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Estimating the Stochastic Discount Factor without a Utility Function

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1

Abstract

Using the Pricing Equation in a panel-data framework, we construct a novel consistent estimator of the stochastic discount factor (SDF) which relies on the fact that its logarithm is the serial-correlation "common feature" in every asset return of the economy. Our estimator is a simple function of asset returns, does not depend on any parametric function representing preferences, is suitable for testing different preference specifications or investigating intertemporal substitution puzzles, and can be a basis to construct an estimator of the risk-free rate.

For post-war data, our estimator is close to unity most of the time, yielding an average annual real discount rate of 2.46%. In formal testing, we cannot reject standard preference specifications used in the literature and estimates of the relative risk-aversion coefficient are between 1 and 2, and statistically equal to unity. Using our SDF estimator, we found little signs of the equity-premium puzzle for the U.S.

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1 Introduction

In this paper, we derive a novel consistent estimator of the stochastic discount factor (or pricing kernel) that takes seriously the consequences of the Pricing Equation established by Harrison and Kreps (1979), Hansen and Richard (1987), and Hansen and Jagannathan (1991), where asset prices today are a function of their expected future payoffs discounted by the stochastic discount factor (SDF). If the Pricing Equation is valid for all assets at all times, it can serve as a basis to construct an estimator of the SDF in a panel-data framework when the number of assets and time periods is sufficiently large. This is exactly the approach taken here.

We start with a general Taylor Expansion of the Pricing Equation to derive the determinants of the *logarithm* of returns once we impose the moment restriction implied by the Pricing Equation. The identification strategy employed to recover the *logarithm* of the SDF relies on one of its basic properties – it is the serial-correlation "common feature," in the sense of Engle and Kozicki (1993), in every asset return of the economy. Under mild restrictions on the behavior of asset returns, used frequently elsewhere, we show how to construct a consistent estimator for the SDF which is a simple function of the arithmetic and geometric averages of asset returns alone, and does not depend on any parametric function used to characterize preferences.

A major benefit of our approach is that we are able to study intertemporal asset pricing without the need to characterize preferences or use of consumption data; see a similar approach by Hansen and Jagannathan (1991) and Campbell (1993). This yields several advantages of our SDF estimator over possible alternatives. First, since it does not depend on any parametric assumptions about preferences, there is no risk of misspecification in choosing an inappropriate functional form for the estimation of the SDF. Moreover, our estimator can be used to test directly different parametric-preference specifications commonly used in finance and macroeconomics. Second, since it does not depend on consumption data, our estimator does not inherit the smoothness observed in previous consumption-based estimates which generated im-

3

portant puzzles in finance and in macroeconomics, such as excess smoothness (excess sensitivity) in consumption, the equity-premium puzzle, etc.; see Hansen and Singleton (1982, 1983, 1984), Mehra and Prescott (1985), Campbell (1987), Campbell and Deaton (1989), and Epstein and Zin (1991). Third, because of the close relationship between the SDF and the risk-free rate, a consistent estimator of the latter can be based on a consistent estimator of the former.

Our approach is related to research done in three different fields. From econometrics, it is related to the common-features literature after Engle and Kozicki (1993). Indeed, we attempt to bridge the gap between a large literature on serial-correlation common features applied to macroeconomics, e.g., Vahid and Engle (1993, 1997), Engle and Issler (1995), Issler and Vahid (2001, 2005), Vahid and Issler (2002), and Hecq, Palm and Urbain (2005), and the financial econometrics literature related to the SDF approach, perhaps best represented by the research on orthonormal polynomials of Chapman (1998), by the kernel estimation of the risk-neutral density of Aït-Sahalia and Lo (1998, 2000), by the work on empirical pricing kernels of Rosenberg and Engle (2002), and by the sieves estimator approach on preferences of Chen and Ludvigson (2004). It is also related respectively to work on common factors in macroeconomics and in finance; see Geweke (1977), Stock and Watson (1989, 1993) Forni et al. (2000), Bai and Ng (2004), and Boivin and Ng (2005) as examples of the former, and a large literature in finance perhaps best exemplified by Fama and French (1992, 1993) and Lettau and Ludvigson (2001) as examples of the latter. From macroeconomics, it is related to the work using panel data for testing optimal behavior in consumption, e.g., Runkle (1991), Blundell, Browning, and Meghir (1994), Attanasio and Browning (1995), Attanasio and Weber (1995), Meghir and Weber (1996), and to the recent work of Mulligan (2002, 2004) on cross-sectional aggregation, intertemporal substitution, and the volatility of the SDF.

The set of assumptions needed to derive our results are common to many papers in financial econometrics: the Pricing Equation is assumed in virtually all studies

4

estimating the SDF, and the restrictions we impose on the stochastic behavior of asset returns are fairly standard. What we see as non-standard in our approach is an attempt to bridge the gap between economic and econometric theory in devising an econometric estimator of a random process which has a straightforward economic interpretation: it is the common feature of asset returns. Once the estimation problem is put in these terms, it is straightforward to apply panel-data techniques to construct a consistent estimator for the SDF. By construction, it will not depend on any parametric function used to characterize preferences, which we see as a major benefit following the arguments in the seminal work of Hansen and Jagannathan (1991).

When applied to quarterly data of U.S.\$ real returns, from 1972:1 through 2002:4, using ultimately thousands of assets available to the average U.S. investor, our estimator of the SDF is close to unity most of the time and bound by the interval $[0.85, 1.15]$, with an equivalent average annual discount factor of 0.9760, or an average annual real discount rate of 2.46%. When we examined the appropriateness of different functional forms to represent preferences, we concluded that standard preference representations cannot be rejected by the data. Moreover, estimates of the relative risk-aversion coefficient are close to what can be expected *a priori* – between 1 and 2, statistically significant, and not different from unity in statistical tests. A direct test of the equity-premium puzzle using our SDF estimator cannot reject that the discounted equity premium in the U.S. has mean zero. If one takes the equity-premium puzzle to mean the need to have incredible parameter values either for the discount factor of future utility or the relative risk-aversion coefficient (or both) in order to achieve a mean-zero discounted equity premium in the U.S., then our results show little signs of the equity-premium puzzle.

