FUNDAÇÃO
GETULIO VARGAS

EPGE
Escola de Pós-Graduação
em Economia

Ensaios Econômicos

Escola de

Pós Graduação

em Economia

da Fundação

Getulio Vargas

Two additions to Lucas´s "inflation and welfare"

Rubens Penha Cysne

Abril de 2004

Two Additions to Lucas's "Inflation and Welfare"[*][†]

Rubens Penha Cysne[‡]

April 3, 2004

Abstract

This work adds to Lucas (2000) by providing analytical solutions to two problems that are solved only numerically by the author. The first part uses a theorem in control theory (Arrow's sufficiency theorem) to provide sufficiency conditions to characterize the optimum in a shopping-time problem where the value function need not be concave. In the original paper the optimality of the first-order condition is characterized only by means of a numerical analysis. The second part of the paper provides a closed-form solution to the general-equilibrium expression of the welfare costs of inflation when the money demand is double logarithmic. This closed-form solution allows for the precise calculation of the difference between the general-equilibrium and Bailey's partial-equilibrium estimates of the welfare losses due to inflation. Again, in Lucas's original paper, the solution to the general-equilibrium-case underlying nonlinear differential equation is done only numerically, and the posterior assertion that the general-equilibrium welfare figures cannot be distinguished from those derived using Bailey's formula rely only on numerical simulations as well.

1   Introduction

This work is divided into two independent parts.

The value function derived in Lucas (2000, Section 5) need not be concave, leading this author to develop numerical analyses to determine the conditions under which the consumer utility is maximal. The first part of the paper uses control methods to derive sufficient conditions for the problem. Besides substituting for the numerical simulations, a second advantage of using control methods is that it does not require proving the differentiability of the value function (which is not done explicitly in the original paper and cannot be performed by standard methods in this nonconcave case) in order to obtain the associated first order conditions[1].

The second part of the work presents a closed-form solution to Lucas's general-equilibrium expression for the welfare costs of inflation in the double logarithmic case that allows for a direct (analytical) comparison with Bailey's (1956) partial-equilibrium measure. In the original paper, both the solution of the underlying nonlinear differential equation and the posterior assertion that the general-equilibrium solution cannot be practically distinguished from Bailey's partial-equilibrium figures are based only on numerical methods.

2   Sufficient conditions

The consumer is supposed to maximize utility from the consumption (c) :

$$\int_0^\infty e^{-gt}\ U(c)dt \tag{1}$$

subject to the households budget constraint (2) and to the transactions-technology constraint (3):

$$\dot{m} = 1 - (c + s) + h - \pi m \tag{2}$$

$$-c + m\phi(s)\ \geq 0 \tag{3}$$

In these equations, s stands for the fraction of the initial endowment spent as transacting time (the total endowment of time being equal to the unity), m for the real quantity of money, π for the rate of inflation, $U(c)$ for a strictly concave utility function, h for the (exogenous) real value of the flow of money transferred to the household by the government, and $g > 0$ for a

---

[1]On the other hand, the traditional derivation of the Euler equations requires interchanging two limits (entering the derivative operator inside the integral), a procedure that poses no problem when the integrand is concave, but requires an alternative justification in this nonconcave case.

discount factor. Following Lucas, in this Section (but not in the next) we particularize the transacting technology by making $\phi(s) = ks, k > 0$.

Since constraint (3) always hold with equality, we use it to substitute for c in the above equations and define the Hamiltonian:

$$H(m, s, \lambda) = U(kms) + \lambda(1 - (kms + s) + h - \pi m) \tag{4}$$

It is well known (Mangasarian , (1966)) that the Pontryagin's (1962) necessary conditions are sufficient for optimality if the Hamiltonian is jointly concave in the state and control variables. However, this condition is not fulfilled in the present case, due to the term sm in H. However, sufficient conditions can be generated with the use of Arrow's (1968) theorem[2]. In Seierstad and Sydsater's (1987, p. 107) version, the theorem reads as follows:

