F U N D A Ç Ã O
GETULIO VARGAS

EPGE
Escola de Pós-Graduação
em Economia

Ensaios Econômicos

Escola de

Pós Graduação

em Economia

da Fundação

Getulio Vargas

Equilibria in security markets with a continuum of agents

A. Araujo
V. F. Martins da Rocha
P. K. Monteiro

Novembro de 2003

EQUILIBRIA IN SECURITY MARKETS WITH A CONTINUUM OF AGENTS

A. ARAUJO, V. F. MARTINS-DA-ROCHA AND P. K. MONTEIRO

ABSTRACT. We prove the existence of a competitive equilibrium for exchange economies with a measure space of agents and for which the commodity space is ℓ^p, $1 < p < +\infty$. A vector $x = (x_n)$ in ℓ^p may be interpreted as a security which promises to deliver x_n units of numeraire at state (or date) n.

Under assumptions imposing uniform bounds on marginal rates of substitution, positive results on core-Walras equivalence were established in Rustichini–Yannelis [21] and Podczeck [20]. In this paper we prove that under similar assumptions on marginal rates of substitution, the set of competitive equilibria (and thus the core) is non-empty.

1. INTRODUCTION

We consider an Arrow–Debreu exchange economy with infinitely many agents and infinitely many commodities. Following the modelisation of perfect competition by Aumann [4] and Hildenbrand [9], the set of agents is a finite positive measure space. The commodity space is the space ℓ^p of sequences $x = (x_k)$ such that $\sum_k |x_k|^p < +\infty$ for $1 < p < +\infty$. Such spaces are relevant to the allocation of resources over time or states of nature. In particular a bundle x in ℓ^p may be interpreted as a security which promises to deliver x_k units of numéraire at date (or state) k.

Under assumptions imposing uniform bounds on marginal rates of substitution, positive results on core-Walras equivalence were established in Rustichini–Yannelis [21] and Podczeck [20]. In this paper we prove that under similar assumptions on marginal rates of substitution, the set of competitive equilibria (and thus the core) is non-empty. More precisely, we provide two frameworks to prove the existence of competitive equilibria. In the first one, existence is proved under an assumption (borrowed from Zame [23] and Podczeck [20]) imposing the existence of a uniform (over consumption) upper and lower bounds on marginal rates of substitution, at each state of nature. In the second one, existence is proved for preference relations represented by separable utility functions. But for this framework, we only require the existence of an upper bound on the marginal rates of substitution at the initial endowment (and not uniformly over consumption) and we require the existence of a uniform (over consumption) lower bound on the marginal rates of substition at only one state of nature.

In the framework of economies with infinitely many agents and infinitely many commodities, there are several existence results: Khan–Yannelis [11], Podczeck [18] and Martins-da-Rocha [14] for separable Banach commodity spaces with an interior point in the positive cone; and Mas-Colell [16], Jones [10], Ostroy–Zame [17], Podczeck [18, 19] and Martins-da-Rocha [15] for economies with differentiated commodities. To the best of our knowledge, this paper is the first to provide an existence result for economies with an ℓ^p-commodity space.

The paper is organized as follows. In Section 2 we define the model of an economy with infinitely many agents and commodities and we set out the main definitions and notations. In Section 3

we give the list of assumptions that economies will be required to satisfy and we present the two existence results. The different assumptions on the marginal rates of substitution are discussed in Section 4. Finally, Section 5 is devoted to the proof of the two theorems.

2. The Model

2.1. **Preliminaries.** For each $1 < p < +\infty$, we denote by ℓ^p the real vector space of sequences $x = (x_k)_k$ in $\mathbb{R}^{\mathbb{N}}$ such that $\lim_n \sum_{k=0}^n |x_k|^p < \infty$ and we denote by $\|x\|_p = (\sum_{k \in \mathbb{N}} |x_k|^p)^{1/p}$. The space ℓ^p endowed with the norm $\|.\|_p$ is a separable Banach space whose dual is ℓ^q where $1 < q < +\infty$ is defined by $1/p + 1/q = 1$. We denote by ℓ^p_+ the natural positive cone defined by $x \in \ell^p_+$ if and only if $x_k \geqslant 0$ for each $k \in \mathbb{N}$. A vector x in ℓ^p_+ is called positive, and if for each k, $x_k > 0$, then x is called striclty positive. As usual, if x, y are two vectors of ℓ^p, $x \geqslant y$ means $x - y \in \ell^p_+$. The weak topology $\sigma(\ell^p, \ell^q)$ is denoted by w, the weak-star topology $\sigma(\ell^q, \ell^p)$ is denoted by w^* and the norm-topology defined by $\|.\|_p$ is denoted by s.

Let τ be a topology on ℓ^p. If $(C_n)_n$ is a sequence of subsets of ℓ^p, the τ-*sequential upper limit* of $(C_n)_n$, is denoted $\tau\text{-}\mathrm{ls}_n C_n$ and is defined by

$$\tau\text{-}\mathrm{ls}_n C_n := \{x \in \ell^p : \ x = \tau\text{-}\lim_k x^k, \quad x^k \in C_{n(k)}\}$$

where $(C_{n(k)})_k$ is a subsequence of $(C_n)_n$.

The Borel σ-algebra of ℓ^p for the norm-topology or for the weak-topology coincide and is denoted by $\mathcal{B}$. A correspondence F from Ω to ℓ^p is said to be graph measurable if $\{(a, x) \in \Omega \times \ell^p : \ x \in F(a)\}$ belongs to $\mathcal{A} \otimes \mathcal{B}$. A correspondence P from Ω to $\ell^p \times \ell^p$ is said to be graph measurable if $\{(a, x, z) \in \Omega \times \ell^p \times \ell^p : \ (x, z) \in P(a)\}$ belongs to $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{B}$.

A mapping s from Ω to ℓ^p is simple if there exist $x_1, x_2, \ldots, x_n$ in ℓ^p and $A_1, A_2, \ldots, A_n$ in $\mathcal{A}$ such that $s = \sum_{i=1}^n x_i \chi_{A_i}$ where $\chi_{A_i}(a) = 1$ if $a \in A_i$ and $\chi_{A_i}(a) = 0$ if $a \notin A_i$. A mapping $x : \Omega \to \ell^p$ is Bochner measurable if there is a sequence of simple mappings $s_n : \Omega \to \ell^p$ such that $\lim_n \|s_n(a) - x(a)\|_p = 0$ almost every where. Since ℓ^p is norm-separable, we know from Pettis' measurability theorem (see [7, Theorem II.1.2, p.42]) that a mapping $x : \Omega \to \ell^p$ is Bochner measurable if and only if for each $B \in \mathcal{B}$, $x^{-1}(B) := \{a \in \Omega : \ x(a) \in B\}$ belongs to $\mathcal{A}$; and moreover x is Bochner measurable if and only if for each $k \in \mathbb{N}$, the function $a \mapsto x_k(a)$ is measurable.

A Bochner measurable mapping x from Ω to ℓ^p is Bochner integrable if there is a sequence of simple mappings $s_n : \Omega \to \ell^p$ such that $\lim_n \int_\Omega \|s_n(a) - x(a)\|_p \, d\mu(a) = 0$. For each measurable set A in $\mathcal{A}$, we denote by $\int_A x d\mu$ the limit $\lim_n \int_A s_n d\mu$. It can easily be shown (see [7, p.45]) that a Bochner measurable mapping x is Bochner integrable if and only if the mapping $a \mapsto \|x(a)\|_p$ is integrable. In particular $\left\|\int_A x d\mu\right\|_p \leqslant \int_A \|x(a)\|_p \, d\mu(a)$. Note that if x is a Bochner integrable mapping from Ω to ℓ^p, then for each $k \in \mathbb{N}$, the function $a \mapsto x_k(a)$ is integrable and if we denote by $v := \int_A x d\mu$ then $v_k = \int_A x_k d\mu$.

