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A CAPM WITH HIGHER MOMENTS: THEORY AND ECONOMETRICS*

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Abstract

We develop portfolio choice theory taking into consideration the first $p \geq 2$ moments of the underlying assets distribution. A rigorous characterization of the opportunity set and of the efficient portfolios frontier is given, as well as of the solutions to the problem with a general utility function and short sales allowed. The extension of classical mean-variance properties, like two-fund separation, is also investigated. A general CAPM is derived, based on the theoretical foundations built, and its empirical consequences and testing are discussed.

* This is a preliminary version based on the presentation made at the 1997 Forecasting Financial Markets Conference, London, UK .

1. Introduction.

Since Markowitz (1952)'s pioneering contribution, several authors have tried to enlarge the portfolio selection model by including the consideration of higher order moments. An early attempt is Jean (1971), strongly opposed by Arditti and Levy (1972), among others. Both his answer and his tentative of improvement, Jean (1972, 1973), lack the appropriate mathematics to tackle the problem. The same can be said of many other contributions, even the interesting one by Ingersoll (1975). A main criticism to such proposals at the time consisted in exhibiting higher order utility functions, implied by the higher moments, which were economic meaningless. Moreover, during the seventies, the perception of the non-normality of many distributions in finance, and of the consequent importance of - at least - the coefficients of asymmetry and kurtosis, was not widespread. Notwithstanding, some had already the intuition that perhaps Markowitz's characterization of the distribution of returns was a bit too simplistic. The best example is probably Samuelson (1970)'s sophisticated argument, which tries to save the quadratic expected utility approximation - as a relatively good (limit) model for practical situations in which the distribution of returns belongs to a compact family - while droping the normality assumption.

However, more things contributed to the attractiveness of the mean-variance formulation: its elegant theoretical develoments and properties, like the separation results, and its CAPM extensions. In Kraus and Litzenberger (1976)'s paper, working with moments up to the third, the concern in trying to develop a theoretical framework as rich as the one of the mean-variance world is evident. Theirs is also the last important representative of this period.

In this paper we set the basis for a utility-based portfolio choice theory, taking into account the p, $p \geq 2$, first moments of the returns distribution. The following section develops the adequate notation and tools for characterizing the main objects, namely the opportunity set and the efficient portfolios frontier. Though the choice of notation has been made so as to keep the mathematics to a minimum, the efficient frontier is constructed under a general utility function, with short sales allowed but no riskless borrowing or lending. In section 3 the riskless asset is introduced, and a two-fund separation result is obtained. Section 4 establishes the techniques for asset pricing, proposes the equivalent of a single index model and tries to interpret the resulting beta(s). Section 5 outlines the econometrics required for testing the existence of moments of order $p \geq 3$ in the actual investment decision process. The paper concludes with some considerations on further developments and the practical consequences of the results already obtained.

2. The efficient frontier.

2.1. Notation.

The first issue in the generalization of the mean-variance approach is a matter of tractability, as the number of parameters raises exponentially. Given a n-dimensional vector of risky assets, while its variance is represented by a n-dimensional *square* matrix - and $n(n+1)/2$ different elements -, its asymmetry involves n^3 numbers, of which $n(n+1)(n+2)/6$ are in principle different.

We shall work with the first p centred moments (supposed of course finite) of the returns distribution of the n individual assets and of the possible portfolios. In statistics, these parameters, together with the cumulants, are easily treated by way of tensor calculus. Considering for instance the third moment of a n-dimensional random vector $r = (r_1, r_2, \ldots, r_n)'$ of returns, beyond the n marginal versions $E(r_i - Er_i)^3$, $i=1,\ldots,n$, several different combinations like

$$E\,(r_i-Er_i)^2(r_j-Er_j) \qquad \text{or} \qquad E\,(r_i-Er_i)(r_j-Er_j)\,(r_k-Er_k)\ ,\quad i,j,k=1,\ldots,n\ ,\quad i\neq j\neq k\ ,$$

figure in the total array of third moments. Without caring for the repetitions, all possible elements may be represented by the tensor τ_{ijk}, $i,j,k=1,\ldots,n$, which the reader should take as an abbreviated representation of the n^3 "third moments". Analogously, the fourth moment will be represented by τ_{ijkl}, $i,j,k,l=1,\ldots,n$, and so on.