Lemma 1 *(Arrow's Sufficiency Theorem): Let $(\bar{x}(t), \bar{u}(t))$ be a pair that satisfies the conditions (7) and (6) below, in the problem of finding a piecewise-continuous control vector $u(t)$ and an associated continuously-differentiable state vector variable $x(t)$, defined on the time interval $[t_0, t_1]$, that maximizes:*

$$\int_{t_0}^{t_1} f_0(x(t), u(t), t)dt \tag{5}$$

subject to the differential equations:

$$\dot{x}_i(t) = f_i(x(t), u(t), t), \ i = 1, 2, ..., n \tag{6}$$

and to the conditions

$$
\begin{aligned}
x_i^0(t_0) &= x_i^0 \ , i = 1, 2, ..., n \\
x_i(t_1) \ free, \ i &= 1, ..., n \\
u(t) &\in U \subset R^r.
\end{aligned}
\tag{7}
$$

Given the Hamiltonian function

$$H(x(t), u(t), p(t), t) = p_0 f_0(x(t), u(t), t) + \sum_{i=1}^{n} p_i f_i(x, u, t)$$

---

[2]Seidseter and Sydersat (1977) argue (p. 370) that the first published demonstration of this theorem, which was presented in Arrow and Kurz (1970), is not satisfactory, and that a correct proof did not seem to be available in the literature till the publication of their work. This theorem was first mentioned in Arrow (1968).

if there exists a piecewise continuously-differentiable function $p(t) = (p_1(t), ..., p_n(t))$ defined on $[t_0, t_1]$ such that the following conditions are satisfied with $p_0 = 1$:

$$H(\bar{x}(t), \bar{u}(t), p(t), t) \geq H(\bar{x}(t), u(t), p(t), t), \; for \; all \; u \in U \; and \; all \; t. \; (8)$$

$$\dot{p}_i(t) = -H_{x^i}(\bar{x}(t), \bar{u}(t), p(t), t), \qquad i = 1, ..., n \qquad (9)$$

$$p_i(t_1) = 0 \qquad i = 1, ..., n \qquad (10)$$

$H^*(x, p(t), t) = \max\limits_{u \in U} H(x, u, p, t)$ exists and is a concave function of x for all t, then, $(\bar{x}(t), \bar{u}(t))$ solves problem (5)-(7) above.

In this theorem, the concavity of the maximized Hamiltonian with respect to the state variables substitutes for the concavity of the Hamiltonian which is required to hold both for the state *and* control variables in the theorem due to Mangasarian. Also, notice that this theorem is written for a finite horizon. As in Lucas (2000), we assume that $\lim_{t \to \infty} m(t) = \bar{m} \in \mathbb{R}$. It is also assumed that $\exists \bar{K} > 0 \mid U(c(s, m)) \leq \bar{K}, \forall (s, m)$. As pointed out by Seierstad and Sydsater's (1987, p. 231), in this special case the above theorem can be trivially extended to the infinite horizon case by reading $lim_{t \to \infty} p_i(t) = 0$, $i = 1, ..., n$ in (10).

In our original problem, s is the control variable (u in the theorem) and m the state variable (x). Relatively to the above theorem, $n = 1$, $U = [0, 1]$, $f_0(x, u, t) = U(c)$, $f_1(x, u, t) = 1 - (kms + s) + h - \pi m$, $[t_0, t_1] = [0, \infty]$. Given the way we wrote the Hamiltonian in (4), equation (9) must read $-\dot{\lambda}(t) + g\lambda = H_m(s, m)$ where, in (10), $i = 1$ and $e^{-gt}\lambda(t)$ substitutes for $p_1(t)$.

In order to derive the maximized Hamiltonian used in the theorem, we make the first derivative of (4), with respect to the control variable s, equal zero for

$$U(c) = c^{1-\sigma}/(1-\sigma), \sigma \neq 1, \sigma > 0$$
$$U(c) = \ln c \; (\text{ case } \sigma = 1)$$

which corresponds to the utility function used by Lucas. This leads, in both cases above, to:

$$s = \frac{1}{km}(\frac{km+1}{km}\lambda)^{-1/\sigma} \qquad (11)$$

where $\lambda = U'(c)km/(1 + km) > 0$. This value of s maximizes the Hamiltonian, since $H_{ss} = U''(c)k^2m^2 < 0$. Using (11) in (4), the maximized Hamiltonian is equal to:

$$H^*(m, \lambda) = \frac{\sigma}{1-\sigma}(\frac{km}{\lambda(1+km)})^{(1-\sigma)/\sigma} + \lambda(1+h-\pi m), \sigma \# 1 \qquad (12)$$

$$H^*(m, \lambda) = \log \frac{km}{\lambda(1+km)} + \lambda(1-1/\lambda+h-\pi m), \text{ (case } \sigma = 1)$$

