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N° 400 ISSN 0104-8910

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Agosto de 2000

URL: http://hdl.handle.net/10438/905

Marques Menezes, Flavio
The Economics of Auctions Chapter 3: The Independent Private Values Model/ Flavio Marques Menezes, Paulo Klinger Monteiro - Rio de Janeiro : FGV,EPGE, 2010
(Ensaios Econômicos; 400)

Inclui bibliografia.

CDD-330

The Economics of Auctions
Chapter 3: The Independent Private Values Model

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August 2000

Chapter 3

The independent Private Values Model

A single object will be sold to one of n bidders. For the moment, assume that the seller's valuation for the object is equal to zero. Each bidder i, $i = 1, ..., n$, receives a signal v_i and her valuation is equal to $u_i(v_i) = v_i$. The implicit assumption here is that buyers are risk-neutral. That is, they are indifferent between a lottery that yields an expected value of x and receiving x for certain.

Each bidder knows her own valuation v_i and that her opponents' valuations are drawn independently from the distribution $F(\cdot)$ with density $f(\cdot) > 0$ in the interval $[0, \bar{v}]$. (Appendix 1 contains an introduction to probability theory.) That is, $F(x)$ denotes the probability that the random variable v is less than or equal to a certain number x.

This is the independent private values model where the value of the object to a bidder depends only on her own signal. Bidding behavior, however, depends on one's expectation about other bidders' valuations and on how they bid. Although the independent private value model is only appropriate to describe the case where the object does not have a resale value (or it is too costly to resell), it allows us to derive several important insights. For simplicity, we assume that the seller sets the reserve price at zero and that there are no entry fees.

In this chapter we will compute the equilibrium bidding strategies and the seller's expected revenue in four distinct types of auctions: first and second-price sealed-bid, English and Dutch auctions. As we have seen in Chapter 1, each bidder submits her bid without observing the bids made by other players in a sealed-bid auction. In a first-price auction, the winner is the bidder with the highest bid and she pays her bid. In a second-price auction the winner is still the bidder with the highest bid but she pays the second highest bid.

A naive commentator would argue that a first-price auction should generate more revenue than the second-price auction as the winner pays her bid in the former and the second highest bid in the latter. This argument fails because bidders behave strategically. We will show below that bidders bid less than their valuations in the unique equilibrium of a first-price auction and bid their valuation in the unique equilibrium of the second-price auction.

The oral English auction is perhaps the most popular auction format; the auctioneer announces a minimum opening bid and requests bids in an increasing fashion. The winner is the bidder with the highest standing bid. The English auction is analytically complex to model. We follow Milgrom and Weber and model it as a button auction: bidders participate by pressing a button. The price increases continuously in a clock. If a participant removes her finger from the button she drops out from the auction and cannot bid again. The winner is the last individual pressing the button and she pays the price at which the next-to-last bidder drop out from the auction. The dutch auction is a descending-price auction where the auction starts at a high price and declines continuously in a clock until one of the bidders stops the clock. This bidder is the winner and pays the price at which she stopped the clock.

3.1 First-price auctions

As we have seen in Chapter 2, we start our search for a symmetric Bayesian Nash equilibrium by analyzing the game from the point of view of one of the players, say Player 1. Suppose player 1 has a valuation v_1 and believes that other players follow a bidding strategy $b(\cdot)$. Knowing only her value and the distribution of the valuations of players $2, ..., n$, Player 1 has to figure out what is her best reply. Suppose bidder $i = 2, \ldots, n$ has valuation v_i. Thus bidder $i \geq 2$ bids $b_i = b(v_i)$. Then if Player 1 bid b_1 the object is won if $b_1 > b(v_i)$ for $i \geq 2$. If $b_1 < \max \{b(v_2), \ldots, b(v_n)\}$ Player 1 does not win the object. Let us suppose that in case of a drawn, $b_1 = \max \{b(v_2), \ldots, b(v_n)\}$ the object is not delivered. Thus Player 1's payoff is

$$\begin{cases} v_1 - b_1 & \text{if } b_1 > \max \{b(v_2), \ldots, b(v_n)\} \\ 0 & \text{if } b_1 \leq \max \{b(v_2), \ldots, b(v_n)\}. \end{cases}$$

