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Too Little Oil, Too Much Coal:
Optimal Carbon Tax and when to Phase in Oil, Coal and Renewables

Frederick van der Ploeg
Cees Withagen

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Abstract

Our main message is that it is optimal to use less coal and more oil once one takes account of coal being a backstop which emits much more CO2 than oil. The way of achieving this is to have a steeply rising carbon tax during the initial oil-only phase, a less-steeply rising carbon tax during the intermediate phase where oil and coal are used alongside each other and the following coal-only phase, and a flat carbon tax during the final renewables-only phase. The "laissez-faire" outcome uses coal forever or starts with oil until it is no longer cost-effective to do so and then switches to coal. We also analyze the effects on the optimal transition times and carbon tax of a carbon-free, albeit expensive backstop (solar or wind energy). Subsidizing renewables to just below the cost of coal does not affect the oil-only phase. The gain in green welfare dominates the welfare cost of the subsidy if the subsidy gap is small and the global warming challenge is acute. Without a carbon tax a prohibitive coal tax leads to less oil left in situ and substantially delays introduction of renewables, but curbs global warming substantially as coal is never used. Finally, we characterize under general conditions what the optimal sequencing oil and coal looks like.

JEL-Code: Q300, Q420, Q540.

Keywords: Hotelling rule, non-renewable resource, dirty backstop, coal, global warming, carbon tax, renewables, tax on coal, subsidy on renewables, transition times, Herfindahl rule.

Frederick van der Ploeg
University of Oxford
Manor Road Building
UK – Oxford OX1 3UQ
England
rick.vanderploeg@economics.ox.ac.uk

Cees Withagen
Department of Economics
VU Amsterdam
De Boelelaan 1105
NL – 1081 HV Amsterdam
The Netherlands
cwithagen@feweb.vu.nl

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1. Introduction

Oil[3] will last at most another half a century after which society has to switch to alternative sources of energy. If no breakthrough on inventing viable and cost-effective carbon-free energy has been realized by then, the world has to make do with coal which could last for another three or four centuries. Oil is relatively expensive to extract compared with coal, but it has the advantage of less CO2 emissions per unit of energy and thus less global warming damages. Wind and solar energy are clean, albeit expensive backstops, but account for at most a few percentage points of global energy supplies. Furthermore, they suffer from intermittence and are difficult to roll out on a large scale. Fig. 1 confirms that oil is much less abundant than coal and that coal is potentially a much bigger threat to global warming than oil.

The key to restructuring from a carbon economy to a carbon-free economy is to get the market to tell the truth about the costs of global warming. Putting a tax on carbon should ensure that carbon-free renewables are phased in more quickly and that the economy phases out fossil fuel more quickly. However, if the economy is still using coal which emits more CO2 per unit of energy than oil, it might make more sense to use less coal and more oil in the phase before the economy relies on carbon-free renewables only. We wish to understand these important public finance issues in an analytical setting with the following features: coal is abundant, relatively cheap but emits a lot of CO2; oil is exhaustible, emits less CO2 and gets more costly as fewer reserves remain in the crust of the earth; renewables are abundant, expensive but emit no CO2; all energy sources are perfect substitutes; marginal global welfare costs rise with the stock of CO2 in the atmosphere so the social cost of oil increases more strongly than the private cost as oil fields are depleted. We abstract from R&D into developing new, clean forms of energy (e.g., Bosetti et al., 2009; Aghion et al., 2009; Acemoglu et al., 2010), so offer a *conservative* guide to what should be done if there is no hope of a technological fix for global warming.

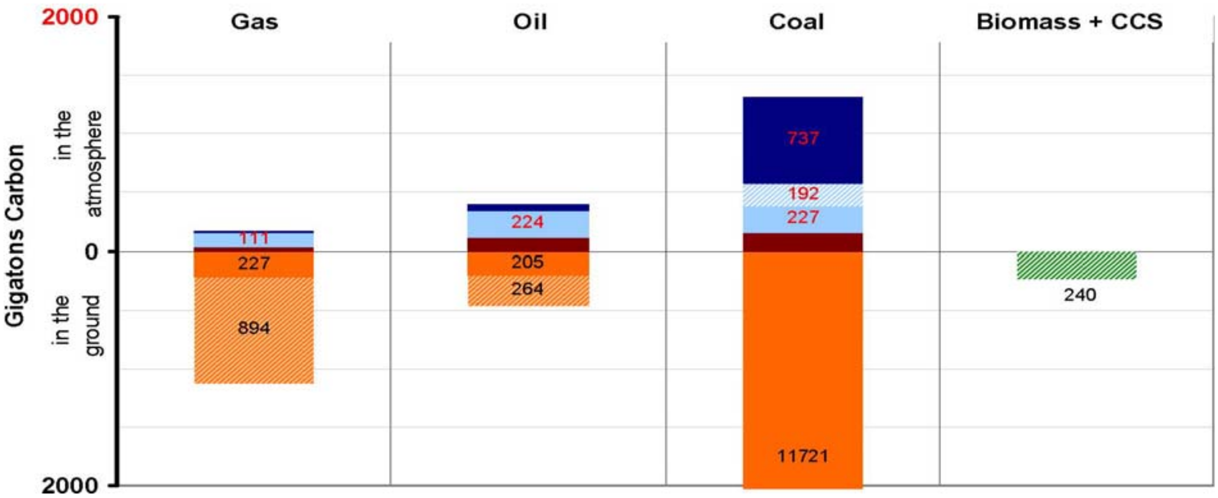
Our first objective is to contrast the social optimum with the "laissez-faire" outcome. We concentrate on the case where the optimum starts with oil, before using oil and coal together, and finally coal. We show that the "laissez-faire" outcome never has simultaneous use of oil and coal and leaves too much oil in situ. It either uses coal forever if it is cheap enough, but more likely it starts with oil until it is no longer profitable to do so and then switches to using only coal.

Our second objective is to analyze what the optimal carbon tax today and in the future will have to look like to ensure the optimal transition from a coal- and oil-based economy to a carbon-free economy. In particular, we are interested in what determines the optimal times of phasing in coal and renewables, the optimal times of phasing out coal and oil, and the optimal amount of oil to be left in situ forever. We find

[3] Throughout this paper we refer to oil when we mean oil or gas.

that the optimal carbon tax rises steeply during the initial phase where only oil is used, then rises less steeply during the intermediate phase where oil and coal are used alongside each other and the subsequent coal-only phase, and finally stays constant in the final renewable-only phase. This time profile of the carbon tax ensures that the market does not quickly and abruptly stop using oil, leaving too much oil in situ, and then moves on to use the relatively dirty coal as its energy source. The optimal carbon tax might not rise if there is natural decay of the atmospheric stock of CO2[4], but the time scale is so long that it seems for all practical intents and purposes relevant and prudent to focus on zero decay. Abstracting from decay in the stock of CO2 also makes the analysis much less cumbersome, since one effectively has to deal with only one state instead of two states.

Figure 1: Coal Reserves dominate Gas and Oil Reserves



Key: The CO2 that has been emitted and will in the future be emitted under the scenario that CO2-equivalent levels are stabilized at 400 parts per million compatible with the objective of limiting the increase in global temperature to a maximum of 2 degrees centigrade over pre-industrial levels is: estimated energy consumption since 1990 has already led to 227, 224 and 111 Giga tons of carbon for coal, oil and gas, respectively; estimated additional consumption to hit targets 737 Gigaton for coal, but almost none for gas and oil and coal plus CCS much less, namely 192 Giga ton of carbon. The remaining bit immediately above the line is Giga tons of carbon emitted in the atmosphere due to cumulative historical energy consumption. Biomass and CCS has and will not lead to CO2 emissions.
Under-the-ground reserves imply the following further future CO2 emissions: probable and proven resources and reserves for gas, respectively, 894 and 227 Giga ton; oil 264 and 205 Giga ton of carbon; reserves for coal 11,721 Giga ton of carbon; biomass plus carbon, capture and storage (CCS) 240 Giga ton of carbon unless it is fully sequestrated in which case there will be zero CO2 emissions.

Source: Edenhofer and Kalkuhl (2009)

[4] Allowing for decay, the time profile of the optimal carbon tax without backstops or stock-dependent oil extraction costs first rises and then falls (Ulph and Ulph, 1994; Hoel and Kverndokk, 1996), falls (Sinclair, 1994; Groth and Schou, 2007) or takes on a variety of shapes (Farzin and Tahvonen, 1996).

Our third objective is to offer a second-best analysis of what might happen in a market economy where it is infeasible to introduce such a rising, optimal carbon tax. We show that subsidizing renewables to just below the market price of coal does not affect the oil-only phase and always increases the present value of the welfare costs of global warming ('green welfare'). However, green welfare net of the welfare costs of the tax needed to finance the renewables subsidy falls if the "laissez-faire" economy is not too far from the social optimum. This occurs if the discount rate is high so that the present value of global warming damages is low, global warming is not that strong, coal emits only a little bit more CO2 than oil, and when the market price of coal is much lower than that of renewables. We also show that a moratorium on coal or a prohibitive tax on coal leads to less or no oil reserves left in situ, substantially delays the introduction of renewables and significantly curbs global warming as coal is never used.

Our final objective is to give a full characterization of all the other regimes that can occur apart from the regime which we focus on, namely the one where it is optimal to start with oil, followed by an oil-coal phase and then a coal-only phase, and finally a carbon-free phase. The Herfindahl rule which says that reserves with lowest cost to extract should be used first (Herfindahl, 1967)[5] need no longer hold. For example, Chakravorty et al. (2008) show that if all resources are non-renewable and have the same (zero) cost of extraction and the economy starts below its CO2 cap, it may be socially optimal to start using coal, then use clean natural gas, and finally use coal again ("preference reversal"). One should thus not start with gas with the lowest CO2 cost but with coal to benefit from natural decay of the stock of CO2 in the atmosphere. Only if gas is abundant will it be used first. Our results in section 5 offer related insights with a very different model in which coal is abundant, CO2 does not follow from a cap but from welfare analysis, oil/gas extraction costs are non-zero and rise as the stock of oil/gas reserves is depleted, and both extraction costs and pollution intensities differ for each energy source. In several regimes it is optimal to use coal and oil alongside each other provided coal is not very much dirtier than oil (cf., Tahvonen, 1979; Kemp and Long, 1980; Chakravorty and Krulce, 1994).

Section 2 analyzes the socially optimal transition from conventional oil to coal with climate externalities and stock-dependent extraction costs. Damages from CO2 emissions can be modelled through a negative externality in production (cf., Heal, 1985; Sinn, 2008ab), but we suppose that they directly damage social welfare. We abstract from capital accumulation.[6] We focus on the situation where it is optimal to start with using oil only and then phasing in coal. Section 3 studies the outcome in a market economy and

[5] This result has been studied in a general equilibrium framework (Kemp and Long, 1980; Lewis, 1982), in a situation with setup costs (Gaudet et al., 2001), and under heterogeneous demands (Chakravorty and Krulce, 1994).
[6] Golosov et al. (2010) and van der Ploeg and Withagen (2010b) also study a general equilibrium growth model of fossil fuels and a backstop fuel with capital accumulation, but do not consider dirty backstops such as coal.

shows how the social optimum can be sustained with an appropriately designed carbon tax. Section 4 analyzes what happens if carbon-free renewables kick in eventually and investigates the effect of lowering the cost of renewables and a coal tax. Section 5 briefly discusses the possibility that it is attractive to start using coal and discusses all the regimes for optimal sequencing of oil extraction and coal use that can occur. Section 6 concludes and discusses policy implications.

2. Social optimum

We study optimal extraction of oil with coal being phased in instead of oil or alongside oil when the social cost of energy becomes high enough. The backstop coal is a perfect substitute for oil and its supply is infinitely elastic. We add a convex function in past CO2 emissions to the felicity function to capture the damage done by accumulated CO2 emissions into the atmosphere from burning oil or coal. We abstract from natural decay of the stock of CO2 in the atmosphere and thus suppose that this stock evolves according to:

(1) $\dot{E}(t) = q(t) + \psi x(t), \quad E(0) = E_0 \text{ given}, \quad \psi > 1,$

where *E*, *q* and *x* denote the CO2 concentration in the atmosphere, oil use, and coal use, respectively.[7] The emission coefficient of oil is normalized to one, so that the emission coefficient of relatively dirty coal ψ is bigger than one. With quasi-linear preferences, the social planner's problem then reads:

(2) $\text{Max} \int_0^\infty e^{-\rho t}[U(q(t)+x(t)) - G(S(t))q(t) - bx(t) - D(E(t))]\text{d}t$

subject to (1), the non-negativity condition $x(t) \geq 0$ and the depletion equation for oil,

(3) $\dot{S}(t) = -q(t), \; q(t) \geq 0, \; S(t) \geq 0, \; S(0) = S_0 \text{ given},$

where ρ denotes the rate of time preference, *G* per unit extraction cost of oil, and *b* cost of supplying coal. Equation (3) implies that total current and future oil depletion cannot exceed oil reserves, $\int_0^\infty q(t)\text{d}t \leq S_0$.

Assumption 1: Instantaneous utility is concave, $U' > 0$ and $U'' < 0$, global warming damages are convex, $D' > 0$ and $D'' > 0$, unit oil extraction costs increase as less reserves remain, $G' < 0$.

So we suppose that marginal global warming damages are high when the atmospheric CO2 concentration is already high and that it becomes more expensive to extract oil as more of the more accessible oil fields have been mined. The necessary conditions for a social optimum are:

[7] Our model extends the model analysed by Hoel and Kverndokk (1996) to allow for a dirty backstop, but we restrict attention to the case where there is no natural decay.

