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Hybrid Auctions I: Theory*

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Abstract

In this paper we examine the properties of a hybrid auction that combines a sealed bid and an ascending auction. In this auction, each bidder submits a sealed bid. Once the highest bid is known, the bidder who submitted it is declared the winner if her bid is higher than the second highest by more than a predetermined amount or percentage. If at least one more bidder submitted a bid sufficiently close to the highest bid (that is, if the difference between this bid and the highest bid is smaller than the predetermined amount or percentage) the qualified buyers compete in an open ascending auction that has the highest bid of the first stage as the reserve price. Qualified bidders include not only the highest bidder in the first stage but also those who bid close enough to her. We show that this auction generates more revenue than a standard auction. Although this hybrid auction does not generate as much revenue as the optimal auction, it is ex-post efficient.**JEL Classification**: C72, D44.

Keywords: hybrid auctions, expected revenue, efficiency.

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1 Introduction

Hybrid auctions, which combine features of different auction formats, are becoming increasingly popular as allocation mechanisms. Klemperer [1] for example discusses the Anglo-Dutch auction - a hybrid of the sealed bid and ascending auctions - that may perform better in terms of the traditional concerns of competition policy such as preventing collusive, predatory and entry deterring behavior.

In this paper we examine the revenue properties of another hybrid auction, one that combines a sealed bid first price auction with an ascending auction. This particular mechanism has been used, for example, in the sale of the companies constituted through the partial division of the Telebras System (The Brazilian Telecom). The sale represented a major step towards the restructuring of the telecommunications sector in the country and it raised in excess of US$ 20 billion.

This hybrid auction works as follows. Each buyer submits a sealed bid. Once the highest bid is known, the bidder who submitted it is declared the winner if her bid is higher than the second highest bid by more than a predetermined amount or percentage. If at least one more bidder submitted a bid sufficiently close to the highest (that is, if the difference between this bid and the highest bid is smaller than the predetermined amount or percentage) the qualified buyers compete in an open ascending auction that has the highest bid of the first stage as the reserve price. Qualified bidders include the highest bidder in the sealed-bid stage and those who bid sufficiently close to her.

We develop a model that captures some of the features of this hybrid auction. We model a situation where three risk neutral bidders compete in a two stage auction. The first stage is a sealed bid first price auction. This is followed by a Vickrey auction as a second stage when there are bids sufficiently close to the highest one in the first price sealed bid auction. We consider a model in which potential buyers' values have both a private and a common component. Of course, special cases include the independent private values model and the pure common values case. For the case of a discrete distribution, we show that the hybrid auction generates more revenue than any standard auction. In sections 3 and 4 we show that this result is robust to relaxing both the risk neutrality and symmetry assumptions. The reason is that we may view this hybrid auction as a Vickrey auction with a reserve price set endogenously at the first stage. As a result, this mechanims is ex-post efficient.

2 The Basic Model

Suppose that three risk neutral bidders compete for a single object. Each bidder's valuation to the object is a function of a private value, specific to the agent, and of an unknown common value. The individual's private and common value components are independently distributed. s_i, *the vector of the signals observed by the bidder has two elements:* $s_i = (s_{i1}, s_{i2})$, *where* s_{i1} *is the private value, and* s_{i2} *is a signal sent by experts that represents an estimate of the common value component of the object.* The realized value of the object to bidder i, gross of her expected payment, is given by

$$v_i = u_i(s_i, v) = s_{i1} + v \qquad i = 1, 2, 3 \tag{1}$$

where v is the common value component. The private value, s_{i1}, may take one of two values,

$$s_{i1} = \{x_0, x_1\} \qquad x_0 < x_1.$$

We suppose further that each player i, $i = 1, 2, 3$, knows her own private value, but knows only that her opponents' values are x_1 with probability p or x_0 with probability $q = (1 - p)$. This structure is common knowledge among players.

The common value, v, is not directly observable by the bidders. In this setting v has one of two possible values. Without loss of generality, suppose that the common value can be either $V_0 = 0$ or $V > 0$. Let p_0 be the probability that the common value is V_0 and p_v the probability that it is V. Even though the bidder does not know the common value component, at the beginning of the first period she has access to an expert's appraisal of it. Therefore, she observes simultaneously both her private value and the signal sent by the experts, s_{i2}. This signal can also take two values, $s_{i2} = \{L, H\}$, indicating an unfavorable result, L, or a favorable result, H, from the experts estimates of the common value factor of the object. Given this information structure, there is a positive probability that the common value is mistakenly estimated. Let $q_{L|0}$ be the probability that the common value is 0 when the bidder receives a low signal and $q_{H|0}$ the probability that it has been mistakenly evaluated. In turn, let $q_{H|V}$ and $q_{L|V}$ be, respectively, the probability that the bidder receives a favorable or an unfavorable result when the item has a positive common value.

Given symmetry, we can restrict attention to the problem faced by one of the bidders, say Bidder 1. Her goal is to choose a bid $b(s_{i1}, s_{i2})$ that maximizes her expected payoff. Let $b^{(t)}$ represent the t^{th} highest bid. Conditional on winning the auction, the expected profit of Bidder 1 who receives simultaneously signals s_{i1} and s_{i2} and bids b is given by

$$\begin{aligned} & E_j\left[\left(u\left(s_{11}, v\right)-b\left(s_{11}, s_{12}\right)\right) 1_{\left\{b^{(2)}\left(s_{j1}, s_{j2}\right)+z<b\left(s_{11}, s_{12}\right)\right\}}\left|s_1, s_2, s_3\right.\right]+ \\ & +E_j\left[\left(u\left(s_{11}, v\right)-b^{(2)}\left(s_{j1}, s_{j2}\right)\right) 1_{\left\{b^{(2)}\left(s_{j1}, s_{j2}\right)+z>b\left(s_{11}, s_{12}\right)\right\}}\left|s_1, s_2, s_3\right.\right] . \end{aligned} \tag{2}$$

An auction is said to be *efficient* if in equilibrium, for all signal values $s_i = (s_1, s_2, s_3)$, the winner is buyer i such that $v_i(s_1, s_2, s_3) \geq v_j(s_1, s_2, s_3)$, $\forall j \neq i$. To aid our intuition, we start our analysis by considering the special case where an individual's value for the object is determined only by her private signal.