A risky portfolio will be a linear combination of the n assets, with weights a_i, $i=1,\ldots,n$, summing up to one. This means that short sales are allowed in the classical sense.[1] If r_p denotes the return of a given risky portfolio, it is immediate to see that its central moments will be

$$Er_p = \sum_i a_i \tau_i ; E(r_p - Er_p)^2 = \sum_{i,j} a_i a_j \tau_{ij} \quad ; \quad E(r_p - Er_p)^3 = \sum_{i,j,k} a_i a_j a_k \tau_{ijk} \quad ;etc$$

However, we shall work differently. For each moment $p \geq 3$, we shall define a $n x n^{p-1}$ matrix M_p formed by the side juxtaposition of n matrices of order $n x n^{p-2}$. The i-th of these matrices is as follows: its j-th row contains all the n^{p-2} p-th moments in which the product of the i-th and j-th components are included together with all possible combinations of p-2 elements of the n indexes. These arrangements generate, for each row, entries in a given order, which will be specified below.

Let us also define the power of a column vector a with respect to the Kronecker product as

[1] And not in Lintner (1965)'s sense.

$$a^{\otimes k} = a \otimes a \otimes \ldots \otimes a \qquad \text{(k times, for } k \geq 2) \; ;$$

$$a^{\otimes 1} = a \qquad ; \qquad a^{\otimes 0} = 1 \qquad .$$

If $k=p-1$, this gives a vector of dimension n^{p-1} which can be looked at as the vertical stacking of n vectors resulting from the multiplication of n identical n^{p-2} vectors each time by a component of a. The order of the indexes of each element in the n^{p-2} vector is well defined, and is the order in which the rows of the n matrices making matrix M_p will be written.

With this notation, and calling $M_1 = Er$ and $M_2 = var(r)$, the centred moments of the portfolio can be rewritten as:

$$\sigma^1 = a'M_1 \; ; \; \sigma^2 = a'M_2\, a \; ; \; \sigma^3 = a'M_3\, a^{\otimes 2} \; ; \; \sigma^4 = a'M_4\, a^{\otimes 3} \; ; \; \ldots ; \; \sigma^p = a'M_p\, a^{\otimes p-1} \; .$$

It is easy to see that, taken as functions of a, each σ^i is a homogenous function of order i. Use of the Euler theorem gives a simple expression for its two first derivatives. These facts are summarized in

Proposition 1.

a) The p-th moment of the risky portfolio, $p \geq 1$, is a homogenous function of order p.

b) For every $p \geq 1$, the first and second derivatives of the p-th moment with respect to a are given by

$$\frac{\partial \sigma^p}{\partial a} = pM_p a^{\otimes p-1} \qquad , \qquad \frac{\partial^2 \sigma^p}{\partial a^2} = p(p-1)M_p a^{\otimes p-2} \otimes I_n$$

Proof. a) Immediate. b) Immediate with the aid of Euler's theorem. Recall that, by convention, M_1 is the vector with the expectations of the returns and M_2 their covariance matrix; and also that $a^{\otimes 1}=a$ and $a^{\otimes 0}=1$. The first derivative is homogenous of degree p-1 and, with the notation used, *a* must left multiply the expression of the second derivative for obtaining Euler's theorem. ☺

Throughout the paper the centred moments will be considered, without standardizing them by dividing by an appropriate power of the standard deviation, as it is sometimes done. However, use of the corresponding roots, i.e., $\sigma_p = (\sigma^p)^{1/p}$, can also be made as this renders all the quantities into the same unit.

We now define the opportunity set:

Definition 1. The opportunity set is a differentiable manifold $\mathbf{M}$ (or $\mathbf{M}^*$) of dimension min(n-1;p) , in p-moments space R^p, defined by

$$\mathbf{M} = \{ (\sigma^1(a), \sigma^2(a), \ldots, \sigma^p(a)) \in R^p \, ; \; a \in R^n \, , \; a'\mathbf{1} = 1 \}$$

$$(\text{or } \; \mathbf{M}^* = \{ (\sigma_1(a), \sigma_2(a), \ldots, \sigma_p(a)) \in R^p \, ; \; a \in R^n \, , \; a'\mathbf{1} = 1 \}) \; .$$