(4) $U'(q+x)-G(S)\le\lambda+\mu,\ q\ge 0,$ c.s., $U'(q+x)-b\le\psi\mu,\ x\ge 0,$ c.s.,

(5) $\dot{\lambda}=\rho\lambda+G'(S)q,$ $\dot{\mu}=\rho\mu-D'(E),$ $\lim_{t\to\infty}e^{-\rho t}\left[\lambda(t)S(t)-\mu(t)E(t)\right]=0,$

where λ is the social value of oil and μ (> 0) the social cost of the stock of CO2 in the atmosphere and c.s. stands for complimentary slackness. The first part of (4) states that if the marginal rent of oil is less than the social value of keeping oil in the ground including the value resulting from less global warming damages, no oil is used. If oil is used, its marginal rent must equal the social value of oil. The second part of (4) states that coal is not used if its rent is less than its global warming cost. If coal is used, its marginal utility must equal the marginal cost of extraction including the damage done by global warming. Equations (5) give the dynamics of the shadow value of oil and the social cost of CO2, and the transversality condition. The social cost of carbon (SCC), i.e., μ, corresponds to the present value of all future marginal global warming damages.

To focus on the most policy-relevant case, we adopt assumption 2 below. The consequences of other assumptions for the sequencing of the various sources of energy are briefly discussed in section 5.

Assumption 2: (i) $b+\psi D'(E_0)/\rho>G(S_0)+D'(E_0)/\rho$; (ii) $\exists\ 0<S_1<S_0$ such that $b+\psi D'(E_0+S_0-S_1)/\rho=G(S_1)+D'(E_0+S_0-S_1)/\rho$; and (iii) $U'(0)>b+\psi D'(E_0+S_0+\hat{Y})/\rho$ where $\hat{Y}$ follows from $G(0)=b+(\psi-1)\dfrac{D'(E_0+S_0+\hat{Y})}{\rho}.$

It follows from (4) and (5) that when coal and oil are used alongside each other, we must have

(6) $b+\psi D'(E)/\rho=G(S)+D'(E)/\rho.$

This says that the marginal cost of extracting coal plus the present value of global warming damages resulting from burning an extra unit of coal must equal the oil extraction cost plus the present value of global warming damages resulting from burning an extra unit of oil. Because assumption 2(i) implies that (6) is not satisfied at the beginning, we rule out simultaneous use of oil and coal at the beginning. Assumption 2(i) ensures that there will be an initial phase where only oil is used.[8]

Assumptions 2(i) and (ii) together ensure that the overall cost advantage of using oil decreases as oil reserves fall and does not last forever. Assumption 2(ii) states that, as the oil extraction cost increases as

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[8] Formally, if we would start with only coal, $U'(q+x)-b=\psi\mu.$ The increase of $b+\psi\mu$ is equal to $\rho U'(x)-\psi D'(E).$ The right-hand side of $U'(q+x)\le G(S)+\lambda+\mu$ increases by more than $\rho(U'(x)-G(S))-D'(E).$ Therefore, the two right-hand sides diverge and no transition to the use of oil will ever take place. But assumption 2(i) indicates that a marginal reduction in coal use at the initial date together with a marginal increase in oil use is better from a cost perspective.

more reserves are depleted, there exists a stock level, denoted by S_1, such that (6) holds and that it becomes attractive to use coal alongside oil. This requires that the overall cost advantage of oil is decreasing with a decrease in the stock. This is not trivial as coal is more polluting. For the functional forms specified in section 3 and used in the policy simulations, the decrease in overall costs is monotonic (see the discussion after (10) in section 5). In general, this may not be so.

After the start of simultaneous supply, two possibilities arise. The first possibility is that the period of simultaneous use continues forever and some oil $\bar{S} > 0$ is left unexploited. This implies that extraction goes to zero as time goes to infinity. Since the damage function is convex, this also holds for coal use and thus accumulated CO2 emissions from burning coal, $Y(t) \equiv \psi \int_0^t x(s)ds,$ approaches a constant $\bar{Y}$. The shadow value of oil, λ, must converge to zero in view of the transversality condition and the shadow cost of pollution, μ, converges to $D'(E_0 + S_0 - \bar{S} + \bar{Y}) / \rho.$ We can thus solve recursively for $\bar{S}$ and $\bar{Y}$ from:

(7) $U'(0) = G(\bar{S}) + \mu(\infty) = G(\bar{S}) + (U'(0) - b)/\psi$ and

$$b + \psi D'(E_0 + S_0 - \bar{S} + \bar{Y}) / \rho = G(\bar{S}) + D'(E_0 + S_0 - \bar{S} + \bar{Y}) / \rho .$$

Hence, if (7) yields non-negative solutions, the phase with simultaneous use will last forever. Else, the simultaneous phase ends in finite time. In this case oil gets fully depleted and coal takes over.

One final proviso needs to be made. The path that we have now was constructed assuming that use of coal and oil was attractive, meaning that the marginal utility of energy is high enough. For this to be warranted, we assume throughout that assumption 2(iii) holds. So, after exhaustion of oil we have a pollution stock equal to $E_0 + S_0 + \hat{Y}.$ At that level of pollution it should still be profitable to use coal. Clearly, the use of coal will vanish in the end, but only asymptotically.

These results are summarized in the following proposition.

Proposition 1: If the solutions to (7) are both positive, i.e., $\bar{S} > 0$ and $\bar{Y} > 0,$ the social optimum has an initial phase with only oil and a subsequent final phase, lasting forever, with simultaneous use of oil and coal. The stock of oil reserves tends to a strictly positive value. Otherwise, the social optimum has an initial phase with only oil, followed by a phase with simultaneous use of oil and coal for a finite period of time, along which all oil is depleted, and a final phase where only coal is used and coal use vanishes asymptotically.

The next proposition characterizes the market outcome.

Proposition 2: In the "laissez-faire" economy (with D = 0), there is no simultaneous use. The outcome is either (i) only coal if $G(S_0) > b$, or (ii) initially only oil and later only coal if $G(S_0) < b < G(0)$, or (iii) only oil if $b > G(0)$. The optimal carbon tax equals the shadow price μ of the socially optimal program.

In case (iii) the amount of oil left unexploited is defined by $b = G(\bar{S})$ and is decreasing in b. The "laissez-faire" outcome thus then leaves more oil in situ if coal is cheap (low b) and extraction of oil is expensive. The market wants to make the transition to coal too soon, and too abrupt. Moreover, no unexploited arbitrage opportunities require continuity of energy prices, $U'(q(T)) = U'(x(T)) = b.$ Hence, the final use of oil also decreases in the extraction cost of oil. In contrast, the socially optimal outcome may fully exhaust oil reserves before having to rely on coal as the sole source of energy (if (7) does not allow a positive solution). But even if it is optimal to leave oil unexploited, less will be left in situ than in the market economy (as can be seen from $G(\bar{S}) = U'(0) - (U'(0) - b)/\psi > b$). The reason is that coal emits more CO2 per unit of energy than oil so it makes sense from the point of view of combating climate change to use up all oil. Also, coal use is high and remains high in the final coal phase of the "laissez-faire" outcome but coal use is lower and vanishes asymptotically in the social optimum.

Proposition 3: Suppose $G(S_0) < b < G(0).$ Then in the "laissez-faire" economy the amount of oil left unexploited follows from $b = G(\bar{S})$ and decreases in b. The unregulated market leaves more oil unexploited and uses more coal than the social optimum. At the transition time T extraction of oil and coal use decrease in b.

3. Which carbon tax sustains the socially optimal outcome?

Proposition 3 provides a qualitative comparison of the market economy and the socially optimal outcome. In the present section we make a more quantitative comparison using specific functional forms.

Assumption 3: $U(x) = \alpha x - \frac{1}{2}\beta x^2$, $G(S) = \gamma - \delta S$ and $D(E) = \frac{1}{2}\kappa E^2$.

Proposition 4: Given assumption 3 the optimum paths for the stock of oil, oil use and coal use satisfy

(8a) $$S(T_1) = \frac{\rho(\gamma - b) - (\psi - 1)\kappa(E_0 + S_0)}{\rho\delta - (\psi - 1)\kappa}, \quad q(T_1-) = q(T_1+) + x(T_1+), \quad Y(T_1) = 0,$$

(8b) $$q(T_2-) + x(T_2-) = x(T_2+), \; S(T_2) = 0, \quad Y(T_2) = \frac{\rho(\gamma - b)}{(\psi - 1)\kappa} - E_0 - S_0,$$

where T_1 and T_2 indicate the times that coal is phased in and that oil is phased out, respectively. The optimal paths for the stock of oil, oil use and coal use in the "laissez-faire" paths are given by

(9a) $$S(t) = \left(\frac{\frac{\alpha - b}{\delta} - \left(S_0 + \frac{\alpha - \gamma}{\delta} \right) e^{\omega_2 T}}{e^{\omega_1 T} - e^{\omega_2 T}} \right) e^{\omega_1 t} + \left(\frac{\left(S_0 + \frac{\alpha - \gamma}{\delta} \right) e^{\omega_1 T} - \frac{\alpha - b}{\delta}}{e^{\omega_1 T} - e^{\omega_2 T}} \right) e^{\omega_2 t} - \left(\frac{\alpha - \gamma}{\delta} \right),$$

(9b) $$q(t) = -\omega_1 \left(\frac{\frac{\alpha - b}{\delta} - \left(S_0 + \frac{\alpha - \gamma}{\delta} \right) e^{\omega_2 T}}{\left[e^{\omega_1 T} - e^{\omega_2 T} \right]} \right) e^{\omega_1 t} - \omega_2 \left(\frac{\left(S_0 + \frac{\alpha - \gamma}{\delta} \right) e^{\omega_1 T} - \frac{\alpha - b}{\delta}}{\left[e^{\omega_1 T} - e^{\omega_2 T} \right]} \right) e^{\omega_2 t}, \quad 0 \le t \le T,$$

(9c) $$x(t) = 0, \quad 0 \le t \le T, \qquad S(t) = \frac{\gamma - b}{\delta}, \quad q(t) = 0, \quad x(t) = \frac{\alpha - b}{\beta}, \quad t \ge T,$$

where $\omega_1 = \tfrac{1}{2}\rho + \tfrac{1}{2}\sqrt{\rho^2 + 4\rho\delta / \beta}$ and $\omega_2 = \rho - \omega_1$. The transition time T is such that $q(T) = (\alpha - b)/\beta$ and increases in b.

Proof: See propositions A4 and A6 in the appendix for the conditions determining the optimal outcome and proposition A8 in the appendix for the solution trajectories of the "laissez-faire" outcome.

We take parameter values such that the socially optimal and the "laissez-faire" outcomes will leave oil in situ at the end of the oil-only phase[9]. The intermediate coal-oil phase of the "laissez-faire" outcome is degenerate, since it is never optimal to use oil and coal simultaneously. The transition times will be denoted by T for the market economy and by T_1 (from only oil to simultaneous use) and T_2 (from simultaneous use to only coal) for the optimal economy.

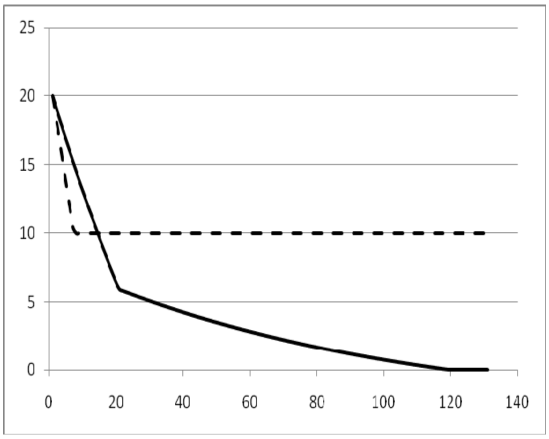
3.1. Comparing the "laissez-faire" and the optimum economy

The insights from propositions 1-3 are confirmed by the simulations presented in fig. 2. The "laissez-faire" economy phases out relatively clean oil and switches to coal much earlier, at instant T = 6.3, than the time coal is phased in and oil is phased out in the optimal economy. The optimal economy has

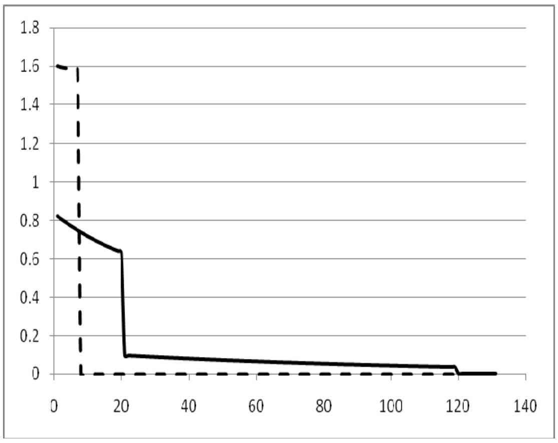
[9] The following parameter values have been used: $S_0 = 20, E_0 = 24, b = 0.4, \gamma = 0.6, \delta = 0.02, \psi = 1.5, \rho = 0.014,$ $\kappa = 0.00006, \alpha = 1.2, \beta = 0.506.$ These values satisfy assumptions 1 and 2. γ and δ have been calibrated so that initial oil extraction costs are half those of coal, $G(S_0) = b/2$, and in the "laissez-faire" outcome half of oil reserves is left in situ, $(\gamma - b)/\delta = S_0/2$. The parameters α = 1.2 and β = 0.51 are obtained by taking a second-order Taylor series approximation around $x = (\alpha - b)/\beta$ of the utility function $U(x) = \frac{x^{1-1/\sigma}}{1-1/\sigma}$, where the intertemporal elasticity of substitution has been set to σ = 0.5 and thus the elasticity of intertemporal inequality aversion to $1/\sigma$ = 2.