2.1 Independent Private Values

In the independent private values setting the agent's value for the object is a function of its private value only. The item has no intrinsic value that is common to all bidders. In the general framework of the last section it amounts to set $v = V = V_0 = 0$ and $v_i(s_1, s_2, s_3) = s_{i1}$. Without loss of generality assume that the seller's evaluation, v_s, is equal to zero. Let $s_{jt}^{(2)}$ be the second highest s_{jt} signal. Conditional on winning the auction, the expected return to the bidder 1 that observes signal s_{i1} and bids b is given by

$$\begin{aligned} \pi\left(s_{i1}, b\left(s_{i1}\right)\right) &= E_j\left[\left(s_{i1} - b\left(s_{i1}\right)\right) 1_{\{b(s_{i1}) > b(s_{j1}) + z\}}\right] + \\ &\quad + E_j\left[\left(s_{i1} - s_{j1}^{(2)}\right) 1_{\left\{s_{j1}^{(2)} < b(s_{i1}) < b(s_{j1}) + z\right\}}\right], \end{aligned} \tag{3}$$

where z is the cutoff value that *implies the occurrence of the contingent second stage*. In this auction game the proper equilibrium notion is that of a Bayesian Nash equilibrium. This concept extends the Nash equilibrium notion to static games of incomplete information. Each player's action is a best response to other players' actions, that is, each individual chooses a strategy that maximizes her expected payoff given that the other players are also choosing strategies to maximize their expected payoff. A strategy for player i in a Bayesian game is defined as a function from her set of types into her set of actions.[1]

We will focus on symmetric equilibrium, an equilibrium in which all bidders choose the same bidding function. Given the discrete nature of the model, we will characterize a mixed strategy equilibrium for this game consisting, for each possible type of bidder, of a support to the strategies, that in the present setting correspond to equilibrium bidding functions, and of the associated distribution functions. Upon learning that her private value is x_0, the lower type bidder never bids higher than her value. By playing a mixed strategy that randomizes in a variety of bids her expected profit would be negative. We will characterize an equilibrium such that, for each player, a bidder observing x_0 bids so as to earn 0 expected return and a bidder having a private value x_1 randomizes according to a continuous distribution function $F(b)$ in $\left[\underline{b}, \overline{b}\right]$. The equilibrium existence is guaranteed by Maskin and Riley [3] who show that with a finite number of types and private values there is an equilibrium to the first price auction when

[1] See Myerson [7].

ties are solved through a Vickrey auction. To characterize the equilibrium it is convenient to consider two separate cases.

2.1.1 $x_1 - x_0 > z$

If a bidder has a value x_0, she only wins the auction when all the other bidders have the same value.

$$U(x_0) = \frac{1}{3}(x_0 - b(x_0))(1-p)^2$$

Her equilibrium bid must guarantee her a zero expected return, that is

$$b(x_0) = x_0.$$

Note that in this case the second stage Vickrey auction will always occur. However at this stage no bidder will raise her bid, otherwise she would pay more than her value, earning a negative expected return. Ties in this stage will be resolved through a random mechanism that assigns the same probability to all participants.

If Bidder 1 has a private value x_1, she wins the first price auction when $b_1(x_1) - b_j(s_{j1}) > z$, $\forall j \neq 1$. She may also win the auction in the second stage. But if at least one of the other bidders has the same private value her expected payoff from the Vickrey auction is zero once the equilibrium bidding function in the second stage is to bid one's value, that is, $\beta^*(s_{i1}) = s_{i1}$, where $\beta^*(\cdot)$ stands for the second stage equilibrium bidding function. Therefore, the expected return of a bidder who bids b when she has value x_1 for the item is

$$U_1(x_1, b) = (x_1 - b)\left[(1-p)^2 + 2p(1-p)F(b-z) + p^2F^2(b-z)\right] \quad (4)$$

As there can be no gaps in the support of the equilibrium bids distribution, it is possible to show that $\underline{b} = b(x_0) = x_0$. In a mixed strategy equilibrium, the player must be indifferent to all bids in the support of her bids distribution. Given that $F(x_0) = 0$, one can find the expected return of a x_1-type bidder who bids b.

$$\overline{U}_1 = U_1(x_1) = (x_1 - x_0 - z)(1-p)^2 \tag{5}$$

Using the fact that $F(\overline{b}) = 1$ in (4) we are able to solve for $\overline{b}$.

$$\overline{b} = \left(1 - (1-p)^2\right)(x_1 - z) + (1-p)^2 x_0$$

The equilibrium bid strategies for the first price auction with a Vickrey auction as second stage when there are bids that are sufficiently close to the higher bid are:

$$b(s_{i1}) = \begin{cases} \cdot \quad x_0 \quad if \quad s_{i1} = x_0; \\ \cdot \quad \text{bid randomly in the interval } [x_0, \overline{b}] \\ \text{according to the bid distribution function} \\ F(b) \quad if \quad s_{i1} = x_1. \end{cases}$$

Proposition 1 *The sealed bid first price auction with a Vickrey auction as the second stage when there are bids that are suficiently close to the highest bid implies higher expected revenue to the seller relatively to standard auction institutions.*

Proof. The seller's expected revenue, ER, is the difference between the expected social surplus,

$$x_0(1-p)^3 + x_1\left(1 - (1-p)^3\right) \tag{6}$$

and the bidders expected return,

$$(1-p)U_l + pU_h = 0 + p(x_1 - x_0 - z)(1-p)^2$$

$$ER = x_0(1-p)^3 + x_1\left(1-(1-p)^3\right) - 3p(x_1 - x_0 - z)(1-p)^2$$

■

The effect of z is to reduce the expected return to the high type bidder and, therefore, to increase the seller's expected revenue. The reason is that the hybrid auction may be viewed as a Vickrey auction with an endogenously determined reserve price. This generates more revenue than any standard auction with a reserve price set at zero (that is, equal to the seller's value).

Recall from the optimal auction literature (e.g., Myerson [6] and Riley and Samuelson [9]) that the auction that maximizes the seller's expected revenue may be implemented by a Vickrey auction with an optimally chosen reserve price. This of course implies that the optimal auction is ex-post inefficient – that is, there is a positive probability that the object is not sold although there is at least one bidder with a value greater than the seller's value. In contrast, the hybrid auction is ex-post efficient as the outcome of the first stage has produced at least one bidder who is willing to pay the highest bid in that stage.