A few considerations are in line, as regards the definition. First, by assuming that the dimension is min(n-1,p) certain special cases are ruled out. Perhaps a most important one is when all (risky) expected returns are equal. In this instance, the condition $a'\mathbf{1} = 1$ implies that $\sigma^1(a)$ is constant for all portfolios and the opportunity set "loses" one dimension. Though this is not a problem for determining the efficient frontier, it can be troublesome when the riskless asset is introduced; so that, to avoid too many technicalities we rule out these cases in this paper. Secondly, there might also exist pathological structures in the moments of the returns that will put in check the differentiability condition. These are also ruled out. Finally, it should be noticed that, in the "usual cases", when assets outnumber the order of moments considered, the manifold will have the same dimension as moments space. In the event of "fewer assets", the manifold will have a smaller dimension until be equal to a surface in R^p, if only three assets are available.

The importance of Definition 1 is that it makes room for clearly understanding the nature of the efficient portfolios in our case. They are simply boundary points of $\mathbf{M}$. This is crucial because, with higher moments, the options for setting the conditions defining an optimal portfolio are multiple, and one can get lost trying to establish a standard procedure.[2] Indeed, the way one defines his optimal portfolio - or rather, "walks" on the boundary of $\mathbf{M}$ - has nothing to do with the set of *all* optimal portfolios.

We now turn to determine the efficient frontier.

2.2. The first-order conditions.

Assume that the agent has a utility function $U = U(m_1, m_2, \ldots, m_p)$ that takes into account the first p moments of the portfolio. U can be thought of as an expected utility function arriving from a Taylor development up to the p-th derivative, which is considered satisfactory. We propose to characterize the efficient frontier by :

Definition 2. The efficient frontier is a manifold

$$\mathbf{E} = \{ (\sigma^1, \sigma^2(a), \ldots, \sigma^p(a)) \in R^p \,;\; a \in R^n ,\; a'\mathbf{1} = 1 ,\, \sigma^1 > 0 \}$$

[2] This actually baffled some of the earlier contributors.

in R^p, where the $\sigma^2(a), \ldots, \sigma^p(a)$ are defined by the following programmes, "indexed" by the values σ^1:

[P1] $\max \ U(\sigma^1, \sigma^2(a), \ldots, \sigma^p(a))$

subject to $a'\mathbf{1} = 1$

$a' M_1 = \sigma^1$.

To solve these programmes, before taking derivatives, we need at least a sufficient condition ensuring the existence of the maximum. This is given by

Proposition 2.

Suppose that $U(\sigma^1, \sigma^2(a), \ldots, \sigma^p(a)) = U(a) = Eu$ is equal to the Taylor development of order p of the expected utility Eu, where $u: R^+ \to R$ is an utility function such that $u''(x)<0$ a.s. for all $x=a'r \in R^+$. *Then*, under regularity conditions for u, U(a) is concave.

Proof: The regularity conditions are supposed to allow to reverse the order between differentiation and integration.[3] If this is valid, one can successively write:

$$\frac{\partial^2 U}{\partial a' \partial a} = \frac{\partial}{\partial a'}\left(\frac{\partial Eu(x)}{\partial a}\right) = \frac{\partial}{\partial a'}\left(E\frac{\partial u(x)}{\partial a}\right) = E\frac{\partial}{\partial a'}\left(u'(x)\frac{\partial x}{\partial a}\right) = E\left(u''(x)\frac{\partial x}{\partial a}\cdot\frac{\partial x}{\partial a'}\right) \quad .$$

[3] They are similar to the ones needed in statistics to obtain the Cramér-Rao lower bound (see, for instance, Silvey(1970), chapter 2).

The symmetric matrices $\frac{\partial x}{\partial a} \cdot \frac{\partial x}{\partial a'}$ = r r' do not depend on a and are always positive definite. The hypothesis then implies that the random matrices between brackets at the right are negative definite a.s., and the result ensues. ☺

When the requirements of Proposition 2 are satisfied, it makes sense to write down the first-order conditions (f.o.c.). Calling U_i , $2 \le i \le p$, the derivative of U with respect to the i-th moment, use of Proposition 1 gives the system

$$\sum_{i=2}^{p} iU_i M_i a^{\otimes i-1} - \lambda_1 \vec{1} - \lambda_2 M_1 = 0$$

$$a'\vec{1} = 1 \qquad (1)$$

$$a' M_1 = \sigma^1$$

where λ_1 , λ_2 are (scalar) Lagrange multipliers.