Figure 2: Simulations of “laissez-faire” and optimum economy

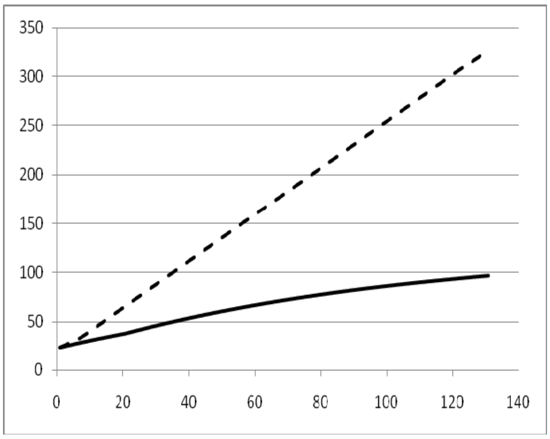
Oil reserves (*S*)



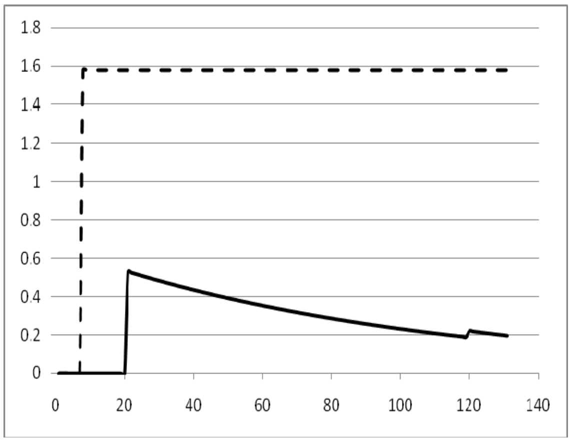
Oil use (*q*)



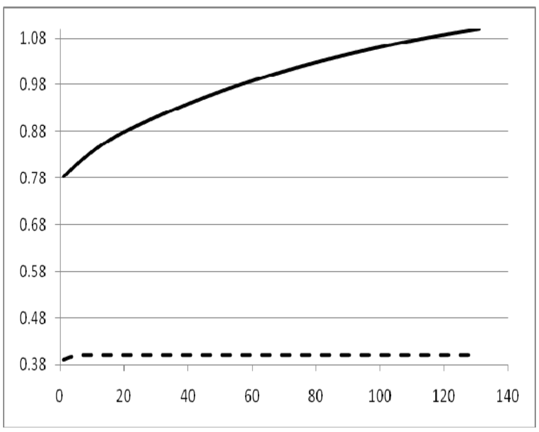
CO2 concentration in the atmosphere (*E*)



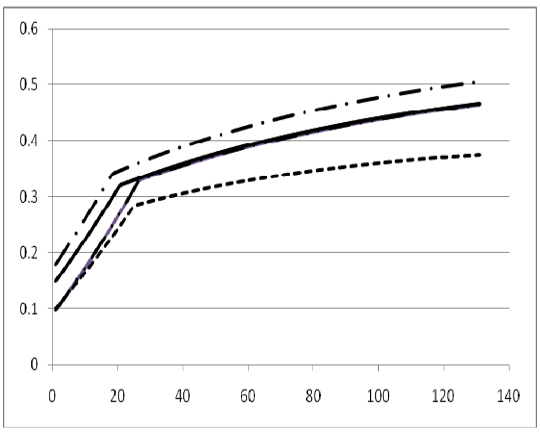
Coal use (*x*)



Price of energy ($U'(q+x)$)



Optimal carbon tax ($\tau = \mu$)



Key: Solid = social optimum; dashes = “laissez-faire” outcome.
Thin solid = discovery of oil ($\Delta S_0 = 5$); short dashes = cheaper coal ($\Delta b = -0.1$);
long dashes and dots = lower EIS ($\Delta(1/\sigma) = -0.5$).

depleted oil reserves from 20 at instant zero to 5.92 at instant T_1 = 19.7 before coal is phased in alongside oil, and then moves to using coal only once oil reserves are fully exhausted at the much later instant T_2 = 118.4. Furthermore, the “laissez-faire” economy uses much more substantial and non-vanishing amounts of dirty coal than the social optimum. Since coal contributes more to global warming than oil per unit of energy, both effects exacerbate global warming and curb social welfare. This is also reflected in fig. 2, which shows a much steeper rise in the concentration of CO2 in the atmosphere resulting from a much lower price of energy in the “laissez-faire” outcome than in the socially optimal outcome. Note that with our chosen parameter values the effect of oil extraction costs rising dominates the effect of marginal global warming damages rising (i.e., $(\psi-1)\kappa-\rho\delta=-0.00025<0$) and therefore the advantage of using oil rather than coal falls (as it does in the “laissez-faire” outcome) as oil reserves are depleted. There are discrete jumps in both oil use and coal use. When in the optimal economy coal is phased in at instant T_1 = 19.7, oil use jumps down by an amount 0.53 and coal use jumps up by the same amount so that there is no jump in the price of energy. Also, oil use jumps from 0.034 to zero at instant T_2 = 118.4 as coal jumps up by the same amount, again to ensure that there is no jump in the price of energy.

In the “laissez-faire” outcome the rent on oil, i.e., the difference between the price of oil and the extraction cost of oil, vanishes at the time the economy switches to the backstop as at that point the extraction cost of oil must equal that of coal (cf., Heal, 1976). In the socially optimal outcome all oil reserves are depleted. Oil still has a positive shadow price as it has lower CO2 emissions per unit of energy than coal.

3.2. The optimal carbon tax

The “laissez-faire” outcome relies in the long run too much on coal instead of oil, and thus leads to too much CO2 emissions compared to the socially optimal outcome. In the short run it uses up oil quickly, although it leaves more oil in situ. The key challenge for policy makers is to design an optimal carbon tax in such a way that the market outcome replicates the socially optimal outcome. Such a tax should persuade private agents to use up all oil reserves, even the more expensive fields, rather than using highly CO2-intensive coal. The optimal carbon tax inevitably rises over time and faces the difficult task of encouraging private agents to use both oil and coal alongside each other for a period of time.

Proposition 5: The optimal carbon tax rises during all three phases. The socially optimal outcome is attained if the carbon tax during the intermediate oil-coal phase and the final phase where only coal is used indefinitely is given by $\tau=[U'(q+x)-b]/\psi$ and during the initial oil-only phase by

$$(10)\qquad \tau(t)=e^{-\rho(T_1-t)}\left[U'\big(q(T_1)+x(T_1)\big)-b\right]/\psi-\int_t^{T_1}D'(E_0+S_0-S(s))e^{-\rho(s-t)}ds,\quad \forall 0\le t\le T_1.$$

With assumption 3 the optimal carbon tax and coal use during the coal-only phase are

(10′) $$\tau(t) = \frac{\alpha - \beta x(t) - b}{\psi} \text{ with } x(t) = \left(\frac{\psi\kappa}{\beta\lambda_1}\right)\left[\left(\frac{\rho}{\psi\kappa}\right)(\alpha - b) - \left(E_0 + S_0\right)\right]e^{\lambda_2(t-T_2)}, \quad t \geq T_2.$$

where $\lambda_1 = ½\rho + ½\sqrt{\rho^2 + 4\psi^2\kappa/\beta} > \rho > 0$ and $\lambda_2 = \rho - \lambda_1 < 0,$ and for the oil-coal phase the optimal carbon tax and energy use are:

(10″) $$\tau(t) = \frac{\alpha - \beta\left[q(t) + x(t)\right] - b}{\psi}, \quad q(t) + x(t) = \left(\frac{\rho\delta + (\psi - 1)^2\kappa}{\psi(\psi - 1)\kappa}\right)q(t), \quad T_1 \leq t \leq T_2.$$

The time path of the carbon tax is concave.

Proof: See appendix.

From (10′) we see that coal use vanishes asymptotically (assumption 2(iii) ensures that it is positive) and the optimal carbon tax rises asymptotically towards $(\alpha - b)/\psi$ during the final phase. Inspection of λ_2 shows that the rise of the carbon tax and the fall in coal use during the final coal-only phase are faster if coal is more CO2 intensive and global warming damages are strong (high ψ or κ), but note from (10′) that the paths of the tax and coal use also depend on the moment when the final phase starts (i.e., T_2) which is typically brought forward if coal is more CO2 intensive and global warming damages are strong. Also, during the final phase less coal is used if the global warming challenge (E_0+S_0) is large. From (10″) we see that during the intermediate phase relatively less CO2-intensive oil than coal is used if the discount rate ρ is small and thus global warming damages are weighed more heavily, the global warming cost parameter κ is large, oil extraction costs do not rise rapidly as oil reserves diminish (small δ), and coal is relatively dirty compared with oil (high ψ).

In the intermediate and final phases where either coal and oil are used alongside each other or only coal is used, the optimal carbon tax (i.e., the present discounted value of all future marginal global warming damages) increases over time as the marginal global warming damages increase over time (as there is no natural decay of the atmospheric stock of CO2). In the initial phase where only oil is used, the use of oil must diminish over time, but that is already encouraged by the rising oil extraction costs and the higher royalties. Hence, the carbon tax can be less high. However, fig. 2 indicates that the optimal carbon tax rises much more steeply during the oil-only phase than during the oil-coal and coal phases. This reflects that the carbon tax has to be designed in such a way as to ensure that the market is giving the correct incentive to switch from using coal to oil in the initial phase, and to ensure that the transition times for phasing in coal and phasing out oil are socially optimal. Also, the optimal carbon tax has to ensure that the market fully exhausts all oil reserves. This requires that the market expects the net price of oil to rise at a rate smaller than the interest rate.

The simulations confirm proposition 5 in that the time path of the optimal carbon tax is concave as the market has an incentive to conserve more oil as oil reserves are depleted and the marginal cost of pumping up more oil increases; hence, the government has to correct less for climate damages.

Furthermore, the optimal carbon tax never falls. In our model coal use eventually vanishes and the cumulated stock of CO_2 in the atmosphere due to coal use and thus the total stock of CO_2 approach asymptotically a finite positive constant. This ensures that the optimal carbon tax never declines in our model. However, if the rate of natural decay of the atmospheric stock of CO_2 is $\chi > 0$ instead of zero, the optimal carbon tax must obey $\dot{\tau} / \tau = \rho + \chi - D'(E) / \tau.$ Natural decay of CO_2 thus pushes like impatience for a *rising* carbon tax whilst a high stock of pollution calls for a *falling* carbon tax (cf., Ulph and Ulph, 1994). A more thorough investigation by Tahvonen (1997) of the effects of linear decay of the stock of CO_2 employs a similar model to ours with stock-dependent oil extraction costs and convex damages, but has a clean instead of a dirty backstop. He shows formally that the time path for the optimal carbon tax has an inverted U shape if the initial stock of CO_2 in the atmosphere is small and declines monotonically if the initial stock of CO_2 is large.[10]

3.3. Sensitivity analysis

Although the qualitative nature of the simulation trajectories is unaffected by small changes in parameter values, table 1 indicates that the optimal transition times of phasing in coal and phasing out oil in the “laissez-faire” and optimal economies and the size of the optimal carbon tax do change. It also gives the long-run values of the optimal carbon tax, social price of energy and atmospheric CO_2 concentration. The last panel of fig. 2 shows the sensitivity of the optimal carbon tax to key parameter values.

A discovery of new oil reserves today ($\Delta S_0 = 5$) does not affect the long-run social price of energy or the long-run optimal carbon tax. The discovery of new oil reserves does not lead to a bigger long-run build-up of CO_2 in the atmosphere either. It does lead to a later phasing in of coal and a somewhat later phasing

[10] Other studies on the time profile of the optimal carbon tax assume there is no backstop and no stock-dependent oil extraction costs, but natural decay of the stock of CO_2 (Ulph and Ulph, 1994; also Hoel and Kverndokk, 1996). This also suggests an inverted U-shape for the optimal carbon tax. If one part of the stock of CO_2 decays and the other does not, the optimal time path can be hump shaped, U shaped or monotonically declining (Farzin and Tahvonen, 1996). Allowing for capital accumulation and global warming damages in production, there may be a declining path for the optimal carbon tax (Sinclair, 1994) but this result has been criticized (Ulph and Ulph, 1994, section 3). In a growth model without a carbon-free backstop and global warming externalities in production, a falling carbon tax is called for (Groth and Schou, 2007). However, in a Ramsey growth model with exhaustible oil, a carbon-free backstop and additive welfare costs of global warming, the optimal carbon tax rises, especially in the early stages of economic development (van der Ploeg and Withagen, 2010).