The next step in the application of the theorem is showing that the maximized Hamiltonian is concave with respect to the state variable m. This is trivially satisfied in the case when $\sigma = 1$ since, given k and λ, $\log \frac{km}{\lambda(1+km)}$ is a composite function of two monotone increasing concave functions. When $\sigma \neq 1$, first notice that the derivative of (12) with respect to m is given by:

$$H_m^*(m, \lambda) = (\frac{km}{\lambda(1+km)})^{(1-\sigma)/\sigma} \frac{1}{m(1+km)} - \lambda\pi$$

Taking the derivative of the above expression, one easily concludes that $H_{mm}^*(m, \lambda) < 0$ iff:

$$\sigma > \frac{1}{2+2km} \qquad (13)$$

This expression can be used as a sufficiency condition. It allows an analytical characterization of the situations in which the absence of concavity is not a problem, adding to the original (numerical) solution. In this case H is strictly concave in m and the interior balanced path is unique.

Also, since $km > 0$, a sufficient condition that does not depend on m is given by

$$\sigma > 1/2$$

As one concludes from (13), lower values of σ may also imply $H_{mm}^*(m, \lambda) < 0$.

To characterize the (unique) stationary point $(\bar{m}, \bar{\lambda})$ as an optimum, it remains only noticing that the transversality condition (10) ($\lim_{t \to \infty} e^{-gt} \lambda(t) m(t) = 0$) is trivially satisfied under the assumed hypotheses.

In Lucas' paper, figures 9 and 10 are used to report numerical calculations designed to check if consumer utility is in fact maximized along the balanced path constructed from the first order conditions of the dynamic program. The numerical simulations are carried out for $k = 400$. From his numerical simulations, Lucas concludes that possible problems could only arise for values of $\sigma < 0.01$. Given the range of values assumed by m in the problem, this value is compatible with those given by (13).

3   A Closed-Form Solution for the Welfare Costs of Inflation

The presentations of Lucas' model in this Section draws on Simonsen and Cysne (2001). As in the first Section, the consumer is supposed to maximize (1) subject to (2) and (3).

In the steady-state solution, m converges to a constant figure, the rate of interest r equals the rate of inflation plus the discount factor $(r = \pi + g)$, the inflation rate equals the rate of monetary expansion and the real transfers (h) equal the inflation tax $(h = \sigma m, \sigma$ standing for the rate of monetary expansion).

In this case, intertemporal optimization leads to the equilibrium equation:

$$\phi\ (s) = rm\phi'(s) \tag{14}$$

and equilibrium in the goods market reads:

$$1 - s = m\phi\ (s) \tag{15}$$

Solving the system given by (14) and (15) for $s = s(r)$ and $m = m(r)$ yields implies $s'(r) > 0$ and $m'(r) < 0$. The problem of deriving $s(r)$ from $m(r)$, without knowing $\phi\ (s)$ is solved by eliminating $\phi\ (s)$ and $\phi'(s)$ using (14) and (15). The result is the differential equation [Lucas (2000, equation5.8)]:

$$s' = -\ \frac{r\,(1 - s)}{1 - s + r\ m}\ m' \tag{16}$$

which determines the welfare cost $s(r)$ as a function of the money-demand $m(r)$.

Lucas (2000) argues that the double-logarithmic functional specification fits the United States data better than the alternative semi-log specification. Consistently with this observation, we concentrate our analysis by making $m\,(r) = Ar^{-a}$, $0 < a < 1$, $A > 0$. In this case (16) leads to:

$$\frac{ds}{dr}\ =\ v(r,s) = -\frac{(1 - s)\,(aAr^{-a})}{1 - s + Ar^{1-a}} \tag{17}$$

$$s(r_0)\ =\ s_0,\ r_0 > 0 \tag{18}$$

Lucas does not provide a closed-form solution for this equation. His comparisons with Bailey's measures are based on numerical simulations.