The first thing to notice about system (1) is that, given the p moment tensors of the returns distribution and a specification for the U_i it is computationally feasible. Though non-linear, the above p+2 equations describe completely the unique vector of proportions a which, under the conditions of Proposition 2, solves the maximization problem. It must also be noted that, under these very conditions

$$U_i = \frac{\partial Eu}{\partial \sigma^i} = \frac{1}{i!} u^{(i)}(\sigma^1) \qquad (2)$$

so that it suffices to specify the function u(x).

With a bit of algebra, the efficient manifold can be implicitly described by the following two alternative equations:

$$\sigma^2 + \sum_{i=3}^{p} \frac{iU_i}{2U_2} \sigma^i = \alpha(\sigma^1)^2 + 2\beta\sigma^1 + \gamma + \left[A\sigma^1 + B\right] \sum_{i=3}^{p} \frac{iU_i}{2U_2} \left(\frac{1}{i} \frac{\partial \sigma^i}{\partial a}\right) \tag{3}$$

where α, β, γ are univariate functions of M_1 and M_2, and A, B are 1xn matrix functions of the same objects;

$$\sigma^2 + (a - v)' \sum_{i=3}^{p} k_i \frac{\partial \sigma^i}{\partial a} = k_1 \tag{4}$$

where the k_i, $3 \leq i \leq p$, are functions of σ^1, $k_1 = \alpha(\sigma^1)^2 + \beta\sigma^1 + \gamma$ is a function of σ^1 and the first two moments tensors and v is a nx1 matrix of first degree polynomials in σ^1 also involving the first two moments of the returns distribution.

The first formula shows that, if p=2, the two summations disappear and actually one ends up with the equation of a second degree curve

$$\sigma^2 = \alpha(\sigma^1)^2 + 2\beta\sigma^1 + \gamma \tag{5}$$

which includes the usual efficient frontier in the mean-variance case.[4]

Equation (4) presents a theoretical interest, allowing for the exploitation of general qualitative properties of the efficient manifold.

Both equations also point out to the relevance of special cases, of which the one in which the marginal elasticity of substitution between the variance and the moment of order i is constant is worth mentioning. It is easy to see that this amounts to:

$$\frac{iU_i\sigma^i}{2U_2\sigma^2} = -\eta_i \qquad or \qquad \frac{U_i\sigma^i}{U_2\sigma^2} = -\varepsilon_i \quad ; \tag{6}$$

[4] As known, the efficient frontier is one half of this parabola in the mean-variance plane.

meaning that the agent will trade (percentage) volatility changes for, e.g., kurtosis changes at a constant rate. Of course, the strict introduction of this requirement may impose unnecessary restrictions on the form of the utility function. In practical terms, however, it can be quite useful if assumed approximately valid for a "large set of utility levels of interest".

3. The riskless asset.

The existence of a riskless asset with a return of r_f changes the programme which determines optimal portfolios. It is now written as

[P2] max $U(\sigma^1, \sigma^2(a), \ldots, \sigma^p(a))$

subject to $a'M_1 + (1-a'1)\, r_f = \sigma^1$.

Assuming that the hypotheses of Proposition 2 are valid, the optimum exists and it makes sense to write down the f.o.c. . They reduce to:

$$\sum_{i=2}^{p} iU_i M_i a^{\otimes i-1} - \lambda(M_1 - r_f \vec{1}) = 0 \tag{7}$$

$$a' M_1 + (1 - a'\vec{1}) r_f = \sigma^1$$

Again, the above system can be put into a form that allows for finding a solution through computable methods:

$$a + \sum_{i=3}^{p} \frac{iU_i}{2U_2} \left[M_2^{-1} M_i a^{\otimes i-1} - \frac{1}{H^2} u' M_i a^{\otimes i-1} u \right] = \frac{\sigma^1 - r_f}{H^2} u \tag{8}$$

where $u = M_2^{-1}(M_1 - r_f \vec{1})$ and $H^2 = u' M_2 u = (M_1 - r_f \vec{1})' M_2^{-1} (M_1 - r_f \vec{1})$.

The quadratic form H^2 is simply the square of the norm of the vector of (expected) excess returns, with weights equal to the inverse of the covariance matrix - a most common metric. From (8) it is possible to see that if condition (6) holds, it suffices to find the solution of the system for the case

$$a'(M_1 - r_f \vec{1}) = \sigma^1 - r_f = 1 \quad . \tag{9}$$

Indeed, this is the object of

Proposition 3.