Table 1: Optimal and “laissez-faire” transition times and the stock of oil left in situ by the market

	Benchmark	$\Delta S_0 = 5$	$\Delta b = -0.1$	$\Delta(1/\sigma) = -0.5$	$\rho = 0.025$	$\kappa = 0.00003$
Optimal economy:						
Long-run carbon tax	0.53	0.53	0.4	0.57	0.53	0.53
Long-run price of energy	1.2	1.2	0.9	1.26	1.2	1.2
Oil left in situ forever	0	0	0	0	0	0
Long-run CO2 concentration	124.4	124.4	93.3	133.4	222.2	248.9
Amount due to coal use	80.4	75.4	49.3	89.4	178.2	204.9
Long-run oil and coal use	0	0	0	0	0	0
Time of phasing in coal, T_1	19.7	25.6	24.1	17.3	11.4	12.2
Time of phasing out oil, T_2	118.4	120.6	85.0	97.9	164.9	205.4
“Laissez-faire” economy:						
Oil left in situ forever	10	10	10	10	10	10
Long-run oil use	0	0	0	0	0	0
Long-run coal use	1.58	1.58	1.83	1.80	1.58	1.58
Time of phasing in coal, T	6.3	7.9	5.5	5.6	6.3	6.3

out of oil in such a way that the increase in carbon emissions resulting from burning the extra oil is exactly offset by the reduction in carbon emissions from using less coal. The last panel of fig. 2 indicates that the optimal carbon tax still rises steeply during the oil-only phase, but will be lower during this initial phase as oil is less scarce and thus cheaper to extract from the earth. Consequently, fewer incentives are needed to persuade the market to use oil rather than coal in this initial phase. Note that the “laissez-faire” outcome leaves more oil in situ and phases coal in later.

A lower cost of coal ($\Delta b = -0.1$) brings forward the date when the economy starts to use coal in the “laissez-faire” economy. Cheaper coal lowers both the long-run social price of energy and the long-run value of the carbon tax. In fact, the last panel of fig. 2 shows that the whole time trajectory of the optimal carbon tax shifts down when coal is cheaper. Interestingly, cheaper coal leads to a later phasing in of coal and a much earlier phasing out of oil in the socially optimal outcome so that the period that relatively dirty coal is used becomes shorter. The oil-coal phase also lasts much shorter. The reason for these intertemporal shifts in the optimal outcomes is that it is better to use oil in the beginning whilst it is still relatively cheap to extract before having to inevitably shift to using the relative dirty, but cheaper coal. The result is that carbon emissions resulting from burning coal are substantially reduced.

A higher elasticity of intertemporal substitution (σ = 2/3) implies that the market can more easily substitute current consumption for future consumption; hence, coal is phased in earlier and carbon emissions resulting from burning coal increase. It corresponds to a lower elasticity of intertemporal inequality aversion ($\Delta(1/\sigma)$ = –0.5), so that the socially optimal outcome phases in coal and phases oil out more quickly. The long-run price of energy and the long-run carbon tax are somewhat higher. In fact, the last panel of fig. 2 indicates that the whole time path of the optimal carbon tax shifts up if society has a lower intertemporal inequality aversion. Still, the long-run concentration of CO2 in the atmosphere is a bit higher. If society cares less about the impact of global warming on future generations, it phases in coal more quickly and ends up with a bigger climate problem.

If the government follows the market and does not apply for precautionary or other reasons a lower rate of discount rate, but a discount rate of say 0.025, it phases in coal more quickly and phases out oil much later. As a result, carbon emissions from burning coal are more than doubled. The optimal carbon tax is then initially much lower and takes much longer to reach its long-run value. Hence, there are more CO2 emissions in the short run which exacerbates global warming. Finally, if the government has a too optimistic view on the costs of global warming (κ = 0.00003 instead of 0.00006), it phases in coal to early and phases out oil too late. As with the too impatient government, this leads a much bigger accumulation of CO2 in the atmosphere which is entirely caused by much more coal being burned.

4. Carbon-free renewables: taxing carbon or a moratorium on coal?

The optimal climate change policy is to set a carbon tax that reflects the marginal environmental damage from using the respective types of fossil fuels. Here we address the important issue of how our results are affected when there are carbon-free renewables as a potential source of energy supply as well. Consider therefore the presence of an infinitely elastic supply of such renewables at a cost *c* bigger than that of coal *b*. A necessary condition for renewables to be worthwhile introducing at some point is that $U'(0) = \alpha > c$. This is not a sufficient condition, because renewables might take over before the coal-only phase starts in view of the already accumulated CO2 stock. This might occur for *c* relatively low, i.e., close to *b*. However, we are interested in the more realistic case that the cost of renewables *c* is not relatively low and that renewables only take over after the oil-coal phase.

A high cost of renewables and a low extraction cost of coal necessitate a high long-run value of the optimal carbon tax. The final stock of accumulated stock of CO2 in the atmosphere resulting from burning coal will then be higher as well, especially if the society is very impatient (high ρ) and the perceived cost of global warming is small (low E_0 and S_0). Since the social cost of carbon will be lower as a result of the availability of a carbon-free backstop, the optimal carbon tax, the final stock of CO2 in the atmosphere and thus the transition time to the oil-only and oil-coal phases will be affected.

Since renewables are more expensive than coal, they will not be phased in under the "laissez-faire" outcome unless there is a subsidy on them or their cost falls with time due to technical progress. However, the social planner which takes account of the rising costs of global warming does phase in carbon-free renewables eventually. Over time the use of coal rapidly increases the marginal damages of global warming and thus the social cost of coal. As soon, as the social cost of coal hits the cost of renewables, the economy stops using coal and switches to renewables.

4.1. Optimal carbon tax with renewables

The introduction of renewables affects both transition times T_1 and T_2. Moreover, we take care that the cost of the carbon-free alternative should not be too low to have a coal-only phase after the exhaustion nof oil. Renewables take over at the instant $T_3 > T_2$ where the cost of coal plus the carbon tax equals the cost of renewables:

$$(11) \qquad b + \psi\tau(T_3) = \alpha - \beta x(T_3) = c, \text{ where } \tau(T_3) = \kappa\left[E_0 + S_0 + Y(T_3)\right] / \rho.$$

The social cost of carbon at the end of the coal-only phase thus equals the present value of marginal global warming damages, which are constant as there is no CO2 pollution anymore once carbon-free renewables have been introduced. The following proposition characterizes the transition to the carbon-free economy.

Proposition 6: Suppose the optimal sequence is only oil until T_1, oil and coal from T_1 to T_2, only coal from T_2 to T_3, and renewables from T_3 onwards. Then, from the moment that oil is phased out, the accumulated stock of CO2 due to past burning of coal grows from $Y(T_2) = \frac{\rho}{\kappa}\left(\frac{\gamma - b}{\psi - 1}\right) - E_0 - S_0$ until the moment that renewables is introduced and then stays constant at the level $Y(T_3) = \frac{\rho}{\kappa}\left(\frac{c - b}{\psi}\right) - E_0 - S_0$.

The long-run optimal carbon tax rate, renewables use, and stock of CO2 in the atmosphere from that moment on are given by:

$$(11') \qquad \tau(t) = (c - b)/\psi, \quad x(t) = \frac{\alpha - c}{\beta}, \quad E(t) = E_0 + S_0 + Y(T_3) = \frac{\rho(c - b)}{\kappa\psi}, \quad t \geq T_3.$$

During the oil-only and the oil-coal phase, the optimal carbon tax must rise.

Proof: At the moment oil is phased out, (6) holds with $S = 0$, i.e., $b+\psi D'(E_0+S_0+Y(T_2))/\rho =$ $G(0)+D'(E_0+S_0+Y(T_2))/\rho$ which yields the expression for $Y(T_2)$. The use of renewables $x(t), t\geq T_3$, as in (9′) follows from $U'(x(t))=c, t\geq T_3$. Continuity of energy use requires $U'(x(T_3-))=b+\psi\tau(T_3)=$ $U'(x(T_3+)=c,$ which yields $\tau(t), t\geq T_3$ as in (11′). $E(t)=E_0+S_0+Y(T_3), t\geq T_3$ as in (9′) and $Y(T_3)$ then follow from $\tau = D'(E_0+S_0+Y(T_3))/\rho.$ Q.E.D.

Hence, a high cost of renewables and a low extraction cost of coal necessitate a high long-run value of the optimal carbon tax. In that case, the final stock of accumulated stock of CO2 in the atmosphere resulting from burning coal will be higher as well, especially if the society is very impatient (high ρ) and the perceived cost of global warming is small (low κ, E_0 and S_0). Since the social cost of carbon will be lower as a result of the availability of a carbon-free backstop, the optimal carbon tax, the final stock of CO2 in the atmosphere and thus the transition time to the oil-only and oil-coal phases will be affected.

4.2. Policy simulations

As we have seen, to have a solution with more accumulated CO2 from coal at the end than at the start of the coal-only phase we need that renewables are relatively expensive, $c > b+\psi(\gamma-b)=0.775$. In fact, our simulations indicate that this lower bound on c is not high enough to ensure that renewables are only introduced after the coal-only phase has started. This requires that coal must be even more expensive. For our chosen parameter values this requires $c > 1.0$. For the core parameter values in our simulations (see footnote 8 in section 3), we thus require values of c in the range (1.0, 1.2).

We simulate the optimal regime for when there is a relatively expensive carbon-free renewable backstop available (at cost $c = 1.1 > b = 0.4$). The solid and dashed-dotted lines in fig. 3 present the resulting optimal and “laissez-faire” simulation trajectories and fig. 3 allows them to be compared with the outcomes when renewables are not available. The eventual introduction of carbon-free renewables lowers the social cost of carbon and hardly brings forward the phasing in of coal from instant T_1 =19.7 if there are no renewables to instant 19.6. The anticipation of the future introduction of renewables and the resulting reduction in the social cost of carbon also brings forward the date that oil is phased out and the economy relies on coal only slightly from instant T_2 = 118.4 to instant of time 115.2.

The length of the oil-coal phase is reduced somewhat from 98.7 to 95.6. Carbon-free renewables become profitable from instant T_3 = 163.3 onwards, so that the coal-only phase has a length of only 48.1 instead of infinity. Since a little more oil is used and less coal, the stock of accumulated CO2 is only 108.9 at the moment coal is phased out and stays at that level from thereafter. The optimal carbon tax rises from 0.14

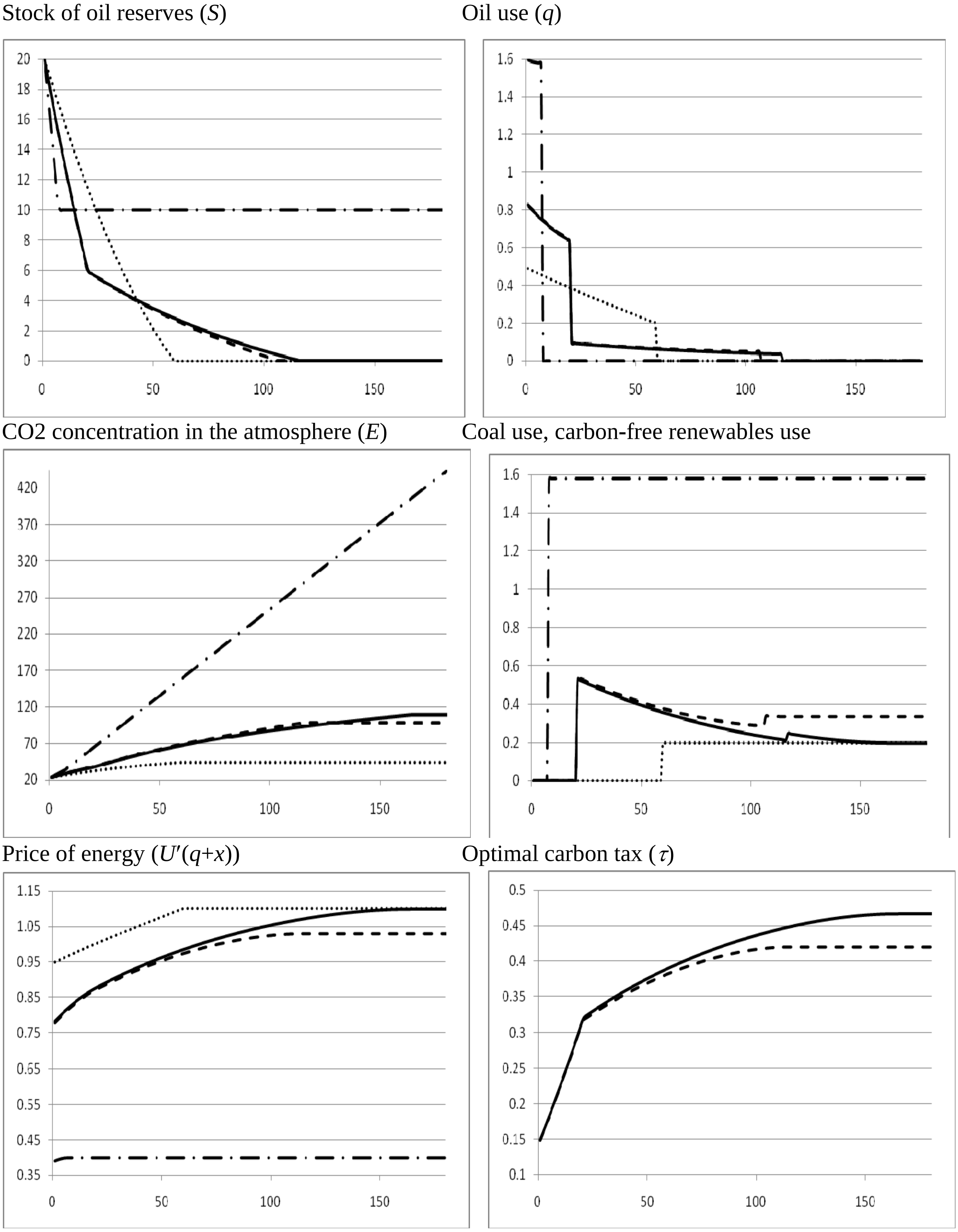
to 0.47 at the time of the switch to renewables; hence, the presence of renewables means that the carbon tax can be lower, rising to 0.47 at the end of the coal-only phase and staying at that level thereafter. The optimal carbon tax and the social price of energy still rises steeply during the oil-only phase (from time zero to 19.6), rises less steeply during the oil-coal phase (from time 19.6 to 115.2) and the coal-only phase (from time 115.2 to 163.3). The final value of the optimal carbon tax has to be maintained at a constant level during the renewables-only phase; else, the economy has an incentive to switch to using coal again. Given the high cost of renewables, no drastic changes are observed during the oil-only and oil-coal phases compared to the situation when there are no renewables, but the coal-only phase comes to an end and thus the rise in the carbon tax comes to an end as well.