We consider solutions for this equations for s and r in a closed, bounded and convex region $D \subset \mathbb{R}^2_{++}$. With r bounded away from zero, it is easy to see that $v(r, s) \in C^1$, and, therefore, by the mean-value theorem, and for a certain constant L, satisfies the Lipschtz condition $|f(r, s_1) - f(r, s_2)| \leq L |s_1 - s_2|$ for each par $(r, s_1), (r, s_2)$ in D. It follows from a standard result in ordinary differential equations (see, e.g., Coddington and Levinson (1955)) that there exists an interval containing r_0 such that a solution to (17) exists, and that this solution is unique. It is also easy to prove that such a solution can be continued to the right to a maximal interval of existence $[r_0, +\infty)$.

It by no means clear, though, that this non-separable, non-linear differential equation presents a closed-from solution. For example, it is well known that a simple equation like $\frac{ds}{dr} = w(r, s) = s^2 - r$ cannot be expressed as a finite combination of elementary functions or algebraic functions and integrals of such functions. We shall show that such a problem does not happen with (17).

- A Closed-Form Solution

Start by considering $r_0 > 0$ and the initial condition

$$s(r_0) = s_0 \tag{19}$$

Suppose $s(r)$ is a solution to (17), given (19). Then, since $s'(r_0) > 0$, by the implicit function theorem, the function $r = r(s)$ inverse to $s(r)$, is defined in a sufficiently small neighborhood of the point s_0 and:

$$\frac{dr}{ds} + \frac{-1}{a(1-s)} r = \frac{1}{aA} r^a \tag{20}$$

This type of equation is generally called a Bernoulli equation, which can be easily solved by an adequate change of coordinates.

Consider the diffeomorphism that associates with each $r > 0$, $t = r^{1-a}$. Then (20) is equivalent to the equation:

$$\frac{dt}{ds} - \frac{(1-a)}{a(1-s)} t = \frac{1-a}{aA}$$

Multiplying both sides of this equation by the integration factor $\exp(-\int \frac{1-a}{a(1-s)} ds)$ and taking into consideration that $s(0) = 0$:

$$t = \frac{a-1}{A}(1-s) + c(1-s)^{\frac{a-1}{a}}$$

Since $t = r^{1-a}$, we get the general-equilibrium expression for the welfare costs of inflation:

$$r = \left[\frac{a-1}{A}(1-s)\left[1 - (1-s)^{-1/a}\right]\right]^{\frac{1}{1-a}} \tag{21}$$

4 A Direct Comparison with Bailey's Measure

Lucas provides numerical simulations in order to compare his general-equilibrium measure and Bailey's partial-equilibrium measure (B) of the welfare costs of inflation. Having obtained a closed-form solution for the former allows us to provide a direct analytical comparison.

Bailey's measure is given by the area-under-the-inverse-demand-curve:

$$dB = -rm'(r)dr, \ B(0) = 0$$

By substituting the double-logarithmic money demand function into the above expression and integrating:

$$r = \left(\frac{B(1-a)}{aA} \right)^{\frac{1}{1-a}} \tag{22}$$

Solving (22) for B and using the value of r given by (21):

$$B - s = a(1-s)(-1 + (1-s)^{\frac{-1}{a}}) - s \tag{23}$$

Both Lucas (2000), through numerical simulations, and Simonsen and Cysne (2001), analytically, have shown that that Bailey's measure is an upper bound to Lucas' general-equilibrium measure, and that the difference between B and s in an increasing function of s. Both conclusions are consistent with equation (23). Indeed, make $B(s) - s = g(s)$. Then, $g(0) = 0$ and

$$g'(s) = \left[(1-s)^{-\frac{1}{a}} - 1 \right] (1-a)$$

Hence, $g\prime(s) > 0$ for any $s > 0$. It follows that $B > s$ for any strictly positive values of r. The fact that $B - s$ increases with s follows from the convexity of $g(s)$.

References

[1] Arrow, K. J. 1968, Applications of control theory to economic growth, in G. B. Dantzig and A. F. Veinott, Jr. eds., Mathematics of the decisions sciences (Providence, R. I. : American Mathematical Society).

