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Nº 651   ISSN 0104-8910

Social Welfare Analysis in a Simple Financial Economy with Risk Regulation

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Setembro de 2007

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September 2007

Abstract

In the last years, regulating agencies of many countries in the world, following recommendations of the Basel Committee, have compelled financial institutions to maintain minimum capital requirements to cover market risk. This paper investigates the consequences of such kind of regulation to social welfare and soundness of financial institutions through an equilibrium model. We show that the optimum level of regulation for each financial institution (the level that maximizes its utility) depends on its appetite for risk and some of them can perform better in a regulated economy. In addition, another important result asserts that under certain market conditions the financial fragility of an institution can be greater in a regulated economy than in an unregulated one.

JEL classification: G18; G20; D60; C68.
Keywords: Basel Capital Accord, VaR, Banking Regulation and Social Welfare.

1 Introduction

In the last two decades, many regulating agencies around the world have introduced formalized capital requirements to control the risks of financial institutions based on the recommendations of the 1988 Basel Accord on capital standards and its later amendments. This Accord was the first well successful attempt to harmonize international rules concerning bank capital requirements[1] and resulted of a long term process carried under the heading of the Basel Committee on Banking Supervision[2]. The 1988 Basel Accord was approved in July 1988 by the member countries of the Committee and established minimum capital requirements for credit risk. Basically, it imposed a capital requirement of at least 8% of the Risk-Adjusted Asset, defined as the sum of asset positions multiplied by asset-specific risk weights.

In January 1996, the Committee released a new document named Amendment to the Capital Accord to Incorporate Market Risks (Basel Committee on Banking Supervision, 1996a)[3] defining criteria for capital requirements to cover market risk. Since then the minimum regulatory capital of a financial institution has been the sum of a charge to cover credit risk and another charge to cover market risk[4]. To gauge market risk the Basel Committee adopts the well know Value-at-Risk (VaR) metric[5].

Regardless of legal requirements, several financial institutions have recently adopted internal VaR-based models for market risk management. Most of this self-discipline process was in fact demanded by stockholders and investors who were concerned with the increase of volatility in a globalized world economy and therefore would like transparency in the management of their resources.

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[1]See Freixas and Santomero (2002) or Santos (2002) for a review of the theoretical justifications for bank capital requirements.

[2]The Basel Committee was set up in 1974 under the auspices of the Bank for International Settlements (BIS) by the central banks of the G10 members.

[3]For an overview of the Amendment to the Capital Accord to Incorporate Market Risk, see the Basel Committee on Banking Supervision (1996b).

[4]Recently, the Basel Committee released another document, commonly known as Basel II, which revises the original framework for setting capital charges for credit risk and introduces capital charge to cover operational risk.

[5]VaR represents the maximum loss to which a portfolio is subject for a given confidence interval and time horizon. For instance, a one-day 99% VaR of R\$ 10 million means that there is only 1 in 100 chance of the portfolio loss to exceed R\$ 10 million at the end of the next business day. For an overview of VaR, see Duffie and Pan (1997).

The aim of the present study is to investigate the welfare properties and the bankruptcy probability of financial institutions in an economy with a VaR-based risk constraint using a simple equilibrium model similar to one proposed by Danielsson and Zigrand (2003)[6].

Many recent studies have addressed the economic implications of the adoption of capital requirements based on the Basel Accord proposals. Rochet (1992) analyzes the consequences of capital requirements on the portfolio choices of banks and showed that the optimal risk weight must be proportional to the systemic risk of the assets (their betas). Jackson et. al (1999) review the empirical evidence of the impact of the 1988 Basel Accord. Blum (1999) points out that, in a dynamic framework, capital intertemporal effects can arise which leads to an increase in bank's risk. Marshall and Venkataraman (1999) use a simple model to evaluate alternative bank capital regulatory proposals for market risk. Basak and Shapiro (2001) investigate the implications of the investment decision problem when the trader is subject to an exogenous VaR limit. Danielsson and Zigrand (2003) use an equilibrium model to study the implications on asset prices and variances due to the introduction of a VaR-based risk regulation. Danielsson et al. (2004) extend the model proposed by Danielsson and Zigrand (2003) to a multiperiod environment and estimate the intensity of adverse impacts of VaR-based risk constraint. Cuoco and Liu (2006) study the behavior of a financial institution subject to capital requirements based on self-reported VaR measures. Leippold et al. (2006) consider the asset-pricing implications of VaR regulation in incomplete continuous-time economies. Alexander and Baptista (2006) examine the economic implications arising from a bank using a VaR-constrained mean-variance model for the selection of its trading portfolio as a consequence of the Basle Accord.

We start by analyzing the welfare effects of the introduction of VaR-based risk constraint. Surprisingly, we show that some institutions can perform in a better way when in a regulated economy (i.e., an economy where all financial institutions must satisfy the risk constraint) than in an unregulated economy (i.e., an economy where there are no risk limits). This is just an

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[6]In the same spirit of Danielsson and Zigrand (2003) we don't model reasons to the presence of risk regulation. We simply suppose that it exists (probably due to a market failure) and assess the economic consequence of it. Alternatively we can think that the reason of risk regulation is just to ensure the soundness of the financial system by decreasing the bankruptcy probability of financial institutions which, at least in principle, is a demand of stockholders and investors.

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equilibrium effect. The introduction of a risk constraint leads to a change of assets prices and consequently a change of the risk premium. Then some financial institutions can be benefited by buying risky assets cheaper than in an unregulated economy. Another important result states that a VaR-based risk regulation can increase the bankruptcy probability of well-capitalized or conservative financial institutions when the market volatility is low.

The structure of the paper is as follows. In Section 2 we present the model. Section 3 describes the VaR-based risk constraint and establishes conditions for the existence of equilibrium. In Section 4 we study the welfare of a financial institution in a regulated economy. Section 5 analyzes the bankruptcy probability of a financial institution before and after the introduction of a VaR-based risk regulation. Section 6 concludes. Proofs and auxiliary results are contained in the Appendix.

2   The Model

Consider a two period economy $(t = 0, 1)$ according to proposed by Danielsson and Zigrand (2003). At $t = 0$ agents (financial institutions) invest in $N + 1$ assets that mature at $t = 1$. The asset 0 is risk-free and yields payoff d_0. The risky assets are nonredundant and promise at $t = 1$ a payoff

$$\boldsymbol{d} = \begin{pmatrix} d_1 \\ \vdots \\ d_N \end{pmatrix},$$

that follows a Gaussian distribution with mean μ and covariance matrix $\boldsymbol{\Sigma}$.

We follow common modeling practice by endowing financial institutions with their own utility functions (such as in Basak and Shapiro, 2001 for instance). There is a continuum of small agents characterized by a constant coefficient of absolute risk aversion (CARA) h. The population of agents is such that h is uniformly distributed on the interval $[\ell, 1]$. To guarantee that all agents are risk-averse, let us suppose that $\ell > 0$.

Let x^h and y_i^h be the number of units of the risk-free asset and of the risky asset i, respectively, held by financial institution of type h (hereafter, refered as financial institution h ou agent h) at $t = 0$. Then the wealth of agent h at time $t = 1$ is

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$$W_1^h = d_0 x^h + \sum_i d_i y_i^h.$$

The agents choose the portfolio that maximizes the expected value of their wealth utility $u^h \left(W_1^h \right)$ subject to budget and risk constraints.

The time-zero wealth of agent h comprises initial endowments in the risk-free asset, θ_0^h, as well in the risky assets, $\theta^h = \left(\theta_1^h, \ldots, \theta_N^h \right)'$.

The budget constraint of institution h at $t = 0$ is

$$q_0 x^h + \sum_i q_i y_i^h \leq W_0^h,$$

where q_i is the price of asset i and $W_0^h = q_0 \theta_0^h + \sum_i q_i \theta_i^h$ is the initial wealth of financial institution h.