Now consider the implications of a lower cost of renewables, c = 1.03 instead of 1.1, which are indicated by the dashed lines in fig. 3. Coal is phased in marginally earlier at instant T_1 = 19.4 rather than 19.6, but oil is phased out rather earlier at instant T_2 = 105.0 instead of 115.2 and renewables are phased in much earlier at instant T_3 = 114.2 rather than 163.3. The lower cost of renewables thus leads initially to a little more and somewhat earlier pumping of oil, but much more importantly to coal being phased in and phased out more quickly. Interestingly, coal use is at a higher level but used for a much shorter period (9.2 instead of 48.1 periods). As a result, the concentration of CO_2 in the atmosphere is not reduced as much as it could have done, but still it falls from 108.9 to 98.0 (about 10 percent) in the long run as renewables are being phased in more quickly. If allowance is made for the optimal phasing in of coal on its own or alongside oil and the earlier phasing in of renewables, we see that the lower cost of renewables implies that it is optimal to have a lower carbon tax in the oil-coal and coal phases. The final level of the carbon tax is also lower (i.e., 0.42 instead of 0.47).

The overall welfare effect of a lower cost of renewables is unambiguously positive. But what about the effect on overall green welfare (the present value of global warming damages)? In the absence of coal, a lower cost of renewables leads to faster exhaustion of oil, thus aggravating CO_2 emissions and climate damage (Sinn, 2008ab; van der Ploeg and Withagen, 2010a). Here the same phenomenon occurs when it comes to oil whilst coal use rises initially. But coal is phased out much earlier, so that damages are much smaller in the long run. In the “laissez-faire” economy nothing changes as long as renewables are more expensive than coal; else, all coal use is banned. Hence, a subsidy on renewables is effective only if is large enough to make them cheaper than coal. But even then oil use is enhanced by the subsidy and green welfare might decrease. We return to this issue in the context of a market economy in the next section.

If the cost of renewables becomes low enough (c about 1.0), the coal-only phase vanishes completely in the social optimum. If the cost of renewables becomes lower than that, renewables will be introduced

Figure 3: Simulations with eventual introduction of renewables



Key: solid lines = socially optimum (c = 1.1); dashed-dotted lines = “laissez-faire” outcome; dashed lines = social optimum with lower cost of renewables (c = 1.03); dotted lines = prohibitive coal tax.

alongside oil and possibly coal. We leave this for further research.

4.3. Second-best policies: taxing coal or subsidizing carbon-free renewables?

A full-fledged carbon tax is politically infeasible in many countries, especially on CO2 emissions resulting from burning coal. In that case, governments often resort to subsidizing carbon-free renewables such as solar or wind energy. The idea behind such a second-best policy is that this will bring forward the date at which coal is phased out, so that this lowers CO2 emissions and curbs global warming. In earlier work on just coal and renewables, thus ignoring the impact of dirty backstops such as coal, concern has been voiced that such a subsidy may be counterproductive, since it merely encourages the market to pump up oil more quickly and thus exacerbate global warming damages. This phenomenon is called the Green Paradox (Sinn, 2008). However, if it is optimal to leave oil in situ and there is no dirty backstop, the subsidy encourages the market to leave more oil in situ and switch more quickly to renewables in which case global warming is curbed and there is no green paradox (van der Ploeg and Withagen, 2010a). To investigate what our context with a dirty backstop implies for the green paradox, we analyze the two second-best policies of either subsidizing renewables or taxing coal *without the introduction of an appropriate carbon tax.*

Subsidizing renewables so that they are marginally cheaper than the market price of coal leads to the same oil-only phase, transition time and stock of oil left in situ as under "laissez-faire"; once the backstop is introduced, the stock of CO2 is stabilized. Total energy use is unaffected at all instants of time.

Proposition 7: With renewables being subsidized to just below the extraction cost of coal, green welfare gains dominate the welfare losses resulting from the lump-sum taxes needed to finance the subsidy if $c < b + (\kappa\psi / \rho)\left[E_0 + S_0 - (\gamma - b) / \delta + \psi(\alpha - b) / \rho\beta\right]$ holds. The subsidy then yields higher welfare than the "laissez-faire" outcome

Proof: See appendix.

Hence, making renewables economically viable improves social welfare if renewables are cheap (c low) and coal is expensive (b high) provided $1 + (\kappa\psi / \rho)\left[1/\delta - \psi\right) / \rho\beta\right] > 0$ as is the case for our parameter values), the climate challenge is acute (high E_0, S_0, ψ), society employs a prudently low discount rate (low ρ), the initial extraction cost of oil is low (low γ) and demand for energy is substantial (high α), although it might do somewhat less well if the subsidy has to be financed by distorting taxes). For our core parameters, this condition is satisfied because c = 1.1 < 1.55. This is why in the simulations the subsidy on renewables yields higher (in fact, much higher) welfare than the "laissez-faire" outcome (-27.7 instead of -136.7).

The subsidy does not affect the extraction of oil, but reduces use of coal. Hence, a subsidy which makes renewables marginally cheaper than coal unambiguously improves green welfare. However, we must take account of the welfare losses arising from the taxes needed to finance the subsidy. Proposition 7 indicates that then such a subsidy does not necessarily increase net green welfare if the market price of coal is much cheaper than that of renewables (b–c small), the climate challenge (E_0+S_0) is small, the damage parameter (κ) is small, the CO2-emission rate of coal (ψ) is small, and future climate damages are discounted more heavily (large ρ). Loosely speaking, subsidizing renewables leads to lower net green welfare if the "laissez-faire" outcome is not far removed from the social optimum.

The alternative second-best policy in the absence of a carbon tax is a prohibitive tax on coal, which makes coal marginally more expensive than renewable. This also has the effect that coal will never be introduced, but now renewables will be introduced at a later date. This policy is equivalent to a moratorium on coal-powered electricity and other uses of coal.

Proposition 8: With a prohibitive coal tax and the cost of renewables more than the cost of extracting the final drop of oil ($c > \gamma$), oil reserves will be fully exhausted. During the oil-only phase, oil reserves and oil use fall monotonically while coal use is zero. Renewables use in the carbon-free economy is given by $x(t) = \dfrac{\alpha - c}{\beta}, \quad t \geq T$. The transition time T follows from $q(T) = (\alpha - c)/\beta$, so shift to carbon-free society is earlier if renewables are cheap (c low).

Proof: See appendix (which also gives the explicit solution trajectories).

The dotted lines in fig. 3 show the effects of taxing coal so that it is just below the cost of renewables, hence the cost of coal has been raised from 0.4 to 1.1. The result of this second-best policy is that the market substantially delays the phasing out of oil from time 6.9 to time 58.7, albeit still quicker than the first-best outcome. So this second-best policy exhausts all oil reserves to a bigger extent (and in our simulation exhausts them fully) and uses oil for a much longer period. Afterwards, renewables rather than coal are phased in. The result of the tax on coal is to push up the price of energy very quickly to somewhere close to the cost of renewables (even higher than in first-best outcome). It is this that forces the market to be much more careful and persistent in the use of oil. The prohibitive tax on coal is extremely successful in terms of fighting global warming, since it reduces the long-run stock of carbon in the atmosphere from 108.9 in the first-best outcome to 44.0. However, social welfare is 7.8 in the second-best outcome compared with -136.7 in the "laissez-faire" outcome in which CO2 damages from coal go on forever, so a prohibitive coal tax is much better than doing nothing. Of course, the first-best policy yields higher social welfare (12.4) than the prohibitive coal tax because the latter damages the private component of social welfare even more than it curbs global warming damages. Interestingly, the first-best outcome does rely on coal both alongside oil and on its own to ensure that the private part of welfare does not fall too much. Finally, we see that in our simulations the subsidy on renewables does considerably worse than the coal tax (as welfare is -27.7 instead of 7.8).

5. Optimal sequencing of using oil, coal or both?

In this section we alter assumption 2 to investigate alternative sequences of oil and coal before renewables are phased in, where we assume that renewables are too expensive to be introduced before the oil and coal phases to be discussed below. Two factors play a role in determining the optimal sequencing of oil and coal. The first is *efficiency*: what is the cheapest option to provide energy, taking account of the full cost, including global warming damages. The second is *profitability*: the marginal utility of energy should be larger than the marginal cost for else the use of a particular source comes to an end. Hence, two types of assumptions are needed. In this section we focus on efficiency considerations, assuming that marginal utility is large enough under all regimes that we consider, but for the regimes with simultaneous use we allow for full as well as partial exhaustion which depends on the utility function (as explained in section 2). Assumptions regarding efficiency depend on the cheapness of coal relative to oil, both in terms of unit extraction costs and the present value of marginal global warming damages, defined by:[11]

$$\text{(12)} \qquad \Omega(S(t),Y(t)) \equiv \rho\left[G(S(t)) - b - (\psi-1)\frac{D'(E_0+S_0-S(t)+Y(t))}{\rho}\right].$$

So a high value of Ω indicates that oil is expensive. A necessary condition for simultaneous use of oil and coal is that $\Omega(S,Y)=0.$ Along such an interval we have $\dot{\Omega}(S,Y)=\Omega_S\dot{S}+\Omega_Y\dot{Y}=0$. The stock of oil is decreasing, the stock of pollution due to the accumulation of carbon from coal is increasing, and $\Omega_Y=(1-\psi)D''(E)<0$, since the damage function is convex and coal is dirtier per unit of energy than oil. We have $\Omega_S=\rho G'(S)+(\psi-1)D''(E)$. So, a necessary condition for simultaneous use is that $\Omega_S<0$.

Since oil is more expensive to extract if oil reserves are low, the private component of the cost advantage of coal rises as oil reserves are depleted (i.e., $\Omega_S=\rho G'(S)<0$ if $D=0$). This is the case for the market outcome, which does not internalize global warming externalities. However, taking account of the social cost of warming, we see that the cost advantage of using coal rather than oil might fall as oil reserves are

[11] Chakravorty et al. (2008), Farzin and Tahvonen (1996) and Tahvonen (1997) use a similar function to $\Omega(.)$ to characterize the various regimes, although the arguments of their function are slightly different. Tahvonen (1997) obtains 11 different regimes. Although we do not have decay of the stock of CO2, we can depending on profitability assumptions get as many regimes as well albeit different ones.

depleted and the CO2 concentration rises (i.e., $\Omega_S > \rho G'(S)$ if $D' > 0$). This highlights a key dilemma for climate change policy: the market might want to switch to coal as oil reserves are depleted but duly taking account of global warming externalities one might want to carry on to use oil rather than switching to dirtier coal. The sign of $\Omega_S = \rho G'(S) + (\psi - 1)D''(E)$ is thus negative if the positive effect of a falling stock of remaining oil reserves on oil extraction costs dominates the mitigating effect on global warming damages of using oil rather than coal; and is positive, else. Under assumption 3 $\Omega_S < 0$ if and only if $\delta > (\psi - 1)\kappa / \rho > 0.$ Hence, a large emission-ratio of coal (ψ), a large marginal damage parameter (κ) and a small rate of time preference (ρ) make oil attractive compared to coal. If oil extraction costs rise rapidly enough with declining reserves (high δ), the cost advantage of coal is increasing over time as the stock of oil falls. So, then there is a case for coal being phased in eventually if we start with oil only. Since we assume that Ω is monotonic in S, assumptions 2(i) and (ii) can be rephrased as $\Omega(S_0, 0) < 0$ and $\Omega(0,0) > 0$, so $\Omega_S < 0$. This warrants coal being phased in alongside oil in the social optimum after the oil-only phase.

Table 2 summarizes the optimal sequences of using oil and coal that can occur in general. In sections 2-4 we derived and discussed regime III which applies if the social cost of oil is initially less than that of coal and if the social cost of using the last drop of oil exceeds that of coal. The phase with simultaneous use of oil and coal may last forever, or just for a finite period of time, after which coal takes over indefinitely. In regime IV coal is initially cheaper from a social perspective than oil and, as in regime III, becomes even cheaper as oil reserves are depleted. Hence, we could in principle again have simultaneous use, but not initially. Initially we should use only coal (see proposition A8 in the appendix). Once we arrive at $\Omega(S_0, Y(t)) = 0$ there will be a phase where oil and coal are used simultaneously. This phase may last forever, or just for a finite period of time until it is no longer cost effective to use oil or until oil reserves are fully exhausted, after which coal takes over indefinitely.