Call $\tilde{a}$ the solution of system (8) under restriction (9), *then* , if (6) is valid, for the general case $\sigma^1 = r_f + k$, the vector $a = k\tilde{a}$ will solve system (8).

Proof. Put $a = k\tilde{a}$ and $\sigma^1 - r_f = k$ in (8). We first show that the term under the summation is homogenous of degree one. For each i, multiply and divide the corresponding term by σ^i/σ^2. The expressions between brackets are homogenous of degree i-1 ; multiplication by the inverse of σ^i/σ^2 leaves one remaining k, as desired. Cancelling now k from both sides, the system is almost in the form that gives the ã solution. The only difference is again in the summation terms, where, for each i, the expression outside the brackets is:

$$\frac{iU_i\sigma^i}{2U_2\sigma^2} \times \frac{\sigma^2(\tilde{a})}{\sigma^i(\tilde{a})}$$

The first element of this product, by hypothesis, does not change if its values are replaced by the corresponding ones at point ã. But this means that the moments cancel each other and the remaining global system is exactly (8) when ã is the solution. ☺

This proposition provides a kind of two-fund separation result. Let Ã denote the sum of the elements of ã and consider the risky portfolio P* with weights $\frac{1}{\tilde{A}}\tilde{a}$. By virtue of Proposition 3, any optimal portfolio with given return $\sigma^1 = r_f + k$, is a convex combination of P* and the riskless asset with weights k.Ã and 1-kÃ , respectively. In Markowitz's world, with p=2, Ã=1 and P* is an efficient portfolio; however for arbitrary p, even under hypothesis (6), we are unable to ensure this property.[5]

The equation implicitly defining the manifold is somewhat simpler to find than in the previous section, and it can be written as

$$a'\frac{\partial\sigma^2}{\partial a} + (a - (\sigma^1 - r_f)u)'\sum_{i=3}^{p}\frac{U_i}{U_2}\frac{\partial\sigma^i}{\partial a} = \frac{(\sigma^1 - r_f)^2}{H^2} \qquad ; \qquad (10)$$

a more abstract version, similar to (4), is also easily obtained.

The solutions to systems like (8) or (10) can become quite involved, as consideration of the higher moments quickly introduces nonlinearities in the systems. To give an idea of this fact we deal now with a case in which p=3, but all the covariances between the n assets are zero and all asymmetries, but the (marginal) one relative to the first asset are also zero. This implies that matrix M_2 is diagonal and that the only non-zero cell in M_3 will be its upper left corner. Equation (8) will be reduced to:

[5] At least at the present stage.

$$a + \frac{3U_3}{2U_2}\left[\begin{bmatrix} a_1^2 \frac{\sigma_{111}}{\sigma_{11}} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} - \frac{1}{H^2} u' \begin{bmatrix} a_1^2 \sigma_{111} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} u\right] = \frac{\sigma^1 - r_f}{H^2} u \qquad , \qquad (11)$$

which, taking into account the expression for u, will give rise to the following system of n equations:

$$\frac{3U_3}{2U_2}\frac{\sigma_{111}}{\sigma_{11}}\left[1 - \frac{1}{H^2}\frac{(\sigma_1 - r_f)^2}{\sigma_{11}}\right] a_1^2 + a_1 - \frac{\sigma^1 - r_f}{H^2}\frac{\sigma_1 - r_f}{\sigma_{11}} = 0 \qquad (12)$$

$$a_i = \frac{1}{H^2}\left((\sigma^1 - r_f) + \frac{3U_3}{2U_2}\frac{\sigma_{111}}{\sigma_{11}} a_1^2 (\sigma_1 - r_f)\right)\frac{\sigma_i - r_f}{\sigma_{ii}} \qquad , \quad 2 \le i \le n \quad .$$

The first is a quadratic equation for a_1 , while the set of n-1 remaining equations gives each a_i as a function of a_1. It is important to notice that if the third moments were of no interest - what would be equivalent to set $\sigma_{111}=0$ - system (12) would give the precise solution to the mean-variance case, in which the partial derivatives of the utility function play no role at all. It is telling, in terms of the complexity that systems like (8) can achieve, how the introduction of a *single* asymmetry coefficient increases the difficulty of the problem. Moreover, two sets of a's may solve the problem and, if the quadratic equation in (12) has complex roots, there is no solution at all.