[2] Arrow, K. and Kurz, 1970, Public investment, the rate of return and optimal fiscal policy, The Johns Hopkins Press.

[3] Bailey, Martin J., 1956, Welfare cost of inflationary finance, Journal of Political Economy 64, 93-110.

[4] Coddington, E. A. and N. Levinson, Theory of Ordinary Differential Equations, McGraw Hill, 1955.

[5] Lucas, R. E. Jr., 2000, Inflation and Welfare. Econometrica 68, No. 62 (March), 247-274.

[6] Mangasarian, O. L., 1966, Sufficient Conditions for the Optimal Control of Nonlinear Sytems, Siam Journal on Control, IV, February, 139-152.

[7] Pontryagin, L. S.; Boltyanskii, V.; Gamkrelidze, R.; and Mischenko, E. 1962, The mathematical theory of optimal process, New York and London, Interscience.

[8] Seierstad, A. and Sydsaeter K., 1977, Sufficient Conditions in Optimal Control Theory, International Economic Review, Vol. 18, Nº2, June.

[9] _____, 1987, Optimal control theory with economic applications, North Holland.

[10] Simonsen, Mario H. and Cysne, Rubens P. , 2001, Welfare costs of inflation and Interest-bearing deposits". Journal of Money Credit and Banking, February, 33-1, 90-101.

ENSAIOS ECONÔMICOS DA EPGE

511. ENDOGENOUS COLLATERAL - Aloísio Araújo; José Fajardo Barbachan; Mario R. Páscoa – Novembro de 2003 – 37 págs.

512. NON-MONOTONE INSURANCE CONTRACTS AND THEIR EMPIRICAL CONSEQUENCES - Aloísio Araujo; Humberto Moreira – Novembro de 2003 – 31 págs.

513. EQUILIBRIA IN SECURITY MARKETS WITH A CONTINUUM OF AGENTS - A. Araujo; V. F. Martins da Rocha; P. K. Monteiro – Novembro de 2003 – 17 pág's.

514. SPECULATIVE ATTACKS ON DEBTS AND OPTIMUM CURRENCY AREA: A WELFARE ANALYSIS - Aloisio Araujo; Márcia Leon – Novembro de 2003 – 50 págs.

515. O MÉTODO GENERALIZADO DOS MOMENTOS(MGM): CONCEITOS BÁSICOS - Renato G. Flôres Jr – Novembro de 2003 – 27 págs.

516. VARIÁVEIS INTRUMENTAIS E O MGM: USO DE MOMENTOS CONDICIONAIS - Renato G. Flôres Jr – Novembro de 2003 – 27 págs.

517. O VALOR DA MOEDA E A TEORIA DOS PREÇOS DOS ATIVOS - Fernando de Holanda Barbosa – Dezembro de 2003 – 17 págs.

518. EMPRESÁRIOS NANICOS, GARANTIAS E ACESSO À CRÉDITO - Marcelo Côrtes Néri; Fabiano da Silva Giovanini - Dezembro de 2003 – 23 págs.

519. DESENHO DE UM SISTEMA DE METAS SOCIAIS - Marcelo Côrtes Néri; Marcelo Xerez - Dezembro de 2003 – 24 págs.

520. A NEW INCIDENCE ANALYSIS OF BRAZILIAN SOCIAL POLICIES USING MULTIPLE DATA SOURCES - Marcelo Côrtes Néri - Dezembro de 2003 – 55 págs.

521. AN INTRA-HOUSEHOLD APPROACH TO THE WELFARE COSTS OF INFLATION - Rubens Penha Cysne – Janeiro de 2004 – 16 págs.

522. CENTRAL LIMIT THEOREM FOR ASYMMETRIC KERNEL FUNCTIONALS - Marcelo Fernandes; Paulo Klinger Monteiro – Fevereiro de 2004 – 23 págs.

523. THE TRADE-OFF BETWEEN INCENTIVES AND ENDOGENOUS RISK - Aloísio Araujo; Humberto Moreira; Marcos H. Tsuchida – Fevereiro de 2004 – 21 págs.

524. DO DIVIDENDS SIGNAL MORE EARNINGS ? - Aloísio Araujo; Humberto Moreira; Marcos H. Tsuchida – Fevereiro de 2004 – 26 págs.