2.1.2 $x_1 - x_0 < z$

We claim that when $x_1 - x_0 < z$,

$$b(v_i) = x_0 \qquad \forall i, \forall v_i$$

is an equilibrium of the proposed auction mechanism. Furthermore, it implies the same expected return to the seller as standard auction institutions.

When $x_1 - x_0 < z$ the second stage always occur and the hybrid mechanism is equivalent to a Vickrey auction. The bidder equilibrium bidding function in the second stage is $\beta^*(s_{i1})$. The bidder's expected return is given by

$$U(x_1) = (x_1 - x_0)(1-p)^2$$

and the seller's expected revenue is then

$$R = x_0 (1-p)^3 + x_1 \left(1 - (1-p)^3\right) - 3(x_1 - x_0) p (1-p)^2$$

In sum, the revenue equivalence theorem still holds in this setting.[2]

[2] >From now on we do not analyze the equilibrium when bidders' valuations differ by a magnitude smaller than z because in this case the hybrid auction is equivalent to a Vickrey auction. This situation is analyzed in Maskin and Riley [2] and in Milgrom and Weber [5].

3 Risk Aversion

In this section we drop the risk neutrality assumption retaining the assumption that the item has no intrinsic value that is common to all bidders. In particular, we extend the hybrid auction model to the case in which a buyer i who values the commodity at v_i and purchases a single item at a price ρ receives a Von Neumann utility $u(v_i - \rho)$, where $u(\cdot)$ is a strictly concave utility function normalized to satisfy $u(0) = 0$. Conditional on winning the auction, the expected payoff of Bidder 1 who observes a private value v_i and bids b in the first price auction of the first stage is given by:

$$\begin{aligned} \pi(v_1, b(v_1)) &= E_j\left[u(v_1 - b(v_1))\, 1_{\{b(v_1)>b(v_j)+z;j\neq 1\}}\right] + \\ & E_j\left[u\left(v_1 - v^{(2)}\right) 1_{\left\{v^{(2)}<b(v_1)<b\left(v^{(2)}\right)+z\right\}}\right] \end{aligned} \tag{7}$$

considering that in the contingent Vickrey Stage bidding one's value remains a bidder's best response even under risk aversion.

In the hybrid auction x_0 bidders continue bidding x_0. Once again the equilibrium is a mixed strategy one. If $F_R(\cdot)$ is the cumulative distribution function through which x_1 bidders randomize in equilibrium, it must satisfy

$$u(x_1, b) = u(x_1 - b)\left[(1-p)^2 + 2(1-p)pF_R(b-z) + p^2 F_R(b-z)^2\right].$$

The same reasoning of the previous section allow us to determine the expected payoff to a bidder observing the high value, $\overline{U}_1$, that is the same for all bids in the support of the mixed strategy equilibrium. This also allows us to determine the supports in which bidders randomize and the cumulative distribution function of bids, F_R, for $b \in \left[\underline{b}, \overline{b}_R\right]$.

$$\overline{U}_1 = u(x_1, \underline{b} + z) = u(x_1 - z)(1-p)^2$$

$$F_R(b-z) = \left(\frac{1-p}{p}\right)\left[-1 + \left(1 - \frac{u(x_1 - z)}{u(x_1 - b)}\right)^{0.5}\right]$$

The equilibrium bid strategies for the hybrid auction with symmetric risk averse bidders are:

$$b(s_{i1}) = \begin{cases} \cdot \quad x_0 \qquad if \qquad s_{i1} = x_0; \\ \cdot \quad \text{bid randomly in the interval } \left[x_0, \overline{b}_R\right] \\ \text{according to the bid distribution function} \\ F_R(b) \quad if \quad s_{i1} = x_1; \end{cases}$$

where

$$\overline{b}_R = (x_1 - z) - \varphi\left(u(x_1 - z)(1-p)^2\right), \qquad \varphi(\cdot) = u(\cdot)^{-1}.$$

The strict concavity of u implies that F_R stochastically dominates F.[3] Considering that when bidders are risk averse the first price auction implies higher expected revenue to the seller than the oral ascending auction, we have the following

Proposition 2 *When bidders are risk averse, the first price auction with a Vickrey auction as second stage when there are bids sufficiently close to the top bid implies higher expected revenue to the seller than standard auction mechanisms.*

Proof. Again bidder's expected utility is reduced by z. ■

4 Asymmetry

In the present section we retain the risk neutrality assumption, while dropping the assumption that bidders' values are identically distributed. We also assume that the item has no intrinsic value that is common to all bidders. Different from the other sections, we handle the two bidders case. Suppose now that two bidders dispute an indivisible item in a hybrid auction. Two cases of asymmetry are considered: in the first both buyers have the same probability of observing the high value, although this high value is different for each bidder; in the second buyers have differing probabilities of observing the high value.

[3] This can be seen by comparing the expression valid to the risk averse case, $u(x_1 - b)\left[(1-p)^2 + 2p(1-p)F_R(b-z) + p^2 F_R(b-z)^2\right] = u(x_1 - z)(1-p)^2$, to the expression valid to the risk neutral case, $(x_1 - b)\left[(1-p)^2 + 2p(1-p)F(b-z) + p^2 F(b-z)^2\right] = (x_1 - z)(1-p)^2$.

4.1 When Bidders Have Distinct High Values

Without loss of generality suppose that bidder 1, the "strong" bidder, can have values $s_{s1} = \{x_0, x_2\}$, $x_2 > x_1$, while the "weak" bidder continues to observe either x_0 or x_1. The probabilities of observing the low and the high values are common to both bidders; that is, $pr\left[s_{i1} = x_0\right] = (1-p)$, $i = s, w$.

