that it would be more appropriate to describe the process of X_t using a set of probability measures, not just one measure such as a geometric Brownian motion with a drift term as often used in real options. In other words, the Knightian version of the real options models differs from the plain vanilla real options model by having an entire set of subjective probability distributions. Modelling Knightian uncertainty is a non-trivial task in general. To incorporate a situation where policymakers are unable to assign a precise probability to future alternatives, we use the Knightian uncertainty modelling approach developed by Nishimura and Ozaki (2007). In their comprehensive representation of Knightian uncertainty, unresolved processes are represented by computationally efficient stochastic-dynamic schemes. We introduce their treatment of Knightian uncertainty below.
To formalize the concept, let $(B_t)_{0\leq t\leq T}$ be a standard Brownian motion on $(\Omega, \mathscr{F}_T, P)$ that is endowed with the standard filtration $(\mathscr{F}_t)_{0\leq t\leq T}$ for (B_t). Consider a real-valued stochastic process $(X_t)_{0\leq t\leq T}$ that is generated by the Brownian motion with drift α and standard deviation σ:

$$dX_t = \alpha X_t dt + \sigma X_t dB_t. \tag{9}$$

In equation (9) the particular probability measure P is regarded as capturing the true nature of the underlying process.[12] This, however, is highly unlikely, as this would imply that policymakers are absolutely certain about the probability distribution that describes the future development of $(X_t)_{0\leq t\leq T}$. Unlike this standard case, Knightian uncertainty describes how policymakers form ambiguous beliefs. Thereby a set $\mathscr{P}$ of probability measures is assumed to comprise likely candidates to map the future dynamics.
Technically spoken, these measures are generated from P by means of density generators, θ.[13] By restricting the density generators to a certain range like a real-valued interval $[-\kappa, \kappa]$, we are enabled to confine the range of deviations from the original measure P. The broader this interval is, the larger the set of probability measures, $\mathscr{P} = \left\{Q^\theta \,|\theta \in [-\kappa, \kappa] \right\}$, and thus the

[12]The Brownian motion in equation (9) is a reasonable approximation and we share this assumption with most of the existing literature.I would give a quotation for the assumption that Brownian motions describe the damages by warming well.

[13]Assume a stochastic process $(\theta)_{0\leq t\leq T}$ that is real-valued, measurable and $(\mathscr{F}_t)$-adapted. Furthermore it is twice integrable, hence $\theta := (\theta)_{0\leq t\leq T} \in \mathcal{L}^2 \subset \mathcal{L}$. Define $\left(z_t^\theta\right)_{0\leq t\leq T}$ by $z_t^\theta = e^{\left(-\frac{1}{2}\int_0^t \theta_s^2 ds - \int_0^t \theta_s dB_s\right)} \quad \forall t \geq 0$. Note that the stochastic integral $\int_0^t \theta_s dB_s$ is well-defined for each t, as $\theta \in \mathcal{L}$. A stochastic process $\theta \in \mathcal{L}$ is a density generator, if $\left(z_t^\theta\right)_{0\leq t\leq T}$ is a $(\mathscr{F}_t)$-martingale. Using a density generator θ another probability measure Q^θ on $(\Omega, \mathscr{F}_T)$ can be generated from P by

$$Q^\theta(A) = \int_A z_T^\theta dP \quad \forall A \in \mathscr{F}_T.$$

Note that any probability measure that is thus defined is called equivalent to P .

higher the degree of ambiguity. This specific notion of confining the density generators to an interval $[-\kappa, \kappa]$ is named κ-ignorance by Chen and Epstein (2002), who applied this to a different field of research.
Endowed with this concept we can now define a stochastic processes $(B_t^\theta)_{0\leq t\leq T}$ by

$$B_t^\theta = B_t + t\theta \tag{10}$$

for each $\theta \in [-\kappa, \kappa]$. As Girsanov's theorem shows, each process $(B_t^\theta)_{0\leq t\leq T}$ defined as above is a standard Brownian motion with respect to Q^θ on $(\Omega, \mathscr{F}_T, Q^\theta)$. Inserting the definition of $(B_t^\theta)_{0\leq t\leq T}$ into equation (9), we obtain for every $\theta \in [-\kappa, \kappa]$

$$dX_t = (\alpha - \sigma\theta)\, X_t dt + \sigma X_t dB_t^\theta. \tag{11}$$

Equation (11) displays all stochastic differential equations and thus all future developments of $(X_t)_{0\leq t\leq T}$ that the decision maker thinks feasible. Note that the implementation of Knightian uncertainty implies different drift but not volatility terms.
Knightian uncertainty allows to assume that the policymaker is uncertainty–averse, which makes her consider the worst case scenario. As $e^{-X_t(\Delta\mathbb{T}_t)^2}GDP_t$ is calculated as GDP net of damages, the worst case scenario is described by the largest value of X_t. As an illustration and in order to gain an intuition we have numerically simulated equation (6) and (9) for a time period of 200 years for $\Delta\mathbb{T}_H = 1.9°\text{C}$ versus $\Delta\mathbb{T}_H = 3.4°\text{C}$ (equivalent to pre-industry levels of 2.5°C versus 4°C) of warming and three alternative drift terms. The character of the impact function (6) for various drift terms is shown in Figure 1. The various graphs indicate the forces at play in our analysis. Two effects must be recognized. First, the highest value of the drift term generates the maximum of $1 - L(\Delta\mathbb{T}_t)$ and therefore the minimum of GDP_t net of damages,, which is in line with the our above made considerations about uncertainty-aversion. Second, as can be seen the function $L(\Delta\mathbb{T}_t)$ spreads out considerably for higher temperature increases. After 100 years and for $\Delta\mathbb{T} = 3.4°\text{C}$ the damage is $0.09154 = 9.15$ percent of GDP.[14]
After understanding the process of X_t, we can now discuss the optimal response to climate change under Knightian uncertainty. If the decision maker conducts no climate policy – referred to as the business asusual approach - and faces Knightian uncertainty in equation (11), then the resulting intertemporal welfare, W^{N}, with consumption growing at a rate g_0

[14] The calibrated damages from warming are in the range of previous estimates. Weitzman (2009b) has assumed damage costs of 1.7 percent of GDP for 2.5°C of warming – a level that is considered to be a threshold for danger. For higher temperature increases he has assumed rapidly increasing damages of 9 (25) percent of GDP for 4°C (5°C) of warming. Millner et al. (2010) have assumed damages of 1.7 (6.5) percent of GDP for 2.5°C (5°C) of warming.