A final remark concerning the efficient frontier (Definition 2) is now in place. If for each programme [P1] or [P2] the solution is unique, i.e., functions $a = a(\sigma^1)$ and $a=a(\sigma^1, r_f)$, resp., can be defined, this implies that **E** will be one-dimensional - a line

- in case [P1] and bidimensional, or a surface, in case [P2]. This is an important consequence of our approach, in which consistency and generality in the treatment of the utility functions has been a major goal. Traders and practitioners in general would perhaps be more receptive to an approach based on preferences for the moments, which bears resemblance to earlier attempts like Ingersoll (1975)'s. In this situation, the utility function "disappears" and after fixing the value of any p-1 moments, one searches the portfolio which maximizes or minimizes the moment let free. If it is a variance, it is minized, if an expected return or a skewness, maximized, and so on. For each choice, a manifold of dimension $d=\min(n-1;p)-1$, contained in the boundary of the opportunity set **M** , would be generated. Further insights on these manifolds can be explored, however, contrary to the mean-variance case, their relationship with **E** - or rather, the relationship of their intersection[6] with **E** - is neither so perfect nor rigorously supported by a general utility approach.

4. The multidimensional CAPM.

Suppose now that a point is chosen in the efficient (risky) portfolios frontier. We know that this point corresponds to an (restricted) optimum of the utility function $U(\sigma^1, \sigma^2, ..., \sigma^p)$. Let σ_m^i , $1 \leq i \leq p$, denote the moments defining the coordinates of a particular such point: the market portfolio. If small changes are made in a neighbourhood of $(\sigma_m^1, \sigma_m^2, ..., \sigma_m^p)$, the condition to keep the utility constant is:

$$\sum_{i=1}^{p} U_i d\sigma_m{}^i = 0 \quad \text{or} \quad U_1 + \sum_{i=2}^{p} U_i \frac{d\sigma_m{}^i}{d\sigma_m{}^1} = 0 \ . \tag{13}$$

6 Which could be used as an alternative, *naive* definition of an efficient frontier.

We need then to evaluate the p-1 ratios of differentials which appear in the formula. For this, we shall follow Sharpe (1964)'s original idea and compute the differentials through ("infinitesimal") changes in the weights of the market portfolio. As two-fund separation applies in the set of risky efficient portfolios, when p=2, Sharpe's (differential) approximation is fully justified. In our case, we must warn, this approach is less generally valid.[7]

Within this context, consider now that a differential proportion $d\pi$ of the first risky asset[8] is added to the portfolio. The variation in expected return will be:

$$d\sigma_m^{\ 1} = (Er_1 - \sigma_m^{\ 1})\, d\pi \quad . \tag{14}$$

To find the differential forms giving the variation in the other moments, one can start with the fact that

$$E(\pi(r_1 - Er_1) + (1-\pi)(r_m - \sigma_m^1))^i = \sum_{j=1}^{i} \binom{j}{i} \pi^j (1-\pi)^{i-j} E[(r_1 - Er_1)^j (r_m - \sigma_m^1)^{i-j}] \quad ;$$

taking the derivative of the above expression w.r.t. π, and setting π=0 yields the desired form:

$$d\sigma_m^{\ i} = i\,(\mathrm{cov}(r_1, (r_m - \sigma_m^{\ 1})^{i-1}) - \sigma_m^{\ i})\, d\pi \quad . \tag{15}$$

Calling $r^*_m = r_m - \sigma_m^{\ 1}$, the combination of (14) and (15) yields:

$$d\sigma_m^{\ i} / d\sigma_m^{\ 1} = i\,(\mathrm{cov}(r_1, (r^*_m)^{i-1}) - \sigma_m^{\ i}) / (Er_1 - \sigma_m^{\ 1}) \quad . \tag{16}$$

If the chosen asset is the riskless asset, with return r_f, $\mathrm{cov}(r_f, (r^*_m)^{i-1}) = 0$, and (16) becomes:

$$d\sigma_m^{\ i} / d\sigma_m^{\ 1} = -\,i\,\sigma_m^{\ i} / (Er_f - \sigma_m^{\ 1}) \quad . \tag{17}$$

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[7] As not always there will be a market portfolio among the efficient points.

[8] The fact that it is the first risky asset is totally irrelevant, this is simply for convenience of notation.