Once again we have a mixed strategy equilibrium that is completely characterized by the cumulative distribution functions of bids of the strong and of the weak bidders, F_s and F_w, respectively, and by the relevant supports. A bidder with a low value bids as to have zero expected payoff in equilibrium. This implies that $b(x_0) = x_0$. The expected payoff to the strong bidder (U_s) who has a high value of the commodity is given by:

$$U_s(x_2, b) = (x_2 - b)\left[(1-p) + pF_w(b-z)\right] + p(x_2 - x_1)(1 - F_w(b-z)) \quad (8)$$

The second term in the right hand side of equation (8) is the payoff to the strong bidder when a contingent Vickrey stage happens, that is, when buyers' bids in the first price auction are close enough. In turn, the expected profit to a weak bidder who values the item at x_1 and bids b is given by:

$$U_w(x_1, b) = (x_1 - b)\left[(1-p) + pF_s(b-z)\right]. \quad (9)$$

In this context the optimal response from a weak bidder with a high value to a strong bidder's equilibrium strategy is to bid higher, implying that $0 = F_w(\underline{b})$, where F_w stands for the weak buyer's cumulative distribution function of bids. This allows us to determine the expected payoff to a strong bidder with a high value when she randomizes in the range $\left[\underline{b}, \overline{b}_s\right]$.

$$\overline{U}_s = U_s(x_2, \underline{b} + z) = (x_2 - z)(1-p) + (x_2 - x_1)p \quad (10)$$

Substituting (10) into (8), we can determine the equilibrium bids distribution of the weak bidder. Once in a mixed strategy equilibrium all bids in the support of the winning bids distribution must guarantee equal payoff to the bidder, including the maximum bid $\overline{b}_i, i = s, w$.

$$F_w(b - z) = \left(\frac{1-p}{p}\right)\left(\frac{b-z}{x_1 - b}\right)$$

$$\overline{b}_w = \left(\frac{p}{1-p}\right)(x_1 - z) \qquad (11)$$

Suppose the weak buyer is bidding according to her equilibrium mixed strategy. If her maximum bid is $\overline{b}_w$, the strong buyer has no incentive to bid higher. If she were to do so, she would do better lowering her bid by an infinitesimal amount, to $\overline{b}_s - \epsilon$. This would increase her expected payoff, because her probability of winning would not change, but her winning bid would be lower. This implies that $\overline{b}_w = \overline{b}_s \equiv \overline{b}_A$, *that is, the maximum equilibrium bid is the same to the strong and to the weak buyer*. Analogously, one can show that the lower point in the winning bids support, $\underline{b}$, is common to both bidders. In fact, as there are no gaps in the equilibrium bids distribution, $\underline{b} = b(x_0)$ .By substituting (11) into (9) one can determine the expected payoff to the weak bidder and then her bids cumulative distribution function, that is the same as the one of the strong bidder.

$$\overline{U}_w = U_w\left(x_1, \overline{b}_A + z\right) = (x_1 - z)(1 - p)$$

In sum, when bidders have distinct high values, but the same probability of being high the equilibrium bid strategies for the hybrid auction are

$$b(s_{i1}) = \begin{cases} \cdot \quad x_0 \qquad if \qquad s_{i1} = x_0; \\ \cdot \quad \text{bid randomly in the interval } \left[x_0, \overline{b}_A\right] \\ \text{according to the bid distribution} \\ \text{function } F_w(b) \quad if \quad s_{i1} = x_1,\ x_2. \end{cases}$$

Compared to the first price auction, the effect of z is to increase the expected revenue to the seller.

Proposition 3 *In the two bidders asymmetric case, when bidders have distinct high values to the item, but the same probability of being high, the first price* *auction with a Vickrey auction as second stage when there are bids that are sufficiently close to the top bid implies higher expected revenue to the seller than standard auction mechanisms.*

Considering that a first price auction implies a greater revenue than the oral ascending auction,[4] under the present kind of asymmetry these auctions can be ranked as $ER^{HA} > ER^{FPA} > ER^{OAA}$.

4.2 When Bidders Have Different Probabilities of Observing the High Value

Suppose now that both bidders have values in the same set, $s_{i1} = \{x_0, x_1\}$, but different probabilities of observing the high value. The strong bidder $(i = s)$ has a value x_1 with a higher probability, p, whilst the weak buyer $(i = w)$ has a value x_1 with probability $\widetilde{q} < p$. In equilibrium a bidder with a value x_0 bids her own value, earning zero expected payoff, which implies $b(x_0) = x_0$. In turn, a high value bidder bids randomly in the interval $\left[\underline{b}_i, \overline{b}_i\right]$, $i = w, s$.

The equilibrium characterization is completed through the cumulative distribution functions, and the interval in which bidders randomize. The expected payoff to the bidder s who bids b when $s_{i1} = x_1$ is given by:

$$U_s(x_1, b) = (x_1 - b)\left[(1 - \widetilde{q}) + \widetilde{q}F_w(b - z)\right]. \tag{12}$$

In turn, the expected payoff to the weak bidder when $s_w = x_1$ is given by

$$U_w(x_1, b) = (x_1 - b)\left[(1 - p) + pF_s(b - z)\right]. \tag{13}$$

As there are no gaps in the equilibrium bids distribution, $\underline{b}$, the minimum bid by a x_1 bidder, is equal to $b(x_0)$. In a mixed strategy equilibrium the expected payoffs to the bidders are

$$\overline{U}_s(x_1, \underline{b} + z) = (x_1 - \underline{b} - z)\left[(1 - \widetilde{q}) + \widetilde{q}F_w(\underline{b})\right] \tag{14}$$

[4]See Maskin and Riley [2].

and

$$\overline{U}_w\left(x_1,\underline{b}+z\right)=\left(x_1-\underline{b}-z\right)\left[(1-p)+pF_s\left(\underline{b}\right)\right]. \tag{15}$$

By the same reasoning of the previous subsection, the maximum bid by a high bidder, $\overline{b}$, must be common to both bidders; that is, $F_w\left(\overline{b}\right)=1=F_s\left(\overline{b}\right)$. From (12) and (13) in equilibrium both bidders have the same expected return, once $\overline{U}_s\left(x_1,\overline{b}+z\right)=\overline{U}_w\left(x_1,\overline{b}+z\right)$. As $p>\widetilde{q}$ from (14) and (15) we see that one cannot have both $F_w\left(\underline{b}\right)$ and $F_s\left(\underline{b}\right)$ equal to zero. Instead we have $F_s\left(\underline{b}\right)\geq F_w\left(\underline{b}\right)=0$. The optimal response from a weak buyer when $s_{i1}=x_1$ is to bid more aggressively. This allows us to determine $\overline{U}_s=\left(x_1-z\right)\left(1-\widetilde{q}\right)$ and the cumulative distribution function of bids to the weak buyer with a high value.

$$F_w\left(b-z\right)=\left(\frac{1-\widetilde{q}}{\widetilde{q}}\right)\left(\frac{b-z}{x_1-b}\right)$$

Substituting the expression $\overline{U}_w$ into equation (13) we determine the cumulative distribution function of bids of a strong buyer with a high value.