The role of the regulating agency consists in limiting the set of investment opportunities in the risky assets. That is, the regulating agency introduces a new constraint (hereafter denominated risk constraint) which can be written as

$$\boldsymbol{y}^h \in \Upsilon, \quad \forall h \in [\ell, 1], \tag{1}$$

for some $\Upsilon \subseteq \mathbb{R}^N$. Of course, the regulating agency's aim is to choose Υ so as to minimize the financial fragility of the market, damaging as little as possible the financial institutions welfare. Different choices for Υ corresponds to different bank capital regulatory proposals.

Therefore, the investment problem of financial institution h is[7]

$$\begin{array}{cc} \text{Max} & \mathcal{E} \left(u^h \left(W_1^h \right) \right) \\ \left(x^h, \boldsymbol{y}^h \right) & \\ \text{s.a.} & q_0 x^h + \sum_{i=1}^N q_i y_i^h \leq q_0 \theta_0^h + \sum_{i=1}^N q_i \theta_i^h, \end{array}$$

$$\boldsymbol{y}^h \in \Upsilon.$$

Since the budget constraint is homogeneous of degree zero in prices, we can normalize the price of risk-free asset to $q_0 = 1$ without loss of generality.

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[7]If $W_1^h < 0$, financial institution h is in default. In this case we interpret $u^h \left(W_1^h \right)$ as a punishment for default (see Geanakoplos, 2005).

Moreover, since u^h is strictly increasing, the budget constraint must be binding. The next lemma is just a consequence of the properties of a continuous function defined on a compact set[8].

Lemma 1 *If Υ is compact and convex, then the problem of financial institution has only one solution.*

A competitive equilibrium for the economy in question is a risky asset price vector $\boldsymbol{q} = (q_1, \ldots, q_N)'$ and a mapping $h \in [\ell, 1] \mapsto \left(x^h, \boldsymbol{y}^h \right)$, such that

1. $\left(x^h, \boldsymbol{y}^h \right)$ solves the problem of financial institution h when assets prices are equal to $(1, \boldsymbol{q})$.

2. Market clearing, that is, $\int_\ell^1 \boldsymbol{y}^h \mathrm{d}h = \theta$ and $\int_\ell^1 x^h \mathrm{d}h = \theta_0$, where $\theta = \int_\ell^1 \theta^h \mathrm{d}h$ is the aggregate amount of risky assets and $\theta_0 = \int_\ell^1 \theta_0^h \mathrm{d}h$ is the aggregate amount of the risk-free asset.

3   VaR-Based Risk Constraint

Market risk is the risk that the value of an investment decreases due to moves in market factors (like equity and commodities prices, interest rates and exchange rate). If the soundness of a financial institution is to be known, then its exposure to market risk has to be measured. In recent years, the risk metric known as VaR has become the major market risk metric for regulatory purposes as well as standard industry tool. Following this trend we suppose that the regulating agency makes use of VaR to limit market risk of financial institutions. VaR is usually defined as

$$VaR_\alpha^h \equiv -\inf \left\{ x \in \mathbb{R}; \mathcal{P} \left[W_1^h - \mathcal{E} \left(W_1^h \right) \leq x \right] > \alpha \right\}, \qquad (2)$$

where $\mathcal{P}$ is the probability measure corresponding to risky assets payoff distribution, $\mathcal{E}$ is the expected value relative to this measure and α is the significance level adopted (the probability of losses exceeding the VaR)[9].

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[8]For an analysis of optimal portfolio choice with compact and convex constraints see Elsinger and Summer (1999).

[9]VaR when defined by (2) is known as relative VaR, while the absolute VaR is the VaR defined without reference to the expected value (see Jorion, 2001).

In a simple way, VaR is the loss, which is exceeded with some given probability, α, over a given horizon. This easy interpretation is one of the reasons that justify the large use of VaR as standard market risk metric[10].

The risk constraint is fixed as a uniform upper bound to VaR, that is,

$$VaR_\alpha^h \leq \overline{VaR} \quad \forall h, \tag{3}$$

where $\overline{VaR}$ is a VaR exogenous bound set by the regulating agency. Using normal distribution properties, we can rewrite the risk constraint as an exogenous upper limit for the portfolio variance

$$\Upsilon = \left\{ \boldsymbol{y} \in \mathbb{R}^N; \boldsymbol{y}'\boldsymbol{\Sigma}\boldsymbol{y} \leq \nu \right\}, \tag{4}$$

where the parameter ν, called the nonseverity of risk constraint, depends on α and $\overline{VaR}$.

The next proposition characterizes the solution of the problem of financial institutions. The demonstration of this proposition can be found in Danielsson and Zigrand (2003).

Proposition 1 *Let $\left(x^h, \boldsymbol{y}^h\right)$ be the solution of the problem of financial institution h when the price vector of risky assets is $\boldsymbol{q}$. We have:*

1. *If $h \geq \sqrt{\frac{\rho}{\nu}}$, then*

$$\boldsymbol{y}^h = \frac{1}{h}\boldsymbol{\Sigma}^{-1}\left(\mu - r_0\boldsymbol{q}\right), \tag{5}$$

   where $\rho = \left(\mu - r_0\boldsymbol{q}\right)'\boldsymbol{\Sigma}^{-1}\left(\mu - r_0\boldsymbol{q}\right)$ and r_0 is the risk-free rate.

2. *If $h < \sqrt{\frac{\rho}{\nu}}$, then*

$$\boldsymbol{y}^h = \sqrt{\frac{\nu}{\rho}}\boldsymbol{\Sigma}^{-1}\left(\mu - r_0\boldsymbol{q}\right). \tag{6}$$

In any case $x^h = \theta_0^h + \sum_i q_i \theta_i^h - \sum_i q_i y_i^h$.

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[10]In spite of its widespread adoption, VaR is not without controversy. Its major problem relies on the fact that it is not a coherent measure of risk (VaR fails the sub-additive property, see Artzner et al., 1999). Besides, Kerkhof and Melenberg (2004) use a backtesting procedure to show that expected shortfall, a coherent measure of risk, produces better results than VaR.

Note that the introduction of the risk constraint prevents optimal risk sharing since all institutions with CARA less than or equal to $\sqrt{\frac{\varrho}{\nu}}$ choose the same portfolio.

After solving the problem of the financial institutions, the market clearing condition automatically provides the equilibrium prices, as presented in the following proposition (again, the demonstration is in Danielsson and Zigrand, 2003).

Proposition 2 *The equilibrium price vector of risky assets is*

$$\boldsymbol{q} = \frac{1}{r_0} \left(\mu - \Psi \Sigma \theta \right), \tag{7}$$

where Ψ is the market price of risk scalar. Denoting by $F(\cdot)$ the non-principal branch of the Lambert correspondence[11], we have

$$\Psi = \begin{cases} \frac{1}{\ln \ell^{-1}} & \text{if} \quad 0 \le \kappa \le \ell \ln \ell^{-1} \\[2mm] -\frac{\kappa + \ell}{\kappa F(-(\kappa+\ell)e^{-1})} & \text{if} \quad \ell \ln \ell^{-1} < \kappa < 1 - \ell \\[2mm] \text{any number} \ge \frac{1}{1-\ell} & \text{if} \quad \kappa = 1 - \ell, \end{cases} \tag{8}$$

where

$$\kappa = \sqrt{\frac{\theta' \Sigma \theta}{\nu}}.$$

An equilibrium fails to exist if $\kappa > 1 - \ell$.

Figure 1 illustrates Ψ as a function of κ. When $\kappa = 1 - \ell$ the equilibrium is undetermined. If there exists equilibrium and the regulation is sufficiently strict so that at least one institution is hitting the risk constraint then $\ell \ln \ell^{-1} < \kappa < 1 - \ell$. Hence, Ψ is a strictly increasing function of κ and, consequently, a strictly decreasing function of ν. As a result, the more tightly regulated the economy is (that is, the smaller ν is) the lower the risky assets equilibrium prices are.

Combining Propositions 1 and 2 we have the risky asset demand of each institution in equilibrium.