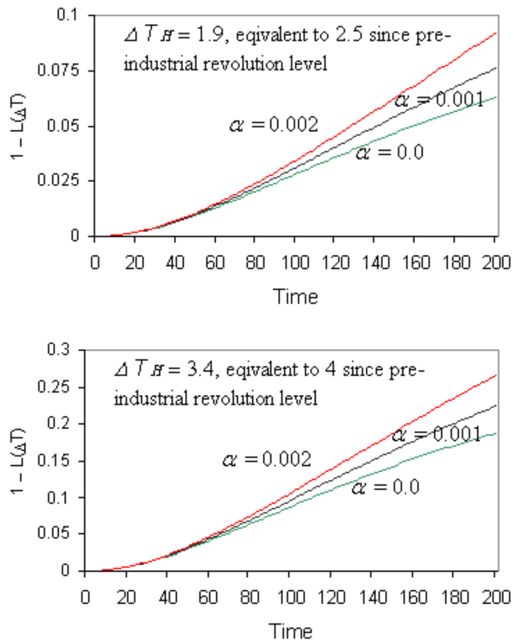

Figure 1: Simulated Damages $1 - L(\Delta\mathbb{T}_t)$ Due To Global Warming in Percent of GDP. The Initial Value for X is $X_0 = 0.008$ and $H = 100$. The Simulated Time Series are Computed Ignoring the Uncertainty Part of Equation (11), i.e. $dX_t = \alpha X_t dt$.

and initial consumption normalised as 1 is determined as

$$
\begin{aligned}
W^{\mathrm{N}}\left(X, \Delta\mathbb{T}; \Delta\mathbb{T}_H\right) &= \min_{Q^\theta \in \mathscr{P}} E^{Q^\theta}\left[\int_{t=0}^{\infty} \frac{\left(e^{-X_s(\Delta\mathbb{T}_s)^2} C_s\right)^{1-\delta}}{1-\delta} e^{-rs} ds \middle| \mathscr{F}_t\right] \\
&= \frac{1}{1-\delta} \min_{Q^\theta \in \mathscr{P}} E^{Q^\theta}\left[\int_{t=0}^{\infty} e^{-X_s(1-\delta)(\Delta\mathbb{T}_s)^2} e^{-(r-(1-\delta)g_0)s} ds \middle| \mathscr{F}_t\right], \qquad (12)
\end{aligned}
$$

s.t. equations (4) and (11), where "N" refers to the no-actions-taken approach, $r - (1-\delta)\, g_0$ is assumed to be positive, and $E^{Q^\theta}\left[\cdot\, |\mathscr{F}_t\right]$ represents the expectation value with respect to $Q^\theta \in \mathscr{P}$ conditional on $\mathscr{F}_t$.[15] The first equation holds as uncertainty aversion implies that the policymaker is interested in the lowest welfare value.[16]

For the sake of analytical tractability, we apply a Taylor series expansion to $e^{-X(1-\delta)\Delta\mathbb{T}^2}$ in

[15] For reasons of mathematical tractability we assume that the continuous Knightian uncertainty is independent of time and therefore the planning horizon is infinite. The reasoning for the perpetual assumption is that the underlying time scales in the natural climate system are much longer than those in the economic system. Technically, we consider $T \to \infty$ in the above made introduction to the concept of Knightian uncertainty.

[16] First the uncertainty-averse politician takes only the probability measure into consideration that creates the worst outcomes for the welfare. Then she strives to find the policy strategy that maximizes this 'worst-case welfare function'. The maxmin nature of the problem links the analysis with contributions on robust control. See, for example, Funke and Paetz (2011).

equation (7) such that

$$e^{-X_s(1-\delta)\Delta\mathbb{T}_s^2} \cong 1 - X_s(1-\delta)\Delta\mathbb{T}_s^2 + \frac{1}{2}\left(X_s(1-\delta)\Delta\mathbb{T}_s^2\right)^2, \tag{13}$$

where $0 < L(\Delta\mathbb{T}_t) \leq 1$ and $\partial L/\partial(\Delta\mathbb{T}) < 0$ still hold. By inserting (13) into (12) we thus obtain

(14)

$$W^{\mathrm{N}}(X,\Delta\mathbb{T};\Delta\mathbb{T}_H)$$
$$= \frac{1}{1-\delta}\min_{Q^\theta\in\mathscr{P}} E^{Q^\theta}\left[\int_{t=0}^{\infty}\left(1 - X_s(1-\delta)\Delta\mathbb{T}_s^2 + \frac{1}{2}\left(X_s(1-\delta)\Delta\mathbb{T}_s^2\right)^2\right)e^{-(r-(1-\delta)g_0)s}ds\middle|\mathscr{F}_t\right],$$

s.t. equation (4). Using Itô's Lemma and following the standard dynamic programming argument, we formulate the problem in terms of the Hamilton-Jacobi-Bellman equation[17]

$$\begin{aligned}(r-(1-\delta)g_0)W^{\mathrm{N}} &= 1 - X^*(1-\delta)\Delta\mathbb{T}^2 + \frac{1}{2}\left(X^*(1-\delta)\Delta\mathbb{T}^2\right)\\ &+\left(\frac{\ln(2)}{H}(2\Delta\mathbb{T}_H-\Delta\mathbb{T})\right)\frac{\partial W^{\mathrm{N}}}{\partial\Delta\mathbb{T}} + (\alpha+\kappa\sigma)X^*\frac{\partial W^{\mathrm{N}}}{\partial X^*} + \frac{1}{2}\sigma^2X^{*2}\frac{\partial^2W^{\mathrm{N}}}{\partial X^{*2}}.\end{aligned} \tag{15}$$

The asterisk represents the density generator $-\kappa$, meaning that Q^* is generated by $-\kappa$ and the stochastic process X^* is defined by inserting $-\kappa$ into equation (11):

$$dX_t^* = (\alpha+\sigma\kappa)X_t^*dt + \sigma X_t^*dB_t^{-\kappa}. \tag{16}$$

Equation (15) describes the model fully. For policies to be optimal, equation (15) must hold. The solution of equation (15) is the sum of a particular and general solution. The particular solution W^{NP} is obtained by integrating the integral for W^{N} of equation (14) without considering possible policy intervention. Therefore, the real options terms are not exercised. It is straightforward to explain W^{NP} as the value of business-as-usual, without the policymaker's intervention through the exercising the real options to reduce the green house gas emissions leading to a cap in future temperature changes $\Delta\mathbb{T}_H$. The general/homogenous solutions or real options solutions W^{NG} are obtained by focusing attention on the homogenous part of equation (15) such that

$$\begin{aligned}(r-(1-\delta)g_0)W^{\mathrm{NG}} &= \left(\frac{\ln(2)}{H}(2\Delta\mathbb{T}_H-\Delta\mathbb{T})\right)\frac{\partial W^{\mathrm{NG}}}{\partial\Delta\mathbb{T}}\\ &+(\alpha+\kappa\sigma)X^*\frac{\partial W^{\mathrm{NG}}}{\partial X^*} + \frac{1}{2}\sigma^2X^{*2}\frac{\partial^2W^{\mathrm{NG}}}{\partial X^{*2}}.\end{aligned} \tag{17}$$

Let's assume that the policymaker is willing to pay annual abatements costs $w(\tau)$ as a percentage of GDP to limit the temperature increase at $t = H$ to less than or equal to τ:

[17]For the derivation please see Appendix A.