2.2. **The Model.** An economy $\mathcal{E}$ is a list

$$\mathcal{E} = ((\Omega, \mathcal{A}, \mu), \ell^p, X, \succeq, e),$$

where X is a correspondence from Ω to ℓ^p, $\succeq$ is a correspondence from Ω to $\ell^p \times \ell^p$ and e is a mapping from Ω to ℓ^p. The space of agents is $(\Omega, \mathcal{A}, \mu)$, a complete finite positive measure space. The commodity space is ℓ^p with $1 < p < +\infty$. For each agent $a \in \Omega$, the consumption set is

$X(a)$, the initial endowment is $e(a) \in \ell^p$ and the preference/indifference relation is $\succeq_a \subset \ell^p_+ \times \ell^p_+$, a reflexive binary relation on ℓ^p_+.

We define the correspondence[1] $P_a : \ell^p_+ \twoheadrightarrow \ell^p_+$ by $P_a(x) = \{x' \in \ell^p_+ : x' \succ_a x\}$. In particular, if $x \in \ell^p_+$ is a consumption bundle, the set $P_a(x)$ is the set of consumption bundles strictly preferred to x by agent a. We let P be the correspondence from Ω to $\ell^p \times \ell^p$ defined for each $a \in \Omega$ by $P(a) = \{(x, x') \in X(a) \times X(a) : x' \succ_a x\}$.

The set of allocations (or plans) of the economy is the set $S^1(X)$ of Bochner integrable selections of X, i.e. $S^1(X)$ is the set of mappings x from Ω to ℓ^p which are Bochner integrable and which satisfies $x(a) \in X(a)$ for almost every $a \in \Omega$. An allocation $x \in S^1(X)$ is feasible if

$$\int_\Omega x d\mu = \int_\Omega e d\mu.$$

We assume that the mapping $e : \Omega \to \ell^p$ is a Bochner integrable mapping and we denote by $\omega := \int_\Omega e d\mu$ the aggregate initial endowment.

The price space is ℓ^q, and if $x = (x_k)_k$ in ℓ^p is a bundle and $p = (p_k)_k$ in ℓ^q is a price then the value of x is defined by the natural dual product

$$\langle p, x \rangle = \sum_{k \in \mathbb{N}} p_k x_k.$$

Definition 2.1. A pair (x, p) consisting of a feasible allocation x and a non-zero price p is said to be a **competitive equilibrium** if for almost every $a \in \Omega$, $\langle p, x(a) \rangle = \langle p, e(a) \rangle$, and $z \in P_a(x(a))$ implies $\langle p, z \rangle > \langle p, x(a) \rangle$.

3. Existence of a competitive Equilibrium

We will maintain in this paper the following assumptions on the economy $\mathcal{E}$.

Assumption 3.1. *For each $a \in \Omega$,*
  (i) *the consumption set is $X(a) = \ell^p_+$;*
  (ii) *the initial endowment is not zero, i.e. $e(a) > 0$;*
  (iii) *$\succeq_a$ is reflexive, transitive and complete;*
  (iv) *$\succeq_a$ is strictly monotone, i.e. for each $x \in X(a)$, if $z > x$ then $z \succ_a x$.*

Assumption 3.2. *For each $a \in \Omega$, for every $x \in X(a)$,*
  (i) *the sets $P_a(x)$ and $P_a^{-1}(x) = \{z \in X(a) : x \succ_a z\}$ are s-open in $X(a)$;*
  (ii) *the set $\{z \in X(a) : z \succeq_a x\}$ is convex.*

Assumption 3.3. *The correspondence P is graph measurable.*

Assumption 3.4. *For each $a \in \Omega$, $e(a)$ belongs to ℓ^1 and the function $a \mapsto \|e(a)\|_1$ is integrable.*

Remark 3.1. Assumptions 3.1–3.3 are standard in the literature dealing with exchange economies with finitely or infinitely many agents. We will see that in our framework, we can not dispense with Assumption 3.4.

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[1]As usual, $y \succ_a x$ means $[y \succeq_a x$ and $x \not\succeq_a y]$. Note that the binary relation $\succ_a$ coincide with the graph of the correspondence P_a.

We provide hereafter two frameworks to prove the existence of competitive equilibria. In the first one, existence is proved under an assumption imposing upper and lower uniform (over agents and consumptions) bounds on marginal rates of substitution. In the second one, existence is proved for preference relations represented by separable utility functions but the assumption on the marginal rates of substitution required for the existence is weaker. We only require a uniform (over agents only) upper bound on the marginal rates of substitution at the initial endowment and a uniform (over agents and consumption) lower bound on the marginal rates of substitution at one state of nature.

3.1. **The general case.** In this section, we consider economies with general preference relations. The following requirement is borrowed from Zame [23]. It is discussed in Section 4.

Definition 3.2. The preference relations $(\succeq)_a$ are said to be **strong-uniformly proper**, if there exist strictly positive prices α and β in ℓ_+^q with $\alpha \leqslant \beta$ and such that for every $a \in \Omega$, whenever $x, u, v \in \ell_+^p$ satisfy $v \leqslant x$ and $\langle \alpha, u \rangle > \langle \beta, v \rangle$ then $x - v + u \succ_a x$.

An economy $\mathcal{E}$ is said strong-uniformly proper if it has strong-uniformly proper preference relations.

We can now state our first result for economies with general preference relations.

Theorem 3.3. *If the economy $\mathcal{E}$ is strong-uniformly proper then there exists a competitive equilibrium.*

Remark 3.4. The strong-uniform properness assumption was already used in Zame [23]. Podczeck in [20] prove the equivalence between the core and the set of competitive equilibria under this assumption. Note that Rustichini–Yannelis [21] also proved the equivalence between the core and the set of competitive equilibria under another properness assumption. Assumption 3.4 is unusual. Following Zame [23] we provide hereafter an example of a strong-uniformly proper economy satisfying Assumptions 3.1–3.3 and not satisfying Assumption 3.4. For this economy the set of competitive equilibria is empty.

Counterexample 3.5. Consider the economy $\mathcal{E}$ where $\Omega = [0, 1]$, $\mathcal{A}$ is the Lebesgue σ-algebra and μ is the Lebesgue measure. For each trader $a \in [0, 1]$, the consumption set coincide with ℓ_+^p, the utility function u_a is defined by

$$u_a(x) = \sum_{n \in \mathbb{N}} (2 + a)^{-n} x_n$$

and the initial endowment is defined by

$$e(a) = (1, 1/2, 1/3, \dots, 1/n, \dots).$$

The economy $\mathcal{E}$ is strong-uniformly proper, it satisfies Assumptions 3.1–3.3 but not Assumption 3.4. It is proved in Zame [23] that $\mathcal{E}$ has no competitive equilibrium.

3.2. **The separable case.** In this section, we consider economies for which preference relations are represented by separable utility functions.

Definition 3.6. A utility function $u : \ell_+^p \to \mathbb{R}$ is called separable if there exists for each n, a function $v_n : [0, +\infty) \to \mathbb{R}$ concave and stricly increasing such that

$$\forall x \in \ell_+^p, \quad u(x) = \sum_{n \in \mathbb{N}} v_n(x_n).$$

The function $v = (v_n)$ is called the kernel of u. The left derivative of v_n in $t > 0$ is noted $v_n^-(t)$ and the right derivative is denoted $v_n^+(t)$. If $x \in \ell_+^p$ then we note $v^-(x) := (v_n^-(x_n))_n$. For $x \in \ell_+^p$ we define $S(x) = \{h \in \ell^p \colon \exists t > 0, \; x + th \geqslant 0\}$ and $I(x) = S(x) \cap -S(x)$. We define $u'(x) \cdot h = \lim_{r \to 0}(1/r)[u(x + rh) - u(x)]$ for each $x \in \ell_+^p$ and each $h \in S(x)$. Note that if $h \geqslant 0$ then $u'(x) \cdot h = \sum_n v_n^+(x_n) h_n$.

For economies with separable utility functions, a weaker condition than the uniform properness will be sufficient to prove the existence of competitive equilibria.

Definition 3.7. An economy $\mathcal{E}$ is said **separably proper** if for each agent $a \in \Omega$, the preference relation $\succ_a$ is represented[2] by a separable utility function u_a which kernel is denoted v_a and there exists a measurable set $\Omega' \in \mathcal{A}$ of full measure,[3] satisfying the following conditions.