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[11]The non-principal branch of the Lambert correspondence is the inverse of the function $f : (-\infty, -1] \mapsto [-e^{-1}, 0)$ defined by $f(x) = xe^x$. For more details and properties of the Lambert correspondence see Corless et al. (1996).

Proposition 3 *Let $\left\{x^h, \boldsymbol{y}^h\right\}_{h \in [\ell,1]}$ be an equilibrium allocation. We have:*

1. *If $h \geq \kappa\Psi$ then $\boldsymbol{y}^h = \dfrac{\Psi\theta}{h}$.*

2. *If $h < \kappa\Psi$ then $\boldsymbol{y}^h = \dfrac{\theta}{h}$.*

Therefore the more risk-averse financial institutions (which, probably, do not hit the VaR constraint) hold riskier portfolios in the presence of VaR constraint than they would otherwise. But if the VaR constraint for financial institution h binds then the stricter is the VaR regulation the less risky is the portfolio of h.

4 The Welfare of Financial Institutions

Analyzing how the welfare of a particular agent varies due to a change in the nonseverity parameter is an interesting exercise both for regulating agencies and financial institutions. From this analysis we can answer some important questions like: (i) Is the welfare of a financial institution an increasing function of the nonseverity parameter? (ii) Would it be possible for a financial institution to increase its welfare in a regulated economy? Proposition 4 (below) states that, under certain conditions, the answer to the last question is positive. The intuition is immediate: At a regulated economy, agents less risk averse decrease their positions in riskier assets, then prices of these assets fall, making it interesting for other agents to buy them and thus increasing this agents' utility. Therefore, each financial institution maximizes its utility for a certain value of the nonseverity parameter that doesn't correspond necessarily to the situation of an unregulated economy ($\nu = \infty$). Before presenting Proposition 4 we are going to establish some preliminary calculations and define some notations.

Denote by $\overline{\nu}$ the maximum value of ν such as at least one institution is hitting the risk constraint and by $\underline{\nu}$ the lowest value of ν for which exists equilibrium, i.e.,

$$\overline{\nu} = \frac{\theta'\Sigma\theta}{\left(\ell\ln\ell^{-1}\right)^2} \quad \text{and} \quad \underline{\nu} = \frac{\theta'\Sigma\theta}{\left(1-\ell\right)^2}.$$

Let $\left\{x^h, \boldsymbol{y}^h\right\}_{h \in [\ell, 1]}$ be an equilibrium allocation with price vector of risky assets $\boldsymbol{q}$. Since we are working under a mean-variance model, if we fix the market parameters ($\boldsymbol{\Sigma}$ and μ), then the welfare of the financial institution h can be measured by the difference between the mean and the variance multiplied by $h/2$ of its wealth at $t = 1$, that is,

$$S^h = \mathcal{E}\left(W_1^h\right) - \frac{h}{2} Var\left(W_1^h\right)$$
$$= r_0 \left(\theta_0^h + \boldsymbol{q}\theta^h - \boldsymbol{q}\boldsymbol{y}^h\right) + \mu\boldsymbol{y}^h - h\frac{\boldsymbol{y}^{h\prime}\boldsymbol{\Sigma}\boldsymbol{y}^h}{2},$$

where S^h is the welfare of institution h and $Var(\cdot)$ represents the variance of a random variable[12]. It must be noted that the welfare of institution h depends on the nonseverity parameter ν. If the aggregate endowment of the risky assets is uniformly distributed across all agents (that is, $\left(\theta_0^h, \theta^h\right) = \left(\frac{\theta_0}{1-\ell}, \frac{\theta}{1-\ell}\right)$) then, after some algebraic manipulations and disdaining the terms independent of ν and h, it is possible to show that analyzing the welfare of institution h as a function of ν is equivalent to study the function $f^h(\nu) : [\underline{\nu}, \overline{\nu}] \mapsto \mathbb{R}$ defined by:

$$f^h(\nu) = \begin{cases} \frac{\Psi^2}{2h} - \frac{\Psi}{1-\ell} & \text{if } \nu \geq g_2^{-1}(h) \\[2mm] \frac{\Psi}{\kappa} - \frac{h}{2\kappa^2} - \frac{\Psi}{1-\ell} & \text{if } \nu < g_2^{-1}(h), \end{cases} \tag{9}$$

where $g_2(\cdot)$ is a decreasing function defined in the Appendix. The higher is $f^h(\nu)$ the higher is the welfare of agent h.

A reduction in ν (which corresponds to a tighter regulation) results in an increase in the risk premium of the assets. The response of the welfare function to this change in ν depends on the appetite for risk of the financial institution. Let ν^h be the value of the nonseverity parameter that maximizes $f^h(\cdot)$. The next proposition presents some aspects about the behavior of ν^h.

Proposition 4 *Let $f^h(\nu)$ be the modified welfare function of financial institution h defined by (9). We have*

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[12]Since agents have a constant absolute risk aversion coefficient, without loss of generality, we can suppose that the utility of institution h has the form $u^h(x) = -\mathrm{e}^{-hx}$. Then S^h is just a monotonic function of $\mathcal{E}\left(u^h\left(W_1^h\right)\right)$.

1. If $h \in [\ell, h^*]$ then f^h has a unique global maximum at the point $\nu^h = g_1^{-1}(h) < \overline{\nu}$, where $g_1(\cdot)$ is a decreasing function and $h^* < 1$ is a constant defined in the Appendix. In other words, if $h \in [\ell, h^*]$ and $\nu = g_1^{-1}(h)$, then financial institution h benefits from VaR-based risk regulation.

2. If $h \in (h^*, 1]$ then f^h has a unique global point at $\nu = \overline{\nu}$, that is, financial institution h prefers an economy without risk regulation.

Figure 3 illustrates the graphs of $f^h(\nu)$ for $h \in [\ell, h^*]$ and $h \in (h^*, 1]$. Figure 4 shows the optimum ν as a function of h. Observe that if $h \leq h^*$ the financial institution h prefers that the regulation would be fixed in a specific level $\nu^h < \overline{\nu}$ and the higher h is the smaller ν^h is. If $h > h^*$ then financial institution h prefers no regulation (that is, $\nu^h = \overline{\nu}$). The intuition is very simple: to get benefit from regulation these financial institutions would like a level of regulation extremely tight ($g_1^{-1}(h)$ very small). But in this case, the institution h holds a very risky portfolio what increases $Var(W_1^h)$ without a proportional increases in $\mathcal{E}(W_1^h)$. Since h is high ($h > h^*$) hence $S^h(g_1^{-1}(h)) < S^h(\overline{\nu})$[13].

From the regulating agency point of view, it is necessary to understand the welfare of financial institutions as a whole and not just to study the behavior of a particular agent in a regulated economy. To measure the financial institutions welfare we suppose that we have a linear-in-utilitily social welfare function, also called Bergson welfare function (see Varian, 1992)[14].

Definition 1 *Let $\left\{ \left(x^h, \boldsymbol{y}^h \right) \right\}_{h \in [\ell, 1]}$ be an equilibrium allocation for the economy under analysis. We define the financial institutions' social welfare function by:*

$$\Lambda_f \left(\nu \right) \equiv \int_\ell^1 \alpha(h) f^h(\nu) \mathrm{d}h,$$

where $\alpha(h)$ is a weight function defined in $[\ell, 1]$ and taking values in $\mathbb{R}^+$ such as the integral above is well defined.

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[13]The results of Proposition 4 suggest a different approach for the problem of risk regulation in which the existence of financial institutions lobbies are taken into account. In other words, we can admit that financial institutions try to persuade the regulating agency that their preferred positions would also serve the regulating agency's interests and perhaps those of the general public. So the nonseverity parameter is endogenous and is only determined by equilibrium conditions.

[14]We study the financial institutions' welfare only for equilibrium allocations.