$\Delta\mathbb{T}_H \leq \tau$.[18] Due to Itô's lemma, the intertemporal welfare function of taking action to reduce the green house gas emission, W^{A}, is then given by

$$
\begin{aligned}
(r-(1-\delta)\,g_0)\,W^{\mathrm{A}} =& 1 - X^*\,(1-\delta)\,\Delta\mathbb{T}^2 + \frac{1}{2}\left(X^*\,(1-\delta)\,\Delta\mathbb{T}^2\right) \\
&+ \left(\frac{\ln(2)}{H}\,(2\tau-\Delta\mathbb{T})\right)\frac{\partial W^{\mathrm{A}}}{\partial\Delta\mathbb{T}} + (\alpha+\kappa\sigma)\,X^*\frac{\partial W^{\mathrm{A}}}{\partial X^*} + \frac{1}{2}\sigma^2 X^{*2}\frac{\partial^2 W^{\mathrm{A}}}{\partial X^{*2}},
\end{aligned} \tag{18}
$$

which is derived from the following integral

(19)

$$
W^{\mathrm{A}}\,(t=0,X,\Delta\mathbb{T};\tau) = \frac{1}{1-\delta}\times
$$
$$
E^{Q^*}\left[(1-w\,(\tau))^{1-\delta}\int\limits_{t=0}^{\infty}\left(1-X^*_s\,(1-\delta)\,\Delta\mathbb{T}_s^2+\frac{1}{2}\left(X^*_s\,(1-\delta)\,\Delta\mathbb{T}_s^2\right)\right)e^{-(r-(1-\delta)g_0)s}dt\,\middle|\,\mathscr{F}_t\right],
$$

s.t. (20) that is

$$
d\Delta\mathbb{T}_s = \frac{\ln(2)}{H}\,(2\tau-\Delta\mathbb{T}_s)\,ds, \tag{20}
$$

where equation (20) is a variant of equation (4) by replacing $\Delta\mathbb{T}_H$ with τ. If climate policy is time-consistent, then the solutions to W^{A} can be obtained by integrating equation (20) directly. In this case, the thresholds for X^* of taking actions to limit the future temperature increase to less than or equal to τ at $t=H$ are then computed from the identity

$$
W\,(\text{taking action}) = W\,(\text{business as usual}) + \text{Real options}. \tag{21}
$$

Substituting, we have

$$
W^{\mathrm{A}}\left(\bar{X},\Delta\mathbb{T};\tau\right) = W^{\mathrm{NP}}\left(\bar{X},\Delta\mathbb{T};\Delta\mathbb{T}_H\right) + W^{\mathrm{NG}}\left(\bar{X},\Delta\mathbb{T};\Delta\mathbb{T}_H\right), \tag{22}
$$

where $\bar{X}$ denotes the thresholds at which the policy-maker would take action by exercising the real options today and committing paying annual abatement costs $w\,(\tau)$ in percent of GDP to limit the future temperature increase to less than τ at $t=H$. On the contrary, exercising of the real options $W^{\mathrm{NG}}\left(\bar{X},\Delta\mathbb{T};\Delta\mathbb{T}_H\right)$ implies that the policymaker forgoes the option to wait and act later as more information about X_t becomes available.

The next step is to solve the particular integrals of W^{NP} and W^{A}, and real options W^{NG}. As there are no uncertain terms for the processes of changes in temperatures $\Delta\mathbb{T}_t$, we can use equation (5) to obtain

$$
\Delta\mathbb{T}_t = 2\tau\left(1-e^{-\frac{\ln 2}{H}t}\right). \tag{23}
$$

[18] In practical terms, this means that the policymaker reduces G_t in equation (1) so that the increase in temperature is limited to less than τ at $t=H$.

As shown in Appendix B the following particular integrals result from Itô's Lemma:

(24)
$$W^{\mathrm{NP}}\left(X,\Delta\mathbb{T};\Delta\mathbb{T}_H\right)=\frac{1}{1-\delta}\left[\frac{1}{r-\left(1-\delta\right)g_0}-4\Delta\mathbb{T}_H^2\left(1-\delta\right)\gamma_1X^*+8\Delta\mathbb{T}_H^4\left(1-\delta\right)^2\gamma_2{X^*}^2\right]$$

(25)
$$W^{\mathrm{A}}\left(X,\Delta\mathbb{T};\tau\right)=\frac{\left(1-w\left(\tau\right)\right)^{1-\delta}}{1-\delta}\left[\frac{1}{r-\left(1-\delta\right)g_0}-4\Delta\tau^2\left(1-\delta\right)\gamma_1X^*+8\Delta\tau^4\left(1-\delta\right)^2\gamma_2{X^*}^2\right].$$

where

$$\begin{aligned}
\gamma_1&=\frac{1}{\eta_1}-\frac{2}{\eta_1+\frac{\ln 2}{H}}+\frac{1}{\eta_1+2\frac{\ln 2}{H}},\\
\gamma_2&=\frac{1}{\eta_2}-\frac{4}{\eta_2+\frac{\ln 2}{H}}+\frac{6}{\eta_2+2\frac{\ln 2}{H}}-\frac{4}{\eta_2+3\frac{\ln 2}{H}}+\frac{1}{\eta_2+4\frac{\ln 2}{H}},\\
\eta_1&=r-\left(1-\delta\right)g_0-\left(\alpha+\kappa\sigma\right),\\
\eta_2&=r-\left(1-\delta\right)g_0-\left(2\left(\alpha+\kappa\sigma\right)+\sigma^2\right).
\end{aligned}$$

Note that it is assumed that both η_1 and η_2 are positive.
After obtaining the analytical particular solutions of equations (24) and (25), we now need to turn our attention to the real options term W^{NG} in equation (17). In Appendix C we show that the general solutions have the forms:

$$W^{\mathrm{NG}}\left(t=0,X,\Delta\mathbb{T};\Delta\mathbb{T}_H\right)=A_1{X^*}^{\beta_1}\left(\Delta\mathbb{T}^2-4\Delta\mathbb{T}_H\Delta\mathbb{T}+4\Delta\mathbb{T}_H^2\right), \tag{26}$$

where β_1 is the positive root of the quadratic characteristic equation

$$\frac{1}{2}\sigma^2\beta\left(\beta+1\right)+\left(\alpha+\kappa\sigma\right)\beta-\left(r-\left(1-\delta\right)g_0+2\left(\frac{\ln\left(2\right)}{H}\right)\right)=0, \tag{27}$$

and A_1 is the unknown parameter to be determined by the value-matching and smooth-pasting conditions. The meaning of equation (26) is straightforward. For a small $\Delta\mathbb{T}_H$ the value of the options to take actions is small – the option of taking action is reduced for less global warming. The effective discount rate for real options is a positive function of $2\ln\left(2\right)/H$. As we know from equation (4), $\ln\left(2\right)/H$ also denotes the adjustment speed of changes in temperature. Higher adjustment speed to the higher temperature (for example, $H=50$ years instead of $H=100$ years) means that the damage is higher and thus the option value is smaller. After obtaining the solutions to equation (22) by applying the value-matching condition, the smooth-pasting condition is given by equalising the derivative of (25) with respect to X^* with the sum of the derivatives of (24) and (26) with respect to X^*. Substituting (24) – (26) back into the value-matching and smooth-pasting conditions yields

$$\begin{aligned}
&4\gamma_1\left(\Delta\mathbb{T}_H^2-\Delta\tau^2\left(1-w\left(\tau\right)\right)^{1-\delta}\right)\bar{X}^*-8\left(1-\delta\right)\gamma_2\left(\Delta\mathbb{T}_H^2-\left(1-w\left(\tau\right)\right)^{1-\delta}\tau^4\right){\bar{X}^*}^2\\
&\qquad=\frac{1-\left(1-w\left(\tau\right)\right)^{1-\delta}}{\left(r-\left(1-\delta\right)g_0\right)\left(1-\delta\right)}+A_1{\bar{X}^*}^{\beta_1}\left(\Delta\mathbb{T}^2-4\Delta\mathbb{T}_H\Delta\mathbb{T}+4\Delta\mathbb{T}_H^2\right),
\end{aligned} \tag{28}$$

and

$$
\begin{aligned}
&4\gamma_1 \left(\Delta\mathbb{T}_H^2 - \Delta\tau^2 \left(1 - w\left(\tau\right)\right)^{1-\delta}\right) - 16\left(1-\delta\right)\gamma_2 \left(\Delta\mathbb{T}_H^2 - \left(1 - w\left(\tau\right)\right)^{1-\delta}\tau^4\right)\bar{X}^* \\
&\qquad = A_1\beta_1 \bar{X}^{*\beta_1 - 1}\left(\Delta\mathbb{T}^2 - 4\Delta\mathbb{T}_H\Delta\mathbb{T} + 4\Delta\mathbb{T}_H^2\right). \qquad (29)
\end{aligned}
$$

So far, our discussion of Knightian uncertainty has been exclusively analytical. With the optimality conditions and the value-matching and smooth-pasting conditions, we can now proceed to the numerical simulations of the model.