The expected profits from bidding b_1 are given by

$$\pi_1(v_1, b_1, b(\cdot)) = (v_1 - b_1)\Pr(b_1 > \max\{b(v_2), \ldots, b(v_n)\})$$

We can rewrite the expression above as

$$\pi_1(v_1, b_1, b(\cdot)) = (v_1 - b_1)\Pr(b_1 > b(v_2), ..., b_1 > b(v_n))$$

For the moment assume that the function $b(\cdot)$ is strictly increasing and differentiable. (We will later verify that our equilibrium strategy is indeed increasing and differentiable in the domain and thus our analysis is justified). As $b(\cdot)$ is assumed to be increasing, we can apply the inverse function to both sides of the inequality between brackets without changing its sign:

$$\pi_1(v_1, b_1, b(\cdot)) = (v_1 - b_1)\Pr[b^{-1}(b_1) \geq v_2, ..., b^{-1}(b_1) \geq v_n] \tag{3.1}$$

Note that we are now able to write Player 1's expected profits as a function of the distribution of the valuations of players $2, ..., n$ as

$$\Pr[b^{-1}(b_1) \geq v_2] = \Pr[v_2 \leq b^{-1}(b_1)] = F\left(b^{-1}(b_1)\right)$$

Since the v_js are independent and identically distributed random variables, we can rewrite (??) as follows:

$$\pi_1(v_1, b_1, b(\cdot)) = (v_1 - b_1)F(b^{-1}(b_1))^{n-1}$$

Now player 1 chooses b_1 to maximize her expected profits. The first-order condition is obtained by taking derivative of the above expression with respect to b_1 and setting it equal to zero:

$$\frac{\partial \pi_1}{\partial b_1} = (v_1 - b_1)(n-1)f(b^{-1}(b_1))F(b^{-1}(b_1))^{n-2}(b^{-1}(b_1))' - \tag{3.2}$$

$$F(b^{-1}(b_1))^{n-1} = 0$$

Recall that we are searching for a symmetric equilibrium, that is, we have $b_1(\cdot) = b(\cdot)$. Thus

$$b^{-1}(b(v)) = v \tag{3.3}$$

The rule of derivative of the inverse function yields:

$$(b^{-1}(b_1))' = \frac{1}{b'(b^{-1}(b_1))} \tag{3.4}$$

Replacing (??) and (??) into (??) we obtain

$$\frac{(v - b(v))\,(n - 1) f(v) F(v)^{n-2}}{b'(v)} = F(v)^{n-1} \tag{3.5}$$

Next we rearrange (??) to obtain

$$\begin{gathered} v(n-1) f(v) F(v)^{n-2} = \\ b(v)(n-1) f(v) F(v)^{n-2} + b'(v) F(v)^{n-1} \end{gathered} \tag{3.6}$$

Note that

$$\left(b(v)F(v)^{n-1}\right)' = b'(v)F(v)^{n-1} + b(v)(n-1)f(v)F(v)^{n-2} \tag{3.7}$$

Replacing (??) into (??) we obtain:

$$\left(b(v)F(v)^{n-1}\right)' = v(n-1)f(v)F(v)^{n-2} \tag{3.8}$$

This differential equation can now be solved by simply integrating both sides of (??):

$$b(v)F(v)^{n-1} = \int_0^v x(n-1)f(x)F(x)^{n-2}dx + k$$

where k is the constant of integration. To be able to find the value of k we need to impose an initial condition. A natural condition is to require a bidder with a zero valuation to submit a bid equal to zero, that is, $b(0) = 0$. By doing so we obtain

$$0 = b(0)F(0)^{n-1} = k$$

That is, the candidate equilibrium bidding strategy is given by

$$b^*(v) = \frac{(n-1)\int_0^v xf(x)F(x)^{n-2}dx}{F(v)^{n-1}} \tag{3.9}$$

We check formally in an appendix to this chapter that $b^*(v)$ is indeed an equilibrium. Note that the interpretation of (??) is quite revealing. From appendix 1, we can conclude that the equilibrium bid of a player with value v is equal to the expected value of the individual with the second highest valuation conditional on v being the highest valuation. If my value v is the highest among all players, then in a symmetric equilibrium where strategies are increasing, it suffices to bid just to outbid the opponent with the second highest valuation.