By virtue of (13), multiplying (16) by U_i and summing the r.h.s. over i, for $i \geq 2$, gives the same result if the similar operation is performed with (17), so that one can write:

$$Er_1 - \sigma_m^1 = [\, \sum i\, U_i (cov(r_1, (r^*_m)^{i-1}) - \sigma_m^i) / \sum i\, U_i \sigma_m^i \,] (\sigma_m^1 - r_f) \qquad (18)$$

Dividing both members of the ratio between brackets by $2U_2\, \sigma_m^2$, and recalling from (6) the definition of η_i , $i \geq 3$, after a little algebra we get :

$$Er_1 - r_f = \left[\frac{\dfrac{cov(r_1, r_m^*)}{\sigma_m^2} - \sum_{i=3}^{p} \eta_i \dfrac{cov(r_1, r_m^{*i-1})}{\sigma_m^i}}{1 - \sum_{i=3}^{p} \eta_i} \right] \left(\sigma_m^1 - r_f \right) \qquad (19)$$

The previous equation is the expression of the asset pricing line for the multidimensional case. The important thing it shows is that, if $p>2$, the angular coefficient β^* of the line will be a linear combination of p-1 betas given by:

$$\beta_i = cov(r_1, (r^*_m)^{i-1}) / \sigma_m^i \quad , \quad i \geq 2 , \qquad (20)$$

associated to each of the moments considered.

This could perhaps be expected: if the objective is to relate the first moments of the (given) asset and of the market portfolio one would end up in principle with one coefficient[9]; the influence of the other moments must then be included in this single coefficient. Actually, formula (19) could have been obtained from equation (8), as shown in the Appendix. Combining both methods, the intuition of the CAPM in the multidimensional case becomes clear. Recalling the discussion at the end of the previous section, if in moments space R^p, given a pair (σ^1, r_f), one draws a straight line joining the

[9] Or, perhaps, two, if a different functional form would have been obtained.

point v of the corresponding efficient portfolio to the point (r_f, 0, 0, ..., 0), on the axis of first moments, this line will belong to the tangent plane to **E** at v.

5. Some econometrics.

Formula (19) may pose an identification problem for the testing of a higher dimensional CAPM if one works only with one asset beyond the market portfolio. However, supposing the separate β_i , $2\leq i\leq p$, known (they can be estimated through time series data), the η_i coefficients can be estimated via non-linear regression. Given a sample of T observations on the returns, and calling $Y_t = r_{1t} - r_f$, and $X_t = r_{mt} - r_f$, the η_i's will be a solution of:

$$\sum_t [Y_t - (\frac{\beta_2 - \sum_{i=3}^{p} \eta_i \beta_i}{1 - \sum_{i=3}^{p} \eta_i}) X_t]^2 \qquad (21)$$

For the case p=3, there is only one parameter, and the solution to (21) is:

$$\eta_3{}^* = (\Sigma_t (Y_t - \beta_2 X_t) X_t)^{-1} (\Sigma_t (Y_t - \beta_3 X_t) X_t) \quad . \qquad (22)$$

In the general case, checks on the precision of the estimators and the needed hypothesis tests can be conducted with the aid of the Gauss-Markov regressions[10].

[10] See Davidson and MacKinnon (1993).

Other approaches are also possible, if one works with a sample of m>p assets. In particular, the above estimation problem can be framed into the GMM technique. Calling now $Y_{jt} = r_{jt} - r_f$, and X_t as before; and evaluating, also as before, the β_{ij}, $2 \le i \le p$, for each asset j, the m moment conditions are:

$$E[Y_{jt} - \frac{\beta_{2j} - \sum_{i=3}^{p} \eta_i \beta_{ij}}{1 - \sum_{i=3}^{p} \eta_i} X_t] = 0 \qquad , 1 \le j \le m , \tag{23}$$

noticing that the η_i depend only on constant characteristics of the market portfolio. Hansen's J test can then be used in a specific to general strategy. Starting with p=3, one can test whether the overidentifying conditions are accepted. Rejection of the test would suggest the incorporation of the next order of moments.