Another alternative to assumption 2(i) and (ii) is to have $\Omega(S, 0) < 0$ for all $0 < S < S_0$, which yields regime I. Since $\Omega_Y < 0$, it follows that it will never be optimal to have simultaneous use. Moreover, coal is too expensive to be taken into exploitation until all oil is fully depleted (see proposition A7 in the appendix). It is thus optimal to start with only oil, exhaust the oil stock and then switch to coal forever. Yet another alternative is regime II which occurs if coal is cheap initially, $\Omega(S_0, 0) > 0$, but where, contrary to regime IV, the cost advantage is declining with a decrease in the oil stock $\Omega(S_0, 0) > \Omega(0,0)$. A constructive argument can be used again to show that it is indeed optimal to start with using coal only (see proposition A8 in the appendix). After this phase we cannot have simultaneous supply because $\Omega_S > 0$. Hence, there is an initial phase of only coal, then oil takes over until exhaustion, which is followed by a final phase with only coal.

Table 2: Regimes for optimal sequences of oil extraction and coal use

I. High coal cost throughout: $\Omega(S,0) < 0$ for all $0 < S < S_0$	First use oil; then switch to coal. Fully exhaust oil reserves.
II. Low coal cost initially; last drop of oil cheap: (a) $\Omega(S_0,0) > 0 > \Omega(0,0)$ (b) $\Omega(S_0,0) > \Omega(0,0) > 0$	First use coal; then use oil if marginal global warming damages become prohibitive; finally switch back to using coal.
III. High coal cost initially; last drop of oil expensive: $\Omega(S_0,0) < 0 < \Omega(0,0)$	First use oil; then use oil and coal together. There will be a switch to using only coal forever if oil becomes fully exhausted.
IV. Low coal cost ; last drop of oil expensive: $0 < \Omega(S_0,0) < \Omega(0,0)$	First use coal; then use coal and oil together. There will be a switch back to using only coal if oil becomes fully exhausted.

6. Conclusions

The main message of this paper is that it is optimal to use less coal and more oil in the time before it is optimal to switch to carbon-free renewables such as solar energy. This requires a steeply rising carbon tax when the economy uses only oil, a less steeply rising carbon tax in the intermediate phase when oil and coal are used alongside other, and finally a high and constant carbon tax in the coal-only phase before renewables are used. Interestingly, a grass roots movement is emerging in the US which has defeated or abandoned 139 coal-fired power plants since 2000. The environmental disaster resulting from the collapse of the Tennessee Valley Authority coal ash containment boosted public opinion against coal in the US even further. Denmark and New Zealand have already banned coal-fired power plants. Even China is surging ahead with renewables and will hopefully reverse its growth in coal-fired power plants. The world gradually seems to be cottoning on to the idea of using less coal and more oil.

The other main message of our paper is that, if it is politically infeasible to levy a rising carbon tax, a moratorium on coal or equivalently a prohibitive tax on coal ensures that the market uses oil for much longer (albeit not as long as the first-best outcome) and exhausts a larger portion of oil reserves before switching to renewables. This second-best policy does surprisingly well and produces an outcome somewhere in between the optimal and the “laissez-faire” outcome. The alternative of subsidizing renewables to just below the extraction cost of coal leaves the oil-only phase, the transition time and the amount of oil left in situ unaffected, but does stabilize the stock of CO2 once renewables are introduced.

It increases green welfare (net of the welfare cost resulting from the taxes needed to finance the subsidies) if society uses a low discount rate, if the problem of global warming is acute and, typically, if the cost price of coal is much cheaper than that of renewables.

We have thus investigated optimal climate policy in the presence of an exhaustible resource, oil, and an abundant resource, coal (or tar sands), where coal contributes more CO2 per unit of energy than oil. We focused on a regime where it is attractive to use oil before phasing in coal. The oil-only phase is followed by an oil-coal phase in the social optimum until oil reserves are fully exhausted and the economy switches to coal completely. In contrast, the “laissez-faire” market outcome never has a phase where coal and oil are used alongside each other. It either uses coal forever or more realistically starts with oil until it is no longer profitable to do so and then switches to using coal. We have shown that choosing the optimal level and rate of growth of the carbon tax appropriately persuades the market to stop switching abruptly from oil to coal thereby leaving a substantial amount of oil in the crust of the earth.

The “laissez-faire” outcome leaves more oil in situ if coal is cheap and extraction of oil is expensive. The market wants to make the transition to coal too soon, and too abrupt. Moreover, final use of oil also decreases in the extraction cost of oil. In contrast, the socially optimal outcome may fully exhaust oil reserves before having to rely on coal as the sole source of energy. But even if it is optimal to leave oil unexploited, less will be left in situ than in the market economy. The reason is that coal emits more CO2 per unit of energy than oil so it makes sense from the point of view of combating climate change to use up all oil. Also, coal use is high and remains high in the final coal phase of the “laissez-faire” outcome but coal use is lower and vanishes asymptotically in the social optimum.

If we allow for renewables and if they are optimally introduced after a coal-only phase, then during the new coal-only phase the social cost of carbon is smaller than before and thus the phasing in of coal and the phasing out of oil are brought forward somewhat. This shortens the length of the oil-coal phase. Since more oil is used and less coal, there will be less global warming and thus the optimal carbon tax can be lower. The time profile of the optimal carbon tax is similar, albeit that it stops increasing once renewables are introduced. A lower cost of carbon-free renewables implies that oil will be pumped slightly more vigorously and quickly, coal phased in more quickly albeit for a shorter period, and renewables phased in earlier. Although the optimal carbon tax is lower, the CO2 concentration and global warming are curbed.

We have also characterized the various ways of sequencing oil and coal and departures from the Herfindahl rule that can occur in the social optimum. Indeed, in the social optimum the ordering does not just depend on extraction costs (which rise as oil reserves are depleted), but also on the social cost of global warming damages. With high enough demand for energy and with very dirty coal, coal will only be used after complete exhaustion of oil, even if coal is cheap to extract. For moderate CO2-emission coefficients of coal, an intermediate period of simultaneous use of oil and coal will be optimal. In the unlikely case where coal is relatively clean, the optimal economy might start with coal, after which there is a simultaneous phase and finally only coal again. So, 'preference reversal' may occur. Simultaneous use only occurs if during the oil-only phase with oil reserves declining, oil extraction costs rise rapidly and CO2 emissions from coal are only moderate so the comparative advantage of oil over coal decreases.

Our analysis can be extended in several directions. First, imperfect substitution in the demand for oil and coal may arise from concerns with security of energy supplies, diversification and/or intermittence of backstops. Second, there may be various types of backstop available at the same time which may be ranked, e.g., clean but competitive (nuclear), clean and expensive (wind, solar, advanced nuclear) and dirty and expensive (tar sands) in which case one might go for the cheapest and cleanest backstops. As we have seen, with dirty and cheap backstops matters are more complicated especially if their supply schedules slope upwards. Third, exhaustibility of coal as well as oil may impact optimal climate policy in which case the challenge is to offer a comprehensive analysis of what the optimal transition times are for phasing in oil, coal and the various types of renewables (cf., Chakravorty et al., 2008). Fourth, many countries still have substantial subsidies on fossil fuel (e.g., gasoline) and on coal. It is interesting to analyze what kind of political distortions keep these in place and what can be done to get rid of them. Fifth, it is of interest to set our analysis within the context of a Ramsey growth model (cf., van der Ploeg and Withagen, 2010b) to investigate why developing countries have a bigger incentive to use coal than oil or renewables. Also, one could apply the theory of exogenous growth or that of endogenous growth and directed technical change to investigate how renewables can be introduced much more quickly (Bosetti et al., 2009; Aghion et al., 2009; Acemoglu et al., 2010). Finally, one could study international aspects such as carbon leakage and ways to sustain international cooperation within the context of a multi-country version of our model (cf., Hoel, 2008; Eichner and Pethig, 2010).

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Web appendix

The current-value Hamiltonian function for the social planner is defined by:

(A1) $H(q,x,S,\lambda,\mu) \equiv U(q+x) - G(S)q - bx - D(E) - \lambda q - \mu\left(q + \psi x\right),$

where λ is the social value of oil and μ (> 0) is the social cost of the stock of CO2 in the atmosphere. The necessary conditions for a social optimum are:

(A2a) $U'(q+x) - G(S) \leq \lambda + \mu,\ q \geq 0,$ c.s.,

(A2b) $U'(q+x) - b \leq \psi\mu,\ x \geq 0,$ c.s.,

(A2c) $\dot{\lambda} = \rho\lambda + G'(S)q,$

(A2d) $\dot{\mu} = \rho\mu - D'(E),$

(A2e) $\lim_{t\to\infty} \exp(-\rho t)\left[\lambda(t)S(t) - \mu(t)E(t)\right] = 0,$

where c.s. stands for complimentary slackness. Conditions (A2) yield (4) and (5) in the paper. Before we get to prove some of the proposition in the paper, it is useful to establish a few preliminary propositions. Propositions A1-A3 are useful for obtaining the explicit solution trajectories for the oil-only, coal-only and oil-coal phases that might occur in the various regimes. Proposition A4 then fleshes out propositions 2 and 3 for the "laissez-faire" economy and gives the corresponding solution trajectories needed for section 3. This appendix also gives the proofs of proposition 7 and an extended version of proposition 8, both stated in section 4.3 of the paper.

Propositions A1, A2, and A3 below describe the solution for the phases where, respectively, only oil, only coal, and oil and coal are used. These propositions use the specific functional forms of assumption 3.

Proposition A1: Oil-only phase

Suppose there exists an interval of time $V = [T_1^o, T_2^o]$ such that $q(t) > 0$ and $x(t) = 0$ along the interval. Then the time paths of the stock of oil reserves and of oil use are given by:

(A3) $S(t) = M_1 e^{\omega_1 t} + M_2 e^{\omega_2 t} - \dfrac{\alpha - \gamma - \kappa(E_0 + S_0 + Y(T_1^o))/\rho}{\delta + \kappa/\rho}$ and $q(t) = -\omega_1 M_1 e^{\omega_1 t} - \omega_2 M_2 e^{\omega_2 t}$,

where $\omega_1 = \tfrac{1}{2}\rho + \tfrac{1}{2}\sqrt{\rho^2 + 4(\rho\delta + \kappa)/\beta}$, $\omega_2 = \rho - \omega_1$ and the constants M_1 and M_2 are to be determined from the boundary conditions.

Proof: Along the interval $\beta\ddot{S}-\rho\beta\dot{S}-(\rho\delta+\kappa)S=\rho(\alpha-\gamma)-\kappa(E_0+S_0+Y(T_1))/\rho,$ which yields (A5). Q.E.D.

Proposition A2: Oil-coal phase

Suppose there exists an interval of time $V=[T_1^s,T_2^s]$ such that $q(t)>0$ and $x(t)>0$ along the interval. Then the time paths for the stock of oil reserves, oil use and coal use are given by:

(A4)
$$S(t)=\frac{(\alpha-\gamma)(1-\psi)-b+\gamma}{\psi\delta}+L_1e^{\pi_1 t}+L_2e^{\pi_2 t},\quad q(t)=-\pi_1L_1e^{\pi_1 t}-\pi_2L_2e^{\pi_2 t},$$
$$x(t)=\frac{\alpha-\left[b+\psi\int_t^\infty \kappa(E_0+S_0-S(t'))e^{-\rho(t'-t)}dt'\right]}{\beta}+\pi_1L_1e^{\pi_1 t}+\pi_2L_2e^{\pi_2 t},\quad t\in V,$$

where $\pi_1=\tfrac{1}{2}\rho+\tfrac{1}{2}\sqrt{\rho^2+4\psi^2\kappa/\beta[1+\kappa(1-\psi)^2/\rho\delta]}>0$, $\pi_2=\rho-\pi_1<0$ and L_1 and L_2 are to be determined from the boundary conditions.

If $(\alpha-\gamma)(1-\psi)-b+\gamma<0$ then any oil-coal phase lasts for a finite time and oil becomes fully exhausted. In the optimal program there will be a final phase where only coal is used. Coal use and accumulated CO_2 emissions from coal use can be calculated from:

(A4′)
$$x(t)=\left(\frac{\rho\delta-(\psi-1)\kappa}{\psi(\psi-1)\kappa}\right)q(t),\quad Y(t)=\frac{\left[(\psi-1)\kappa-\rho\delta\right]S(t)+\rho(\gamma-b)}{(\psi-1)\kappa}-E_0-S_0,\quad t\in V.$$

If $(\alpha-\gamma)(1-\psi)-b+\gamma>0$ then the oil-coal phase may last forever.

Proof: Using the necessary conditions and the fact that $\Omega(S,Y)=0$ along V gives the differential equation $\beta\ddot{S}-\rho\beta\dot{S}+\dfrac{\psi^2\delta\kappa}{-\delta-\kappa(1-\psi)^2/\rho}S=\dfrac{\psi\kappa\left[(\alpha-\gamma)(1-\psi)-b+\gamma\right]}{-\delta-\kappa(1-\psi)^2/\rho}$, which yields the expression for $S(t)$ in (A4). Differentiating this expression yields the expression for $q(t)$ in (A4). The expression for $x(t)$ in (A4) then comes from the optimality conditions (A2b) and (A2d). Suppose exhaustion occurs at some instant of time T. Suppose $(\alpha-\gamma)(1-\psi)-b+\gamma<0$.Then we cannot have $T_2=\infty$. Oil will be fully depleted at some instant of time T and $\gamma-b+\kappa(1-\psi)(E_0+S_0+Y(T))/\rho=0.$ This implies $\alpha>b+\kappa\psi(E_0+S_0+Y(T))/\rho,$ so that after the coal-oil phase there will be a final phase where only coal is used. The expression for Y in (45′) comes from solving $\Omega(S,Y)=0$ and the expression for x in (A5′) comes from $x=\dot{Y}/\psi$ and using $\dot{S}=-q.$ Q.E.D.