Furthermore, the equilibrium bidding strategy in (??) is strictly increasing in v (the numerator increases with v and the denominator decreases with v) and differentiable so that our analysis is justified. It is also possible to integrate expression (??) by parts. Recall that the rule for integration by parts is:

$$\int_a^b u dz = uz\Big|_a^b - \int_a^b z du$$

Letting $z = F(x)^{n-1}$ implies that $dz = (n-1)F(x)^{n-2}f(x)$. Similarly, letting $du = dx$ implies (by integration) that $u = x$. Therefore

$$(n-1)\int_0^v xf(x)F(x)^{n-2}dx = \tag{3.10}$$

$$\int_0^v udz = xF(x)^{n-1}\Big|_0^v - \int_0^v F(x)^{n-1}dx =$$

$$vF(v)^{n-1} - \int_0^v F(x)^{n-1}dx$$

Replacing (??) into (??) we obtain

$$b^*(v) = v - \frac{\int_0^v F(x)^{n-1}dx}{F(v)^{n-1}} \tag{3.11}$$

It is clear from (??) that $b^*(v) < v$ for $v > 0$. The difference $v - b^*(v)$ indicates the amount of shading in equilibrium. Finally, one can infer from either (??) or (??) that $b(v) > 0$ and that $b(\cdot)$ is indeed differentiable and strictly increasing.

Now that we have a prediction for how bidders will behave in a first-price auction, it is possible to ask what is the expected revenue for the seller from a first-price auction, denoted by R^1. The expected revenue is simply the expected value of the highest bid, that is

$$R_1 = E[\max\{b^*(v_1), ..., b^*(v_n)\}] =$$

$$E[b^*(\max\{v_1, ..., v_n\})]$$

From the viewpoint of the seller, buyers are ex-ante identical. Thus, the probability that all valuations are below a given value v is simply $F(v)^n$ and its density is $nF(v)^{n-1}f(v)$. (See Appendix 1). As a result, the expected revenue can be written as:

$$R^1 = \int_0^{\bar{v}} nb^*(v)F(v)^{n-1}f(v)dv \tag{3.12}$$

In the remainder of this section we investigate individual behavior and compute the seller's expected revenue from a Dutch auction. We will need the following definition.

Definition 1 Two games with the same set of players and the same strategy space are said to be <u>strategically equivalent</u> if each player's expected profits under one of the games is identical to her expected profits in the other game.

A bidding strategy in a Dutch auction is a function $b(\cdot) : [0, \bar{v}] \to \mathbb{R}_+$. For example, consider the strategies profile $(b_1^*, ..., b_n^*)$. Suppose b_1^* is the highest bid. In a first-price auction, player 1 wins the object and her profits are $v_1 - b_1^*$, while the profits of all other players are equal to zero. In a Dutch auction, if player 1 is the one stopping the clock at price b_1^*, her profits are equal to $v_1 - b_1^*$, while the profits of all other players are equal to zero. Player 1, however, was chosen arbitrarily. The conclusion is that for any player with the highest bid, if the same profile of strategies is used in both auctions, this profile yields the same profits for all players. That is, the first-price auction and the Dutch auction are strategically equivalent. Thus, these two auction formats yield the same expected revenue given by (??).