3 Numerical Simulations and Results

While the preceding section has laid out the modelling framework, we now focus on a thorough numerical analysis of the model. Several problems occur when mapping the theoretical framework presented above into real-world climate data. Where possible, parameter values are drawn from empirical studies. The determination of some parameters, however, requires the use of judgement, i.e. they reflect a back-of-the-envelope calculation.[19] Therefore, for each parameter a sensitivity analysis over a sufficiently wide grid is performed.[20]

The unit time length corresponds to one year. Our base parameters are $\sigma = 0.075$, $\kappa = 0.02$, $r = 0.04$, $\alpha = 0.0$, $g_0 = 0.01$, $\delta = 0.0$, and $H = 100$. $\Delta\mathbb{T}_H$ is assumed to be 3.4°C which is equivalent to 4 degrees of warming since the pre-industrial level. τ is assumed to be 1.4°C by assumption which is equivalent to 2 degrees of warming compared with the pre-industrial level. Special attention has to be paid to the calibration of $w(\tau)$. The term $w(\tau)$ represents the achievability and costs of climate targets. What are the economic costs of reaching the target of climate stabilisation at no more than 2°C above pre-industrial level by the end of this century? In oder to assess this question, Edenhofer et al. (2010) have compared the energy-environment-economy models MERGE, REMIND, POLES, TIMER and E3MG in a model comparison exercise. In order to improve model comparability, the macroeconomic drivers in the five modelling frameworks employed were harmonised to represent similar economic developments. On the other hand, different views of technology diffusion and different structural assumptions regarding the underlying economic system across the models remained. This helps to shed light on how different modelling assumptions translate into differences in mitigation costs. Low stabilisation crucially depends upon learning and technologies available. Despite different structures employed in the models, four of the five models show a

[19] Despite the increasingly detailed understanding of climate processes from a large body of research, various parameters involved inevitably remain inestimable, except in retrospect.

[20] The calibrated model is not based on detailed time series data in the way econometric models are and does not have the predictive power of the latter. Note, however, that the goal of this paper is not to derive precise quantitative estimates of the impact of Knightian uncertainty, but rather to illustrate the scale of the Knightian uncertainty impact, and to see what we can learn from this framework. We address this point clearly and frankly knowing that economics ultimately is an empirical science. Without empirical evidence, many beautiful theories would be merely that beautiful.

similar pattern in mitigations costs in order to achieve the first-best 400ppm CO2 concentration pathway. After allowing for endogenous technical change and carbon capture and storage with a storage capacity of at least 120 GtC, the mitigation costs are estimated to be approximately 2 percent of worldwide GDP. These costs turned out to be of a similar order of magnitude across the models. We therefore assume that $w(\tau) = 0.02$.

In the real option literature the problem we must solve is referred to as "optimal stopping". The idea is that at any point in time the value of temperature reductions is compared with the expected value of waiting dt, given the available information set and the knowledge of the stochastic processes. First, we consider the thresholds for adopting climate policies, i.e. we calculate the optimal timing of adopting climate policies. The optimal strategy is to stop and adopt the climate policy right now if $X_t^* \geq \bar{X}^*$ and to continue waiting if $X_t^* < \bar{X}^*$, where $\bar{X}^*$ is the threshold value.[21] To start with, in Figure 2 and 3 we focus our attention on the sensitivity of the optimal thresholds of a risk-neutral and uncertainty-averse policymaker with respect to the degree of Knightian uncertainty κ and changes in the degree of risk, i.e. the volatility of the geometric Brownian motion process σ. The solutions for $\kappa = 0$ characterize the situation of a single probability measure and therefore the situation without Knightian uncertainty as in a traditional real option framework.

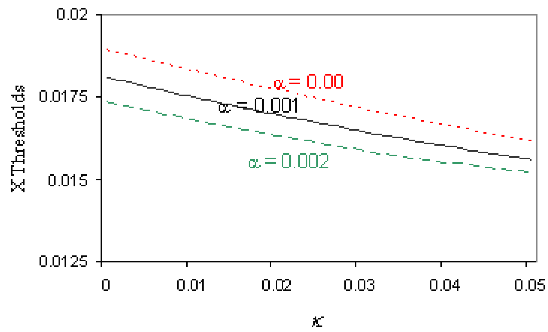

Figure 2: The Climate Policy Thresholds for Alternative κ's and α's

Figure 2 provides a sensitivity analysis of the threshold with respect to κ. The numerical results indicate an acceleration of climate policy for higher degrees of Knightian uncertainty, i.e. increasing ambiguity has an unequivocally positive impact upon the timing of optimal climate policy and shrinks the continuation region where exercising climate policy is sub-optimal. In contrast, Figure 3 indicates that the threshold value at which climate policy is implemented is increasing in the noisiness level σ even though the policymaker is risk neutral. The intuition is that the policymaker can counteract the impact from additional risk by a wait and see attitude for the time being. The case $\kappa = 0$ again represents the case of no Knightian uncertainty. As expected, increased Knightian uncertainty tends to accelerate

[21] It is worth conjecturing that the existence of the no action area sheds light on why policymakers often deem it desirable to stay put, contrary to intuition which stems from thinking in terms of a simple cause and effect framework..

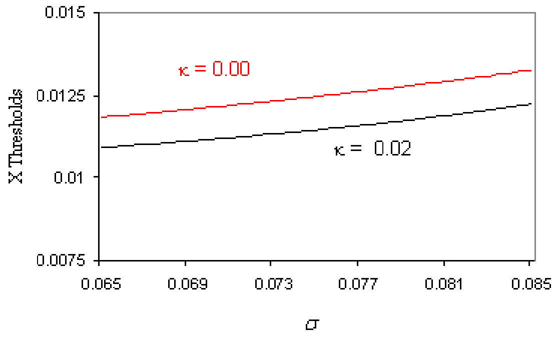

Figure 3: The Climate Policy Thresholds for Alternative σ's and κ's

optimal timing, while increasing risk leads to the opposite response.

Additional observations emerge from a bird's eye examination of the 3-dimensional Figure 4 which helps to visualize the parameter space. The perspective is such that the viewer is looking from the origin from a point high in the positive orthant., i.e. from a low value for all three axis variables. Figure 4 tells essentially the same story. The qualitative result is that as κ increases or/and σ decreases, the threshold plunges downward. Furthermore it is evident from Figure 4 that an increase in κ has a comparatively smaller impact on the climate policy threshold.