A simple and natural choice of the weight function is $\alpha(h) = 1$ for all h (egalitarian welfare function)[15]. In this case we have

$$\Lambda_f(\nu)|_{\alpha(h)=1} = \frac{1}{4\kappa^2} \left[(\kappa\Psi)^2 - 2(\ell + \kappa)\kappa\Psi + \ell^2 \right].\qquad(10)$$

The next proposition shows that for this special choice of the weight function, the financial institutions' welfare is an increasing function of ν. In other words, Proposition 5 tells us that the tighter is the risk regulation the lower is the financial institutions' welfare as whole.

Proposition 5 *The financial institutions welfare function defined by (10) is increasing in ν.*

Proposition 4 states that some financial institutions can perform better in a regulated economy while others can perform worse. This fact leads us to a classic and fundamental question in economic theory: are equilibrium allocations Pareto efficient? That is, if the economy is in equilibrium, is it possible, using only the initial endowments, to reorganize the distribution of assets such that some agents are better off without making some another agent worse? If the answer is positive, then the equilibrium is not efficient. In an unregulated economy the first welfare theorem guarantees that equilibrium allocations are Pareto efficient. But what happens in economy with risk constraint? In the sequel we propose a definition of Pareto efficiency accordingly to the intuition explained above and show that for the economy with VaR-based risk constraint, the equilibrium allocation complies this criterion.

Definition 2 *An allocation $\left\{ \left(x^h, \boldsymbol{y}^h \right) \right\}_{h \in [\ell, 1]}$ is feasible if $\int_\ell^1 x^h\, dh \leq \theta_0$ and $\int_\ell^1 \boldsymbol{y}^h\, dh \leq \theta$[16].*

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[15]Of course, other choices of the weight functions can produce different shapes of the financial institutions welfare functions. For example for $\alpha(h) = h$ and $\alpha(h) = 1/h$ we have $\Lambda_f(\nu)|_{\alpha(h)=h} = -\frac{\kappa\Psi^3}{6} + \frac{\Psi^2}{2} - \frac{(\ell^2 + \kappa\ell + \kappa)\Psi}{2\kappa} + \frac{\ell^3}{6\kappa^2}$ and $\Lambda_f(\nu)|_{\alpha(h)=1/h} = -\frac{\Psi^2}{2} + \Psi\left(\frac{\ln\ell}{1-\ell} + \frac{1-\ln\ell}{\kappa} \right) - \frac{2\kappa - \ell}{2\kappa^2}$. The weight function $\alpha(h) = h$ is more favorable to the more risk averse agents than $\alpha(h) = 1$. In Section 4 we show that the more risk averse institutions prefer the softest regulation. Then, since $\Lambda_f(\nu)|_{\alpha(h)=1}$ is increasing in ν we can expect that $\Lambda_f(\nu)|_{\alpha(h)=h}$ would be increasing in ν too. Figure 5 shows $\Lambda_f(\nu)|_{\alpha(h)=h}$ for some values of ℓ. Observe that all functions are decreasing in κ and consequently increasing in ν. On the other hand, $\Lambda_f(\nu)|_{\alpha(h)=1/h}$ can be increasing, decreasing or even a non monotonic function depending on the value of ℓ as demonstrated in Figure 6.

[16]If $x, y \in \mathbb{R}^N$, then $x \leq y$ means that $x_i \leq y_i$ for all i.

Definition 3 *A feasible allocation $\left\{ \left(x^h, \boldsymbol{y}^h \right) \right\}_{h \in [\ell, 1]}$ is Pareto efficient if there is no other feasible allocation $\left\{ \left(\hat{x}^h, \hat{\boldsymbol{y}}^h \right) \right\}_{h \in [\ell, 1]}$ such as $\mathcal{E} \left[u^h \left(\hat{x}^h, \hat{\boldsymbol{y}}^h \right) \right] \geq \mathcal{E} \left[u^h \left(x^h, \boldsymbol{y}^h \right) \right]$ for all h, and strict inequality holds for $h \in H \subseteq [\ell, 1]$ with $\mathcal{L} \left(H \right) > 0$, where $\mathcal{L}$ is the Lebesgue measure on $[\ell, 1]$.*

The next proposition asserts that, although the risk constraint introduces a friction on the market, the equilibrium is, if it exists, still efficient.

Proposition 6 *Suppose that there exists an equilibrium for the economy with VaR constraint and that $\boldsymbol{q} \geq 0$. Then the equilibrium allocation is Pareto efficient.*

The introduction of VaR-based risk constraint maintains the market efficient in the Pareto sense. Then it is natural to investigate if it can bring some benefits to the economy. Of course, to analyze this question in details we need a more general model that includes other agents like stockholders or small savers. Probably, these agents would like that the fragility of financial institutions to be small. Despite of its simplicity, our model can be used to determine the bankruptcy probability of an institution and, therefore, to find out possible consequences of the regulation to the soundness of the financial system. We do it in the next section[17].

5   Bankruptcy Probability

The financial institution h will be in default if its wealth at $t = 1$ is less or equal zero. If equilibrium exists and at least one institution is hitting the risk constraint, the probability of agent h to be declared insolvent is

$$pb^h \equiv \mathcal{P} \left[W_1^h < 0 \right] = \Phi \left(-\frac{m^h}{s^h} \right),$$

where $m^h = r_0 W_0^h + \Psi \theta' \boldsymbol{\Sigma} \boldsymbol{y}^h$ and $s^h = \sqrt{y^{h\prime} \boldsymbol{\Sigma} y^h}$ are, respectively, the mean and the standard deviation of W_1^h, and Φ represents the cumulative standard normal distribution function. Since Φ is strictly increasing, to analyze the behavior of pb^h as a function of the nonseverity parameter ν, it is enough

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[17]We stress that to study this problem in details a more general model than the one proposed here is necessary. Our intention is just to give some insights into benefits/harms of risk regulation.

to study how $\frac{m^h}{s^h}$ varies when the regulating agency modifies ν. The higher is this quotient, the lower is the default probability of institution h. Using Proposition 3 it is easy to see that in equilibrium we have

1. If $h < \kappa\Psi$ then
$$\frac{m^h}{s^h} = \frac{\kappa r_0 W_0^h}{\sqrt{\theta'\Sigma\theta}} + \Psi\sqrt{\theta'\Sigma\theta}.$$

2. If $h \geq \kappa\Psi$ then
$$\frac{m^h}{s^h} = \frac{r_0 W_0^h h}{\Psi\sqrt{\theta'\Sigma\theta}} + \Psi\sqrt{\theta'\Sigma\theta}.$$

For comparison purpose, the value of this quotient in an unregulated economy is

$$\frac{m^h}{s^h} = \frac{r_0 W_0^h h \ln \ell^{-1}}{\sqrt{\theta'\Sigma\theta}} + \frac{\sqrt{\theta'\Sigma\theta}}{\ln \ell^{-1}} \quad \forall h.$$

Proposition 7 *Assume that there exists equilibrium and at least one institution hits the risk constraint. Let ν^* be the nonseverity parameter value such as $\Psi = \Psi^*$, where $\Psi^* \equiv \sqrt{\frac{h r_0 W_0^h}{\theta'\Sigma\theta}}$. That is, considering Ψ as function of ν we have $\nu^* = \Psi^{-1}\left(\Psi^*\right)$ (if $\Psi^* \leq \frac{1}{\ln \ell^{-1}}$ set $\nu^* = \overline{\nu}$ and if $\Psi^* \geq \frac{1}{1-\ell}$ set $\nu^* = \underline{\nu}$). Then $\frac{m^h}{s^h}$ is a decreasing function of ν on the interval $\left[\underline{\nu}, \max\left\{g_2^{-1}(h), \nu^*\right\}\right]$ and increasing on the interval $\left[\max\left\{g_2^{-1}(h), \nu^*\right\}, \overline{\nu}\right]$.*

Since Ψ is a decreasing function of ν we have that the lower is the ratio $\frac{h r_0 W_0^h}{\theta'\Sigma\theta}$ the higher is ν^*.