3.2 Second-Price Auctions

In a second-price sealed-bid auction, players submit their bids simultaneously without observing the bids made by other players. We now explain Vickrey's

(1961) original insight that in such auction it is in a bidder's best interest to always bid her own valuation. We will need the following definitions:

Definition 2 A strategy $b_i \in [0, \bar{v}]$ is a <u>dominant strategy</u> for player i if

$$\pi_i(v_i, b_i, b_{-i}) \geq \pi_i(v_i, \hat{b}_i, b_{-i})$$

for all $b_i \in [0, \bar{v}]$, all $\hat{b}_i \in [0, \bar{v}]$ and for all $b_{-i} \in [0, \bar{v}]^{n-1}$, where $b_{-i} = (b_1, ..., b_{i-1}, b_{i+1}, ..., b_n)$.

In words, b_i is a dominant strategy for player i if it maximizes i's expected profits for any strategies of the other players. An equilibrium in dominant strategies is one where every bidder players her dominant strategy. Formally,

Definition 3 An outcome $(b_1^*, ..., b_n^*)$ is said to be an <u>equilibrium in dominant strategies</u> if b_i^* is a dominant strategy for each player i, $i = 1, ..., n$.

Next we are going to show that bidding one's true valuation is a dominant strategy equilibrium in a second-price auction. This is quite a remarkable fact. A player always bid their true valuation regardless of her beliefs about other players' strategies! First we explain intuitively why truth telling is a dominant strategy in a second-price auction and then we present a formal proof that $b(v) = v$ is in equilibrium bidding strategy.

Let's look at Bidder 1 who has valuation equal to v_1. Denote by $\hat{b}$ the highest bid among players $2, ..., n$. Assume first that Bidder 1 bids $b_1 < v_1$. If $b_1 > \hat{b}$ then Bidder 1 wins the object as she would have won with a bid equal to v_1. However, if $b_1 < \hat{b} < v_1$ then Bidder 1 loses the auction. By bidding her valuation she would have won the auction and earned expected profits equal to $v_1 - \hat{b}$. Therefore, Bidder 1 does not gain by bidding less than her valuation and could possibly lose. That is, her expected profits decrease with a bid $b_1 < v_1$.

Now suppose that Bidder 1 bids $b_1 > v_1$ If $b_1 < \hat{b}$, then Bidder 1 loses the auction as she would have lost if she had bid her valuation. However, if $v_1 < \hat{b} < b_1$, then Player 1 wins the object and pays more than her valuation. That is, she loses $\hat{b} - v$. Therefore, Bidder 1 does not gain by bidding more than her valuation but could possibly lose. Thus, her expected profits decrease with a bid $b_1 > v_1$.

We now show formally that telling the truth is an equilibrium bidding strategy. We examine the auction from the viewpoint of Bidder 1, who has a value equal to v_1, chooses a bid b_1 to maximize her expected profits given that players $2, ..., n$ follow some strategy $b(\cdot)$. Bidder 1's expected profits can be written as

$$\pi_1\left(v_1, b_1, b(.)\right) = E[(v_1 - b(z))\, I_{(b_1 \geq b(z))}] \tag{3.13}$$

where $I_{(b_1 \geq b(z))}$ denotes an indicator variable that is equal to 1 when $b_1 \geq b(z)$ and taking the value 0 otherwise. Moreover, we suppose that Bidder 1 assumes that she gets the object in case of a draw and we let z denote the highest valuation among players $2, ..., n$. That is, Bidder 1's expected profits is equal to the expected value of the difference between 1's valuation and the second highest bid for the case when 1's bid is greater than $b(z)$. The distribution function of the highest among $n-1$ samples is simply $F(x)^{n-1}$. (See Appendix 1). Therefore, we can take the expected value in (??) to obtain

$$\pi_1\left(v_1, b_1, b(\cdot)\right) = \int_0^{b^{-1}(b_1)} (n-1)\left(v_1 - b(x)\right) f(x) F(x)^{n-2} dx \tag{3.14}$$