  (a) There exists $\beta \in \ell^q$ such that for each $a \in \Omega'$,

$$v_a^-(e(a)) \leqslant \beta.$$

  (b) There exists $k \in \mathbb{N}$ and $\alpha_k > 0$ such that $\omega_k \alpha_k > 0$ and for each $a \in \Omega'$,

$$0 < \alpha_k \leqslant \inf\{v_{a,k}^+(t) \colon t \geqslant 0\} = \lim_{t \to +\infty} v_{a,k}^+(t).$$

We can now state our second existence result for economies with preference relations represented by utility functions.

Theorem 3.8. *If the economy $\mathcal{E}$ is separably proper then there exists a competitive equilibrium.*

The two proper conditions are not comparable. Obviously, not all strong-uniformly proper economies are separably proper. Moreover, we provide hereafter an example of an economy which is separably proper but not strong-uniformly proper.

Example 3.9. Consider the economy $\mathcal{E}$ where $\Omega = [0,1]$, $\mathcal{A}$ is the Lebesgue σ-algebra and μ is the Lebesgue measure. For each trader $a \in [0,1]$, the consumption set coincide with ℓ_+^p, the utility function u_a is defined by

$$u_a(x) = x_0 + \sum_{n \geqslant 1} \frac{1 - \exp(-ax_n n^2)}{n^2}$$

and the initial endowment is defined by

$$e(a) = (1, 1/2, 1/3, \dots, 1/n, \dots).$$

Following Example 4.5 the economy $\mathcal{E}$ satisfies Assumptions 3.1–3.4. Moreover this economy is separably proper but not strong-uniformly proper.

4. Proper economies

We discuss in this section the notions of properness used in Theorem 3.3 and 3.8.

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[2]That is $x' \succ_a x$ if and only if $u_a(x') > u_a(x)$.
[3]That is $\mu(\Omega \setminus \Omega') = 0$.

4.1. **Strong-uniformly proper economies.** We recall that the preference relations $(\succeq)_a$ are said to be strong-uniformly proper, if there exist strictly positive prices α and β in ℓ_+^q with $\alpha \leqslant \beta$ and such that for every $a \in \Omega$, whenever $x, u, v \in \ell_+^p$ satisfy $v \leqslant x$ and $\langle \alpha, u \rangle > \langle \beta, v \rangle$ then $x - v + u \succ_a x$.

This properness condition is borrowed from Zame [23]. Note that this is a requirement on preferences that is uniform over agents as well as over consumption. We refer to Zame [23] for a discussion of this condition as well as for corresponding examples. Following Podczeck [20], it may be seen that if for each $a \in \Omega$, $\{y \in X(a) \colon y \succeq_a x\}$ is convex then uniform properness is equivalent to the following statement: *There are strictly positive prices $\alpha, \beta \in \ell_+^q$, such that given any $a \in \Omega$ and $x \in X(a)$ there is a price p in the order interval $[\alpha, \beta]$ such that $\langle p, x \rangle \leqslant \langle p, y \rangle$ for all $y \in X(a)$ with $y \succeq_a x$.* Since supporting prices are measures of marginal rates of substitution, the strong-uniform properness assumption is a condition that puts strong bounds on these rates.

We recall the notion of uniform properness introduced by Yannelis–Zame [22] for economies with finitely many agents.

Definition 4.1. The preference relations $(\succeq)_a$ are said to be v-**uniformly proper** with $v \in \ell^p$, if there exists a s-open 0-neighborhood $V \subset \ell^p$ such that for each $a \in \Omega$, for each $x \in \ell_+^p$, $(x + \Gamma) \cap \ell_+^p \subset P_a(x)$ where $\Gamma = \cup_{t > 0} t(v + V)$.

Remark 4.2. The strong-uniform properness assumption on the preference relations implies that

$$\forall a \in \Omega, \quad \forall x \in \ell_+^p, \quad (x + \Gamma) \cap \ell_+^p \subset P_a(x),$$

where Γ is the convex and s-open cone defined by $\Gamma = \{x \in \ell^p \colon \alpha(x^+) > \beta(x^-)\}$. It follows that the strong-uniform properness assumption is a particular case of the uniform properness assumption.

Example 4.3. Consider the case of positive separable utility functions $u_a \colon \ell_+^p \to \mathbb{R}$, defined by the formula $u_a(x) = \sum_n v_{a,n}(x_n)$ where for each n, the function $v_{a,n} \colon [0, +\infty) \to \mathbb{R}$ is continuous, the derivative $v_{a,n}'(t)$ exists for each $t > 0$. Suppose that there exist α and β two strictly positive functionals in ℓ^q such that

$$\forall a \in \Omega, \quad \forall t > 0, \quad \alpha_n \leqslant v_n'(t) \leqslant \beta_n.$$

Then the preference relations defined by the utility functions $(u_a)_{a \in \Omega}$ are strong-uniformly proper. Indeed, let $x, y, z \in \ell_+^p$ satisfying $y \leqslant x$ and $\alpha(z) > \beta(y)$. Using the mean value theorem, we see that for each n there exists $t_n > 0$ such that

$$v_{a,n}(x_n - y_n + z_n) - v_n(x) = v_{a,n}'(t_n)[z_n - y_n].$$

But $v_{a,n}'(t_n)[z_n - y_n] \geqslant \alpha_n z_n - \beta_n y_n$, in particular

$$u(x - y + z) - v(x) \geqslant \alpha(z) - \beta(y) > 0.$$

We refer to Araujo–Monteiro [2], Le Van [12] and Aliprantis [1] for precisions about proper conditions for separable utility functions.

4.2. **Separably proper economies.** Following Aliprantis [1], we introduce the following notion of separable utility function.

Definition 4.4. A separable utility function $u \colon \ell_+^p \to \mathbb{R}$, where $u(x) = \sum_n v_n(x_n)$, is said to be **rational** if for each $n \in \mathbb{N}$,

    (a) $v_n(0) = 0$;
    (b) v_n is positive, continuous and concave on $[0, +\infty)$; and

(c) v_n is differentiable on $(0, +\infty)$ with $v'_n(t) > 0$ for each $t > 0$.

Now let u be a rational separable utility function. The components of the lower and upper gradient sequences $\underline{v}' = (\underline{v}'_1, \underline{v}'_2, \dots)$ and $\overline{v}' = (\overline{v}'_1, \overline{v}'_2, \dots)$ are given by

$$\underline{v}'_n = \lim_{t \to +\infty} v'_n(t) \quad \text{and} \quad \overline{v}'_n = \lim_{t \to 0} v'_n(t).$$

Following Aliprantis [1, Theorem 6.7], we have the following result.

Proposition 4.1. *Let $u : \ell^p_+ \to \mathbb{R}$ be rational utility function given by $u(x) = \sum_n v_n(x_n)$. If the preference relations represented by u are ω-uniformly proper for some $\omega \in \ell^p$ strictly positive, then*

    (a) *the lower gradient $\underline{v}'$ is non-zero and belongs to ℓ^q_+; and*

    (b) *there exists some $k \in \mathbb{N}$ such that*

$$(0, 0, \dots, 0, \overline{v}'_k, \overline{v}'_{k+1}, \dots) \in \ell^q_+.$$

It follows that if $\mathcal{E}$ is an economy with rational separable utility function such that $\mathcal{E}$ is ω-uniformly proper and ω is striclty positive, then $\mathcal{E}$ is separably proper. We provide hereafter an example of a rational separable utility function which is separably proper but which is not uniformly proper.

Example 4.5. Consider the rational separable utility function $u : \ell^p_+ \to \mathbb{R}$ defined by

$$v_0(t) = t \quad \text{and} \quad \forall n \geqslant 1, \quad v_n(t) = \frac{1 - \exp(-tn^2)}{n^2}.$$

For each $n \geqslant 1$, $v'_n(t) = \exp(-n^2 t)$. It follows that

$$\underline{v}' = (1, 0, 0, \dots) \quad \text{and} \quad \overline{v}' = (1, 1, 1, \dots).$$

It follows that u is not uniformly proper. However if $e = (e_n)_n$ is defined by $e_0 = 1$ and for each $n \geqslant 1$ by $e_n = 1/n$ then

$$v'(e) = (e^{-n})_n \in \ell^p_+.$$

Hence u is separably proper.