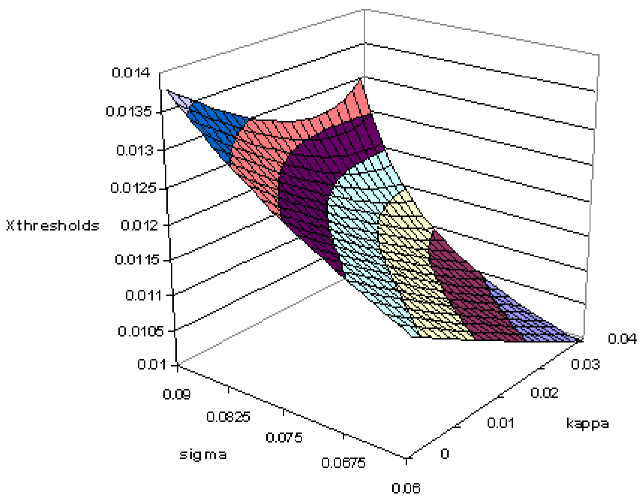

Figure 4: The Impact of Simultaneous Changes in κ and σ Upon the Threshold

The fundamental explanation to this finding lies in the fact that higher Knightian uncertainty decreases the confidence of the policymaker on the credibility of the probability distribution describing the stochastic behaviour of the underlying state variable X_t. Consequently, a rational policymaker becomes more reluctant to postpone the timing of climate policy further into the future on the basis of this vaguer probability distribution. We now put the spotlight on the discount rate

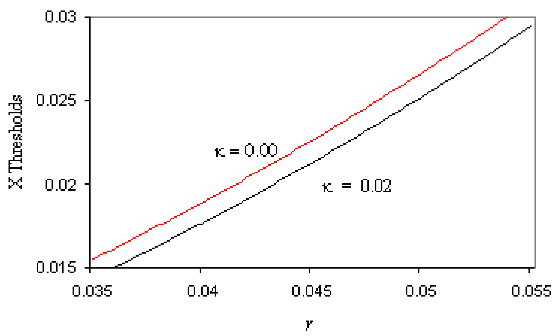

Figure 5: The Climate Policy Thresholds for Alternative Discount Rates r and κ's

To explore the sensitivity to alternative discounting assumptions, we employ a range of $0.035 < r < 0.055$. As expected, the results in Figure 5 affirm the view that higher discount rates will bolster the reasons for taking a "wait and see" attitude" towards climate policy. This is due to the fact that for a small value of r the particular integral is a good deal bigger and therefore the intertemporal damage is substantially larger. Conversely, a higher discounting factor will trigger a later adoption and a lower intensity of climate policy. This highlights the importance of attaining a consensus on the discount rate before an appraisal on the optimal timing of policy implementation can be achieved. Once again, we also find that if policymakers face a higher level of Knightian uncertainty, then the option value is lower and the policymaker exercises the option earlier.

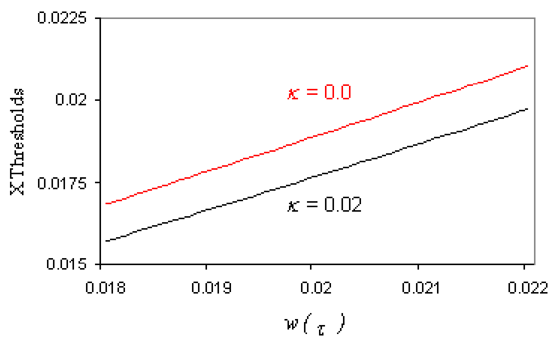

Figure 6: The Climate Policy Thresholds for Alternative Costs of Climate Stabilisation

Figure 6 provides a sensitivity analysis of the thresholds with respect to $w(\tau)$, i.e. we illustrate the impact of alternative climate stabilisation costs upon the threshold. The major result of the simulations is that higher climate stabilisation costs lead to an increase of the no action area, i.e. an increasing $w(\tau)$ increases the climate policy threshold. Intuitively, this makes perfect sense. Higher costs make climate policies less attractive for policymakers, so policymakers hesitate to perform them in the first place. However, the option value of the climate policy opportunity is again lower under Knightian uncertainty than in the standard model. Therefore, an uncertainty averse policymaker acts earlier.

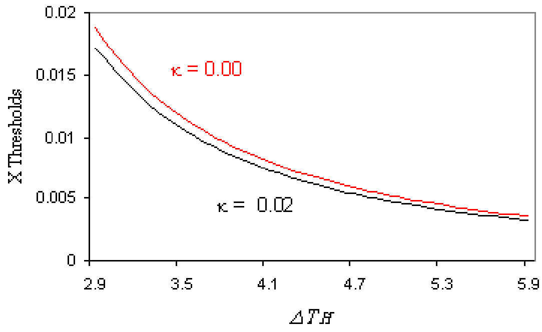

Figure 7: The Impact of Changes in $\Delta\mathbb{T}_H$ Upon the Threshold

Finally we analyse how different expected degrees of warming, i.e. changes in $\Delta\mathbb{T}_H$, affect the threshold. Figure 7 clearly indicates that the tactic to keep options open and await new information rather than undertake climate policy today becomes less attractive. In other words, higher $\Delta\mathbb{T}_H$ values accelerate climate policies by shrinking the no action area.

4 Conclusions

The modelling of Knightian uncertainty is a relatively unchartered area of climate research. In spite of its clear climate policy relevance, few authors have explored the topic yet. While the paper will be of interest to specialists in real option theory, given the policy importance of the issue in hand we also believe that our assessment of the central question motivating our analysis will be of interest to a wider audience of climate scientists and policymakers.

A unifying message from our paper could be stated as follows: We have demonstrated that Knightian uncertainty affects irreversible climate policies in a way which radically differs from the impact of risk, and that Knightian uncertainty accelerates climate policy.[22] This insight holds non-trivial value for decision making. We believe that our application of Knightian uncertainty comes with an advantage and a disadvantage. The advantage is that it allows one to recognize the difference between risk and uncertainty.[23] Thus it provides a more realistic grounding for assessing current climate policy and to derive optimal and rational policy trajectories when fundamental uncertainties and ambiguities are involved.[24] On the

[22]It should be mentioned that the acceleration of climate policy depends upon the underlying dynamics being characterised by a geometric Brownian motion. This model feature is not necessarily valid any more for more general stochastic processes.

[23]To quote from Mastrandrea and Schneider (2004, p. 571) "we do not recommend that our quantitative results be taken literally, but we suggest that our probabilistic framework and methods be taken seriously". Also see Schneider and Mastrandrea (2005).

[24]Some readers may find the ambiguity and the additional layer of uncertainty psychologically disturbing. But if the previously agreed modeling framework was wrong and the certainty about appropriate climate policy unjustified, it seems an improvement.

other hand one has to admit that the comparative static results also have their limitations. First, the numerical results do not account for the fundamental dynamic nature of abatement and mitigation policies.[25] Second, we have focussed on Knightian uncertainty in the damage function. However, there may be further layers of uncertainty in complex climate models about which we have ambiguous beliefs. Our analysis may therefore be considered as a first step and it may be refined in several ways. One future research question is the possibility of tipping points. In addition to a high level of complexity, the major challenge of this extension is the need to incorporate thresholds, discontinuities and sudden switches which remain poorly understood on a theoretical level.[26] Another interesting direction goes towards a more detailed analysis of decadal climate predictions.[27] We hope to take up some of these tasks in our future work and we consider it probable that this research agenda and the conceptual follow-up issues will continue to warrant substantial research effort in the future.

Appendix

A Derivation of equation (12)

First, we to show that the $Q^\theta \in \mathscr{P}$ that minimises the expectation value in equation (14) is generated by $\theta = -\kappa$.

We know that $X_s (1-\delta) \Delta\mathbb{T}_s^2$ has a small value so that $\frac{1}{2}\left(X_s (1-\delta) \Delta\mathbb{T}_s^2\right)^2$ only adds insignificantly to the term in equation (14). We therefore neglect the quadratic term when minimising the expectation value in the following.