Bidder 1's problem is to choose a b_1 to maximize (??). Since b_1 appears only in the upper limit of the integral, the derivative is obtained by replacing x with b_1 on the integrand as follows

$$\frac{\partial \pi_1}{\partial b_1} = (n-1)\left(v_1 - b_1)\right) f(v_1) F(v_1)^{n-2}. \tag{3.15}$$

If $b_1 < v_1$ then $\frac{\partial \pi_1}{\partial b_1} > 0$ and therefore π_1 is increasing when $b_1 < v_1$. Similarly, π_1 is decreasing increasing when $b_1 > v_1$ so that $\frac{\partial \pi_1}{\partial b_1} < 0$. Therefore, $b_1 = v_1$ maximizes π_1.

What is the expected revenue generated by the second-price auction? Given that each bidder bids her true valuation, the expected revenue is the expected value of the second highest valuation. From Appendix 1, the probability that a certain value v is one of the two highest valuations is $F(v)^n + nF(v)^{n-1}[1 - F(v)]$. The first term of the sum denotes the probability that v is the highest valuation and the second term of the sum presents the probability that v is the second highest value (there are n ways to choose the highest valuation, $F(v)^{n-1}$ represents the probability of $n-1$ valuations being smaller than que v, and $[1 - F(v)]$ denotes the probability of exactly one valuation being higher than v.) Therefore, the density is $n(n-1)F(v)^{n-2}[1 - F(v)]f(v)$ and the seller's expected revenue given by

$$R_2 = \int_0 n(n-1)vF(v)^{n-2}[1-F(v)]f(v)dv \tag{3.16}$$

Is it possible to analyze bidding behavior in oral English auctions? These auctions are very complex. For example, it is not uncommon for bidders to signal their bids by raising a hand or nodding to the auctioneer instead of calling out their bids. However, for analytical purposes we will refer to the following version (sometimes referred to as Japanese auctions): each bidder presses a button while the price increases continuously. A participant drops out when she takes her hand off the button. The auction ends when there is only one bidder left pressing the button. This bidder wins the auction and pays the price at which the next-to-last player stopped pressing the button. A strategy in this auction is a function from $[0, \bar{v}]$ into the nonnegative real numbers.

Consider a strategy profile $(b(v_1), ..., b(v_n)) = (v_1, ..., v_n)$. Suppose that $b(v_1)$ is the highest bid and that $b(v_2)$ is the second highest bid. In a second-price auction, Bidder 1 wins the auction and has profits equal to $v_1 - v_2$. Player $2, ..., n$ receive zero profits. In the oral auction – represented by the button auction – Bidder 1 is the last pressing the button, while Bidder 2 takes her hand off the button when the price reaches v_2. Bidder 1's profits are equal to $v_1 - v_2$, while bidders $2, ..., n$ earn zero profits. Note that the choice of players 1 and 2 was completely arbitrary. Thus, the same profile of strategies in both auctions yields the same profits for all players. That is, oral auctions and second-price auctions are strategically equivalent. The expected revenue generated by both is given by (??).

3.3 Revenue Equivalence

Among the four types of auctions considered above, first and second-price, Dutch and English auctions, which one generates the highest expected revenue for the seller? It turns out that with independent private values, these four auction formats generate the same expected revenue! This result is actually quite general as we will see in the next section and is a by-product of the revelation principle. A direct proof of the result below can be provided by just comparing expressions (??) and (??).

Theorem 1 (Revenue Equivalence) *With private independent values, the four auction formats analyzed, first and second-price, Dutch and Oral, yield the same expected revenue.*

Proof: $R^1 = \int_0 b^*(x) n F^{n-1}(x) f(x)\,dx =$

$$\int_0 \left(\frac{\int_0^x (n-1)\, y F(y)^{n-2} f(y)\,dy}{F(x)^{n-1}} \right) n F^{n-1}(x) f(x)\,dx =$$

$$\int_0 \left[\int_0^x n(n-1)\, y F(y)^{n-2} f(y)\,dy \right] f(x)\,dx.$$

Changing the order of integration in the last integral (Given that $0 < y < x$ and $0 < x < \bar{v}$, by changing the order we obtain we obtain $\int_y^{\bar{v}} f(x)\,dx = 1 - F(y)$.)