Proposition 7 gives interesting conclusions about the efficiency of the risk regulation (efficiency understood here as the reduction of the bankruptcy probability). The higher is ν^* the more efficient is the risk regulation. When $\nu^* = \overline{\nu}$ then VaR-based risk regulation is completely efficient since, in this case, $\frac{m^h}{s^h}$ is a decreasing function of ν on $[\underline{\nu}, \overline{\nu}]$. Therefore we have:

1. If a financial institution is well-capitalized (i.e. if W_0^h is high), the regulation can increase its bankruptcy probability. On the other hand, if the net worth of an institution is sufficiently small, then, from the regulating agency point of view, the regulation is always beneficial, since the more severe it is, the lower the default probability of the institution is.

2. The more tense the market, i.e. the higher market volatility (measured by $\theta'\Sigma\theta$), the more effective the regulation is.

3. The regulation is more effective for the institutions less risk averse (small h). If the institution is very conservative then the regulation can increase its bankruptcy probability.

4. The higher is r_0 the less efficient is the risk regulation.

Figure 7 presents the graphs of $\frac{m^h}{s^h}$ (solid line) for cases 1 and 3 of Proposition 7. The horizontal dash-dot line represents the same relation in unregulated economy.

Evidently, the regulating agency must also consider the financial system as a whole and not only a particular institution. Therefore, it is interesting to analyze the total bankruptcy probability (a measure of financial fragility), defined as the sum (integral) of the default probability of all institutions[18],

$$pgb \equiv \int_\ell^1 pb^h \mathrm{d}h. \tag{11}$$

Directly related (and more treatable from the algebraic point of view) with the metric defined by (11) is the integral in h of the quotient $\frac{m^h}{s^h}$,

$$\Lambda_s(\nu) \equiv \int_\ell^1 \frac{m^h}{s^h} \mathrm{d}h. \tag{12}$$

If the initial endowment of the assets is uniformly distributed across all agents, then $W_0^h = W_0$ for all h. In this case

$$\Lambda_s(\nu) = \frac{r_0 W_0}{\sqrt{\theta'\Sigma\theta}} \left(\frac{\kappa^2 \Psi}{2} + \frac{1}{2\Psi} - \kappa\ell \right) + \Psi(1-\ell)\sqrt{\theta'\Sigma\theta}. \tag{13}$$

The first and the second terms of the right-hand side of (13) are, respectively, increasing and decreasing functions of ν. Then the phenomenon already individually observed happens again in the aggregate level: If the level of capitalization of the financial institutions is high or the market volatility is

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[18]We could consider as a measure of financial fragility any linear combination of pb^h in the same way that was done to the financial institutions welfare (see Definition 1). However, since the choice of the weight function is completely arbitrary, this kind of analysis is controversial. Therefore we decided to present a simple example where the weight function is equals to 1 for all h.

low, then regulation can have effects which are contrary to the planned ones (increasing the financial fragility of the institutions). On the other hand, if the institutions have a small initial wealth or the market is tense, then the risk regulation presents the benefit of diminishing the number of bankruptcies. Figure 8 shows these two situations.

6 Conclusion

The primary aim of this work was to analyze the welfare properties in an economy where financial institutions are subject to a VaR-based risk regulation. The simplicity of the model studied in this paper allowed us to implement a kind of comparative statics analysis, in which we examined the change in the welfare and in the bankruptcy probability of a financial institution in reponse to a change in the level of regulation.

First, we determined for each institution the level of regulation that maximizes its utility. We showed that this level is not necessarily equivalent to the absence of regulation. Less risk averse financial institutions prefer a level of regulation which depends on its coefficient of risk aversion (the higher is the coefficient of risk aversion, the tighter is the optimum level of regulation). But institutions sufficiently risk averse (i.e., institutions with coefficient of risk aversion greater than a specific value determined by market parameters) prefer no VaR-based risk regulation. We also saw that despite the VaR-based risk constraint introduces a friction in the market, the equilibrium in a regulated economy is, if it exists, still efficient in the Pareto sense.

Second we analyzed the bankruptcy probability of financial institutions as a function of the level of regulation. We showed that if a financial institution is well-capitalized or if the market volatility is small or yet if the institution is very risk averse, then the VaR-based risk regulation can increase its bankruptcy probability. This fact suggests that the regulating agency should use different schemes of risk regulation across all institutions.

Appendix - Proofs of Propositions

Proof of Proposition 4

Consider the following functions:

1. $g_1(\nu) : [\underline{\nu}, \overline{\nu}] \mapsto [\ell, 1]$, defined by $g_1(\nu) = \kappa\Psi + \kappa^3\Psi'(\kappa)\left(\frac{1}{1-\ell} - \frac{1}{\kappa}\right)$,

2. $g_2(\nu) : [\underline{\nu}, \overline{\nu}] \mapsto [\ell, 1]$, defined by $g_2(\nu) = \kappa\Psi$ and

3. $g_3(\nu) : [\underline{\nu}, \overline{\nu}] \mapsto \left[\frac{1-\ell}{\ln \ell^{-1}}, 1\right]$, defined by $g_3(\nu) = (1 - \ell)\Psi$;

where $\Psi'(\kappa)$ is the derivate of Ψ, that is,

$$\Psi'(\kappa) =$$
$$\frac{1}{\kappa F\left(-(\kappa+\ell)\,\mathrm{e}^{-1}\right)}\left[\frac{\ell}{\kappa} + \frac{1}{F\left(-(\kappa+\ell)\,\mathrm{e}^{-1}\right)+1}\right] =$$
$$\frac{\Psi}{\kappa}\cdot\frac{\kappa\Psi - \ell}{1 + \ell - \kappa\Psi}.$$

It is easy to see that $g_1(\underline{\nu}) = g_2(\underline{\nu}) = g_3(\underline{\nu}) = 1$. Since κ and Ψ are strictly decreasing functions of ν we have that g_2 and g_3 are strictly decreasing function of ν too. Moreover, g_1 is also a strictly decreasing function of ν[19] and $g_1 \geq g_2 \geq g_3$. Figure 2 shows the graphs of these three functions.

The demonstration of Proposition 4 is a direct consequence of the following lemmas.

Lemma 2 *Let $\left\{\left(x_h, \boldsymbol{y}^h\right)\right\}_{h\in[\ell,1]}$ be an equilibrium allocation with price vector of risky assets $\boldsymbol{q}$.*

1. *For $\frac{1-\ell}{\ln \ell^{-1}} < h \leq 1$ we have*

   - *If $g_3^{-1}(h) < \nu \leq \overline{\nu}$, then $f^h(\nu)$ is strictly increasing.*
   - *If $g_1^{-1}(h) < \nu \leq g_3^{-1}(h)$, then $f^h(\nu)$ is strictly decreasing.*
   - *If $\underline{\nu} < \nu \leq g_1^{-1}(h)$, then $f^h(\nu)$ is strictly increasing.*

2. *For $\ell \leq h \leq \frac{1-\ell}{\ln \ell^{-1}}$ we have*

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[19]Substituing Ψ' in the definition of g_1 we have $g_1(\kappa) = \kappa\Psi + \kappa\Psi\cdot\frac{\kappa\Psi-\ell}{1+\ell-\kappa\Psi}\cdot\frac{\kappa-1+\ell}{1-\ell}$, that is, g_1 is the sum of a increasing function of κ and a product of three increasing functions of κ, then g_1 is a increasing function of κ. Therefore g_1 is a decreasing function of ν.

- If $g_1^{-1}(h) < \nu \leq \overline{\nu}$ then $f^h(\nu)$ is strictly decreasing.

- If $\underline{\nu} < \nu \leq g_1^{-1}(h)$ then $f^h(\nu)$ is strictly increasing.

In any case $f^h(\overline{\nu}) = \frac{1}{\ln \ell^{-1}} \left(\frac{1}{2h \ln \ell^{-1}} - \frac{1}{1-\ell} \right)$ and $f^h(\underline{\nu}) = -\frac{h}{2(1-\ell)^2}$.

Proof of Lemma 2

Since κ is a strictly decreasing function of ν, to verify the intervals where $f^h(\nu)$ is increasing or decreasing it is enough to analyze f^h as a function of κ.