5. Proof of Theorem 3.3 and Theorem 3.8

The space ℓ^q is norm-separable. Let $(p_i)_{i \in \mathbb{N}}$ be a norm-dense sequence in the closed unit ball of ℓ^q and define for each x, y in ℓ^p,

$$d(x, y) = \sum_{i \in \mathbb{N}} \frac{|\langle p_i, x - y \rangle|}{2^i}.$$

The topology defined by this distance coincide with the w^*-topology on norm-bounded subsets of ℓ^p. Moreover, the d-topology is separable and the Borel σ-algebra generated by d coincide with the Borel σ-algebra $\mathcal{B}$ generated by the norm-topology and the w^*-topology.

Let $\mathcal{E}$ be an economy satisfying Assumptions 3.1–3.4. Suppose that $\mathcal{E}$ is either strong-uniformly proper or separably proper. The correspondence X is graph measurable. Applying Theorem B.1, there exists a sequence $(f_k)_k$ of measurable selections of X such that for each $a \in \Omega$, $X(a) = s\text{-}\mathrm{cl}\{f_k(a) : k \in \mathbb{N}\}$. For every $k \in \mathbb{N}$, we let R_k be the correspondence from Ω into ℓ^p, defined by $R_k(a) = \{x \in \ell^p_+ : x \succeq_a f_k(a)\}$. For each $\nu \in \mathbb{N}$, we let $X_\nu : a \mapsto X_\nu(a) := X(a) \cap \nu B$ and $R_{k,\nu} : a \twoheadrightarrow R_{k,\nu}(a) := R_k(a) \cap \nu B$, where B is the closed unit ball in ℓ^p.

Claim 5.1. *There exists[4] a sequence $(\sigma^n)_n$ of measurable partitions $\sigma^n = (A_i^n)_{i \in S^n}$ of $(\Omega, \mathcal{A})$, and a sequence $(A^n)_n$ of finite sets $A^n = \{a_i^n \ : \ i \in S^n\}$ subordinated to the measurable partition σ^n, satisfying [5] for each $a \in \Omega$,*

(i)
$$\lim_n \|e^n(a) - e(a)\|_p = 0 \quad and \quad \forall k \in \mathbb{N}, \quad \lim_n \|f_k^n(a) - f_k(a)\|_p = 0;$$

(ii) *for each $\nu \in \mathbb{N}$, for each sequence $(x^n)_n$ of ℓ^p, d-converging to $x \in \ell^p$ and for every $k \in \mathbb{N}$,*
$$\lim_n d(x^n, R_{k,\nu}^n(a)) = d(x, R_{k,\nu}(a));$$

(iii) *if we pose $g(a) := \|e(a)\|_1$ then g is an integrable function satisfying*
$$\forall n \in \mathbb{N}, \quad \|e^n(a)\|_1 \leqslant 1 + g(a).$$

In particular, if we pose for each $n \in \mathbb{N}$, $\omega^n := \int_\Omega e^n$ then $\lim_n \|\omega^n - \omega\|_p = 0$.

Proof. If f is a function from Ω to ℓ^p, then we let $\{f(.)\}$ be the correspondence from Ω into ℓ^p defined for each $a \in \Omega$, by $\{f(.)\}(a) := \{f(a)\}$. Note that if f is measurable then f is Bochner integrable if and only if $\|f(.)\| : a \mapsto \|f(a)\|$ from Ω to $\mathbb{R}_+$ is integrable.

Let $Z := \ell_p \times \ell_p$ and consider the following distance δ on Z defined for each $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in Z by
$$\delta(x, y) := \|x_1 - y_1\|_p + d(x_2, y_2).$$

The metric space (Z, δ) is complete and separable. Let $\mathcal{G} := \{\|e(.)\|_1\}$ and[6]
$$\mathcal{F} := \{\{e(.)\} \times \{0\}, \{f_k(.)\} \times \{0\}, \{0\} \times R_k \ : \ k \in \mathbb{N}\}.$$

Now apply Theorem B.6. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We will now construct a sequence $(\mathcal{E}^n)_n$ of economies with finitely many consumers *converging* to $\mathcal{E}$. We let
$$\mathcal{E}^n := \left((I^n, 2^{I^n}, \sigma), \ell^p, X^n, \succ^n, e^n \right),$$

where $I^n = \{i \in S^n \colon \mu(A_i^n) \neq 0\}$ is the finite set of consumers; σ is the counting measure on I^n; for each agent $i \in I^n$, the consumption set is defined by $X_i^n := \mu(A_i^n)X(a_i^n) = \ell_+^p$, the initial endowment is defined by $e_i^n := \mu(A_i^n)e(a_i^n)$ and the preference relation is defined by $x' \succ_i^n x$ if and only if $(x'/\mu(A_i^n)) \succ_{a_i^n} (x/\mu(A_i^n))$. In particular, the correspondence of strictly prefered bundles P_i^n from X_i^n to X_i^n is defined by $P_i^n(x) = \mu(A_i^n)P_{a_i^n}(x/\mu(A_i^n))$, for each $x \in X_i^n$.

Claim 5.2. *There exists a feasible allocation $(x_i^n)_{i \in I^n}$ for the finite economy $\mathcal{E}^n$, a non-zero price p^n and a w^*-compact set $K \subset \ell^q$ such that*

(1) *for each $i \in I^n$, $\langle p^n, x_i^n \rangle = \langle p^n, e_i^n \rangle$, and $z \in P_i^n(x_i^n)$ implies $\langle p^n, z \rangle \geqslant \langle p^n, x_i^n \rangle$; and*

(2) *$p^n \in K$ with $\langle p^n, \omega \rangle = 1$.*

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[4]We refer to Appendix B for definitions and notations.

[5]Following notations of Section B.2, if f is function from Ω to ℓ^p, then for each n, $\{f(.)\}^n = \{f^n(.)\}$.

[6]If F and G are two correspondences from Ω to ℓ^p, then we let $F \times G$ be the correspondence from Ω to $\ell^p \times \ell^p$, defined for each $a \in \Omega$ by $(F \times G)(a) = F(a) \times G(a)$.

Proof. If the economy $\mathcal{E}$ is strong-uniformly proper then each economy $\mathcal{E}^n$ satisfies the assumptions of Theorem A.1. In particular if we let $K := \{q \in \ell^q : \langle q, \omega \rangle = 1 \text{ and } \langle q, \Gamma \rangle > 0\}$, where $\Gamma = \{x \in \ell^p : \langle \alpha, x^+ \rangle > \langle \beta, x^- \rangle\}$, then Claim 5.2 is proved.

If the economy $\mathcal{E}$ is separably proper then for each n large enough, $\alpha_k \omega_k^n > 0$. Hence the economy $\mathcal{E}^n$ satisfies the assumptions of Theorem A.3. In particular for each n, $\|p^n\|_q \leqslant (1/\alpha_k \omega_k^n) \|\beta\|_q$. Since $(\omega_k^n)_n$ is norm-convergent to ω^k if follows that the sequence $(p^n)_n$ lies in a norm-bounded set $K \subset \ell^q$. In particular, Claim 5.2 is proved. $\qquad\square$

If we denote by x^n and e^n the Bochner integrable mappings defined by

$$x^n := \sum_{i \in I^n} x_i^n \chi_{A_i^n} \quad \text{and} \quad e^n := \sum_{i \in I^n} e_i^n \chi_{A_i^n}$$

then

$$\text{(5.1)} \qquad \int_\Omega x^n d\mu = \int_\Omega e^n d\mu$$

$$\text{(5.2)} \quad \text{for a.e. } a \in \Omega, \quad \langle p^n, x^n(a) \rangle = \langle p^n, e^n(a) \rangle \quad \text{and} \quad z \in P_a^n(x^n(a)) \Rightarrow \langle p^n, z \rangle \geqslant \langle p^n, e^n(a) \rangle$$

$$\text{(5.3)} \qquad p^n \in K \quad \text{and} \quad \langle p^n, \omega \rangle = 1.$$

The set K is w^*-compact. Since ℓ^p is norm-separable, passing to a subsequence if necessary, we can suppose that $(p^n)_n$ converge to a non-zero price $p \in \ell^q$ which satisfies $\langle p, \omega \rangle = 1$.