$$R^1 = n(n-1) \int_0 y\,(1 - F(y))\, F(y)^{n-2} f(y)\,dy = R^2.$$

∎

From inspection of (??) or (??), we have

Corollary 1 *The seller's expected revenue in any of the four auction formats increases with the number of participants.*

The Revenue Equivalence Theorem is really quite remarkable. In its general form it establishes that any auction that allocates the object to the bidder with the highest valuation (and satisfies a technical condition on assigning zero expected profits to the player with the lowest possible valuation) yields the same expected revenue. The astute reader, however, will point out that in the introduction we gave several examples of objects that are sold exclusively by oral auctions (e.g., houses, paintings, wool, etc.), objects that are sold by first-price (e.g., government purchases), objects that are sold exclusively by Dutch auctions (e.g., flowers) and that second-price auctions are extremely rare. The Revenue Equivalence Theorem would predict that the auction mechanism does not matter so we would expect to see flowers, for example, being sold by different auction formats.

One could argue that tradition plays an important role in the establishment of the auction format, but this argument is difficult to justify as in some cases these are new markets (such as auctions of used cars). Although we do observe changes in auction formats (for example, wool in Australia will be sold by electronic auctions) in some markets, there are several examples of little experimentation of other auction formats. This leads us to conclude that there may be other factors at work that are not captured by the independent private values model.

Indeed in the next chapters we will examine several extensions of the independent private values model where revenue equivalence breaks down. For example, this is the case when bidders are risk averse or when their valuations are correlated.

Although the revenue equivalence result is not robust, some of the insights developed above are robust and have being applied successfully to the design of several markets. In the next chapter we will pursue a more abstract approach and analyze the private independent values model under the realm of the revelation principle.

3.4 Exercises

1. Compute the equilibrium bidding strategy in both first and second-price auctions when the seller sets a reserve price equal to v_0. That is, the seller only accepts bids that are greater or equal to v_0. What is the seller's expected revenue in both auctions? Does revenue equivalence still hold?

2. (Riley and Samuelson, 1981): Consider an auction with two buyers with valuations drawn independently from the uniform [0,1] distribution. The seller sets a reserve price equal to 1/2 and she employs the following auction rules:

   (a) There is a single round of bidding. Buyer 1 is given the opportunity to quote a price $b_1 \geq 1/2$.

(b) If buyer 1 makes a bid, buyer 2 can match it, if he chooses, obtaining the good for this price. If buyer 1 makes no bid, buyer 2 can obtain the good at price 1/2 if he so chooses.

i. Does this auction resemble any selling mechanism that you know of?

ii. Can you compute buyers's equilibrium bidding strategies and the seller's expected revenue?

iii. Is the object in equilibrium always allocated to the individual with the highest valuation?

iv. Compare the expected revenue generated by this auction with the expected revenue generated by a second-price auction with reserve price equal to 1/2.

3. The reasoning leading to equation (??) is incomplete as b_1 may not be in the range of $b(\cdot)$. Reduce the general case to the case where $b_1 \in b([0, \bar{v}])$.

4. Compute the seller's expected revenue as the number of bidders goes to infinity?

5. In the text we assumed that in case of a tie the object is not delivered to any bidder. Show that the equilibrium strategies obtained above (for both first and second-price auctions) still hold under any tie breaking rule.

6. The proof that $b_1 = v_1$ is a dominant strategy equilibrium in a second-price auction, obtained from equality (??), did not use the first-order condition. $\frac{\partial \pi}{\partial b}(v_1) = 0$ but only the sign of $\frac{\partial \pi}{\partial b}(b_1)$ for $b_1 \neq v_1$. Give an example of a function that increases for values greater than v_1, decreases for values smaller than v_1 and is not differentiable at v_1. Do we need the expected profit function to be differentiable?