If $\underline{\nu} \leq \nu \leq g_2^{-1}(h)$ then $h \leq g_2(\nu) = \kappa \Psi$, hence $f^h(\nu) = \frac{\Psi}{\kappa} - \frac{h}{2\kappa^2} - \frac{\Psi}{1-\ell}$ and

$$\frac{\partial f^h}{\partial \kappa} = \Psi'(\kappa) \left(\frac{1}{\kappa} - \frac{1}{1-\ell} \right) + \frac{h}{\kappa^3} - \frac{\Psi}{\kappa^2}.$$

If $\nu \leq g_1^{-1}(h)$ then $\frac{\partial f^h}{\partial \kappa} < 0$ and f^h is a strictly decreasing function of κ and therefore a strictly increasing function of ν. If $g_1^{-1}(h) \leq \nu \leq g_2^{-1}(h)$, a similar argument shows that f^h is a strictly decreasing function of ν.

If $g_2^{-1}(h) < \nu \leq \overline{\nu}$ then

$$\frac{\partial f^h}{\partial \kappa} = \Psi'(\kappa) \left(\frac{\Psi}{h} - \frac{1}{1-\ell} \right).$$

We have to consider two cases:

1. If $\ell \leq h \leq \frac{1-\ell}{\ln \ell^{-1}}$ then $g_3(\nu) > h$. Therefore $\frac{\partial f^h}{\partial \kappa} > 0$ which implies that f^h is a strictly increasing function of κ and a strictly decreasing of ν.

2. If $\frac{1-\ell}{\ln \ell^{-1}} \leq h \leq 1$ then the equation $g_3(\nu) = h$ has only one solution. Therefore, if $g_3^{-1}(h) < \nu \leq \overline{\nu}$ then f^h is a strictly increasing function of ν. On the other hand, if $g_2^{-1}(h) < \nu \leq g_3^{-1}(h)$ then f^h is a strictly decreasing function of ν.

$\square$

The next lemma shows that between the tightest level ($\nu = \underline{\nu}$) and the softest level ($\nu = \overline{\nu}$) of regulation, all financial institutions prefer the last one.

Lemma 3 *For all h we have $f^h(\overline{\nu}) \geq f^h(\underline{\nu})$.*

Proof of Lemma 3

It is sufficient to show that

$$\frac{1}{2}\left(\frac{1}{h\left(\ln \ell^{-1}\right)^2} + \frac{h}{(1-\ell)^2}\right) \geq \frac{1}{(1-\ell)\left(\ln \ell^{-1}\right)}.$$

But the left-hand side of the previous equation has a minimum at $h = \frac{1-\ell}{\ln \ell^{-1}}$ which is equal to $\frac{1}{(1-\ell)(\ln \ell^{-1})}$. $\qquad\square$

By Lemma 2 we have that if $\ell \leq h \leq \dfrac{1-\ell}{\ln \ell^{-1}}$ then $f^h(\nu)$ has a maximum at $\nu = g_1^{-1}(h)$. However, if $\dfrac{1-\ell}{\ln \ell^{-1}} < h \leq 1$ there are two possible candidates for the maximum of $f^h(\nu)$: the same $g_1^{-1}(h)$ or $\overline{\nu}$. The next lemma gives conditions that allow us to decide in which of these points the function $f^h(\nu)$ assumes its maximum.

Lemma 4 *Let* $t(h) : \left[\frac{1-\ell}{\ln \ell^{-1}}, 1\right] \mapsto \mathbb{R}$ *be defined by*

$$t(h) = \frac{\Psi}{\kappa} - \frac{h}{2\kappa^2} - \frac{\Psi}{1-\ell} - \frac{1}{\ln \ell^{-1}}\left(\frac{1}{2h\ln \ell^{-1}} - \frac{1}{1-\ell}\right),$$

where κ *and* Ψ *are calculated at* $\nu = g_1^{-1}(h)$. *The function* $t(h)$ *is strictly decreasing and has only one root. Denoting by* h^* *this root we have*

1. *If* $\frac{1-\ell}{\ln \ell^{-1}} \leq h \leq h^*$ *then* $f^h(\nu)$ *has a maximum at* $\nu = g_1^{-1}(h)$.

2. *If* $h^* \leq h \leq 1$ *then* $f^h(\nu)$ *has a maximum at* $\nu = \overline{\nu}$.

Proof of Lemma 4

The function $t(h)$ is continuous and using elementary differential calculus it is possible, after tedious manipulation, to prove that:

1. $t\left(\frac{1-\ell}{\ln \ell^{-1}}\right) > 0$ and

2. $t(1) < 0$.

By the Bolzano's theorem (Apostol, 1967) the function $t(h)$ has at least one real root on the interval $\left[\frac{1-\ell}{\ln \ell^{-1}}, 1\right]$. To show that it is the only root we have to prove that $t(h)$ is strictly decreasing. We can write $t(h)$ as the difference between two functions: $t(h) = t_2(h) - t_1(h)$ where

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$$t_1(h) = \frac{1}{\ln \ell^{-1}} \left(\frac{1}{2h \ln \ell^{-1}} - \frac{1}{1 - \ell} \right) \quad \text{and}$$

$$t_2(h) = \frac{\Psi}{\kappa} - \frac{h}{2\kappa^2} - \frac{\Psi}{1 - \ell}$$

with κ and Ψ computed at $\nu = g_1^{-1}(h)$. Therefore,

$$\frac{\partial t_1}{\partial h} = -\frac{1}{2\left(h \ln \ell^{-1}\right)^2} \quad \text{and}$$

$$\frac{\partial t_2}{\partial h} = -\frac{1}{2\kappa^2},$$

where to compute the last derivate we use the fact that at $\nu = g_1^{-1}(h)$, $\frac{\partial t_2}{\partial \kappa} = 0$. Hence we must demonstrate that $\frac{\partial t_2}{\partial h} \leq \frac{\partial t_1}{\partial h}$. But this occurs because

$$\text{Max}_{h} \ \frac{\partial t_2}{\partial h} = \text{Min}_{h} \ \frac{\partial t_1}{\partial h} = -\frac{1}{2(1-\ell)^2}.$$

The other claims of the lemma are immediate consequences of the behavior of $t(h)$. □

Proof of Proposition 5

Since κ is a decreasing function of ν, to show that Λ_f is an increasing function of ν is sufficient to show that

$$f(\kappa) = (\kappa\Psi)^2 - 2(\ell + \kappa)\kappa\Psi + \ell^2$$

is a decreasing function of κ. Consider the quadratic polynomial $p(x) = x^2 - 2(\ell + \kappa)x + \ell^2$. This polynomial has two positive real roots:

$$x_1 = \ell + \kappa - \sqrt{\kappa^2 + 2\kappa\ell} \quad \text{and} \quad x_2 = \ell + \kappa + \sqrt{\kappa^2 + 2\kappa\ell}.$$

When κ increases x_1 decreases and x_2 increases (see Figure 9). Since $\kappa\Psi$ is an increasing function of κ and $\kappa\Psi < \kappa + \ell$ we have that when κ increases, $(\kappa\Psi)^2 - (\ell + \kappa)\kappa\Psi + \ell^2$ decreases. □

Proof of Proposition 6

The demonstration follows the usual procedure. Assume that the equilibrium allocation $\left\{\left(x^h, \boldsymbol{y}^h\right)\right\}_{h \in [\ell, 1]}$ with prices $\boldsymbol{q}$ is not Pareto eficient. Hence,

there is another feasible allocation $\left\{ \left(\hat{x}^h, \hat{\boldsymbol{y}}^h \right)_{h \in [\ell,1]} \right\}$ that dominates $\left\{ \left(x^h, \boldsymbol{y}^h \right)_{h \in [\ell,1]} \right\}$ in the Pareto sense. That is, $\mathcal{E}\left[u^h \left(\hat{x}^h, \hat{\boldsymbol{y}}^h \right) \right] \geq \mathcal{E}\left[u^h \left(x^h, \boldsymbol{y}^h \right) \right]$ for all h and there is $H \subseteq [\ell,1]$ with $\mathcal{L}(H) > 0$ such as if $h \in H$ then the strict inequality holds.