For each $z \in \ell_+^p$, $\|z\|_p \leqslant \|z\|_1$; and for each $y \in \ell_+^p$, $\|z + y\|_1 = \|z\|_1 + \|y\|_1$. Hence from (5.1), we have

$$\int_\Omega \|x^n(a)\|_p \, d\mu(a) \leqslant \int_\Omega \|x^n(a)\|_1 \, d\mu(a) = \int_\Omega \|e^n(a)\|_1 \, d\mu(a).$$

Applying Claim 5.1, the sequence of Bochner integrable mappings $(x^n)_n$ is mean norm-bounded, i.e.

$$\sup_n \int_\Omega \|x^n(a)\|_p \, d\mu(a) < +\infty.$$

Passing to a subsequence if necessary, we can suppose that $w\text{-}\lim_n \int_\Omega x^n d\mu$ exists in ℓ^p. Applying Fatou's Lemma (Theorem C.1) of Cornet–Martins-da-Rocha [6], there exists a Bochner integrable mapping x from Ω to ℓ^p such that

$$\text{(5.4)} \qquad \int_\Omega x d\mu \leqslant w\text{-}\lim_n \int_\Omega x^n d\mu$$

$$\text{(5.5)} \qquad x(a) \in \overline{\text{co}} \, w\text{-}\text{ls}_n\{x_n(a)\} \quad \text{a.e.}$$

Claim 5.3. *The Bochner integrable mapping x satisfies $\int_\Omega x d\mu \leqslant \int_\Omega e d\mu$.*

Proof. This is a direct consequence of Claim 5.1, (5.1) and (5.4). $\qquad\square$

Claim 5.4. *For almost every $a \in \Omega$, $x(a) \in \ell_+^p$ and $z \in P_a(x(a))$ implies $\langle p, z \rangle \geqslant \langle p, e(a) \rangle$.*

Proof. Consider $\Omega_0 = \bigcup_{n \in \mathbb{N}} \Omega \setminus (\cup_{i \in I^n} A_i^n)$, then $\mu(\Omega_0) = 0$. Let Ω' be a measurable subset of $\Omega \setminus \Omega_0$ with $\mu(\Omega \setminus \Omega') = 0$ and such that all *almost everywhere* assumptions and properties are satisfied for each $a \in \Omega'$.

Since $X(a) = \ell_+^p$ is closed convex, we have that for each $a \in \Omega'$, $x(a) \in X(a)$. We will now prove that for each $a \in \Omega'$, if $z \in P_a(x(a))$ then $\langle p, z \rangle \geqslant \langle p, e(a) \rangle$. Let $a \in \Omega'$ and let $z \in P_a(x(a))$. Since $\ell_+^p = s\text{-}\mathrm{cl}\{f_k(a) : k \in \mathbb{N}\}$, we can suppose (extracting a subsequence if necessary) that $(f_k(a))_k$ is s-convergent to z. But $P_a(x(a))$ is s-open in ℓ_+^p, thus there exists $k_0 \in \mathbb{N}$, such that for each $k \geqslant k_0$, $f_k(a) \in P_a(x(a))$. To prove that $\langle p, z \rangle \geqslant \langle p, e(a) \rangle$, it is sufficient to prove that for each k large enough, $\langle p, f_k(a) \rangle \geqslant \langle p, e(a) \rangle$. Now, let $k \geqslant k_0$.

Claim 5.5. *There exists an increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ such that*

$$\forall n \in \mathbb{N}, \quad f_k^{\varphi(n)}(a) \in P_a^{\varphi(n)}\left(x^{\varphi(n)}(a)\right).$$

Proof. Suppose that for each increasing function $\varphi : \mathbb{N} \to \mathbb{N}$, there exists an increasing function $\phi : \mathbb{N} \to \mathbb{N}$, such that:

$$\forall n \in \mathbb{N}, \quad x^{\varphi \circ \phi(n)}(a) \in R_k^{\varphi \circ \phi(n)}(a).$$

Let $\ell \in w\text{-}\mathrm{ls}\{x^n(a) : n \in \mathbb{N}\}$, then there exists a subsequence $(x^{\varphi(n)}(a))_n$ w-converging to ℓ. In particular $(x^{\varphi \circ \phi(n)}(a))_n$ is w-convergent to ℓ. It follows that there exists $\nu > 0$ such that for each n, $x^{\varphi \circ \phi(n)}(a)$ belongs to νB. In particular, $d(x^{\varphi \circ \phi(n)}(a), R_{k,\nu}^{\varphi \circ \phi(n)}(a)) = 0$. Applying Claim 5.1, it follows that $d(\ell, R_{k,\nu}(a)) = 0$. Since $R_{k,\nu}(a)$ is w-closed and d coincide with w on νB, we have that $\ell \in R_k(a)$. Thus $w\text{-}\mathrm{ls}\{x^n(a)\} \subset R_k(a)$, and under Assumption 3.2, this implies that $\overline{\mathrm{co}}\, w\text{-}\mathrm{ls}_n\{x_n(a)\} \subset R_k(a)$. It follows that $x(a) \in R_k(a)$, i.e. $f_k(a) \notin P_a(x(a))$: contradiction. $\qquad\square$

With Claim 5.5 and (5.2), for each n, $\left\langle p^{\varphi(n)}, f_k^{\varphi(n)}(a) \right\rangle \geqslant \langle p^{\varphi(n)}, e^{\varphi(n)}(a) \rangle$. Passing to the limit, we get that $\langle p, f_k(a) \rangle \geqslant \langle p, e(a) \rangle$. $\qquad\square$

Now let $\bar{x}$ be the Bochner integrable mapping from Ω to ℓ^p defined by

$$\forall a \in \Omega, \quad \bar{x}(a) = x(a) + (1/\mu(\Omega)) \int_\Omega (e - x) d\mu.$$

Claim 5.6. *The pair $(\bar{x}, p)$ is an equilibrium of $\mathcal{E}$.*

Proof. Since $\int_\Omega (e - x) d\mu \geqslant 0$, Assumption 3.1 implies that $\bar{x}(a) \in \ell_+^p$ and $P_a(\bar{x}(a)) \subset P_a(x(a))$. In particular the allocation $\bar{x}$ is feasible and for each $a \in \Omega'$, if $z \in P_a(\bar{x}(a))$ then $\langle p, z \rangle \geqslant \langle p, e(a) \rangle$. Since $\succ_a$ is monotone, it follows that $\bar{x}(a)$ belongs to the norm-closure of $P_a(\bar{x}(a))$, in particular $\langle p, \bar{x}(a) \rangle \geqslant \langle p, e(a) \rangle$. But $\int_\Omega \bar{x} d\mu = \int_\Omega e d\mu$, it follows that for almost every $a \in \Omega$, $\langle p, \bar{x}(a) \rangle = \langle p, e(a) \rangle$. To prove that $(\bar{x}, p)$ is an equilibrium, it is now sufficient to prove that for almost every $a \in \Omega$,

$$\inf\{\langle p, z \rangle : z \in \ell_+^p\} < \langle p, e(a) \rangle.$$

Let $B := \{a \in \Omega' : \langle p, e(a) \rangle > 0\}$. The set B is measurable and since $\langle p, \omega \rangle = 1$, $\mu(B) \neq 0$. Now for each $a \in B$, $\inf\{\langle p, z \rangle : z \in \ell_+^p\} < \langle p, e(a) \rangle$ and $z \in P_a(\bar{x}(a)) \Rightarrow \langle p, z \rangle \geqslant \langle p, \bar{x}(a) \rangle$. It follows that

$$\forall a \in B, \quad z \in P_a(\bar{x}(a)) \Rightarrow \langle p, z \rangle > \langle p, \bar{x}(a) \rangle.$$

The preference relation $\succ_a$ is monotone, i.e. for each $z > 0$, $\bar{x}(a) + z \in P_a(\bar{x}(a))$. It follows that for each $z > 0$, $\langle p, z \rangle > 0$. Now from Assumption 3.1, for each $a \in \Omega$, $e(a) > 0$, hence

$$\inf\{\langle p, z \rangle : z \in \ell_+^p\} = 0 < \langle p, e(a) \rangle.$$

$\square$

APPENDIX A. FINITELY MANY CONSUMERS

We suppose in this section that the economy is finite in the sense that the set of consumers $(\Omega, \mathcal{A}, \mu)$ is $(I, 2^I, \sigma)$ where I is a finite set, 2^I is the σ-algebra of all subsets of I and σ is the counting measure.