The next Section presents basic theoretical results and our estimation techniques, discussing first consistency and then efficiency in estimation. Section 3 shows how to use our estimator to evaluate the Consumption-based Capital Asset-Pricing Model (CCAPM) with formal statistical methods. Section 4 presents empirical results using

the techniques proposed here, and Section 5 concludes.

2 Economic Theory and Our SDF Estimator

2.1 A Simple Consistent Estimator

Harrison and Kreps (1979), Hansen and Richard (1987), and Hansen and Jagannathan (1991) describe a general framework to asset pricing, associated to the stochastic discount factor (SDF), which relies on the Pricing Equation[1]:

$$\mathbb{E}_t \{M_{t+1} x_{i,t+1}\} = p_{i,t}, \qquad i = 1, 2, \dots, N, \text{ or} \tag{1}$$

$$\mathbb{E}_t \{M_{t+1} R_{i,t+1}\} = 1, \qquad i = 1, 2, \dots, N, \tag{2}$$

where $\mathbb{E}_t(\cdot)$ denotes the conditional expectation given the information available at time t, M_t is the stochastic discount factor, $p_{i,t}$ denotes the price of the i-th asset at time t, $x_{i,t+1}$ denotes the payoff of the i-th asset in $t+1$, $R_{i,t+1} = \frac{x_{i,t+1}}{p_{i,t}}$ denotes the gross return of the i-th asset in $t+1$, and N is the number of assets in the economy.

The existence of a SDF M_{t+1} that prices assets in (1) is obtained under very mild conditions. In particular, there is no need to assume a complete set of security markets. Uniqueness of M_{t+1}, however, requires the existence of complete markets. If markets are incomplete, i.e., if they do not span the entire set of contingencies, there will be an infinite number of stochastic discount factors M_{t+1} pricing all traded securities. Despite that, there will still exist a unique discount factor M_{t+1}^*, which is an element of the payoff space, pricing all traded securities. Moreover, any discount factor M_{t+1} can be decomposed as the sum of M_{t+1}^* and an error term orthogonal to payoffs, i.e., $M_{t+1} = M_{t+1}^* + \nu_{t+1}$, where $\mathbb{E}_t(\nu_{t+1} x_{i,t+1}) = 0$. The important fact here is that the pricing implications of any M_{t+1} are the same as those of M_{t+1}^*, also known as the mimicking portfolio.

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[1]See also Rubinstein(1976) and Ross(1978).

We now state the four basic assumptions needed to construct our estimator:

Assumption 1: The Pricing Equation (2) holds.

Assumption 2: The stochastic discount factor obeys $M_t > 0$. The same holds for the mimicking portfolio M_t^*.

Assumption 3: There exists a risk-free rate, labelled R_{t+1}^f, which is measurable with respect to the sigma-algebra generated by the conditioning set used in computing conditional moments $\mathbb{E}_t(\cdot)$.

Assumption 4: Let $\mathbf{R}_t = (R_{1,t}, R_{2,t}, ... R_{N,t})'$ be an $N \times 1$ vector stacking all asset returns in the economy. Then, the vector process $\{\ln(M_t\mathbf{R}_t)\}$ is assumed to be covariance-stationary with finite first and second moments. Let $\varepsilon_{i,t} = \ln(M_tR_{i,t}) - \mathbb{E}_{t-1}\{\ln(M_tR_{i,t})\}$ denote the innovation in predicting $\ln(M_tR_{i,t})$. Then, we further assume that $\lim_{N \to \infty} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} |\mathbb{E}(\varepsilon_{i,t}\varepsilon_{j,t})| = 0$.

Assumption 1 is present, either implicitly or explicitly, in virtually all studies in finance and macroeconomics dealing with asset pricing and intertemporal substitution; see, e.g., Hansen and Singleton (1982, 1983, 1984), Mehra and Prescott (1985), Epstein and Zin (1991), Fama and French (1992, 1993), Attanasio and Browning (1995), Lettau and Ludvigson (2001) and Mulligan (2002). The Pricing Equation (2) is essentially equivalent to the "law of one price" – where securities with identical payoffs in all states of the world must have the same price. Although its validity implies mild restrictions on preferences as noted by Cochrane (2001), it does not imply any parametric choice for preferences.

Assumption 2 is required because we will take logs of M_t in proving our asymptotic results. $M_t > 0$ implies that there are no-arbitrage opportunities, which is slightly stronger than the law of one price associated with Assumption 1. All CCAPM studies implicitly assume $M_t > 0$, since $M_t = \beta \frac{u'(c_t)}{u'(c_{t-1})} > 0$, where c_t is consumption, $\beta \in (0,1)$ and $u'(\cdot) > 0$.

Assumption 3 is necessary wherever there is a discussion about the equity-premium puzzle; see, *inter alia*, Hansen and Singleton (1982, 1983, 1984), and Mehra and Prescott (1985). The existence of a risk-free rate also augments the payoff space, since there will be one asset that pays the same amount in every state of nature.

Assumption 4 controls the degree of time-series and cross-sectional dependence in the data. In empirical work, asset returns have clear signs of conditional heteroskedasticity; see Bollerslev, Engle and Wooldridge (1988), Engle, Ito and Lin (1990), Harvey, Ruiz and Shepard (1994), and Engle and Marcucci (2005). Of course, weak stationary processes can display conditional heteroskedasticity as long as second moments are finite; see Engle (1982) and Bollerslev (1986). Therefore, Assumption 4 allows for conditional heteroskedasticity in all returns used in computing our estimator, being an important feature of our assumptions. The condition $\lim_{N\to\infty} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} |\mathbb{E}\left(\varepsilon_{i,t}\varepsilon_{j,t}\right)| = 0$ simply controls the degree of cross-sectional dependence present in the data. It guarantees convergence in probability of cross-sectional means. Similar conditions, although in a less restrictive format, can be found in Bai and Ng (2004). One important case where this condition is violated is when the error terms have a common component of the form $\varepsilon_{i,t} = \xi_t + \upsilon_{i,t}$, where we assume that ξ_t is orthogonal to $\upsilon_{i,t}$ at all leads and lags. Violation happens also under perfect correlation among errors, although this case is less realistic.