Additionally Fubini's theorem for conditional expectations transforms $W^{\mathrm{N}}(X, \Delta\mathbb{T}; \Delta\mathbb{T}_H)$ to

$$\text{(A.1)} \qquad \frac{1}{1-\delta} \min_{Q^\theta \in \mathscr{P}} \int_{t=0}^{\infty} e^{-(r-(1-\delta)g_0)s} E^{Q^\theta}\left[1 - X_s (1-\delta) \Delta\mathbb{T}_s^2 \,|\mathscr{F}_t\right] \, ds.$$

By applying Itô's Lemma to the logarithm of X_s we obtain $\forall\ s \geq 0$:

$$\text{(A.2)} \qquad X_s = X_0 e^{(\alpha - \frac{1}{2}\sigma^2 - \sigma\theta)s + \sigma B_s^\theta} = X_0 e^{(\alpha - \frac{1}{2}\sigma^2 - \sigma\theta)s} e^{\sigma B_s^\theta}.$$

[25] One may also follow a different strategy. Instead of tailoring policies towards one future in particular, one may find institutional arrangements, regulatory policies and technologies of adapting to many possible future climate scenarios.

[26] The climate literature on tipping points is, indeed, a fast growing industry. Unfortunately, there are not any models yet incorporating such nonlinearities into micro-founded decision-making frameworks with Knightian uncertainty. It must be emphasised that the model described here is sufficiently general to study various tipping points. It is only necessary to fine-tune the framework for specific nonlinearities and to embed further stochastic processes.

[27] In Figure 2 – 6 the impact of Knightian uncertainty is "statically" addressed. Hence, we may next aim to study the temporal implications of Knightian uncertainty, and the impact of less medium-run ambiguity resulting from more reliable decadal predictions upon optimal climate policies.

Obviously it holds that

$$X_s = X_0 e^{(\alpha - \frac{1}{2}\sigma^2 - \sigma\theta)s} e^{\sigma B_s^\theta} \leq X_0 e^{(\alpha - \frac{1}{2}\sigma^2 + \sigma\kappa)s} e^{\sigma B_s^\theta} \quad \forall\, s \geq 0, \quad \forall\, \theta \in [-\kappa, \kappa]\,. \tag{A.3}$$

Due to the monotonicity of the conditional expectation value, we obtain

$$\begin{aligned}
&E^{Q^\theta}\left[1 - X_0 e^{(\alpha - \frac{1}{2}\sigma^2 - \sigma\theta)s} e^{\sigma B_s^\theta} (1-\delta)\, \Delta\mathbb{T}_s^2 \,|\mathscr{F}_t\right] \\
&\geq E^{Q^\theta}\left[1 - X_0 e^{(\alpha - \frac{1}{2}\sigma^2 + \sigma\kappa)s} e^{\sigma B_s^\theta} (1-\delta)\, \Delta\mathbb{T}_s^2 \,|\mathscr{F}_t\right] \\
&= \left(1 - X_0 e^{(\alpha - \frac{1}{2}\sigma^2 + \sigma\kappa)s}\right) (1-\delta)\, \Delta\mathbb{T}_s^2 E^{Q^\theta}\left[e^{\sigma B_s^\theta} \,|\mathscr{F}_t\right] \\
&= \left(1 - X_0 e^{(\alpha - \frac{1}{2}\sigma^2 + \sigma\kappa)s}\right) (1-\delta)\, \Delta\mathbb{T}_s^2\; e^{\frac{1}{2}\sigma^2 s} \\
&= \left(1 - X_0 e^{(\alpha - \frac{1}{2}\sigma^2 + \sigma\kappa)s}\right) (1-\delta)\, \Delta\mathbb{T}_s^2\; E^{Q^{-\kappa}}\left[e^{\sigma B_s^{-\kappa}} \,|\mathscr{F}_t\right] \qquad \forall \mathrm{s} \geq 0, \forall\theta \in [-\kappa,\kappa]\,.
\end{aligned} \tag{A.4}$$

Thus, the measure $Q^{-\kappa} \in \mathscr{P}$ minimises the expectation value in (14), which we therefore denote as Q^*. Consequently the process X that results from implementing $\theta = -\kappa$ into equation (11) shall be called X^*.

For the following considerations let $W^{\mathrm{N}}(X, \Delta\mathbb{T}; \Delta\mathbb{T}_H)$ be conveniently abbreviated by W^{N}. The corresponding Hamilton-Jacobi-Bellman equation to equation (14) is as follows (see for example chapter 3.1. in Stokey (2009) as an introduction to the Hamilton-Jacobi-Bellman equation):

$$(r - (1-\delta)\, g_0)\, W^{\mathrm{N}} = 1 - X^*(1-\delta)\, \Delta\mathbb{T}^2 + \frac{1}{2}\left(X^*(1-\delta)\, \Delta\mathbb{T}^2\right) + \frac{1}{dt} E^{Q^*}\left[dW^N \,|\mathscr{F}_t\right]. \tag{A.5}$$

W^{N} is obviously differentiable at least once in $\Delta\mathbb{T}$ and twice in X^*, which allows to apply Itô's Lemma:

$$\begin{aligned}
dW^{\mathrm{N}} &= \frac{\partial W^{\mathrm{N}}}{\partial \Delta\mathbb{T}} d\Delta\mathbb{T} + \frac{\partial W^{\mathrm{N}}}{\partial X^*} dX^* + \frac{\partial^2 W^{\mathrm{N}}}{\partial X^{*2}} (dX^*)^2 \\
&= \frac{\ln(2)}{H}(2\Delta\mathbb{T}_H - \Delta\mathbb{T}_t) \frac{\partial W^{\mathrm{N}}}{\partial \Delta\mathbb{T}} dt + \frac{\partial W^{\mathrm{N}}}{\partial X^*}\left[(\alpha + \sigma\kappa)\, X_t^* dt + \sigma X_t^* dB_t^{-\kappa}\right] \\
&\quad + \frac{1}{2}\sigma^2 X^{*^2} \frac{\partial^2 W^{\mathrm{N}}}{\partial X^{*2}} dt,
\end{aligned} \tag{A.6}$$

by using equation (4) in the text. Taking expectation of (A6) and dividing by dt we obtain

$$\frac{E\left[dW^{\mathrm{N}}\right]}{dt} = \frac{\ln(2)}{H}(2\Delta\mathbb{T}_H - \Delta\mathbb{T}_t) \frac{\partial W^{\mathrm{N}}}{\partial \Delta\mathbb{T}} + (\alpha + \kappa\sigma)\, X_t^* \frac{\partial W^{\mathrm{N}}}{\partial X^*} + \frac{1}{2}\sigma^2 X^{*^2} \frac{\partial^2 W^{\mathrm{N}}}{\partial X^{*2}}. \tag{A.7}$$

Substituting (A7) back to the Hamilton-Jacobi-Bellman equation (A5) gives

$$\begin{aligned}
(r - (1-\delta)\, g_0)\, W^{\mathrm{N}} &= 1 - X^*(1-\delta)\, \Delta\mathbb{T}^2 + \frac{1}{2}\left(X^*(1-\delta)\, \Delta\mathbb{T}^2\right) \\
&\quad + \left(\frac{\ln(2)}{H}(2\Delta\mathbb{T}_H - \Delta\mathbb{T})\right) \frac{\partial W^{\mathrm{N}}}{\partial \Delta\mathbb{T}} + (\alpha + \kappa\sigma)\, X^* \frac{\partial W^{\mathrm{N}}}{\partial X^*} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 W^{\mathrm{N}}}{\partial X^{*2}},
\end{aligned} \tag{A.8}$$

which is equation (15) in the text.