A.1. **The general case.** If $\mathcal{E}$ is strong-uniformly proper then we denote by Γ the norm-open convex cone defined by $\Gamma = \{x \in \ell^p : \langle \alpha, x^+ \rangle > \langle \beta, x^- \rangle\}$.

Theorem A.1. *Let $\mathcal{E}$ be a finite economy satisfying Assumptions 3.1–3.4. If $\mathcal{E}$ is strong-uniformly proper then there exists a pair (x, p) consisting of a feasible allocation x and a non-zero price p such that*

(1) *for each $i \in I$, $\langle p, x_i \rangle = \langle p, e_i \rangle$, and $z \in P_i(x_i)$ implies $\langle p, z \rangle \geqslant \langle p, x_i \rangle$;*
(2) *$\langle p, \omega \rangle = 1$ and $\langle p, \Gamma \rangle > 0$.*

Proof. Since order intervals $[0, x] = \{y \in \ell^p : 0 \leqslant y \leqslant x\}$ are w-compact, following Florenzano [8], there exists a feasible allocation $x = (x_i)_i$ such that[7]

$$0 \notin G(x) := \mathrm{co} \bigcup_{i \in I} [P_i(x_i) - e_i].$$

Lemma A.2. $G(x) \cap -\Gamma = \emptyset$.

Proof. To see this,[8] assume by way of contradiction that $G(x) \cap -\Gamma \neq \emptyset$. Then there exist $\gamma \in \Gamma$, $(\lambda_i)_i$ with $\lambda_i \geqslant 0$, $\sum_i \lambda_i = 1$ and $(z_i)_i$ with $z_i \in P_i(x_i)$ such that

$$\sum_i \lambda_i z_i + \gamma = \sum_i \lambda_i e_i.$$

Suppose first that $\gamma \geqslant 0$. For each $i \in I$, we set $y_i := z_i + \gamma$. Then $y_i \succ_i z_i$ for each i since preference relations are strictly monotone, whence $y_i \succ_i x_i$ by transitivity. One the other hand,

$$\sum_i \lambda_i y_i = \sum_i \lambda_i e_i$$

and we have thus got a contradiction.

Thus suppose that $\gamma^- \neq 0$. We must have $\gamma^- \leqslant \sum_i \lambda_i z_i$, so by the Riesz decomposition theorem there exist elements $u_i \geqslant 0$ such that $u_i \leqslant z_i$ and $\sum_i \lambda_i u_i = \gamma^-$. Set for each i,

$$v_i = \frac{\langle \beta, u_i \rangle}{\langle \beta, \gamma^- \rangle} \gamma^+.$$

Since $\langle \alpha, \gamma^+ \rangle > \langle \beta, \gamma^- \rangle$ by definition of Γ,

$$\langle \alpha, v_i \rangle = \frac{\langle \beta, u_i \rangle}{\langle \beta, \gamma^- \rangle} \langle \alpha, \gamma^+ \rangle \geqslant \langle \beta, u_i \rangle,$$

---

[7]In fact x is an Edgeworth equilibrium of $\mathcal{E}$.
[8]The argument given in the sequel to establish this lemma is taken from Zame [23] and Podczeck [20].

with stirct inequality if $u_i \neq 0$. Hence because $u_i \leqslant z_i$ and $v_i \geqslant 0$, we have $z_i - u_i + v_i \succeq_i z_i$ for each i (in fact, $z_i - u_i + v_i \succ_i z_i$ in case $u_i \neq 0$), and therefore by transitivity of preference relations, $z_i - u_i + v_i \succ_i x_i$. Also

$$\sum_i \lambda_i(z_i - u_i + v_i) = \sum_i \lambda_i z_i - \gamma^- + \frac{\langle \beta, \sum_i \lambda_i u_i \rangle}{\langle \beta, \gamma^- \rangle} \gamma^+ = \sum_i e_i,$$

again we get a contradiction. $\qquad\square$

Following Lemma A.2, since Γ is s-open, it follows from the separation theorem that there exists some non-zero linear functional $p \in \ell^q$ with $\langle p, g \rangle \geqslant - \langle p, \gamma \rangle$ for each $g \in G(x)$ and $\gamma \in \Gamma$. It is now routine to prove that (x, p) satisfies properties (1) and (2) of Theorem A.1. $\qquad\square$

A.2. **The separable case.** We recall that a utility function $u : \ell^p_+ \to \mathbb{R}$ is called separable if there exists for each n, a function $v_n : [0, +\infty) \to \mathbb{R}$ concave and strticly increasing such that

$$\forall x \in \ell^p_+, \quad u(x) = \sum_{n \in \mathbb{N}} v_n(x_n).$$

Theorem A.3. *Let $\mathcal{E}$ be a finite economy satisfying Assumptions 3.1–3.4. If $\mathcal{E}$ is separably proper then there exists a pair (x, p) consisting of a feasible allocation x and a non-zero price p such that*

  (1) *for each $i \in I$, $\langle p, x_i \rangle = \langle p, e_i \rangle$, and $z \in P_i(x_i)$ implies $\langle p, z \rangle \geqslant \langle p, x_i \rangle$;*
  (2) *$\langle p, \omega \rangle = 1$ and $\|p\|_q \leqslant (1/\alpha_k \omega_k) \|\beta\|_q$.*

The proof of Theorem A.3 is mostly inspired by the proof of Theorem 3 in Araujo–Monteiro [3].

Proof. We prove Theorem A.3 in two steps. For the first step, we suppose that the economy satisfies an additional assumption on the initial endowments.

Step 1: Strictly positive initial endowments. Suppose that for each i, e_i is strictly positive. Let E^ω be the vector space of all $z \in \ell^p$ such that there exists $r > 0$ satisfying $-r\omega \leqslant z \leqslant r\omega$. From Lemma 1 in Araujo–Monteiro [3], there exists a pair (x, p) consisting of a feasible allocation[9] x and a non-zero linear functional $p : E^\omega \to \mathbb{R}$ such that p is positive, i.e. $\langle p, z \rangle \geqslant 0$ for each $z \in E^\omega_+$; $\langle p, \omega \rangle = 1$ and such that

$$\forall i \in I, \quad \langle p, x_i \rangle = \langle p, e_i \rangle \quad \text{and} \quad z \in P_i(x_i) \cap E^\omega \Rightarrow \langle p, z \rangle \geqslant \langle p, e_i \rangle.$$

Now there exists i with $\langle p, e_i \rangle > 0$, and since u_i is strictly monotone, p is strictly positive, i.e. $\langle p, z \rangle > 0$ for each $0 \neq z \in E^\omega_+$. In particular $\langle p, e_i \rangle > 0$ for each $i \in I$. By the *concave alternative* (see Lemma 5 in [3]), for each i there exists $\lambda_i > 0$ such that

(A.1) $$\forall z \in E^\omega_+, \quad u_i(z) - u_i(x_i) \leqslant \lambda_i \langle p, z - x_i \rangle.$$

For $z \in \ell^p_+$ we define $S(z) = \{h \in \ell^p : \exists t > 0, \ z + th \geqslant 0\}$ and $I(z) = S(z) \cap -S(z)$. Using (A.1) like in [3], we have

(A.2) $$\forall h \in I(x_i) \cap E^\omega_+, \quad \lambda_i \langle p, h \rangle \leqslant \sum_n v^-_{i,n}(x_{i,n}) h_n.$$

---

[9]Note that if x is a feasible allocation then $x_i \in E^\omega$.

Since e_i is striclty positive for each i, we have that $b := \inf\{e_i : i \in I\}$ is strictly positive. Hence E^b is norm-dense in ℓ^p. From this we conclude that if p is norm-continuous on E^b then we can extend it to a linear functional still noted p in ℓ^q, such that (x, p) satisfies

$$\forall i \in I, \quad \langle p, x_i \rangle = \langle p, e_i \rangle \quad \text{and} \quad z \in P_i(x_i) \Rightarrow \langle p, z \rangle \geqslant \langle p, e_i \rangle.$$

So let us prove that p is norm-continuous on E^b.