To construct a consistent estimator for $\{M_t\}$ we consider a second-order Taylor Expansion of the exponential function around x, with increment h, as follows:

$$e^{x+h} = e^x + he^x + \frac{h^2 e^{x+\lambda(h)\cdot h}}{2}, \tag{3}$$

$$\text{with } \lambda(h) : \mathbb{R} \to (0,1). \tag{4}$$

For the expansion of a generic function, $\lambda(\cdot)$ would depend on x and h. However, dividing (3) by e^x:

$$e^h = 1 + h + \frac{h^2 e^{\lambda(h)\cdot h}}{2}, \tag{5}$$

shows that (5) does not depend on x. Therefore, we get a closed-form solution for $\lambda(\cdot)$ as function of h alone:

$$\lambda(h) = \begin{cases} \frac{1}{h} \times \ln\left[\frac{2 \times \left(e^h - 1 - h\right)}{h^2}\right], & h \neq 0 \\ \\ 1/3, & h = 0, \end{cases}$$

where $\lambda(\cdot)$ maps from the real line into $(0, 1)$. To connect (5) with the Pricing Equation (2), we impose $h = \ln(M_t R_{i,t})$ in (5) to obtain:

$$M_t R_{i,t} = 1 + \ln(M_t R_{i,t}) + \frac{[\ln(M_t R_{i,t})]^2 \, e^{\lambda(\ln(M_t R_{i,t})) \cdot \ln(M_t R_{i,t})}}{2}, \tag{6}$$

which shows that the behavior of $M_t R_{i,t}$ will be governed solely by that of $\ln(M_t R_{i,t})$. It is useful to define the random variable $z_{i,t} \equiv \frac{1}{2} \times [\ln(M_t R_{i,t})]^2 \, e^{\lambda(\ln(M_t R_{i,t})) \cdot \ln(M_t R_{i,t})}$. Taking the conditional expectation of both sides of (6), imposing the Pricing Equation, and rearranging terms, gives:

$$\mathbb{E}_{t-1}\left(z_{i,t}\right) = -\mathbb{E}_{t-1}\left\{\ln(M_t R_{i,t})\right\}. \tag{7}$$

Notice that $\mathbb{E}_{t-1}\left(z_{i,t}\right)$ will be a function of $\mathbb{E}_{t-1}\left\{\ln(M_t R_{i,t})\right\}$ alone if and only if the Pricing Equation holds, otherwise it will also be a function of $\mathbb{E}_{t-1}(M_t R_{i,t})$. Moreover, $z_{i,t} \geq 0$ for all (i, t). Therefore, $\mathbb{E}_{t-1}\left(z_{i,t}\right) \equiv \gamma_{i,t}^2 \geq 0$, and we denote it as $\gamma_{i,t}^2$ to stress the fact that it is non-negative. Let $\boldsymbol{\gamma}_t^2 \equiv \left(\gamma_{1,t}^2, \gamma_{2,t}^2, ..., \gamma_{N,t}^2\right)'$ and $\boldsymbol{\varepsilon}_t \equiv \left(\varepsilon_{1,t}, \varepsilon_{2,t}, ..., \varepsilon_{N,t}\right)'$ stack respectively the conditional means $\gamma_{i,t}^2$ and the forecast errors $\varepsilon_{i,t}$. Then, from the definition of ε_t we have:

$$\ln(M_t \mathbf{R}_t) = \mathbb{E}_{t-1}\{\ln(M_t \mathbf{R}_t)\} + \boldsymbol{\varepsilon}_t$$

$$= -\boldsymbol{\gamma}_t^2 + \boldsymbol{\varepsilon}_t. \tag{8}$$

Denoting by $\mathbf{r}_t = \ln(\mathbf{R}_t)$ and by $m_t = \ln(M_t)$, and using these definitions in (8), we

get the following system of equations[2]:

$$r_{i,t} = -m_t - \gamma_{i,t}^2 + \varepsilon_{i,t}, \qquad i = 1, 2, \ldots, N. \tag{11}$$

The system (11) shows that the (log of the) SDF is the serial-correlation common feature, in the sense of Engle and Kozicki (1993), of all (logged) asset returns. For any two economic series, a common feature exists if it is present in both of them and can be removed by linear combination. An early example of common features is cointegration, where the feature is a common unit root component that is removed by the cointegrating vector. Serial-correlation common features are discussed in great detail in Vahid and Engle (1993, 1997) and in Hecq, Palm and Urbain (2005).

Looking at (11), asset returns are decomposed into three terms: the first is the logarithm of the SDF, m_t, which is common to all returns and is a random variable. The second is $\gamma_{i,t}^2 = \mathbb{E}_{t-1}(z_{i,t})$. It is idiosyncratic and, given past information, also deterministic. The third is $\varepsilon_{i,t}$. It is also idiosyncratic and unforecastable, and therefore has no serial correlation. Hence, disregarding deterministic terms, the only source of serial correlation is m_t. Notice that m_t can be removed by linearly combining returns: for any two assets i and j, $r_{i,t} - r_{j,t}$ will not contain the feature m_t, which makes $(1, -1)$ a "cofeature vector" (vector removing the feature) for all asset pairs.[3]

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[2]We could have obtained (11) following Blundell, Browning, and Meghir (1994, p. 60, eq. (2.10)), writing the Pricing Equation as:

$$M_{t+1}R_{i,t+1} = 1 \times u_{i,t+1}, \qquad i = 1, 2, \ldots, N, \tag{9}$$

where $\mathbb{E}_t(u_{i,t+1}) = 1$, $i = 1, 2, \ldots, N$. Taking now logs of (9), decomposing $\ln(u_{i,t+1}) = \mathbb{E}_t\{\ln(u_{i,t+1})\} + \varepsilon_{i,t+1}$, we get:

$$r_{i,t+1} = -m_{t+1} + \mathbb{E}_t\{\ln(u_{i,t+1})\} + \varepsilon_{i,t+1}, \qquad i = 1, 2, \ldots, N. \tag{10}$$

Blundell, Browning, and Meghir argue that, in general, $\mathbb{E}_t\{\ln(u_{i,t+1})\}$ will be a function of higher-order moments of $\ln(u_{i,t+1})$. Our expansion (6) just makes clear the exact way in which $\mathbb{E}_t\{\ln(u_{i,t+1})\}$ depends on the higher-order moments of $\ln(u_{i,t+1})$ if the Pricing Equation holds, and our consistency proof follows directly from this connection.