B Particular solutions to W^{NP} for W^{A}

Using equations (14) and (5) yields the following particular integral,

(B.1)

$$W^{\mathrm{NP}}\left(X,\Delta\mathbb{T};\Delta\mathbb{T}_H\right)=\frac{1}{1-\delta}\times$$
$$\int\limits_{t=0}^{\infty}\left[1-\sum_{i=1}^{2}\frac{(-1)^{i+1}}{i!}X^{*^i}e^{\left[i(\alpha+\kappa\sigma)+\frac{1}{2}i(i-1)\sigma^2\right]s}(1-\delta)^i\left(2\Delta\mathbb{T}_H\left(1-e^{-\frac{\ln 2}{H}s}\right)\right)^{2i}\right]e^{-(r-(1-\delta)g_0)s}ds.$$

In the same manner we employ equation (19) and (23) to derive

(B.2)

$$W^{\mathrm{A}}\left(X,\Delta\mathbb{T};\tau\right)=\frac{\left(1-w\left(\tau\right)\right)^{1-\delta}}{1-\delta}\times$$
$$\int\limits_{t=0}^{\infty}\left[1-\sum_{i=1}^{2}\frac{(-1)^{i+1}}{i!}X^{*^i}e^{\left[i(\alpha+\kappa\sigma)+\frac{1}{2}i(i-1)\sigma^2\right]s}(1-\delta)^i\left(2\Delta\mathbb{T}_H\left(1-e^{-\frac{\ln 2}{H}s}\right)\right)^{2i}\right]e^{-(r-(1-\delta)g_0)s}ds.$$

Equations (B1) and (B2) result from Itô's Lemma which means that equation (A2) with $\theta=-\kappa$ is applied to equation (14) and (19), respectively. Furthermore please note that $E^{Q^{-\kappa}}\left[e^{\sigma B_s^{-\kappa}}|\mathscr{F}_t\right]=e^{\frac{1}{2}\sigma^2 s}$. By expanding the terms

$$\left(1-e^{-\frac{\ln 2}{H}t}\right)^2=1-2e^{-\frac{\ln 2}{H}t}+e^{-2\frac{\ln 2}{H}t} \tag{B.3}$$

and

$$\left(1-e^{-\frac{\ln 2}{H}t}\right)^4=1-4e^{-\frac{\ln 2}{H}t}+6e^{-2\frac{\ln 2}{H}t}-4e^{-3\frac{\ln 2}{H}t}+e^{-4\frac{\ln 2}{H}t}, \tag{B.4}$$

we obtain

(B.5)

$$\left[1-\sum_{i=1}^{2}\frac{(-1)^{i+1}}{i!}X^{*^i}e^{\left[i(\alpha+\kappa\sigma)+\frac{1}{2}i(i-1)\sigma^2\right]s}(1-\delta)^i\left(2\Delta\mathbb{T}_H\left(1-e^{-\frac{\ln 2}{H}s}\right)\right)^{2i}\right]e^{-(r-(1-\delta)g_0)s}$$
$$=e^{-(r-(1-\delta)g_0)s}-4\Delta\mathbb{T}_H^2(1-\delta)X^*e^{(\alpha+\kappa\sigma)s}\left(1-2e^{-\frac{\ln 2}{H}s}+e^{-2\frac{\ln 2}{H}s}\right)e^{-(r-(1-\delta)g_0)s}$$
$$+8\Delta\mathbb{T}_T^4(1-\delta)^2X^{*^2}e^{\left[2(\alpha+\kappa\sigma)+\sigma^2\right]s}\times$$
$$\left(1-4e^{-\frac{\ln 2}{H}s}+6e^{-2\frac{\ln 2}{H}s}-4e^{-3\frac{\ln 2}{H}s}+e^{-4\frac{\ln 2}{H}s}\right)^4e^{-(r-(1-\delta)g_0)s}.$$

Substituting (B5) back into (B1) and integrating yields

$$W^{\mathrm{NP}}\left(X,\Delta\mathbb{T};\Delta\mathbb{T}_H\right)=\frac{1}{1-\delta}\left[\frac{1}{r-(1-\delta)g_0}-4\Delta\mathbb{T}_H^2(1-\delta)X^*\left(\frac{1}{\eta_1}-\frac{2}{\eta_1+\frac{\ln 2}{H}}+\frac{1}{\eta_1+2\frac{\ln 2}{H}}\right)\right.$$
$$\left.+8\Delta\mathbb{T}_H^4(1-\delta)^2X^{*^2}\left(\frac{1}{\eta_2}-\frac{4}{\eta_2+\frac{\ln 2}{H}}+\frac{6}{\eta_2+2\frac{\ln 2}{H}}-\frac{4}{\eta_2+3\frac{\ln 2}{H}}+\frac{1}{\eta_2+4\frac{\ln 2}{H}}\right)\right], \tag{B.6}$$

where

$$\eta_1 = r - (1-\delta) g_0 - (\alpha + \kappa\sigma),$$
$$\eta_2 = r - (1-\delta) g_0 - \left(2(\alpha + \kappa\sigma) + \sigma^2\right).$$

Similarly, we have

$$W^{\mathrm{A}}(X, \Delta\mathbb{T}; \tau) = \frac{(1-w(\tau))^{1-\delta}}{1-\delta}\left[\frac{1}{r-(1-\delta)g_0} - 4\Delta\tau^2(1-\delta)X^*\left(\frac{1}{\eta_1} - \frac{2}{\eta_1 + \frac{\ln 2}{H}} + \frac{1}{\eta_1 + 2\frac{\ln 2}{H}}\right)\right.$$

$$\left. +8\Delta\tau^4(1-\delta)^2 X^{*^2}\left(\frac{1}{\eta_2} - \frac{4}{\eta_2 + \frac{\ln 2}{H}} + \frac{6}{\eta_2 + 2\frac{\ln 2}{H}} - \frac{4}{\eta_2 + 3\frac{\ln 2}{H}} + \frac{1}{\eta_2 + 4\frac{\ln 2}{H}}\right)\right], \tag{B.7}$$

which are equations (24) and (25) in the text, respectively.

C General Solution W^{NG} for W^{N}

We guess the solution to equation (17) has the following functional form:

$$W^{\mathrm{NG}}(t=0, X, \Delta\mathbb{T}; \Delta\mathbb{T}_H) = AX^{*^\beta}\left(\Delta\mathbb{T}^2 + C\Delta\mathbb{T} + D\right). \tag{C.1}$$

where A, C, D are some parameters. Calculating derivatives, we obtain

$$\frac{\partial W^{\mathrm{NG}}}{\partial \Delta\mathbb{T}} = AX^{*^\beta}(2\Delta\mathbb{T} + C), \tag{C.2}$$

$$X^*\frac{\partial W^{\mathrm{NG}}}{\partial X^*} = \beta A X^{*^\beta}\left(\Delta\mathbb{T}^2 + C\Delta\mathbb{T} + D\right) \quad \text{and} \tag{C.3}$$

$$X^{*^2}\frac{\partial^2 W^{\mathrm{NG}}}{\partial X^{*^2}} = \beta(\beta-1)AX^{*^\beta}\left(\Delta\mathbb{T}^2 + C\Delta\mathbb{T} + D\right). \tag{C.4}$$