$$F_S\left(b-z\right)=\left(\frac{1}{p}\right)\left[\left(1-\widetilde{q}\right)\left(\frac{x_1-z}{x_1-b}\right)-(1-p)\right]$$

In sum, when both bidders have the same high values but different probabilities of being high, the equilibrium strategy in the hybrid auction is

$$b\left(s_{i1}\right)=\begin{cases}\cdot \quad x_0 \qquad if \qquad s_{i1}=x_0; \\ \cdot \quad \text{bid randomly in the interval } \left[x_0,\overline{b}_{\hat{A}}\right] \\ \text{according to the bid distribution function} \\ F_i\left(b\right) \quad if \quad s_{i1}=x_1,\ i=s,w,\end{cases}$$

where $\overline{b}_{\hat{A}}=\left(1-\widetilde{q}\right)\underline{b}+\widetilde{q}\left(x_1-z\right)$. Once we have $\overline{U}_s=\left(x_1-z\right)\left(1-\widetilde{q}\right)=\overline{U}_w$,the expected revenue to the seller in the hybrid auction is higher than in the first price auction.

$$ER^{HA}=\left[1-(1-p)\left(1-\widetilde{q}\right)\right]x_1-\left(p+\widetilde{q}\right)\left(x_1-z\right)\left(1-\widetilde{q}\right)$$

In the present setting the equilibrium strategy in the oral ascending auction is to bid one's value. So the seller's expected revenue is equal to $p\widetilde{q}x_1$. The difference between these auction mechanisms in terms of expected revenue is given by

$$\widetilde{\Delta} = ER^{HA} - ER^{OAA} = -\widetilde{q}\,(p - \widetilde{q}) + (p + \widetilde{q})\,(1 - \widetilde{q})\,z.$$

If z is large enough, that is, if $\widetilde{\Delta} > 0$, these auction mechanisms are ranked as $ER^{HA} > ER^{OAA} > ER^{FPA}$ in terms of expected revenue .

5 Mixed Values

In the section 2 we examined the independent private value case and showed that the hybrid auction generates more revenue than any standard auction. We show that this is also true outside the independent private values paradigm.[5] Now each bidder can observe one of four possible combinations of signals: $s^0 = (x_0, L)$, $s^1 = (x_0, H) = (x_1, L)$, $s^2 = (x_1, H)$. For simplicity, we assume that bidders who observe the intermediate signals have a similar pattern of bidding.

We look for an equilibrium in mixed strategies such that a bidder 1 who receives signals $s^0 = (x_0, L)$ wins the auction only when bidders 2 and 3 observe the same signals. The expected payoff of one such bidder if she bids b is given by

$$U\left((x_0, L), b\right) = [x_0 + \pi_0 - b]\, p^2 \pi_{(L,L|L)}$$

where

$\pi_{(x,y|w)} = \Pr\{s_{22} = x, s_{32} = y | s_{12} = w\}$

π_t, $t = 0, 1, 2, 3$, is the expected value of the common factor given that t buyers observe a H signal.

Once again to determine her bid we can use the fact that in equilibrium the lower type bidder earns zero expected return.

$$\underline{b} = b\,(x_0, L) = x_0 + \pi_0$$

[5] Bidders are assumed to be symmetric and risk neutral.

The other bidders' types, those who observe signals s^1 and s^2, bid probabilistically. The monotonicity and continuity properties[6] allow us to determine the equilibrium returns of those buyers.[7] The monotonicity property implies that the set of possible bids in mixed strategy to a bidder who observes signal s^{t+1} must be at least as great as the set of possible bids when she observes signal s^t. Let $\overline{b}_1$ be the largest possible bid to a s^1-type buyer. By continuity there can be no gaps in the winning bids distribution so $\overline{b}_1$ is also the lowest possible bid of an agent who observes signal s^2.

The equilibrium of one such auction consists of supports $\left[\underline{b}, \overline{b}_1\right]$ and $\left[\overline{b}_1, \overline{b}_2\right]$ to bidders observing s^1 and s^2, respectively, and the associated distribution functions. Let the winning bids distribution function of the s^1-type bidders be G_1 with support $\left[\underline{b}, \overline{b}_1\right]$. The expected return to a bidder who observes signal s^1 and bids b is given by:

$$\begin{aligned} U_1\left(\left(x_1, L\right), b\right) = \left(x_1 + \pi_0 - b\right) p^2 \pi_{(L,L|L)} + \left[2\left(x_1 + \pi_1 - b\right) p^2 \pi_{(L,H|L)} + \right. \\ \left. 2\left(x_1 + \pi_0 - b\right) pq\pi_{(L,L|L)}\right] G_1\left(b - z\right) + \left[\left(x_1 + \pi_2 - b\right) p^2 \pi_{(H,H|L)} + \right. \\ \left. 2\left(x_1 + \pi_1 - b\right) pq\pi_{(L,H|L)} + \left(x_1 + \pi_0 - b\right) q^2 \pi_{(L,L|L)}\right] G_1^2\left(b - z\right). \end{aligned} \quad (16)$$

Using the fact that $G_1\left(\underline{b}\right) = 0$, one may define the expected payoff of a s^0-type bidder.

$$\overline{U}_1 = U_1\left(\underline{b} + z\right) = \left(x_1 + \pi_0 - \underline{b} - z\right) p^2 \pi_{(L,L|L)} \quad (17)$$

Equating (16) and (17), considering that $G_1\left(\overline{b}_1\right) = 1$, it is possible to find $\overline{b}_1$.

$$\begin{aligned} \overline{b}_1 \quad = \quad & \left(\underline{b} p^2 \pi_{(L,L|L)} + \left(x_1 + \pi_0 - z\right)\left(1 - p^2\right)\pi_{(L,L|L)} + \right. \\ & \left. 2\left(x_1 + \pi_1 - z\right) p\pi_{(L,H|L)} + \left(x_1 + \pi_2 - z\right) p^2 \pi_{(H,H|L)}\right) / \\ & \left(\pi_{(L,L|L)} + 2p\pi_{(L,H|L)} + p^2 \pi_{(H,H|L)}\right) \end{aligned}$$

[6] By continuity we mean that there can be neither mass points (points of strictly positive probability) nor gaps in the distribution of the equilibrium bids.

[7] We begin by assuming that these two properties hold and then verify that they are in fact satisfied in equilibrium.