[3]There is a serial-correlation common feature between any two $I(0)$ random variables x_t and y_t, if both have serial correlation but, for some constant number $\widetilde{\alpha}$, $x_t - \widetilde{\alpha}y_t$ does not have serial

We now state our most important result.

Proposition 1 *If the vector process* $\{\ln(M_t\mathbf{R}_t)\}$ *satisfies assumptions 1, 2, 3 and 4, the realization of the SDF at time t, denoted by M_t, can be consistently estimated as $N, T \to \infty$, using:*

$$\widehat{M_t} = \frac{\overline{R}_t^G}{\frac{1}{T}\sum_{j=1}^{T}\left(\overline{R}_j^G\overline{R}_j^A\right)},$$

where $\overline{R}_t^G = \prod_{i=1}^{N} R_{i,t}^{-\frac{1}{N}}$ and $\overline{R}_t^A = \frac{1}{N}\sum_{i=1}^{N} R_{i,t}$ are respectively the geometric average of the reciprocal of all asset returns and the arithmetic average of all asset returns.

Proof. Because $\ln(M_t\mathbf{R}_t)$ is weakly stationary, for every one of its elements $\ln(M_tR_{i,t})$, there exists a Wold representation of the form:

$$\ln(M_tR_{i,t}) = \mu_i + \sum_{j=0}^{\infty} b_{i,j}\varepsilon_{i,t-j} \tag{12}$$

where, for all i, $b_{i,0} = 1$, $\mu_i < \infty$, $\sum_{j=0}^{\infty} b_{i,j}^2 < \infty$, and $\varepsilon_{i,t}$ is white noise. Taking the unconditional expectation of (7), in light of (12), leads to:

$$\gamma_i^2 \equiv \mathbb{E}(z_{i,t}) = -\mathbb{E}\{\ln(M_tR_{i,t})\} = -\mu_i, \tag{13}$$

which is well defined and time-invariant under Assumption 4. Taking conditional

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correlation. This is usually verified using the condition: $\mathbb{E}_{t-1}(x_t - \widetilde{\alpha}y_t) = 0$. However, if there are deterministic terms in $x_t - \widetilde{\alpha}y_t$ such as constant term, for example, this will prevent $\mathbb{E}_{t-1}(x_t - \widetilde{\alpha}y_t) = 0$ from holding, even if the serial-correlation component of x_t and y_t is eliminated by the linear combination $x_t - \widetilde{\alpha}y_t$. That is the reason why any deterministic components have to be subtracted from the cofeature linear combination to use this conditional moment restriction. The term $\gamma_{i,t}^2$ is deterministic given past information, since it is a $(t-1)-$adapted series. Therefore:

$$r_{i,t} + \gamma_{i,t}^2 = -m_t + \varepsilon_{i,t}, \qquad i = 1, \ldots, N,$$

makes clear that m_t is the serial-correlation common feature of $r_{i,t} + \gamma_{i,t}^2$, which are (log) returns adjusted for the deterministic terms $\gamma_{i,t}^2$, or simply "risk-adjusted returns," because $\gamma_{i,t}^2$ is a conditional moment that involves the square of returns.

expectations $\mathbb{E}_{t-1}(\cdot)$ of (12), using $\varepsilon_{i,t} = \ln(M_t R_{i,t}) - \mathbb{E}_{t-1}\{\ln(M_t R_{i,t})\}$, yields:

$$r_{i,t} = -m_t - \gamma_i^2 + \varepsilon_{i,t} - \sum_{j=1}^{\infty} b_{i,j}\varepsilon_{i,t-j}, \qquad i = 1, 2, \ldots, N, \tag{14}$$

which is just a different way of writing (11), where it becomes obvious that:

$$\gamma_{i,t}^2 = \gamma_i^2 + \sum_{j=1}^{\infty} b_{i,j}\varepsilon_{i,t-j}.$$

We consider now a cross-sectional average of (14):

$$\frac{1}{N}\sum_{i=1}^{N} r_{i,t} = -m_t - \frac{1}{N}\sum_{i=1}^{N}\gamma_i^2 + \frac{1}{N}\sum_{i=1}^{N}\varepsilon_{i,t} - \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{\infty} b_{i,j}\varepsilon_{i,t-j}, \tag{15}$$

and examine convergence in probability of $\frac{1}{N}\sum_{i=1}^{N} r_{i,t} + m_t$ using (15). Because every term $\ln(M_t R_{i,t})$ has a finite mean $\mu_i = -\gamma_i^2$, even in the limit, the limit of their average must be finite, i.e.,

$$\lim_{N\to\infty} -\frac{1}{N}\sum_{i=1}^{N}\gamma_i^2 \equiv -\gamma^2 < \infty.$$

Assumption 4 is a sufficient condition to apply a Markov Law-of-Large-Numbers to $\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{i,t}$, since:

$$\mathrm{VAR}\left(\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{i,t}\right) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbb{E}\left(\varepsilon_{i,t}\varepsilon_{j,t}\right) \leq \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left|\mathbb{E}\left(\varepsilon_{i,t}\varepsilon_{j,t}\right)\right|,$$

but, by Assumption 4, we obtain,

$$\lim_{N\to\infty}\mathrm{VAR}\left(\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{i,t}\right) = 0, \text{ and}$$

$$\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{i,t} \xrightarrow{p} 0.$$

The last term in (15), $-\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{\infty}b_{i,j}\varepsilon_{i,t-j}$, involves averages of the non-contemporaneous $MA\left(\infty\right)$ terms in the Wold representation (12). There is no cross correlation between $\varepsilon_{i,t}$ and $\varepsilon_{j,t-k}$, $k=1,2,\ldots$. Therefore, in computing the variance of $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{\infty}b_{i,j}\varepsilon_{i,t-j}$, we need only to consider the sum of the variances of terms of the form $\frac{1}{N}\sum_{i=1}^{N}b_{ik}\varepsilon_{i,t-k}$. These are given by:

$$\mathrm{VAR}\left(\frac{1}{N}\sum_{i=1}^{N}b_{i,k}\varepsilon_{i,t-k}\right)=\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}b_{i,k}b_{j,k}\mathbb{E}\left(\varepsilon_{i,t}\varepsilon_{j,t}\right),\qquad(16)$$

due to weak stationarity of $\boldsymbol{\varepsilon}_t$. We now examine the limit of the generic term in (16) with detail:

$$\mathrm{VAR}\left(\frac{1}{N}\sum_{i=1}^{N}b_{i,k}\varepsilon_{i,t-k}\right)=\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}b_{i,k}b_{j,k}\mathbb{E}\left(\varepsilon_{i,t}\varepsilon_{j,t}\right)\leq$$

$$\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}|b_{i,k}b_{j,k}\mathbb{E}\left(\varepsilon_{i,t}\varepsilon_{j,t}\right)|=\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}|b_{i,k}b_{j,k}|\,|\mathbb{E}\left(\varepsilon_{i,t}\varepsilon_{j,t}\right)|\leq\qquad(17)$$

$$\left(\max_{i,j}|b_{i,k}b_{j,k}|\right)\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}|\mathbb{E}\left(\varepsilon_{i,t}\varepsilon_{j,t}\right)|.\qquad(18)$$