We define for each n, $I_n := \{i \in I : x_{i,n} \geqslant e_{i,n}\}$ and for each i we define $N_i = \{n \in \mathbb{N} : i = \min I_n\}$. Since x is a feasible allocation, we have $I_n \neq \emptyset$ for every n and $(N_i)_i$ is a partition of $\mathbb{N}$. Take $h \in E^b$, there exists $r > 0$ such that $-re_i \leqslant h \leqslant re_i$ for each i. Now let $h^i \in E^b$ be defined by

$$h^i = (h^i_n)_n \quad \text{where} \quad h^i_n = \left\{ \begin{array}{ll} |h_n| & \text{if} \quad n \in N_i \\ 0 & \text{if} \quad n \notin N_i. \end{array} \right.$$

As h^i belongs to $I(x_i)$, it follows from (A.2) that[10]

$$\lambda_i \langle p, h^i \rangle \leqslant \sum_{n \in N_i} v^-_{i,n}(x_{i,n})|h_n| \leqslant \sum_{n \in N_i} v^-_{i,n}(e_{i,n})|h_n|.$$

Since p is positive, we have that $|\langle p, h \rangle| \leqslant \sum_i \langle p, h^i \rangle$. It follows that

$$|\langle p, h \rangle| \leqslant \sum_i (1/\lambda_i) \sum_{n \in N_i} v^-_{i,n}(e_{i,n})|h_n|.$$

We define $u'_i(z) \cdot h = \lim_{r \to 0}(1/r)(u_i(z + rh) - u_i(z))$ for each $z \in \ell^p_+$ and each $h \in S(z)$. It follows from (A.1) and separable properness that

$$0 < \alpha_k \omega_k < u'_i(x_i) \cdot \omega \leqslant \lambda_i.$$

In particular

$$|\langle p, h \rangle| \leqslant (1/\alpha_k \omega_k) \sum_i \sum_{n \in N_i} v^-_{i,n}(e_{i,n})|h_n| \leqslant (1/\alpha_k \omega_k) \sum_{n \in \mathbb{N}} \beta_n |h_n| \leqslant (1/\alpha_k \omega_k) \|\beta\|_q \|h\|_p.$$

From separable properness we have that $\|\beta\|_q < +\infty$. This proves the norm-continuity of p on E^b.

Step 2: Positive initial endowments. Let $\mathcal{E}$ be a separably proper finite economy satisfying Assumptions 3.1–3.4. Let v be a strictly positive vector of ℓ^p and consider $\mathcal{E}^n$ the economy defined by $\mathcal{E} = (I, \ell^p, X, \succ, e^n)$ where $e^n_i := e_i + (1/n)v$. Since $v^-_i(e^n_i) \leqslant v^-_i(e_i)$, the economy $\mathcal{E}^n$ is separably proper and satisfies Asssumptions 3.1–3.4. Applying Step1, there exists a pair (x^n, p^n) consisting of a feasible allocation x^n and a non-zero price p^n such that for each $i \in I$, $\langle p, x^n_i \rangle = \langle p, e^n_i \rangle$, $z \in P_i(x^n_i)$ implies $\langle p^n, z \rangle \geqslant \langle p^n, x^n_i \rangle$, $\langle p^n, \omega^n \rangle = 1$ and $\|p^n\|_q \leqslant (1/\alpha_k \omega^n_k) \|\beta\|_q$.

Since the sequence $(\omega^n_k)_n$ is norm-convergent to ω_k, it follows that the sequence $(p^n)_n$ is norm-bounded, and passing to a subsequence if necessary, we can suppose that the sequence $(p^n)_n$ is w^*-convergent to a price $p \in \ell^q$ with $\|p\|_q \leqslant (1/\alpha_k \omega_k) \|\beta\|_q$. Moreover, since $\langle p^n, \omega^n \rangle = 1$ it follows that $\langle p, \omega \rangle = 1$.

For each i, x^n_i belongs to the interval $[0, \omega + v]$, in particular, passing to a subsequence if necessary, we can suppose that $(x^n_i)_n$ is w-convergent to $x_i \in \ell^p_+$. Moreover, since $\sum_i x^n_i = \omega^n$, we have that x is a feasible allocation for the economy $\mathcal{E}$. It is now routine to prove that for each $i \in I$, $\langle p, x_i \rangle = \langle p, e_i \rangle$, and $z \in P_i(x_i)$ implies $\langle p, z \rangle \geqslant \langle p, x_i \rangle$. $\qquad\square$

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[10]Note that $v^-_{i,n}$ is a decreasing function.

APPENDIX B. MEASURABLE CORRESPONDENCES

We consider $(\Omega, \mathcal{A}, \mu)$ a finite complete measure space and (D, d) a separable metric space. We recall that a function $f : \Omega \to D$ is *measurable* if for each open set $G \subset D$, $f^{-1}(G) \in \mathcal{A}$ where $f^{-1}(G) := \{a \in \Omega : f(a) \in G\}$. A correspondence $F : \Omega \twoheadrightarrow D$ is *graph measurable* if $G_F := \{(a, x) \in \Omega \times D : x \in F(a)\} \in \mathcal{A} \otimes \mathcal{B}(D)$, where $\mathcal{B}(D)$ is the σ-algebra of Borelian subsets of D.

B.1. **Measurable selections.** Following Aumann [5], graph measurable correspondences have measurable selections.

Theorem B.1. *Consider F a graph measurable correspondence from Ω into D with non-empty values. If (D, d) is complete then there exists a sequence $(z_n)_n$ of measurable selections of F, such that for each $a \in \Omega$, $(z_n(a))_n$ is d-dense in $F(a)$.*

B.2. **Discretization of measurable correspondences.**

Definition B.2. A partition $\sigma = (A_i)_{i \in I}$ of Ω is a *measurable partition* if for each $i \in I$, the set A_i is non-empty and belongs to $\mathcal{A}$. A finite subset A^σ of Ω is *subordinated to the partition* σ if there exists a family $(a_i)_{i \in I} \in \prod_{i \in I} A_i$ such that $A^\sigma = \{a_i : i \in I\}$.

Given a couple (σ, A^σ) where $\sigma = (A_i)_{i \in I}$ is a measurable partition of Ω, and $A^\sigma = \{a_i : i \in I\}$ is a finite set subordinated to σ, we consider $\phi(\sigma, A^\sigma)$ the application which maps each measurable function f to a simple measurable function $\phi(\sigma, A^\sigma)(f)$, defined by

$$\phi(\sigma, A^\sigma)(f) := \sum_{i \in I} f(a_i) \chi_{A_i},$$

where χ_{A_i} is the characteristic[11] function associated to A_i.

Definition B.3. A function $s : \Omega \to D$ is called *a simple function subordinated* to f if there exists a couple (σ, A^σ) where σ is a measurable partition of Ω, and A^σ is a finite set subordinated to σ, such that $s = \phi(\sigma, A^\sigma)(f)$.

Given a couple (σ, A^σ) where $\sigma = (A_i)_{i \in I}$ is a measurable partition of Ω, and $A^\sigma = \{a_i : i \in I\}$ is a finite set subordinated to σ, we consider $\psi(\sigma, A^\sigma)$, the application which maps each measurable correspondence $F : \Omega \twoheadrightarrow D$ to a simple measurable correspondence $\psi(\sigma, A^\sigma)(F)$, defined by

$$\psi(\sigma, A^\sigma)(F) := \sum_{i \in I} F(a_i) \chi_{A_i}.$$

Definition B.4. A correspondence $S : \Omega \to D$ is called a *simple correspondence subordinated* to a correspondence F if there exists a couple (σ, A^σ) where σ is a measurable partition of Ω, and A^σ is a finite set subordinated to σ, such that $S = \psi(\sigma, A^\sigma)(F)$.