The winning bids distribution function for a buyer observing s^1, $G_1(.)$, can then be completely determined through equations (16) and (17). Its expressions is provided in the appendix. In turn, a s^2-type buyer bids in the range $\left[\overline{b}_1, \overline{b}_2\right]$. As all the strategies played with positive probability in a mixed strategy equilibrium must guarantee her equal expected payoff, it is possible to determine the expected return of one such buyer using the fact that when bidding $\overline{b}_1$ a bidder observing s^2 only wins with positive probability when all her opponents observe lower signals. If she bids b, her expected return is

$$\begin{gathered} U_2\left(\left(x_1, H\right), b\right)=\left(x_1+\pi_1-b\right) p^2 \pi_{(L, L \mid H)}+\left[2\left(x_1+\pi_2-b\right) p^2 \pi_{(L, H \mid H)}+\right. \\ \left.2\left(x_1+\pi_1-b\right) p q \pi_{(L, L \mid H)}\right] G_1(b-z)+\left[\left(x_1+\pi_3-b\right) p^2 \pi_{(H, H \mid H)}+\right. \\ \left.2\left(x_1+\pi_2-b\right) p q \pi_{(L, H \mid H)}+\left(x_1+\pi_1-b\right) q^2 \pi_{(L, L \mid H)}\right] G_1^2(b-z)+ \\ \left\{\left(1-G_1(b-z)\right)\left[\left(x_1+\pi_1-\beta\left(s^1\right)\right)\left(1-p^2\right) \pi_{(L, L \mid H)}+\right.\right. \\ \left.\left.\left(x_1+\pi_2-\beta\left(s^1\right)\right) 2 p \pi_{(L, H \mid H)}+\left(x_1+\pi_3-\beta\left(s^1\right)\right) p^2 \pi_{(H, H \mid H)}\right]\right\}. \end{gathered} \tag{18}$$

The expected payoff to a s^2-type buyer that bids b allows us to rewrite (18) as

$$U_2(b)=\overline{U}_1+K+\psi G_1(b-z)+\theta G_1^2(b-z), \quad \psi<0,$$

where

$$\psi=2\left(\pi_3-\pi_2\right) p \pi_{(L, H \mid H)}-\left(\pi_1-\pi_0\right)\left(1-p^2\right) \pi_{(L, L \mid L)}-\left(x_1-x_0\right) p^2 \pi_{(H, H \mid H)}$$

and

$$\theta=\left(x_1-x_0\right)\left(q^2 \pi_{(L, L \mid L)}+2 p q \pi_{(L, H \mid H)}+p^2 \pi_{(H, H \mid H)}\right).$$

Ordinarily the monotonicity property would imply that the expected payoff function to a bidder observing signal s^{t+1} would increase monotonically in the support of the winning bids distribution function of an agent observing signal s^t. This would imply that the lowest possible bid for an s^{t+1} buyer would be the largest possible bid for an s^t agent. In turn, in the present setup a s^2-type bidder that bids in the range $\left[\overline{b}_1, Es^1\right]$ competing with at least one s^1 bidder and no s^2 buyer has a chance to go to a second stage once her opponents bid close enough. If this happens, in the Vickrey auction she raises her bid to $\beta(s^2) = Es^2$ whilst her opponents raise their bids to their conditional expected value for the item, that is $\beta(s^t) = Es^t$. As a result, the s^2 bidder wins the auction earning a payoff of $(x_1 + \pi_1 - \beta(s^1))$. This set of events is expressed enclosed in braces in equation (18). In summary, the expected return function to a bidder observing s^2 does not increase monotonically in the range $[\underline{b}, \overline{b}_1]$ when three is the number of possible types. In this range the expected payoff function is a convex function once condition

$$\left(\psi G_1(b-z) + \theta G_1^2(b-z)\right)\left(\overline{b}_1 - \underline{b}\right) \leq (b - \underline{b})(\psi + \theta) \tag{19}$$

is satisfied.[8] Considering the expected return of a s^2-type bidder in both limits of these support, it is possible to exclude $\underline{b}$ as an equilibrium. If this were the bid to one such buyer, a s^1 bidder could beat a s^2 bidder in the first stage of the mechanism through a bid $b(s_1) = \underline{b} + z$. So,

$$b(s_i) = \begin{cases} \underline{b} & if \quad s_i = s^0, s^2 \\ \underline{b} + z & if \quad s_i = s^1 \end{cases}$$

cannot be an equilibrium, as it would imply an expected return of

$$U_2(b, (x_1, H)) = (x_1 + \pi_1 - \underline{b})\, p^2 \pi_{(L,L|H)}$$

to a bidder observing (x_1, H). Note that this expected return is strictly lower than the one earned by the buyer in the proposed equilibrium. We can then conclude that if condition (19) is satisfied, a bidder observing s^2 strictly prefers bidding $\overline{b}_1$ than any lower value. So the monotonicity property holds in equilibrium.[9] The

[8] The convexity condition stems from $\frac{U_2(b) - U_2(\underline{b})}{b - \underline{b}} \leq \frac{U_2(\overline{b}_1) - U_2(\underline{b})}{\overline{b}_1 - \underline{b}}$.

[9] It is not hard to verify that the continuity property also holds in equilibrium.

expected return to a s^2-type bidder is then

$$U_2 = \left(x_1 + \pi_1 - \overline{b}_1 - z\right) p^2 \pi_{(L,L|H)} + 2\left(x_1 + \pi_2 - \overline{b}_1 - z\right) p^2 \pi_{(L,H|H)} + \tag{20}$$
$$+2\left(x_1 + \pi_1 - \overline{b}_1 - z\right) pq \pi_{(L,L|H)} + \left(x_1 + \pi_3 - \overline{b}_1 - z\right) p^2 \pi_{(H,H|H)} +$$
$$+2\left(x_1 + \pi_2 - \overline{b}_1 - z\right) pq \pi_{(L,H|H)} + \left(x_1 + \pi_1 - \overline{b}_1 - z\right) q^2 \pi_{(L,L|H)}.$$