Hence:

$$\lim_{N\to\infty}\mathrm{VAR}\left(\frac{1}{N}\sum_{i=1}^{N}b_{i,k}\varepsilon_{i,t-k}\right)\leq\lim_{N\to\infty}\left(\max_{i,j}|b_{i,k}b_{j,k}|\right)\times$$

$$\lim_{N\to\infty}\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}|\mathbb{E}\left(\varepsilon_{i,t}\varepsilon_{j,t}\right)|=0,$$

since the sequence $\{b_{i,j}\}_{j=0}^{\infty}$ is square-summable, yielding $\lim_{N\to\infty}\left(\max_{i,j}|b_{i,k}b_{j,k}|\right)\leq\infty$, and Assumption 4 imposes $\lim_{N\to\infty}\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}|\mathbb{E}\left(\varepsilon_{i,t}\varepsilon_{j,t}\right)|=0$. Thus all variances are zero in the limit, as well as their sum, which gives:

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{\infty}b_{i,j}\varepsilon_{i,t-j}\xrightarrow{p}0,\text{ and, }\frac{1}{N}\sum_{i=1}^{N}r_{i,t}+m_t\xrightarrow{p}-\gamma^2.$$

13

Therefore, a consistent estimator for $e^{\gamma^2} \times M_t = \widetilde{M}_t$ is given by:

$$\widehat{\widetilde{M}}_t = \prod_{i=1}^{N} R_{i,t}^{-\frac{1}{N}}. \tag{19}$$

We now show how to estimate e^{γ^2} consistently and therefore to find a consistent estimator for M_t. Multiply the pricing equation by e^{γ^2} to get:

$$e^{\gamma^2} = \mathbb{E}_{t-1}\left\{ e^{\frac{\gamma^2}{2}} M_t R_{i,t} \right\} = \mathbb{E}_{t-1}\left\{ \widetilde{M}_t R_{i,t} \right\}.$$

Take now the unconditional expectation and average across $i = 1, 2, ..., N$ to get:

$$e^{\gamma^2} = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}\left\{ \widetilde{M}_t R_{i,t} \right\}.$$

It is now straightforward to obtain a consistent estimator for e^{γ^2} using (19):

$$\widehat{e^{\gamma^2}} = \frac{1}{N} \sum_{i=1}^{N} \left(\frac{1}{T} \sum_{t=1}^{T} \widehat{\widetilde{M}}_t R_{i,t} \right) = \frac{1}{T} \sum_{t=1}^{T} \left(\prod_{i=1}^{N} R_{i,t}^{-\frac{1}{N}} \frac{1}{N} \sum_{i=1}^{N} R_{i,t} \right) = \frac{1}{T} \sum_{t=1}^{T} \overline{R}_t^G \overline{R}_t^A.$$

We can finally propose a consistent estimator for M_t:

$$\widehat{M}_t = \frac{\widehat{\widetilde{M}}_t}{\widehat{e^{\gamma^2}}} = \frac{\overline{R}_t^G}{\frac{1}{T} \sum_{j=1}^{T} \overline{R}_j^G \overline{R}_j^A},$$

a simple function of asset returns. ∎

There are three important features of $\widehat{M}_t$: (i) it provides a fully non-parametric way of consistently estimating the realizations of the SDF at a very low computational cost; (ii) asset returns used in computing $\widehat{M}_t$ are allowed to be heteroskedastic, which widens the application of this estimator to (ultra) high-frequency data; (iii) there are important special cases of Proposition 1: under conditional log-Normality of $\ln(M_t \mathbf{R}_t)$, with or without a time-varying variance-covariance matrix for ε_t. As a

14

consequence of Proposition 1, we can now propose a way of consistently estimating the risk-free rate R_t^f:

Proposition 2 *Using $\widehat{M}_t$ as in Proposition 1 above offers a consistent estimator of the risk-free rate, R_t^f, as follows:*

$$\widehat{R_t^f} = \frac{1}{\mathbb{E}_{t-1}\left\{\widehat{M}_t\right\}}.$$

Proof. Because R_{t+1}^f is measurable:

$$1 = \mathbb{E}_t\left\{M_{t+1}R_{t+1}^f\right\} = R_{t+1}^f \mathbb{E}_t\left\{M_{t+1}\right\}, \text{ or } R_{t+1}^f = \frac{1}{\mathbb{E}_t\left\{M_{t+1}\right\}},$$

which offers an immediate consistent estimator for the risk-free rate R_{t+1}^f:

$$\widehat{R_{t+1}^f} = \frac{1}{\mathbb{E}_t\left\{\widehat{M_{t+1}}\right\}},$$

which can be implemented based on an econometric model for $\mathbb{E}_t\left\{\widehat{M_{t+1}}\right\}$. ∎

2.2 An Alternative Equivalent Estimator

In several contexts it is convenient to assume homoskedasticity of asset returns, especially for the sake of simplicity. This is very unrealistic for high-frequency data (daily, weekly or even monthly), but may be reasonable as the frequency of observations decreases to quarterly or annual data; see Drost and Nijman (1993), and Meddahi and Renault (2002) for a discussion. Under homoskedasticity and a log-Normal distribution for $\ln\left(M_t\mathbf{R}_t\right)$, (11) is:

$$r_{i,t} = -m_t - \frac{1}{2}\sigma_i^2 + \varepsilon_{i,t}, \qquad i = 1, 2, \cdots, N, \tag{20}$$

which decomposes (logged) returns in a time-invariant fixed effect, $-\frac{1}{2}\sigma_i^2$, a purely time-varying common component, $-m_t$, and a normally distributed unforecastable error, $\varepsilon_{i,t}$, with mean zero and variance σ_i^2.