Note that for all h we should have $\hat{x}^h + \boldsymbol{q}\hat{\boldsymbol{y}}^h \geq W_0^h$, where $W_0^h = \theta_0^h + \boldsymbol{q}\theta^h$ is the initial wealth of agent h, since on the contrary, for $\epsilon > 0$ sufficiently small, $\left(\hat{x}^h + \epsilon, \hat{\boldsymbol{y}}^h \right)$ belongs to the restriction set of institution h problem with prices $\boldsymbol{q}$. Since u^h is strictly increasing it would result that $\left(\hat{x}^h + \epsilon, \hat{\boldsymbol{y}}^h \right)$ is preferable to $\left(x^h, \boldsymbol{y}^h \right)$ that would be in contradiction with $\left(x^h, \boldsymbol{y}^h \right)$ to be the optimum of institution h problem with prices $\boldsymbol{q}$. Moreover we must have $\hat{x}^h + \boldsymbol{q}\hat{\boldsymbol{y}}^h > W_0^h$ for all $h \in H$.

Since $\left(\hat{x}^h, \hat{\boldsymbol{y}}^h \right)$ is feasible, we have:

$$\theta_0 + \boldsymbol{q}\theta \geq \int_\ell^1 \hat{x}^h \mathrm{d}h + \boldsymbol{q}\int_\ell^1 \hat{\boldsymbol{y}}^h \mathrm{d}h =$$

$$\int_H \hat{x}^h \mathrm{d}h + \int_{[\ell,1]-H} \hat{x}^h \mathrm{d}h + \boldsymbol{q}\left(\int_H \hat{\boldsymbol{y}}^h \mathrm{d}h + \int_{[\ell,1]-H} \hat{\boldsymbol{y}}^h \mathrm{d}h \right) >$$

$$\int_\ell^1 W_0^h \mathrm{d}h = \theta_0 + \boldsymbol{q}\theta.$$

The contradiction demonstrates the desired result. $\qquad\square$

Proof of Proposition 7

If $\nu < g_2^{-1}(h)$, then

$$\frac{\partial \frac{m^h}{s^h}}{\partial \nu} = \frac{r_0 W_0^h}{\sqrt{\theta'\boldsymbol{\Sigma}\theta}} \frac{\partial \kappa}{\partial \nu} + \sqrt{\theta'\boldsymbol{\Sigma}\theta} \frac{\partial \Psi}{\partial \nu},$$

since $\frac{\partial \kappa}{\partial \nu} < 0$ and $\frac{\partial \Psi}{\partial \nu} < 0$ we have $\frac{\partial \frac{m^h}{s^h}}{\partial \nu} < 0$.

If $\nu \geq g_2^{-1}(h)$, then

$$\frac{\partial \frac{m^h}{s^h}}{\partial \nu} = \frac{\partial \Psi}{\partial \nu}\left(-\frac{r_0 W_0^h h}{\Psi^2 \sqrt{\theta'\boldsymbol{\Sigma}\theta}} + \sqrt{\theta'\boldsymbol{\Sigma}\theta} \right).$$

Hence, when $\nu \leq \nu^*$ we have $\frac{\partial \frac{m^h}{s^h}}{\partial \nu} < 0$ and when $\nu \geq \nu^*$ we have $\frac{\partial \frac{m^h}{s^h}}{\partial \nu} > 0$. $\square$

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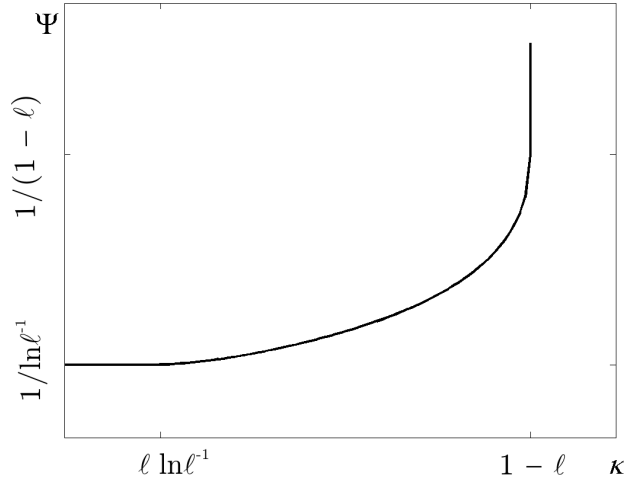Varian, H. R. (1992). *Microeconomic Analysis*. 3th ed. New York: W. W. Norton & Company.

Figure 1: Illustration of Ψ.
This picture shows the market-price of risk scalar (Ψ) as function of κ.
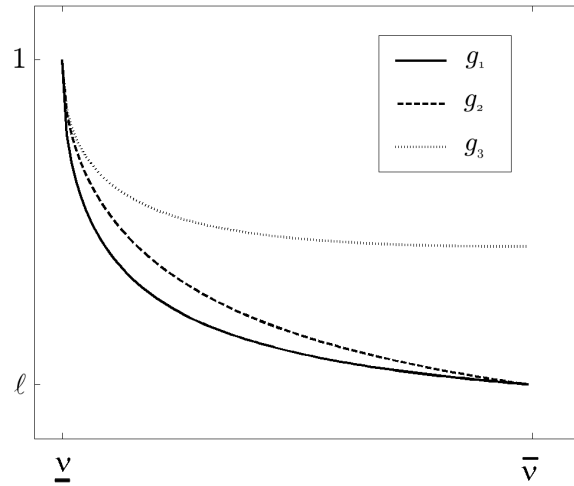