The equilibrium bid strategies for the hybrid auction with symmetric risk neutral buyers who have mixed values for a single indivisible object are:

$$b\left(s_{i1}, s_{i2}\right) = \begin{cases} \cdot \quad \underline{b} \quad if \quad s_i = s^0; \\ \cdot \quad \text{bid randomly in the interval } \left[\underline{b}, \overline{b}_1\right] \\ \text{according to the bid distribution function} \\ G_1(b) \quad if \quad s_i = s^1; \\ \cdot \quad \text{bid randomly in the interval } \left[\overline{b}_1, \overline{b_2}\right] \\ \text{according to the bid distribution function} \\ G_2(b) \quad if \quad s_i = s^2; \end{cases}$$

where

$$\begin{aligned} \overline{b}_2 &= \left[\left(\pi_{(L,L|H)} + 2p\pi_{(L,H|H)} + p^2\pi_{(H,H|H)}\right)\overline{b}_1 + 2\left(x_1 + \pi_2 - z\right)(1-p)\pi_{(L,H|H)} + \right. \\ &\quad \left. \left(x_1 + \pi_3 - z\right)\left(1 - p^2\right)\pi_{(H,H|H)}\right] / \left(\pi_{(L,L|H)} + 2p\pi_{(L,H|H)} + p^2\pi_{(H,H|H)}\right). \end{aligned}$$

Assuming that the convexity condition is satisfied,[10] we have

Proposition 4 *The first price auction with a Vickrey auction as a second stage when there are bids sufficiently close to the top bid guarantees a higher expected revenue to the seller as compared to standard auction mechanisms.*

Proof: z decreases the expected return of the bidders that observe signals s^1 and s^2, as can be seen by expressions (17) and (20) but the expected social value does not change with z.

[10] The winning bids distribution for a buyer observing s^2, $G_2(.)$, is provided in the Appendix A.

6 Continuous Types

A natural concern is the choice of a discrete types model. In this section we show that with continuous types thy hybrid auction does not in general admit pure strategy equilibrium. In what follows we assume two players but in appendix B we generalize the argument to $n \geq 2$ bidders.

For simplicy, assume that bidder's values are uniformly distributed on the interval $[0, 1]$. We seek a monotone symmetric equilibrium.

Suppose bidder i bids the amount b, and her rival bids according to a monotone increasing equilibrium strategy $b(y)$. Then bidder i wins if her bid is higher than her rival's bid by more than z, obtaining a payoff $(v - b)$. Otherwise both bidders dispute the object in a Vickrey auction, where bidding one's value is a dominant strategy. If she wins, her payoff is $(v - y)$. But if $y < b < b(y) + z$, bidder i's payoff is $(v - b)$.Thus bidder i's problem is to find a bidding functions $b(v)$ that maximizes her expected payoff – that is expressed in equation (21).

$$\pi(v, b, b(y)) = E\left[(v - b)\, 1_{\{b > b(y) + z\}} + (v - Max\{b, y\})^{+}\, 1_{\{b < b(y) + z\}}\right] \tag{21}$$

Thereby $\lambda(b)$ is the inverse of $b(v)$, which indicates the valuation that leads to bidding b when strategy $b^{*}(v)$ is to be played.

$$= E\left[(v - b)\, 1_{\{y < \lambda(b - z)\}} + (v - Max\{b, y\})^{+}\, 1_{\{y > \lambda(b - z)\}}\right]$$

$$= (v - b)\, F(\lambda(b - z)) + E\left[(v - Max\{b, y\})^{+}\, 1_{\{y > \lambda(b - z)\}}\right]$$

$$= (v - b)\, F(\lambda(b - z)) + (v - b) \int_{\lambda(b - z)}^{b} f(y)\, dy +$$

$$+ \int_{Max\{b, \lambda(b - z)\}}^{1} (v - y)^{+} f(y)\, dy$$

In general there are two cases:

Case 1: $b < \lambda(b-z)$

$$y > \lambda(b-z) \Rightarrow y > b$$

$$\pi(v,b,b(y)) = (v-b)\,F(\lambda(b-z)) + \int_{\lambda(b-z)}^{1} (v-y)^{+} f(y)\,dy$$

Case 2: $b \geq \lambda(b-z)$

$$\pi(v,b,b(y)) = (v-b)\,F(b) + \int_{b}^{1} (v-y)^{+} f(y)\,dy$$

In both cases

$$\pi(v,b,b(y)) = (v-b)\,F(Max\{\lambda(b-z),b\}) + \int_{Max\{b,\lambda(b-z)\}}^{1} (v-y)^{+} f(y)\,dy.$$

Let $g(b) = Max\{\lambda(b-z), b\}$ .Then

$$\frac{\partial \pi}{\partial b} = -F(g(b)) + (v-b)\,f(g(b))\,g'(b) - (v-g(b))^{+} f(g(b))\,g'(b)$$

The first order condition is

$$(g(b)-b)\,g'(b) = \frac{F(g(b))}{f(g(b))}.$$

For $v_i \sim U[0,1]$, the first order differential equation reduces to:

$$(g(b)-b)\,g'(b) = g(b)\,. \tag{22}$$

Solving the differential equation (22) one obtains

$$g(b) = b + \sqrt{b^2+c^2}. \tag{23}$$

The initial condition

$$\lambda(0) = z \tag{24}$$

means that the type z bidder is the one who bids z. Substituting (24) in (23) one can determine the constant c and then obtain the candidate equilibrium bidding function $b^*(v)$.

We claim that

$$b^*(v) = \begin{cases} \dfrac{\left((v-z)^+\right)^2}{2v} & if \quad v > z \\ 0 & otherwise \end{cases}$$

is not a pure bidding strategy equilibrium in the first stage.

Proof: It suffices to show that by deviating from b^* a player 1 bidder with a private value $v_1 < z$ can earn a positive payoff. Suppose that bidder 1 follows $\widetilde{b}(v) = v/2$. There is a positive probability that the other player has a private value smaller than z also. Assume that $v_2 < v_1 < z$. If bidder 2 follows $\widetilde{b}$, bidder 1 may earn a positive payoff equal to $(v_2 - v_1) > 0$ at the end of the second stage - that in this case is mandatory once $\dfrac{v_2 - v_1}{2} < v_2 - v_1 < z$.If bidder $2's$ value is larger than z, bidder 1 will loose nothing either if she looses in the first price sealed bid auction, $b^*(v_2) > \widetilde{b}(v_1) + z$, or if she disputes the object in the contingent second stage, that is, if $b^*(v_2) < \widetilde{b}(v_1) + z$. So $\widetilde{b}(.)$ can be a best response to a bidder with a private value lower than z and $b^*(.)$ is not an equilibrium.