Panel-data regression equations (20) correspond to a standard unobserved fixed-effect model with no explanatory variables other than time dummies, also known as the two-way fixed-effect model (see Wallace and Hussain, 1969; Amemiya, 1971). Stacking the least-square estimates of the coefficients of the time dummies then provides a consistent estimate for the log-SDFs, whereas the fixed effects capture the individual heterogeneity that stem from the variances of logged returns.

We now discuss least-square estimation of (20). Define: $y_{it} \equiv r_{i,t}$, $a_i \equiv -\frac{1}{2}\sigma_i^2$, $\beta_t \equiv -m_t$. Then, model (20) is:

$$y_{i,t} = a_i + \beta_t + \varepsilon_{i,t},$$

and we are interested in an estimate of $\{\beta_t\}_{t=1}^{T}$. Here, since regressors are time dummies and constants, the assumption that they are strictly exogenous for the a_i's and β_t respectively can be comfortably made. Regressors are deterministic, therefore, current and past values of returns cannot explain their behavior. Using standard notation,

$$\mathbf{y}_i = \boldsymbol{\iota}_T \mathbf{a}_i + \mathbf{I}_T \boldsymbol{\beta} + \boldsymbol{\varepsilon}_i, \qquad i = 1, ..., N, \tag{21}$$

where, $\mathbf{y}_i = \begin{bmatrix} y_{i,1} & y_{i,2} & ... & y_{i,T} \end{bmatrix}'$, $\boldsymbol{\beta} = \begin{bmatrix} \beta_1 & \beta_2 & ... & \beta_T \end{bmatrix}'$, $\boldsymbol{\varepsilon}_i = \begin{bmatrix} \varepsilon_{i,1} & \varepsilon_{i,2} & ... & \varepsilon_{i,T} \end{bmatrix}'$, where $\mathbf{I}_T$ is an identity matrix of order T and $\boldsymbol{\iota}_T$ is a $T \times 1$ vector of ones.

Denoting by $\mathbf{Q}$ the standard "time-demeaning transformation" $\mathbf{Q} = \mathbf{I_T} - \boldsymbol{\iota}_T(\boldsymbol{\iota}_T'\boldsymbol{\iota}_T)^{-1}\boldsymbol{\iota}_T'$, where $\mathbf{Q}$ is idempotent and symmetric, we get:

$$\mathbf{Q}\mathbf{y}_i = \mathbf{Q}\boldsymbol{\iota}_T \mathbf{a}_i + \mathbf{Q}\mathbf{I}_T\boldsymbol{\beta} + \mathbf{Q}\boldsymbol{\varepsilon}_i = \mathbf{Q}\boldsymbol{\beta} + \mathbf{Q}\boldsymbol{\varepsilon}_i.$$

The pooled OLS estimate of $\boldsymbol{\beta}$ is not identified, since the identification condition is:

$$\text{rank}\left(\mathbb{E}\left[\mathbf{Q}'\mathbf{Q}\right]\right) = T,$$

but we have $\mathbb{E}\left[\mathbf{Q}'\mathbf{Q}\right] = \mathbf{Q}$ and $\text{rank}(\mathbf{Q}) = \text{tr}(\mathbf{Q}) = T - 1$, showing that $\mathbb{E}\left[\mathbf{Q}'\mathbf{Q}\right]$ is rank-deficient.

Nevertheless, a reparameterization can be made in order to get a consistent estimator. Consider,

$$y_{i,t} = a_i + \widetilde{\beta}_t + \beta_1 + \varepsilon_{i,t}, \tag{22}$$

where $\widetilde{\beta}_t = \beta_t - \beta_1 = -\left(m_t - m_1\right) = -\ln\left(\frac{M_t}{M_1}\right)$, i.e., is the (log of) the "normalized" SDF, relative to its initial value. It is obvious that $\widetilde{\beta}_1 = 0$, and the model can be rewritten as:

$$\mathbf{y}_i = \boldsymbol{\iota}_T\left(a_i + \beta_1\right) + \mathbf{X}\widetilde{\boldsymbol{\beta}} + \boldsymbol{\varepsilon}_i \tag{23}$$

where: $\underset{T\times(T-1)}{\mathbf{X}} = \left[\begin{array}{cc} \mathbf{0}' & \mathbf{I}_{T-1} \end{array}\right]'$, and $\underset{(T-1)\times 1}{\widetilde{\boldsymbol{\beta}}} = \left[\begin{array}{ccc} \widetilde{\beta}_2 & ... & \widetilde{\beta}_T \end{array}\right]'$. Applying $\mathbf{Q}$ to (23) yields:

$$\mathbf{Q}\mathbf{y}_i = \mathbf{Q}\boldsymbol{\iota}_T\left(a_i + \beta_1\right) + \mathbf{Q}\mathbf{X}\widetilde{\boldsymbol{\beta}} + \mathbf{Q}\boldsymbol{\varepsilon}_i$$
$$= \mathbf{Q}\mathbf{X}\widetilde{\boldsymbol{\beta}} + \mathbf{Q}\boldsymbol{\varepsilon}_i. \tag{24}$$

It is straightforward to show that $\mathbf{Q}\mathbf{X}$ is now full-column rank, and therefore we can compute the pooled OLS estimator of (24):

$$\widehat{\widetilde{\boldsymbol{\beta}}} = \left[\sum_{i=1}^{N}\mathbf{X}'\mathbf{Q}\mathbf{X}\right]^{-1}\left[\sum_{i=1}^{N}\mathbf{X}'\mathbf{Q}\mathbf{y}_i\right] = \frac{1}{N}\sum_{i=1}^{N}\mathbf{y}_i - \left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}y_{it}\right)\boldsymbol{\iota}_{T-1}, \tag{25}$$

which yields, in more familiar notation,

$$-\widehat{\widetilde{\beta}}_t = \widehat{m_t - m_1} = \frac{1}{NT}\sum_{j=1}^{N}\sum_{t=1}^{T}r_{j,t} - \frac{1}{N}\sum_{i=1}^{N}r_{i,t} = \ln\left[\bar{R}\cdot\bar{R}_t^G\right], \tag{26}$$

17

where $\bar{R} = \prod_{i=1}^{N} \prod_{t=1}^{T} (R_{i,t})^{\frac{1}{NT}}$ is the overall geometric average of returns and $\bar{R}_t^G = \prod_{i=1}^{N} (R_{i,t})^{-\frac{1}{N}}$ is the geometric average of the reciprocal of returns. Equation (26) shows that the estimator of $m_t - m_1$ depends exclusively on appropriate averages of the asset gross returns, viz. $\ln\left[\bar{R} \cdot \bar{R}_t^G\right]$. We are now ready to state the asymptotic distribution of $\widehat{\widetilde{\boldsymbol{\beta}}}$.