Another possibility evaluates, for each asset, the corresponding β^*_j. The following coefficients can then be defined:

$$c_2 = 1 / (1+\eta_3+...+\eta_p) \quad , \quad c_i = \eta_i / (1+\eta_3+...+\eta_p) \quad , \ 3 \le i \le p \ . \tag{24}$$

The separate betas for each asset, β_{ij}, $2 \le i \le p$, $1 \le j \le n$, can, as said, be estimated through time series data (probably the same series used in computing β^*_j). The term inside brackets in (18), and the preceding definitions, imply that:

$$\beta^*_j = c_2 \beta_{2j} + c_3 \beta_{3j} + ... + c_p \beta_{pj} \ . \tag{25}$$

The variables in this equation, i.e., all the betas, are measured with error. Moreover, for each given portfolio j, *the estimators of* β^*_j and β_{2j} *coincide.* Using the symbol ' to indicate an estimator, and calling e_j, v_{2j}, ..., v_{pj}, the errors in β^*_j, β_{2j}, ..., β_{pj}, resp., the model to be tested is:

$$(1-c_2)\,\beta^*_j{}' - c_3\,\beta_{3j}{}' - \dots - c_p\,\beta_{pj}{}' = e_j + c_2\,v_{2j} + \dots + c_p\,v_{pj} = w_j \quad . \tag{26}$$

As the c's add up to one, (26) can be further simplified into:

$$c_3\,(\beta^*_j{}' - \beta_{3j}{}') + \dots + c_p\,(\beta^*_j{}' - \beta_{pj}{}') = w_j \quad . \tag{27}$$

Model (27) can be estimated through GMM. Taking the number of conditions to be equal to the number of parameters (p-2), p-3 instruments would be needed. These could be the first p-3 moments of each asset. The important null to be tested is:

$$H_0:\ c_3 = \dots = c_p = 0 \qquad ,$$

which, if valid, means that the agents take into account only the mean-variance pair.

Though all methods must still be checked with different data sets, a key point is already evident. A classical way of testing the influence of higher order moments has been to regress the asset's returns on powers of the returns of the market portfolio. The developments above show that this has no theoretical support within the context of an utility-based portfolio choice model.

6. Extensions and conclusions.

Several recent empirical findings in the finance literature have stressed not only the importance of the third moment, but also, and rather emphatically, that of the kurtosis. The theoretical problems raised by all these enlargements are multiple. Mean-variance, or quadratic utility or normal returns portfolio theory contains many nice and interesting results - most of them connected with the special geometry of the efficient frontier in mean-variance space - not necessarily valid within an enlarged set of moments.

Given the connection between second order stochastic analyses and linear models, various insightful shortcuts to the classical problem, as single index or multifactor models, could also be developed. A counterpart to these approaches, in the general framework, is not evident either.

This paper proposed a general way of tackling the portfolio choice model, and capital asset pricing, taking into consideration moments of the distribution of returns up to the p-th order. It is our view that the theoretical framework presented sets the right structure where the answers to the problems still open must be searched. From one side, a deeper insight on the possible shapes of the efficient frontier seems to be a mandatory research line. On the other hand - given the urgent needs of practitioners, and the many empirical extensions performed with sheer (and sometimes outrageous) disregard to theoretical consistency - empirical work based on developments related to section 4 is also seriously needed.

Appendix : Another derivation of formula (19) .

Formula (19) can be easily obtained from equation (8), which characterizes the optimal a, solution to programme [P2]. Supposing that $\sigma^1=\sigma_m^1$, i.e., that the optimal a defines the market portfolio, two equations can be derived from (8): one by right multiplying it by $a'M_2$ and another by right multiplying it by $e_j'M_2$, where e_j is the j-th element of the canonical basis in R^n . Noticing that

$$a'M_2 u = a' (M_1 - r_f \mathbf{1}) = \sigma_m^1 - r_f$$

$$e_j'M_2 u = e_j' (M_1 - r_f \mathbf{1}) = Er_j - r_f \quad ,$$

and that $$e_j'M_i a^{\otimes i-1} = cov(r_j, (r^*_m)^{i-1}) \quad ,$$

after a little algebra, subtracting one resulting equation from the other one arrives exactly at (19), with, of course, the j-th asset instead of the first one.

The importance of this derivation is that an approach like the one at the beginning of section 4, inspired in Sharpe's technique for the classical case, *is not necessary.* As said at the end of that section, the only needed hypothesis is the existence of a market portfolio which is efficient (in the sense of Definition 2). It also stresses the usefulness of formulae like (8) or (9), for obtaining further results.

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