Substituting equations (C1) - (C4) back to equation (17) and rearranging yields

$$2\left(\frac{\ln(2)}{H}\right)AX^{*^\beta}\left(\Delta\mathbb{T}^2 - \left(2\Delta\mathbb{T}_H - \frac{C}{2}\right)\Delta\mathbb{T} - C\Delta\mathbb{T}_H\right)$$

$$= \left[-\left(r-(1-\delta)g_0\right) + (\alpha+\kappa\sigma)\beta + \frac{1}{2}\sigma^2\beta(\beta-1)\right]AX^{*^\beta}\left(\Delta\mathbb{T}^2 + C\Delta\mathbb{T} + D\right). \tag{C.5}$$

Solving (C5) requires $\Delta\mathbb{T}^2 - \left(2\Delta\mathbb{T}_H - \frac{C}{2}\right)\Delta\mathbb{T} - C\Delta\mathbb{T}_H = \left(\Delta\mathbb{T}^2 + C\Delta\mathbb{T} + D\right)$. Thus, we have

$$C = -4\Delta\mathbb{T}_H \quad \text{and} \tag{C.6}$$

$$D = -C\Delta\mathbb{T}_H = 4\Delta\mathbb{T}_H^2. \tag{C.7}$$

Plugging (C6) and (C7) into (C5), we obtain

$$\left[-\left(r-(1-\delta)g_0 + 2\left(\frac{\ln(2)}{H}\right)\right) + (\alpha+\kappa\sigma)\beta + \frac{1}{2}\sigma^2\beta(\beta-1)\right]W^{\mathrm{NG}} = 0, \tag{C.8}$$

where $W^{\mathrm{NG}} = AX^{*^{\beta}} \left(\Delta\mathbb{T}^2 - 4\Delta\mathbb{T}_H\Delta\mathbb{T} + 4\Delta\mathbb{T}_H^2\right)$. The solution of (C8) requires

$$(\alpha + \kappa\sigma)\,\beta + \frac{1}{2}\sigma^2\beta\,(\beta - 1) - \left(r - (1-\delta)\,g_0 + 2\left(\frac{\ln(2)}{H}\right)\right) = 0. \tag{C.9}$$

Let β_1 and β_1 be the positive and negative roots of the above characteristic function, respectively. By some manipulations, this leads to

$$W^{\mathrm{NG}} = A_1X^{*^{\beta_1}} \left(\Delta\mathbb{T}^2 - 4\Delta\mathbb{T}_H\Delta\mathbb{T} + 4\Delta\mathbb{T}_H^2\right) - A_2X^{*^{\beta_2}} \left(\Delta\mathbb{T}^2 - 4\Delta\mathbb{T}_H\Delta\mathbb{T} + 4\Delta\mathbb{T}_H^2\right). \tag{C.10}$$

As we only consider the option to take action, we need to set the boundary condition such that $\lim_{X\to 0} W^{\mathrm{NG}}(X) = 0$, which is tantamount to a zero option value of a climate policy, if climate change causes no damages that reduce the GDP. Therefore, the general solution with the negative root can be ignored. Consequently, we obtain

$$W^{\mathrm{NG}} = A_1X^{*^{\beta_1}} \left(\Delta\mathbb{T}^2 - 4\Delta\mathbb{T}_H\Delta\mathbb{T} + 4\Delta\mathbb{T}_H^2\right). \tag{C.11}$$

References:

Asano, T. (2010) “Precautionary Principle and the Optimal Timing of Environmental Policy Under Ambiguity", *Environmental and Resource Economics* 47, 173-196.

Bertola, G. (2010) “Options, Inaction, and Uncertainty”, *Scottish Journal of Political Economy* 57, 254-271.

Brennan, M.J. and E.S. Schwartz (1978) “Finite Difference Methods and Jump Processes Arising in the Pricing of Contingent Claims: A Synthesis”, *Journal of Financial and Quantitative Analysis* 13, 461-474.

Chen, Z. and L. Epstein (2002) “Ambiguity, Risk and Asset Returns in Continuous-Time”, *Econometrica*, 70, 1403–1443.

Cox, J.C. and S.A. Ross (1976) “The Valuation of Options for Alternative Stochastic Processes”, *Econometrica* 53, 385-408.

Dixit, A. and R. Pindyck (1994) *Investment Under Uncertainty*, Princeton (Princeton University Press).

Edenhofer, O., Knopf, B., Barker, T., Baumstark, L., Bellevrat, E., Chateau, B., Criqui, P., Isaac, M., Kitous, A., Kypreos, S., Leimbach, M., Lessmann, K., Magne, B., Scrieciu, S. Turton, H. and D.P. van Vuuren (2010) “The Economics of Low Stabilization: Model Comparison of Mitigation Strategies and Costs”, *The Energy Journal* 31, 11-48.

Funke, M. and M. Paetz (2011) “Environmental Policy Under Model Uncertainty: A Robust Optimal Control Approach”, *Climatic Change* (DOI: 10.1007/s10584-010-9943-1).

Gilboa, I. and D. Schmeidler (1989) “Maxmin Expected Utility with Nonunique Prior”, *Journal of Mathematical Economics* 18, 141-153.

Golub, A., Narita, D. and M.G.W. Schmidt (2011) "Uncertainty in Integrated Assessment Models of Climate Change: Alternative Analytical Approaches", *Fondazione Eni Enrico Mattei Working Paper No.* 553, Venice.

IPCC (2007) *Climate Change 2007: The Physical Science Basis*, Cambridge (Cambridge University Press).

Leahy, J.V. (1993) "Investment in Competitive Equilibrium: The Optimality of Myopic Behavior", *The Quarterly Journal of Economics* 108, 1105-1133.

Mastrandrea M and S. Schneider (2004) "Probabilistic Integrated Assessment of 'Dangerous' Climate Change", *Science* 304, 571–575.

Millner, A., Dietz, S. and G. Heal (2010) "Ambiguity and Climate Policy", *NBER Working Paper No. 16050*, Cambridge (State Mass.).

Nishimura K. and H. Ozaki (2007) "Irreversible Investment and Knightian Uncertainty", *Journal of Economic Theory* 136, 668-694.

Pindyck, R.S. (2009) "Uncertain Outcomes and Climate Change Policy", *NBER Working Paper No.* 15259, Cambridge (State Mass.).

Pindyck, R.S. (2010) "Modeling the Impact of Warming in Climate Change Economics", *NBER Working Paper No.* 15692, Cambridge (State Mass.).

Roe, G.H. and M.B. Baker (2007) "Why is Climate Sensitivity So Unpredictable?", *Science* 318, 629-632.

Schneider, S. and M. Mastrandrea (2005) "Probabilistic Assessment of 'Dangerous' Climate Change and Emission Pathways", *PNAS* 102, 15728-15735.

Stokey, NL (2009) *The Economics of Inaction - Stochastic Control Models with Fixed Costs.* Princeton (Princeton University Press).

Trojanowska, M. and P.M. Kort (2010) "The Worst Case for Real Options", *Journal of Optimization Theory and Applications* 146, 709-734.

Vadas, G. and A. Xepapadeas (2010) "Model Uncertainy, Ambiguity and the Precautionary Principle: Implications for Biodiversity Management", *Environmental and Resource Economics* 45, 379-404.

Weitzman, M.L. (2009a) "Additive Damages, Fat-Tailed Climate Dynamics, and Uncertain Discounting", *Economics E-Journal,* 3, 2009-39 (October 22).

Weitzman, M.L. (2009b) "On Modeling and Interpreting the Economics of Catastrophic Climate Change", *Review of Economics and Statistics* 91, 1-19.