Proposition 3 *Under Assumptions 1, 2, 3, and 4, if the error terms $\boldsymbol{\varepsilon}_t$ are homoskedastic and Normally distributed, then $\sqrt{N}\left(\widehat{\widetilde{\boldsymbol{\beta}}} - \boldsymbol{\beta}\right)$ converges weakly to a multivariate Gaussian distribution with mean zero and covariance matrix:*

$$\boldsymbol{\Omega} = (\mathbf{X}'\mathbf{Q}\mathbf{X})^{-1}\mathbf{P}_{\boldsymbol{\iota_N}}\mathbf{X}'\mathbf{Q}\left(\boldsymbol{\Sigma} \otimes \mathbf{I}_T\right)\mathbf{Q}\mathbf{X}\mathbf{P}_{\boldsymbol{\iota_N}}(\mathbf{X}'\mathbf{Q}\mathbf{X})^{-1}, \tag{27}$$

where $\boldsymbol{\Sigma}$ denotes the covariance matrix of $\boldsymbol{\varepsilon}_t$, and $\mathbf{P}_{\boldsymbol{\iota_N}} = \boldsymbol{\iota}_N(\boldsymbol{\iota}'_N\boldsymbol{\iota}_N)^{-1}\boldsymbol{\iota}'_N$, where $\boldsymbol{\iota}_N$ is a $N \times 1$ vector of ones.

Proof. Rewriting (25) in terms of the projection matrix $\mathbf{P}_{\boldsymbol{\iota_N}} = \boldsymbol{\iota}_N(\boldsymbol{\iota}'_N\boldsymbol{\iota}_N)^{-1}\boldsymbol{\iota}'_N$ yields

$$\widehat{\widetilde{\boldsymbol{\beta}}} = (\mathbf{X}'\mathbf{Q}\mathbf{X})^{-1}\mathbf{P}_{\boldsymbol{\iota_N}}\mathbf{X}'\mathbf{Q}\mathbf{y} = \boldsymbol{\beta} - (\mathbf{X}'\mathbf{Q}\mathbf{X})^{-1}\mathbf{P}_{\boldsymbol{\iota_N}}\mathbf{X}'\mathbf{Q}\boldsymbol{\varepsilon},$$

where $\mathbf{y} = (\mathbf{y}'_1, \ldots, \mathbf{y}'_T)'$ and $\boldsymbol{\varepsilon} = (\boldsymbol{\varepsilon}'_1, \ldots, \boldsymbol{\varepsilon}'_T)'$ are $NT \times 1$ vectors. Then, the covariance matrix of $\widehat{\widetilde{\boldsymbol{\beta}}}$ is:

$$\begin{aligned}
\boldsymbol{\Omega} &= \mathbb{E}\left[\left(\widehat{\widetilde{\boldsymbol{\beta}}} - \boldsymbol{\beta}\right)\left(\widehat{\widetilde{\boldsymbol{\beta}}} - \boldsymbol{\beta}\right)'\right] \\
&= \mathbb{E}\left[(\mathbf{X}'\mathbf{Q}\mathbf{X})^{-1}\mathbf{P}_{\boldsymbol{\iota_N}}\mathbf{X}'\mathbf{Q}\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'\mathbf{Q}\mathbf{X}\mathbf{P}_{\boldsymbol{\iota_N}}(\mathbf{X}'\mathbf{Q}\mathbf{X})^{-1}\right] \\
&= (\mathbf{X}'\mathbf{Q}\mathbf{X})^{-1}\mathbf{P}_{\boldsymbol{\iota_N}}\mathbf{X}'\mathbf{Q}\left(\boldsymbol{\Sigma} \otimes \mathbf{I}_T\right)\mathbf{Q}\mathbf{X}\mathbf{P}_{\boldsymbol{\iota_N}}(\mathbf{X}'\mathbf{Q}\mathbf{X})^{-1}.
\end{aligned}$$

Asymptotic normality follows from standard panel-data results. ∎

Up to this point we have a consistent estimator for $\widetilde{\boldsymbol{\beta}} = \beta_t - \beta_1$, $t = 2, ..., T$. However, we are not interested in an estimator for $m_t - m_1$ but for M_t or m_t. To get

it we start with a consistent estimator for $\frac{M_t}{M_1}$, easily obtained from (26):

$$\widehat{\frac{M_t}{M_1}} = \bar{R} \cdot \bar{R}_t^G.$$

Dividing lagged Pricing Equation (2) by the realization M_1, which is measurable with respect to the information set used by the agent[4], taking unconditional expectations and averaging across assets, yields:

$$\frac{1}{M_1} = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \left\{ \frac{M_t}{M_1} R_{i,t} \right\},$$

which leads to the following consistent estimator for M_1:

$$\widehat{M_1} = \frac{1}{\frac{1}{N} \sum_{i=1}^{N} \frac{1}{T} \sum_{t=1}^{T} \widehat{\frac{M_t}{M_1}} R_{i,t}} = \frac{1}{\bar{R} \cdot \frac{1}{T} \sum_{t=1}^{T} \bar{R}_t^G \bar{R}_t^A}.$$

Finally, a consistent estimator for M_t, $t = 2, 3, \cdots, T$, identical to that in Proposition 1, is obtained:

$$\widehat{M_t} = \widehat{\frac{M_t}{M_1}} \cdot \widehat{M_1} = \frac{\bar{R} \cdot \bar{R}_t^G}{\bar{R} \cdot \frac{1}{T} \sum_{j=1}^{T} \bar{R}_j^G \bar{R}_j^A} = \frac{\bar{R}_t^G}{\frac{1}{T} \sum_{j=1}^{T} \bar{R}_j^G \bar{R}_j^A}. \tag{28}$$