We thus see from the first part of (A4′) that during the oil-coal phase more coal than oil is used if society is impatient and cares little about global warming (high ρ, low κ), oil extraction costs rise rapidly as reserves are depleted (high δ), and coal generates little more CO2 per unit of energy than oil (low ψ).

Proposition A3: Coal-only phase

Suppose there exists an interval of time $V=[T_1^c,T_2^c]$ with $T_2^c<\infty$ such that $q(t)=0$ and $x(t)>0$ along the interval . Then the time paths of the stock of CO2 and of coal use are given by:

(A5) $$Y(t)=\frac{\rho}{\psi\kappa}(\alpha-b)-\left[E_0+S_0-S(T_1^c)\right]+\hat{K}_1e^{\lambda_1 t}+\hat{K}_2e^{\lambda_2 t} \text{ and } x(t)=K_1e^{\lambda_1 t}+K_2e^{\lambda_2 t}$$

where $\lambda_1=\frac{1}{2}\rho+\frac{1}{2}\sqrt{\rho^2+4\psi^2\kappa/\beta}>\rho>0$ and $\lambda_2=\rho-\lambda_1<0,$ and K_1 and K_2 are to be determined by the boundary conditions and $\lambda_1\hat{K}_1=\psi K_1$ and $\lambda_2\hat{K}_2=\psi K_2$.

Suppose there exists an interval of time $V=[T_1^c,T_2^c]$ with $T_2^c=\infty$ such that $q(t)=0$ and $x(t)>0$ along the interval . Then the time paths of the stock of CO2 and of coal use are given by:

(A5′) $$Y(t)=Y^*+[Y(T_1^c)-Y^*]e^{\lambda_2(t-T_1^c)},\quad x(t)=\left(\frac{\psi\kappa}{\beta\lambda_1}\right)[Y^*-Y(T_1^c)]e^{\lambda_2(t-T_1^c)}$$

with $Y^*\equiv\left(\frac{\rho}{\psi\kappa}\right)(\alpha-b)-\left[E_0+S_0-S(T_1^c)\right].$

Proof: Along V we have $\alpha-\beta x-b=\psi\mu$ and $\dot{\mu}(t)=\rho\mu(t)-\kappa(E_0+S_0-S(T_1)+Y(t))$. This yields $\dot{x}=-\frac{\rho}{\beta}(\alpha-\beta x-b)+\frac{\psi\kappa}{\beta}\left(E_0+S_0-S(T)+Y\right)$ and $\dot{Y}=\psi x.$ This system displays saddlepoint stability with one positive characteristic root, λ_1, and one negative characteristic root, λ_2. We obtain $-\beta\ddot{x}+\rho\beta\dot{x}+\psi^2\kappa x=0$, which readily gives the second part of (A5). For Y we get $\beta\ddot{Y}-\rho\beta\dot{Y}-\kappa\psi^2Y=-\rho\psi(\alpha-b)+\kappa\psi^2(E_0+S_0-S(T_1))$ yielding the first part of (A5). If $T_2=\infty$, it should be the case that Y is finite. Since $\lambda_1>0$, we thus have $K_1=0.$ Making use of $\lambda_1\lambda_2=-\psi^2\kappa/\beta,$ we obtain (A5′). To ensure that $Y^*>0$, we require that $\alpha>b+\psi\kappa\left[E_0+S_0-S(T_1)\right]/\rho$. Hence, $x(t)\to 0$ and $Y(t)\to Y^*$ as $t\to\infty.$ The rest is straightforward. Q.E.D.

The rate at which use of the backstop falls increases with the rate at which marginal global warming damages increase (κ), the emission intensity of the backstop (ψ), and the sensitivity of demand for energy with respect to the price ($1/\beta$). Total use of the backstop (Y^*) is less if past use of oil has already led to a high concentration of CO2 emissions in the atmosphere and if the unit cost of the backstop (b) is high, but more if autonomous demand for energy (α) is high. Coal use starts especially low if marginal global warming damages are high (high ψ, κ, E_0, S_0, low S(T_1)). If global warming externalities are not internalized, as would be the case in the "laissez-faire" market economy, coal use would be higher.

With the classification of regimes given in table 1 of section 5 and the above propositions A1-A3, we can fully characterize the various regimes and obtain the explicit solution trajectories. We will illustrate this for the regime introduced in proposition 1.

Proposition A4: Suppose assumption 3 holds and that it is optimal to start with using only oil until time T_1 where oil stock and oil use trajectories are $T_1^o = 0$ and $T_2^o = T_1$ given by (A3) in proposition A1, then have a phase from time T_1 to T_2 where both oil and coal are used alongside each other until oil is fully exhausted where the solution trajectories are described by (A4) with $T_1^s = T_1$ and $T_2^s = T_2$ in proposition A2, and finally a phase from time T_2 onwards where only coal is used and coal use is described by (A5) or (A5′) with $T_1^c = T_2$ and $T_2^c = \infty$ in proposition A3. The constants of integration, $(K_1, K_2, L_1, L_2, M_1, M_2)$, and transition times, T_1 and T_2, necessary to obtain the full optimal solution trajectories are determined from the following eight boundary conditions:

(A6) $$S_0 = -\frac{\alpha - \gamma - \kappa(E_0 + S_0)/\rho}{\delta + \kappa/\rho} + M_1 + M_2$$

(A7) $$S(T_1) = -\frac{\alpha - \gamma - \kappa(E_0 + S_0)/\rho}{\delta + \kappa/\rho} + M_1 e^{\omega_1 T_1} + M_2 e^{\omega_2 T_1}$$

(A8) $$S(T_1) = \frac{(\alpha - \gamma)(1-\psi) - b + \gamma}{\psi\delta} + L_1 e^{\pi_1 T_1} + L_2 e^{\pi_2 T_1}$$

(A9) $$S(T_1) = \frac{\rho(\gamma - b) - (\psi - 1)\kappa(E_0 + S_0)}{\rho\delta - (\psi - 1)\kappa}$$

(A10) $$S(T_2) = \frac{(\alpha - \gamma)(1-\psi) - b + \gamma}{\psi\delta} + L_1 e^{\pi_1 T_2} + L_2 e^{\pi_2 T_2} = 0$$

(A11) $$K_1 = 0, K_2 = e^{-\lambda_2 T_2}\frac{\psi\kappa}{\beta\lambda_1}\left(\frac{\rho(\alpha - b)}{\psi\kappa} - \frac{\rho(\gamma - b)}{(\psi - 1)\kappa}\right)$$

(A12) $q(T_1-) = -\omega_1 M_1 e^{\omega_1 T_1} - \omega_2 M_2 e^{\omega_2 T_1} = \left(\frac{\rho\delta - (\psi-1)\kappa}{\psi(\psi-1)\kappa} + 1 \right)(-\pi_1 L_1 e^{\pi_1 T_1} - \pi_2 L_2 e^{\pi_2 T_1})$

(A13) $q(T_2-) = -\pi_1 L_1 e^{\pi_1 T_2} - \pi_2 L_2 e^{\pi_2 T_2} = \left(\frac{\psi(\psi-1)\kappa}{\rho\delta + (\psi-1)^2 \kappa} \right) \frac{\psi\kappa}{\beta\lambda_1} \left(\frac{\rho(\gamma-\beta)}{(\psi-1)\kappa} - \frac{\rho(\alpha-b)}{\psi\kappa} \right)$.

Proof: Equations (A6) and (A7) follow from (A3). Equation (A8) follows from (A4). Equation (A9) follows from the fact that $\Omega(S(T_1), Y(T_1)) = 0$ and $Y(T_1) = 0$: at T_1 the simultaneous phase starts and no CO_2 has been emitted yet due to the use of coal. Equation (A10) follows from (A4) and from the fact that all oil is depleted. The first part of (A11) is needed to have coal use forever after depletion of oil (see (A5) and note that $\lambda_1 > 0$. The second part follows from the fact that $\Omega(0, Y(T_2)) = 0$ implies $Y(T_2) = \frac{\rho(\gamma-b)}{(\psi-1)\kappa} - E_0 - S_0$. (A5) or (A5′) gives $x(T_2+) = K_2 e^{\lambda_2 T_2} = \frac{\psi\kappa}{\beta\lambda_1}\left[\left(\frac{\rho}{\psi\kappa} \right)(\alpha-b) - \frac{\rho(\gamma-b)}{(\psi-1)\kappa} \right]$. The first equality of (A12) follows from (A3). The second equality is a consequence of continuity: $q(T_1-) = q(T_1+) + x(T_1+)$, with, from (A4′), $x(T_1+) = \left(\frac{\rho\delta - (\psi-1)\kappa}{\psi(\psi-1)\kappa} \right) q(T_1+)$ and $q(T_1+)$ following from (A4). Finally, the first part of (A13) follows from (A4). The second equality is a consequence of continuity: $x(T_2+) = q(T_2-) + x(T_2-)$, with, from (A4′), $x(T_2-) = \left(\frac{\rho\delta - (\psi-1)\kappa}{\psi(\psi-1)\kappa} \right) q(T_2-)$ and $x(T_2+)$ given above. Hence, we end up with 8 equations (after inserting (A9) in (A7) and (A8) and noting that (A11) contains two equations) for the 8 variables to be determined. Q.E.D.

Similarly, the integration constants and the transition times for all other regimes can be found. Let us now consider the solution paths for the “the laissez-faire” economy, with some oil left unexploited.

Proposition A5: Suppose assumption 3 holds and $\gamma > b$. The “laissez-faire” economy only uses oil up to instant of time *T*, at a decreasing rate. After *T* only coal is used, at a constant rate. Oil use and coal use are given by:

(A14a) $S(t) = \left(\frac{\frac{\alpha-b}{\delta} - \left(S_0 + \frac{\alpha-\gamma}{\delta} \right) e^{\omega_2 T}}{e^{\omega_1 T} - e^{\omega_2 T}} \right) e^{\omega_1 t} + \left(\frac{\left(S_0 + \frac{\alpha-\gamma}{\delta} \right) e^{\omega_1 T} - \frac{\alpha-b}{\delta}}{e^{\omega_1 T} - e^{\omega_2 T}} \right) e^{\omega_2 t} - \left(\frac{\alpha-\gamma}{\delta} \right),$

$$\text{(A14b) } q(t) = -\omega_1 \left(\frac{\frac{\alpha - b}{\delta} - \left(S_0 + \frac{\alpha - \gamma}{\delta} \right) e^{\omega_2 T}}{\left[e^{\omega_1 T} - e^{\omega_2 T} \right]} \right) e^{\omega_1 t} - \omega_2 \left(\frac{\left(S_0 + \frac{\alpha - \gamma}{\delta} \right) e^{\omega_1 T} - \frac{\alpha - b}{\delta}}{\left[e^{\omega_1 T} - e^{\omega_2 T} \right]} \right) e^{\omega_2 t}, \quad 0 \le t \le T,$$

$$\text{(A14c) } x(t) = 0, \quad 0 \le t \le T, \qquad S(t) = \frac{\gamma - b}{\delta}, \quad q(t) = 0, \quad x(t) = \frac{\alpha - b}{\beta}, \quad t \ge T,$$

where $\omega_1 = \frac{1}{2}\rho + \frac{1}{2}\sqrt{\rho^2 + 4\rho\delta / \beta}$ and $\omega_2 = \rho - \omega_1$. The transition time T is such that $q(T) = (\alpha - b)/\beta$ and increases in b.

Proof: If there is simultaneous use of oil and coal, $G(S) = b$. This implies a constant value of S and thus $q = 0$, which contradicts the assumption of simultaneous use. To establish that q falls during the oil-only phase, note that for the market μ plays no role so that $U'(q) = G(S) + \lambda,$ which implies $U''(q)\dot{q} = G'(S)\dot{S} + \rho\lambda + G'(S)q = \rho\lambda$ and thus (as $\lambda > 0$) we have $\dot{q} < 0.$ From proposition A1 the initial oil phase is described by $S(t) = M_1 e^{\omega_1 t} + M_2 e^{\omega_2 t} - \frac{\alpha - \gamma}{\delta}$ and $q(t) = -\omega_1 M_1 e^{\omega_1 t} - \omega_2 M_2 e^{\omega_2 t}$. From proposition A3 the final coal phase implies that coal use is $x(t) = (\alpha - b)/\beta$, $t \ge T$ as $\lambda_1 = \rho$ and $\lambda_2 = 0$ if $\kappa = 0$ and $Y(T) = 0$. Solving for M_1 and M_2 and the transition time T for the case $\gamma > b$ from $S(0) = S_0$ and $S(T) = (\gamma - b)/\delta,$ we get $M_1 = \frac{\alpha - b - (\delta S_0 + \alpha - \gamma)e^{\omega_2 T}}{\delta\left[e^{\omega_1 T} - e^{\omega_2 T} \right]}$ and $M_2 = \frac{(\delta S_0 + \alpha - \gamma)e^{\omega_1 T} - (\alpha - b)}{\delta\left[e^{\omega_1 T} - e^{\omega_2 T} \right]},$ and thus the expressions for $S(t)$ and $q(t)$ given in (A7). We get the transition time by requiring continuity:

$$q(T) = \frac{(\omega_1 - \omega_2)(\delta S_0 + \alpha - \gamma) - (\gamma - b)\left(\omega_1 e^{-\omega_2 T} - \omega_2 e^{-\omega_1 T} \right)}{\delta\left(e^{-\omega_2 T} - e^{-\omega_1 T} \right)} = \frac{\alpha - b}{\beta} = x(T).$$

The denominator increases in T and, as $\gamma > b$, the numerator decreases in T, hence $q'(T) < 0$. Since ω_1 and ω_2 do not depend on α or b, we see that T increases with b. Q.E.D.