Figure 2: Graphs of functions g_1, g_2 and g_3.
This picture presents graphs of the auxiliary functions defined by $g_1(\nu) = \kappa\Psi + \kappa^3\Psi'(\kappa)\left(\frac{1}{1-\ell} - \frac{1}{\kappa}\right)$, $g_2(\nu) = \kappa\Psi$ and $g_3(\nu) = (1-\ell)\Psi$.
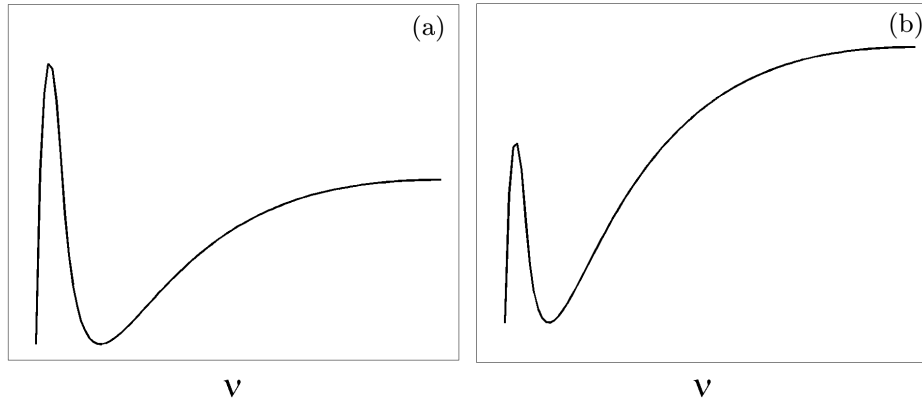
Figure 3: Function f^h.
This picture shows the graphs of $f^h(\nu)$ for different values of h. The function $f^h(\nu)$ is a monotonic transformation of the expected value of the institution h wealth utility. (a) $h \in [\ell, h^*]$ (b) $h \in (h^*, 1]$.
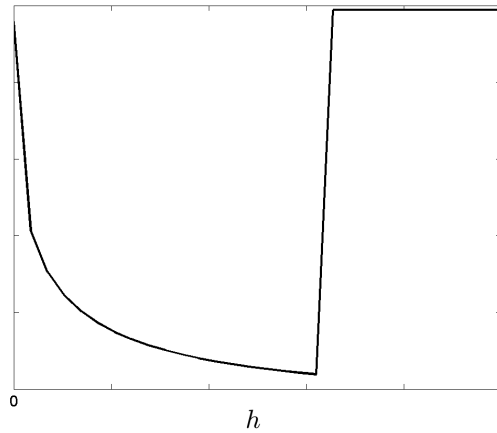
Figure 4: Optimum level of regulation $\left(\nu^h\right)$.
This picture shows the relationship between the optimal level of regulation (ν^h) and the coefficient of risk aversion (h). For each h, the optimal level of regulation is the value of ν that maximizes the expected value of the institution h wealth utility.
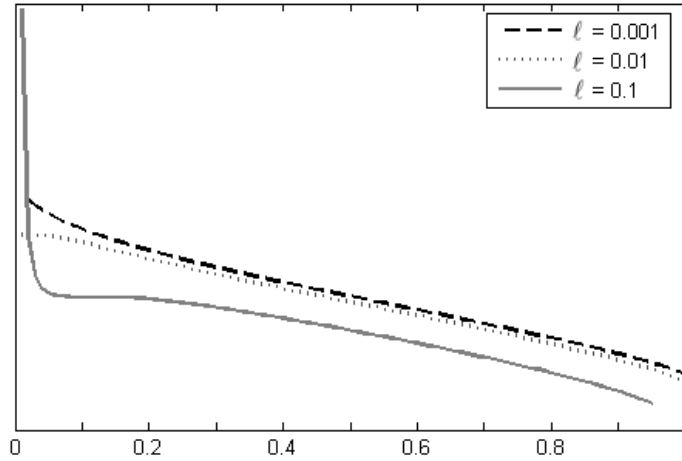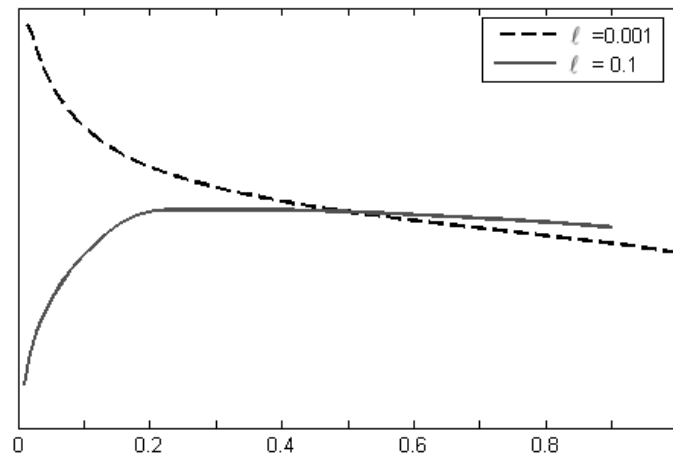
Figure 5: Function $\Lambda_f(\nu)$ for $\alpha(h) = h$.

This picture presents the social welfare function when the weight function is equals to $\alpha(h) = h$ for some values of ℓ. Solid, dotted and dashed lines are the graphs for $\ell = 0.1$, $\ell = 0.01$ and $\ell = 0.001$, respectively.

Figure 6: Function $\Lambda_f(\nu)$ for $\alpha(h) = 1/h$.

This picture presents the social welfare function when the weight function is equals to $\alpha(h) = h$ for some values of ℓ. Solid and dashed lines are the graphs for $\ell = 0.1$ and $\ell = 0.001$, respectively.
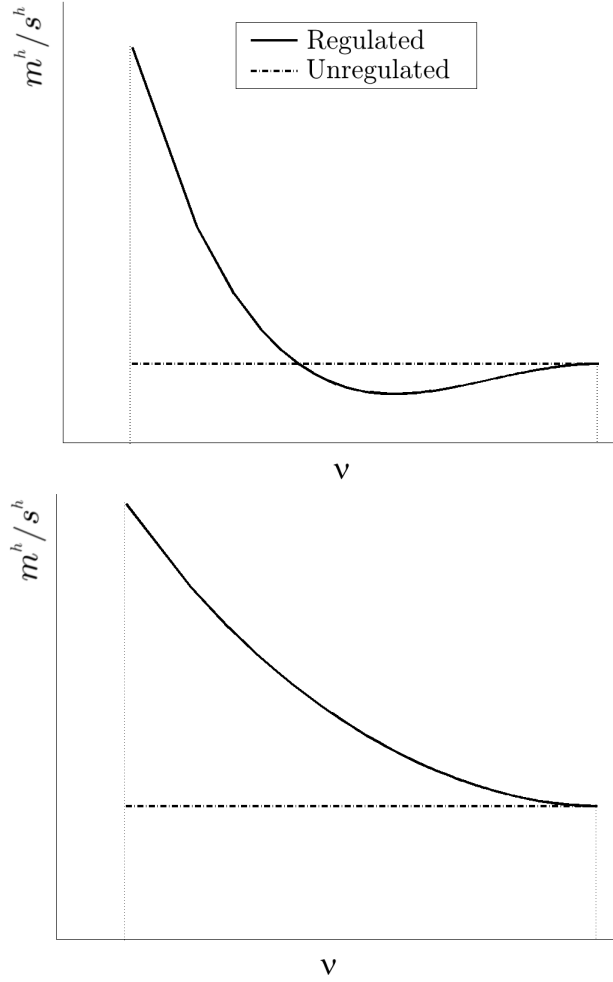
Figure 7: Graphs of the function $\frac{m^h}{s^h}$.
This picture presents the quotient between the mean (m^h) and standard deviation (s^h) of institution h wealth at $t = 1$. The solid represents m^h/s^h in a regulated economy and the dash-dot line represents the same variable in an unregulated economy $(\nu = \infty)$. (a) $\tilde{\nu} \leq g_2^{-1}(h)$ (b) $\tilde{\nu} > \overline{\nu}$.
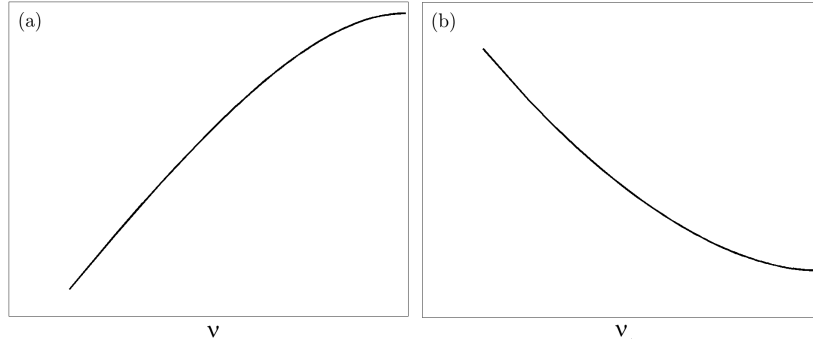
Figure 8: Graphs of the function Λ_s.
This picture shows the total bankruptcy probability (a measure of financial fragility) as a function of the nonseverity parameter (ν). In (a) the level of capitalization of the financial institutions is high and in (b) the opposite occurs.
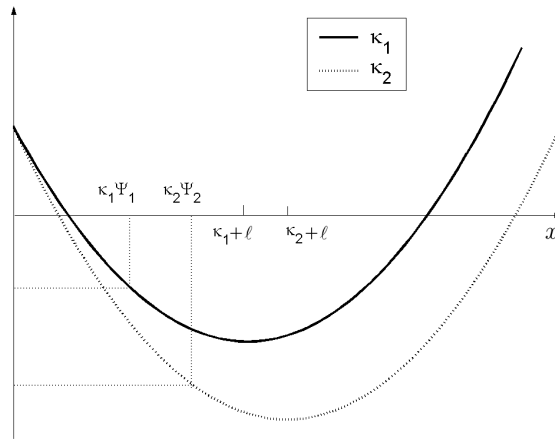

Figure 9: Polynomial $p(x) = x^2 - 2(\ell + \kappa)x + \ell^2$ $\left(\kappa_1 < \kappa_2\right)$.

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