7 Conclusion

We have demonstrated that the hybrid auction generates more revenue than any standard auction and that it is ex-post efficient. Additional research is needed to study the properties of such mechanisms in terms of the traditional concerns of competition policy such as preventing collusive, predatory and entry deterring behavior. For example, the sealed bid stage may help to deter tacit collusion, a common phenomenon in ascending auctions (see, for example, Menezes [4]). On the other hand, the Vickrey auction stage may work towards increasing revenue. In the Brazilian Telecom auctions, the average number of bidders was equal to four bidders. Of the twelve auctions, only two were followed by a second stage. Additionally, in both cases when there was a second stage, the winner of the second stage was the bidder who submitted the second highest bid in the first price auction!

References

[1] Klemperer, P., 2000, "What Really Matters in Auction Design," Working Paper, Nuffield College, Oxford University. Available at www.nuff.ox.ac.uk/economics/people/klemperer.htm.

[2] Maskin, Eric S. and John G. Riley (1985), "Auction Theory with Private Values," *American Economic Review 75*, 150-64.

[3] Maskin, Eric S. and John G. Riley (1999), "Equilibrium in Sealed High Bid Auctions," *Review of Economic Studies*, forthcoming.

[4] Menezes, Flavio M. (1996), "Multiple-Unit English Auctions, " *European Journal of Political Economy* 12, 671-684.

[5] Milgrom, Paul R. and Robert J. Weber (1982), "A Theory of Auctions and Competitive Bidding," *Econometrica 50*, 1089-1122.

[6] Myerson, Roger (1981), "Optimal Auction Design," *Mathematics of Operations Research 6*, 58-73.

[7] Myerson, Roger (1991), *Game Theory: analysis of conflict.* Harvard University Press: Cambridge, Massachussetts.

[8] Riley, John (1989), " Expected Revenue from Open and Sealed Bid Auctions," *Journal of Economic Perspectives* , 41-50.

[9] Riley, John and William Samuelson (1981), "Optimal Auctions," *American Economic Review* 71, pp.381-92.

8 Appendix

A Mixed Values

In the mixed values equilibrium the s^1-type buyers bid randomly in the interval $\left[\underline{b}, \overline{b}_1\right]$ according to the winning bids distribution function $G_1(b-z)$.

$$G_1(b-z) = \Big(2\ \Big(\ (x_1+\pi_2-b)\, p^2\pi_{(H,H|L)} + 2(x_1+\pi_1-b)\, pq\pi_{(L,H|L)} + (x_1+\pi_0-b)\, q^2\pi_{(L,L|L)}\ \Big)\Big)^{-1} \Big\{\ -2(x_1+\pi_1-b)\, p^2\pi_{(L,H|L)} + -2(x_1+\pi_0-b)\, pq\pi_{(L,L|L)}\ +\ \Big[\ \big(2(x_1+\pi_1-b)\, p^2\pi_{(L,H|L)} + 2(x_1+\pi_0-b)\, pq\pi_{(L,L|L)}\big)^2\ -\ 4\big((x_1+\pi_2-b)\, p^2\pi_{(H,H|L)} + +2(x_1+\pi_1-b)\, pq\pi_{(L,H|L)} + (x_1+\pi_0-b)\, q^2\pi_{(L,L|L)}\ (\underline{b}+z-b)\, p^2\pi_{(L,L|L)}\Big]^{\frac{1}{2}}\Big\}$$

In turn, the s^2-type buyers bid randomly in the range $\left[\overline{b}_1, \overline{b}_2\right]$ according to the $G_2(b-z)$ distribution function.

B Inexistence of pure strategy equilibrium with n≥ 2 bidders and continuous types

Let $y = \underset{i \neq j}{Max}\{v_j\}$. Then the expected payoff to the bidder who values the item to be auctioned at v and bids b when the maximum bid from her opponents is $b(y)$ is

$$\pi(v,b,b(y)) = E\left[(v-b)\, 1_{\{b>b(y)+z\}} + (v - Max\{b,y\})^+\, 1_{\{b<b(y)+z\}}\right]$$

$$= (v-b)\, F\left(Max\{\lambda(b-z), b\}\right)^{n-1} + \int\limits_{Max\{b,\lambda(b-z)\}}^{1} (v-y)^+\, F(y)^{n-2} f(y)\, dy.$$

Let $g(b) = Max\{\lambda(b-z), b\}$.Then

$$\pi(v,b,b(y)) = (v-b)\, F(g(b))^{n-1} + \int\limits_{g(b)}^{1} (v-y)^+\, F(y)^{n-2} f(y)\, dy.$$

The first order condition is then

$$(n-1)\left(g\left(b\right)-b\right)g'\left(b\right)=\frac{F\left(g\left(b\right)\right)}{f\left(g\left(b\right)\right)}.$$

For $v\sim U\left[0,1\right]$,

$$(n-1)\left(g\left(b\right)-b\right)g'\left(b\right)=g\left(b\right).$$

To solve the differential equation we multiply both sides of it by $g(b)^{n-2}$ and integrate. We get for some positive c:

$$g(b)^n-\frac{n}{(n-1)}g\left(b\right)^{n-1}b=c.$$

Using initial condition $g(z)=z$,

$$(n-1)\,g\left(b\right)^n-ng\left(b\right)^{n-1}b+z^n=0.$$

With the assumption that $\lambda(.)$ is an increasing function, it is easy to see that $g(b)=\lambda(b-z)\geq b$. Thus we get the following equation for $b(v)$:

$$(n-1)v^n-nv^{n-1}b(v)+z)+z^n=0$$

and therefore

$$b(v)=\frac{(n-1)v^n\;-nv^{n-1}z+z^n}{nv^{n-1}}.$$

This matches the expression we obtained before for $n=2$. It remains to show that

$$b^*\left(v\right)=\begin{cases}\frac{(n-1)v^n-nv^{n-1}z+z^n}{nv^{n-1}}, & if \quad v>z\\ 0 & if \quad v<z\end{cases}$$

is not a pure strategy equilibrium. This follows from the fact that the bid function cannot contain flat portions in some interval $[v_1,v_2]$ in the support $[0,1]$. Otherwise a bidder with some valuation in the flat interval could raise her expected payoff by marginally raising her bid once she could always win when the second highest valuation were in the same interval. The same reasoning of the two-bidders case applies.