Next we consider the case with renewables.

Proposition A6: Suppose assumption 3 holds and it is optimal to start using only oil until time T_1 where oil stock and oil use trajectories are $T_1^o = 0$ and $T_2^o = T_1$ given by (A3) in proposition A1, then the optimal outcome has a phase from time T_1 to T_2 where both oil and coal are used alongside each other until oil is fully exhausted with the solution trajectories described by (A4) with $T_1^s = T_1,$ $T_2^s = T_2$ and $S(T_2) = 0$ in proposition A2, then there is a phase from time T_2 to T_3 where only coal is used with coal use described by (A5) with $T_1^c = T_2$ and $T_2^c = T_3$ in proposition A3, and finally, there is from time T_3 onwards a phase where only the backstop is used. The constants of integration, $(K_1, K_2, L_1, L_2, M_1, M_2)$, and the transition times, T_1, T_2, and T_3, necessary to obtain the optimal paths are determined from (A6)-(A10), (A12) and

(A15) $$x(T_3-) = K_1 e^{\lambda_1 T_3} + K_2 e^{\lambda_2 T_3} = \frac{\alpha - c}{\beta},$$

(A16) $$\frac{\rho(\gamma - b)}{(\psi - 1)\kappa} = \frac{\rho(\alpha - b)}{\psi\kappa} + \left(\frac{\psi K_1}{\lambda_1}\right) e^{\lambda_1 T_2} + \left(\frac{\psi K_2}{\lambda_2}\right) e^{\lambda_2 T_2},$$

(A17) $$q(T_2-) = -\pi_1 L_1 e^{\pi_1 T_2} - \pi_2 L_2 e^{\pi_2 T_2} = \left(\frac{\psi(\psi - 1)\kappa}{\rho\delta + (\psi - 1)^2 \kappa}\right)(K_1 e^{\lambda_1 T_2} + K_2 e^{\lambda_2 T_2}).$$

Proof: Clearly, (A6)-(A10) and (A12) must hold, as in proposition A4. Equation (A15) follows from (A5) with $x(T_3-) = \frac{\alpha - c}{\beta}$. As before, we have $Y(T_2) = \frac{\rho(\gamma - b)}{(\psi - 1)\kappa} - E_0 - S_0$. Using this in (A5) yields (A16). We also have $q(T_2-) + x(T_2-) = x(T_2+)$, with $x(T_2-) = \left(\frac{\rho\delta - (\psi - 1)\kappa}{\rho\delta + (\psi - 1)^2 \kappa}\right) x(T_2+)$. Hence (A17) holds. We have 9 equations with 9 unknowns. Q.E.D.

Proof of proposition 5: If coal is used (alone or in combination with oil), (4) gives the carbon tax $\tau = \mu = [U'(q + x) - b]/\psi$. If only oil is used, equation (10) follows from solving the ordinary differential equation (5) for $\tau = \mu$ with the terminal condition $\mu(T_1) = \left[U'\left(q(T_1) + x(T_1)\right) - b\right]/\psi$. The solutions for $x(t)$ in (10′) follow from (A5′) in proposition A3 with the boundary condition $S(T_2) = 0$. Similarly, the solution for $x(t)$ in (10″) follows from (A4) in proposition A2.

To prove that the time path of τ is concave, consider the following. Since $\dot{\tau} = \rho\tau - \kappa E$, we have $\ddot{\tau} = \rho\dot{\tau} - \kappa(q + \psi x)$. We know that τ and thus $\dot{\tau}$ is continuous everywhere and also that $\dot{\tau} > 0$ throughout. If only coal is used it follows from (A2b) that $U''\dot{x} = -\beta\dot{x} = \psi\dot{\tau} > 0$ so that $\dot{x} < 0.$ In the oil-only phase we have from (A2a-A2c) that $U''\dot{q} = -\beta\dot{q} = \rho\lambda + \rho\tau - \kappa E > 0$ so that $q + \psi x = \frac{\rho\delta}{(\psi - 1)\kappa} q$ $\dot{q} < 0$. In the simultaneous phase we have from (A2b) that $\dot{q} + \dot{x} < 0.$ Moreover, it follows from (A4′) that $\dot{q} < 0$ if and only if $\dot{x} < 0$ implying $\dot{q} < 0$. In addition, it follows from (A4′) that $q + \psi x = \frac{\rho\delta}{(\psi - 1)\kappa} q$. Therefore $\dot{q} + \psi\dot{x} < 0$. So, $\dot{q} + \psi\dot{x} < 0$ throughout. At the transition from oil only to simultaneous use and from simultaneous use to coal only, we have $q(T_1-) = q(T_1+) + x(T_1+),$ and

$x(T_2+) = q(T_2-) + x(T_2-),$ respectively. Now suppose that $\ddot{\tau} > 0$ somewhere along the oil-only phase. Then $\ddot{\tau} > 0$ for the remainder of the oil-only phase as q is declining. At the transition to simultaneous use, we also have $\ddot{\tau} > 0$ as we have a downward jump, i.e., $q(T_1-) < q(T_1+) + x(T_1+) + (1-\psi)x(T_1+).$ Since $\dot{q} + \psi\dot{x} < 0$ in the oil-coal and coal-only phases, the time path of τ is convex. However, it follows from (10′) that in the coal-only phase $\ddot{x}(t) = \lambda_2{}^2\left(\frac{\psi\kappa}{\beta\lambda_1}\right)\left[\left(\frac{\rho}{\psi\kappa}\right)(\alpha-b) - \left(E_0 + S_0\right)\right]e^{\lambda_2(t-T_2)} > 0,$

and thus $\ddot{\tau}(t) = -\lambda_2{}^2\left(\frac{\kappa}{\lambda_1}\right)\left[\left(\frac{\rho}{\psi\kappa}\right)(\alpha-b) - \left(E_0 + S_0\right)\right]e^{\lambda_2(t-T_2)} < 0,\ t \geq T_2.$

We thus obtain a contradiction, so the time path of τ must be concave. Q.E.D.

Proof of proposition 7: Since in the oil-only phase *T* and *S*(*T*) are the same, we need only consider the backstop phase. In the "laissez-faire" outcome, the green welfare loss for that phase is:

$$\Theta^{LF} \equiv \int_T^\infty D(E(t))e^{-\rho(t-T)}dt = \frac{\kappa}{2}\int_T^\infty \left[E_0 + S_0 - \frac{\gamma-b}{\delta} + \psi\left(\frac{\alpha-b}{\beta}\right)(t-T)\right]^2 e^{-\rho(t-T)}dt =$$

$$\frac{\kappa}{2\rho}\left[E_0 + S_0 - \frac{\gamma-b}{\delta}\right]^2 + \frac{\kappa}{\rho^2}\left[E_0 + S_0 - \frac{\gamma-b}{\delta}\right]\psi\left(\frac{\alpha-b}{\beta}\right) + \frac{\kappa}{\rho^3}\psi^2\left(\frac{\alpha-b}{\beta}\right)^2.$$

If renewables are subsidized until they are just economically viable, the green welfare loss is:

$$\Theta^{RS} \equiv \int_T^\infty D(E(t))e^{-\rho(t-T)}dt = \frac{\kappa}{2}\int_T^\infty \left[E_0 + S_0 - \frac{\gamma-b}{\delta}\right]^2 e^{-\rho(t-T)}dt = \frac{\kappa}{2\rho}\left[E_0 + S_0 - \frac{\gamma-b}{\delta}\right]^2 < \Theta^{LF}.$$

If the reduction in green welfare loss thus obtained outweighs the welfare cost of the lump-sum taxes needed to finance the subsidy, i.e., if

$$\Theta^{LF} - \Theta^{RS} = \frac{\kappa}{\rho^2}\left[E_0 + S_0 - \frac{\gamma-b}{\delta}\right]\psi\left(\frac{\alpha-b}{\beta}\right) + \frac{\kappa}{\rho^3}\psi^2\left(\frac{\alpha-b}{\beta}\right)^2 > \int_T^\infty (c-b)x(t)e^{-\rho(t-T)}dt = \left(\frac{c-b}{\rho}\right)\left(\frac{\alpha-b}{\beta}\right),$$

it makes sense to subsidize renewables in this way. This gives rise to the required condition. Q.E.D.

Extended version of proposition 8: With a prohibitive coal tax and the cost of renewables more than the cost of extracting the final drop of oil ($c > \gamma$), oil reserves will be fully exhausted. During the oil-only phase, oil reserves and oil use fall monotonically and coal use is zero. The paths for the oil-only phase are given by:

(A8a′) $$S(t)=\frac{\left[\frac{\alpha-\gamma}{\delta}-\left(S_0+\frac{\alpha-\gamma}{\delta}\right)e^{\omega_2 T}\right]}{e^{\omega_1 T}-e^{\omega_2 T}}e^{\omega_1 t}+\frac{\left[\left(S_0+\frac{\alpha-\gamma}{\delta}\right)e^{\omega_1 T}-\frac{\alpha-\gamma}{\delta}\right]}{e^{\omega_1 T}-e^{\omega_2 T}}e^{\omega_2 t}-\frac{\alpha-\gamma}{\delta},$$

(A8b′) $$q(t)=-\omega_1\frac{\left[\frac{\alpha-\gamma}{\delta}-\left(S_0+\frac{\alpha-\gamma}{\delta}\right)e^{\omega_2 T}-S_0 e^{\omega_2 T}\right]}{e^{\omega_1 T}-e^{\omega_2 T}}e^{\omega_1 t}-\omega_2\frac{\left[\left(S_0+\frac{\alpha-\gamma}{\delta}\right)e^{\omega_1 T}-\frac{\alpha-\gamma}{\delta}\right]}{e^{\omega_1 T}-e^{\omega_2 T}}e^{\omega_2 t},\quad 0\le t\le T.$$

Renewables use in the carbon-free economy is given by $x(t)=\frac{\alpha-c}{\beta},\quad t\ge T.$ The transition time T follows from $q(T)=(\alpha-c)/\beta$, so shift to carbon-free society is earlier if renewables are cheap (c low).

Proof of proposition 8: The proof parallels proposition A5. Since the condition $G(S(T)) = c$ yields a negative value of $S(T)$ with $c > \gamma$, we must have $S(T) = 0$. We obtain (A8a′) and (A8b′) from (A5) after solving for M_1 and M_2. Also, from $q(T)=\frac{(\omega_1-\omega_2)\left(S_0+\frac{\alpha-\gamma}{\delta}\right)e^{\rho T}-\left(\omega_1 e^{\omega_1 T}-\omega_2 e^{\rho T}\right)\frac{\alpha-\gamma}{\delta}}{e^{\omega_1 T}-e^{\omega_2 T}}$ we have $q'(T) > 0$, so T increases with c. Q.E.D.

Proposition A7: If $\Omega(S,Y)<0,$ oil reserves are fully exhausted before coal is phased in.

Proof: Suppose $\Omega(S,Y)<0$ always and $x(t)>0$ for some interval of time along which $S>0.$ Hence, $U'(x)=b+\psi\mu\le G(S)+\lambda+\mu$ and thus $G(S)-b+\lambda+(1-\psi)\mu\ge 0.$ The time derivative of the left-hand side is $G'(S)\dot{S}+\dot{\lambda}+(1-\psi)\dot{\mu}=\rho(\lambda+(1-\psi)\mu-(1-\psi)D'(E)/\rho)\ge -\Omega(S,Y)>0.$ Along this part of the program $G(S)$ is constant and $D'(E)$ is increasing. No transition to simultaneous use or only oil use can take place. Therefore $G(S(t))-b+\lambda(t)+(1-\psi)\mu(t)\to\infty$ as $t\to\infty.$ But $\mu(t)\ge 0$ for all $t\ge 0.$ Hence, $\dot{\lambda}(t)/\lambda(t)=\rho$. But this contradicts $S(t)>0$ for all $t\ge 0.$ Q.E.D.

Proposition A8: If $\Omega(S_0,0)>0,$ it is optimal to start with only coal.

Proof: We define $\widehat{Y}$ by $\Omega(S_0,\widehat{Y})=0$. Then we consider two problems. The first problem is to maximize $\int_0^T e^{-\rho t}(U(x)-bx-D(E))dt$ subject to $\dot{E}=\psi x$ and $E(T)=E_0+\widehat{Y}.$ The second problem is to maximize $\int_T^\infty (e^{-\rho t}U(q+x)-bx-G(S)q-D(E))dt$ subject to $\dot{E}=q+\psi x,\ E(T)=E_0+\widehat{Y},\ \dot{S}=-q, S(T)=S_0$ and $E(T)=E_0+\widehat{Y}.$ Both problems have a solution. Generally, maximization with respect to T yields a positive T. Necessary is $U(x(T-))-bx(T-)-\mu(T-)\psi x(T-)=U(q(T+)-G(S_0)q(T+)-\lambda(T+)q(T+)-\mu(T+)q(T+)$. Imposing continuity of total energy supply yields $b+\mu(T)\psi=G(S_0)+\lambda(T)+\mu(T).$ Hence, it is optimal to start with using coal only, until we optimally arrive in a point where $\Omega=0.$ Q.E.D.