525. Biased managers, organizational design, and incentive provision - Cristiano M. Costa; Daniel Ferreira; Humberto Moreira – Fevereiro de 2004 – 11 págs.

526. Land taxes in a Latin American context - Juliano J. Assunção; Humberto Moreira – Fevereiro de 2004 - 19 págs.

527. Indicadores coincidentes de atividade econômica e uma cronologia de recessões para o Brasil - Angelo J. Mont'alverne Duarte; João Victor Issler; Andrei Spacov - Fevereiro de 2004 – 41 págs.

528. TESTING UNIT ROOT BASED ON PARTIALLY ADAPTIVE ESTIMATION - Zhijie Xiao; Luiz Renato Lima – Março de 2004 – 27 págs.

529. DO SHOCKS PERMANENTLY CHANGE OUTPUT? LOCAL PERSISTENCY IN ECONOMIC TIME SERIES - Luiz Renato Lima; Zhijie Xiao – Março de 2004 – 21 págs.

530. A NEW PERSPECTIVE ON THE PPP HYPOTHESIS - Soyoung Kim; Luiz Renato Lima – Março de 2004 – 36 págs.

531. TRADE LIBERALIZATION AND INDUSTRIAL CONCENTRATION: EVIDENCE FROM BRAZIL - Pedro Cavalcanti Ferreira; Giovanni Facchini – Março de 2004 - 25 págs.

532. REGIONAL OR EDUCATIONAL DISPARITIES? A COUNTERFACTUAL EXERCISE - Angelo José Mont'Alverne; Pedro Cavalcanti Ferreira; Márcio Antônio Salvato – Março de 2004 – 25 págs.

533. INFLAÇÃO: INÉRCIA E DÉFICIT PÚBLICO – Fernando de Holanda Barbosa – Março de 2004 – 16 págs.

534. A INÉRCIA DA TAXA DE JUROS NA POLÍTICA MONETÁRIA – Fernando de Holanda Barbosa – Março de 2004 – 13 págs.

535. DEBT COMPOSITION AND EXCHANGE RATE BALANCE SHEET EFFECTS IN BRAZIL: A FIRM LEVEL ANALYSIS - Marco Bonomo; Betina Martins ; Rodrigo Pinto – Março de 2004 – 39 págs.

536. THE SET OF EQUILIBRIA OF FIRST-PRICE AUCTIONS - Paulo Klinger Monteiro – Março de 2004 – 6 págs.

537. OPTIMAL AUCTIONS WITH MULTIDIMENSIONAL TYPES AND THE DESIRABILITY OF EXCLUSION - Paulo Klinger Monteiro ; Benar Fux Svaiter; Frank H. Page Jr – Março de 2004 – 8 págs.

538. INCOME INEQUALITY IN A JOB-SEARCH MODEL WITH HETEROGENEOUS TIME PREFERENCES – Rubens Penha Cysne – Março de 2004 – 35 págs.

539. IMPOSTO INFLACIONÁRIO E TRANSFERÊNCIAS INFLACIONÁRIAS NO BRASIL: 1947-2003 - Rubens Penha Cysne; Paulo C. Coimbra-Lisboa – Março de 2004 – 7 págs.

540. ON THE STATISTICAL ESTIMATION OF DIFFUSION PROCESSES – A survey - Rubens Penha Cysne – Abril de 2004 – 26 págs.

541. ROBUSTNESS OF STATIONARY TESTS UNDER LONG-MEMORY ALTERNATIVES - Luiz Renato Lima; Zhijie Xiao – Abril de 2004 – 23 págs.

542. MONETARY UNION AND PRODUCTIVITY DIFFERENCES IN MERCOSUR COUNTRIES – Mariam Camarero; Renato G. Flôres, Jr.; Cecilio R. Tamarit – Abril de 2004 – 21 págs.

543. TWO ADDITIONS TO LUCAS´S "INFLATION AND WELFARE" - Rubens Penha Cysne – Abril de 2004 – 9 págs.

544. THE IMPLICATIONS OF EMBODIMENT AND PUTTY-CLAY TO ECONOMIC DEVELOPMENT - Samuel de Abreu Pessoa; Rafael Rob – Abril de 2004 – 70 págs.