Remark B.5. If f is a function from Ω to D, let $\{f\}$ be the correspondence from Ω into D, defined for each $a \in \Omega$ by $\{f\}(a) := \{f(a)\}$. We check that

$$\psi(\sigma, A^\sigma)(F) = \{\phi(\sigma, A^\sigma)(f)\}.$$

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[11]That is, for each $a \in \Omega$, $\chi_{A_i}(a) = 1$ if $a \in A_i$ and $\chi_{A_i}(a) = 0$ elsewhere.

The space of all non-empty subsets of D is noted $\mathcal{P}^*(D)$. We let τ_{W_d} be the Wisjman topology on $\mathcal{P}^*(D)$, that is the weak topology on $\mathcal{P}^*(D)$ generated by the family of distance functions $(d(x,.))_{x \in D}$.

Hereafter we assert that for a countable set of graph measurable correspondences, there exists a sequence of measurable partitions *approximating* each correspondence. The proof of the following theorem is given in Martins-da-Rocha [13].

Theorem B.6. *Let $\mathcal{F}$ be a countable set of graph measurable correspondences with non-empty values from Ω into D and let $\mathcal{G}$ be a finite set of integrable functions from Ω into $\mathbb{R}$. There exists a sequence $(\sigma^n)_n$ of finer and finer measurable partitions $\sigma^n = (A_i^n)_{i \in I^n}$ of Ω, satisfying the following properties.*

(a) *Let $(A^n)_n$ be a sequence of finite sets A^n subordinated to the measurable partition σ^n and let $F \in \mathcal{F}$. For each $n \in \mathbb{N}$, we define the simple correspondence $F^n := \psi(\sigma^n, A^n)(F)$ subordinated to F. Then for each $a \in \Omega$, $F(a)$ is the Wijsman limit of the sequence $(F^n(a))_n$, i.e.,*

$$\forall a \in \Omega, \quad \forall x \in D, \quad \lim_n d(x, F^n(a)) = d(x, F(a)).$$

(b) *There exists a sequence $(A^n)_n$ of finite sets A^n subordinated to the measurable partition σ^n, such that for each n, if we let $f^n := \phi(\sigma^n, A^n)(f)$ be the simple function subordinated to each $f \in \mathcal{G}$, then*

$$\forall f \in \mathcal{G}, \quad \forall a \in \Omega, \quad |f^n(a)| \leqslant 1 + \sum_{g \in \mathcal{G}} |g(a)|.$$

In particular, for each $f \in \mathcal{G}$,

$$\lim_{n \to \infty} \int_\Omega |f^n(a) - f(a)| d\mu(a) = 0.$$

Remark B.7. The property (a) implies in particular that, if $(x^n)_n$ is a sequence of D, d-converging to $x \in D$, then

$$\forall a \in \Omega, \quad \lim_n d(x^n, F^n(a)) = d(x, F(a)).$$

It follows that if F is non-empty closed valued, then property (a) implies that

$$\forall a \in \Omega, \quad \mathrm{ls}_n F^n(a) \subset F(a).$$

Appendix C. Fatou's Lemma

The proof of the following theorem is given in Cornet–Martins-da-Rocha [6].

Theorem C.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite positive complete measure space. Let $(f_n)_n$ be a sequence of Bochner integrable mappings from Ω to ℓ_+^p, which is mean norm-bounded, i.e.*

$$\sup_n \int_\Omega \|f_n(a)\|_p \, d\mu(a) < +\infty.$$

Suppose that $w\text{-}\lim_n \int_\Omega f_n d\mu$ exists in ℓ^p then there exists a Bochner integrable mapping f such that

$$\int_\Omega f d\mu \leqslant w\text{-}\lim_n \int_\Omega f_n d\mu$$

and

$$f(a) \in \overline{\mathrm{co}} \; w \text{-ls}_n \{f_n(a)\} \quad \text{a.e.}$$

Moreover

$$\int_\Omega \|f(a)\|_p \, d\mu(a) \leqslant \sup_n \int_\Omega \|f_n(a)\|_p \, d\mu(a).$$

REFERENCES

1. C.D. Aliprantis, *Separable utility functions*, Journal of Mathematical Economics **28** (1997), no. 4, 415–444.
2. A. Araujo and P.K. Monteiro, *Equilibrium and dynamics: Essays in honor of David Gale*, ch. General equilibrium with infinitely many goods: the case of separable utilities, Macmillan, London, 1989.
3. A. Araujo and P.K. Monteiro, *Equilibrium without uniform conditions*, Journal of Economic Theory **48** (1989), 416–427.
4. R. J. Aumann, *Existence of a competitive equilibrium in markets with a continuum of traders*, Econometrica **34** (1966), 1–17.
5. _____, *Measurable utility and measureable choice theorem*, La Décision Centre National de la Recherche Scientifique, Paris (1969), 15–26.
6. B. Cornet and V.F. Martins-da-Rocha, *Fatou's lemma for unbounded Gelfand integrable mappings*, Working Paper, Université Paris I, 2002.
7. J. Diestel and J. Uhl, *Vector measure*, Mathematical Surveys, 1977.
8. M. Florenzano, *On the non-emptiness of the core of a coalitional production economy without ordered preferences*, Journal of Mathematical Analysis and Application **141** (1989), 484–490.
9. W. Hildenbrand, *Existence of equilibria for economies with production and a measure space of consumers*, Econometrica **38** (1970), 608–623.
10. L.E. Jones, *Existence of equilibrium with infinitely many consumers and infinitely many commodities*, Journal of Mathematical Economics **12** (1983), 119–138.
11. M. A. Khan and N.C. Yannelis, *Equilibrium in markets with a continuum of agents and commodities*, Equilibrium Theory in Infinite Dimensional Spaces (M. A. Khan and N.C. Yannelis, eds.), Berlin Heildelberg New-York: Springer-Verlag, 1991.
12. C. Le Van, *Complete characterization of Yannelis–Zame and Chichilnisky–Kalman–Mas-Colell properness for separable concave functions defined in L_+^p and L^p*, Economic Theory **8** (1996), no. 1, 155–166.
13. V.F. Martins-da-Rocha, *Existence of equilibria for economies with a measurable space of agents and non-ordered preferences*, Working paper Université Paris 1, 2001.
14. _____, *Equilibria in large economies with a separable banach commodity space and non-ordered preferences*, Journal of Mathematical Economics **39** (2003), no. 8, 863–889.
15. _____, *Equilibria in large economies with differentiated commodities and non-ordered preferences and uniform properness*, Economic Theory **23** (2003), no. 3, 529–552.
16. A. Mas-Colell, *A model of equilibrium with differentiated commodities*, Journal of Mathematical Economics **2** (1975), 263–295.
17. J. Ostroy and W.R. Zame, *Nonatomic economies and the boundaries of perfect competition*, Econometrica **62** (1994), 593–633.
18. K. Podczeck, *Markets with infinitely many commodities and a continuum of agents with non-convex preferences*, Economic Theory **9** (1997), 585–629.
19. _____, *Note on the Core-Walras equivalence problem when the commodity space is a banach lattice*, Working Paper University of Vienna, 2001.
20. _____, *Core and Walrasian equilibria when agents' characteristics are extremely dispersed*, Economic Theory **22** (2003), no. 4, 699–725.
21. A. Rustichini and N.C. Yannelis, *Edgeworth's conjecture in economies with a continuum of agents and commodities*, Journal of Mathematical Economics **20** (1991), 307–326.
22. N.C. Yannelis and W.R. Zame, *Equilibria in banach lattices without ordered preferences*, Journal of Mathematical Economics **15** (1986), 75–110.
23. W.R. Zame, *Markets with a continuum of traders and infinitely many commodities*, Working paper, SUNY at Buffalo, 1986.

Instituto Nacional de Matemtica Pura e Aplicada - IMPA and EPGE/FGV, Estrada Dona Castorina 110, CEP 22460, Rio de Janeiro, RJ, Brazil
  E-mail address: aloisio@impa.br

Ceremade, Univerisité Paris-IX Dauphine, Place du Marchal de Lattre de Tassigny, 75775 Paris Cedex 16, France
  E-mail address: martins@ceremade.dauphine.fr

EPGE/FGV - Praia de Botafogo 190 - 10 andar Botáfogo, 22250-040 Rio de Janeiro, RJ, Brazil
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