There are several interesting points to note. First, because of the normality assumption for errors $\varepsilon_{i,t}$, if the $\varepsilon_{i,t}$'s are uncorrelated in the cross-sectional dimension, then pooled OLS yields a consistent and fully efficient estimator for $\ln\left(\frac{M_t}{M_1}\right)$. The term M_1 serves here as a mean correction for the SDF. Second, in the more realistic case where returns have correlated shocks, then the estimator for $\ln\left(\frac{M_t}{M_1}\right)$ will still be consistent but will not be fully efficient. In this case, a GLS approach will produce a fully efficient estimator. However, for the latter to be feasible, we need:

$$T > \frac{N(N+1)}{2},$$

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[4]Since it is a realization, it can be treated as a constant rather than as a random variable.

and asymptotic results will require T to grow at rate N^2, which will certainly be a binding constraint. An alternative to GLS is to use OLS with a robust estimator for the variance-covariance matrix of estimated parameters along the lines of Newey and West (1987). Finally, in either case, uncorrelated or correlated shocks to returns, we do not have a fully efficient estimator for M_t, or for m_t, but only for a corrected version of the latter, and even so only at the cost of assuming homoskedastic normal errors.

2.3 Properties of the M_t Estimator

The first characteristic of our estimator of M_t, labelled $\widehat{M_t}$, is that it is a function of asset-return data alone. No assumptions whatsoever about preferences or even about any finance theories have been made so far. Second, our estimator is unique, since it can be seen as a pooled OLS estimate, as shown in the previous section. Moreover, looking at the normalized SDF estimator – the estimator of $-\ln\left(\frac{M_t}{M_1}\right)$ in (26) – shows that it is a linear combination of logged returns. Using the logarithmic approximation $\ln(1+x) \simeq x$, $\ln\left(\frac{\widehat{M_t}}{M_1}\right) \simeq \frac{\widehat{M_t}}{M_1} - 1$, and $r_{i,t} \simeq R_{i,t} - 1$, makes it clear that the normalized SDF estimator $\frac{\widehat{M_t}}{M_1}$ is a linear function of $R_{i,t+1}$. Hence, it lies in the space of payoffs. Because M_1 only acts as mean correction, our estimator $\widehat{M_t}$ will lie in the space of payoffs if that space contains an asset that pays the same amount in every state of nature. That is why we must assume the existence of a risk-free asset (Assumption 3).

Since there is a unique SDF that lies in the space of payoffs – the mimicking portfolio M_t^* – our estimator identifies it. It is important to stress that with incomplete markets there exists an infinite number of SDF's, all of which can be written as $M_{t+1} = M_{t+1}^* + \nu_{t+1}$, where $\mathbb{E}_t(\nu_{t+1}x_{i,t+1}) = 0$. In this context, any econometric technique could only hope to identify M_t^*, which is what we do here.

Third, because $\widehat{M_t}$ is a consistent estimator, it is interesting to discuss what it converges to. Of course, the SDF is a stochastic process: $\{M_t\}$. Since convergence

in probability requires a limiting degenerate distribution, our estimator $\widehat{M}_t$ converges to the realization of M at time t.

Fourth, from a different angle, it is straightforward to verify that our estimator was constructed to obey:

$$\plim_{N,T\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{1}{T} \sum_{t=1}^{T} \widehat{M_{t+1}} R_{i,t+1} = 1, \tag{29}$$

which is a natural property arising from the moment restrictions in (2), when populational means of the time-series and of the cross-sectional distributions are replaced by sample means.

2.4 Comparisons with the Literature

As far as we are aware of, early studies in finance and macroeconomics dealing with the SDF did not try to obtain a direct estimate of it as we do: we treated $\{M_t\}$ as a stochastic process and constructed an estimate $\widehat{M}_t$, such that $\widehat{M}_t - M_t \xrightarrow{p} 0$. Conversely, most of the previous literature estimated the SDF indirectly as a function of consumption data from the National Income and Product Accounts (NIPA), using a parametric function to represent preferences; see Hansen and Singleton (1982, 1983, 1984), Brown and Gibbons (1985) and Epstein and Zin (1991). As noted by Rosenberg and Engle (2002), there are several sources of measurement error for NIPA consumption data that can pose a significant problem for this type of estimate. Even if this were not the case, there is always the risk that an incorrect choice of parametric function used to represent preferences will contaminate the final SDF estimate.

One of the major features of early estimates of the SDF was that their correlation with the equity premium was not large and negative, generating the equity-premium puzzle; see Hansen and Singleton (1982), Mehra and Prescott (1985), and the latest discussion in Mulligan (2004). Epstein and Zin propose a functional form for M_t that makes it depend on the reciprocal of the return on the optimal portfolio. Because

returns are strongly positively-correlated, this specification will make the correlation between M_t and equity premium to be large and negative, which can perhaps serve as a remedy for earlier estimates with regard to the equity-premium puzzle. Our expression for M_t in Proposition 1 naturally delivers a negative correlation between M_t and the equity premium: the denominator $\frac{1}{T} \sum_{t=1}^{T} \left(\overline{R}_t^G \overline{R}_t^A \right)$ will be approximately constant for large T, but the numerator $\prod_{i=1}^{N} (R_{i,t})^{-\frac{1}{N}} = \prod_{i=1}^{N} \left(\frac{1}{R_{i,t}} \right)^{\frac{1}{N}}$ is the geometric average of the reciprocals of returns, which should generate a strong negative correlation between $\widehat{M}_t$ and the equity premium.

Hansen and Jagannathan (1991) point out that early studies imposed potentially stringent limits on the class of admissible asset-pricing models. They avoid dealing with a direct estimate of the SDF, but note that the SDF has its behavior (and in particular its variance) bounded by two restrictions. The first is Pricing Equation (2) and the second is $M_t > 0$. They exploit the fact that it is always possible to project M onto the space of payoffs, which makes it straightforward to express M^*, the mimicking portfolio, only as a function of observables, as we do in (28):

$$M_{t+1}^* = \boldsymbol{\iota}_N' \left[\mathbb{E}_t \left(\mathbf{R}_{t+1} \mathbf{R}_{t+1}' \right) \right]^{-1} \mathbf{R}_{t+1}, \tag{30}$$

where $\boldsymbol{\iota}_N$ is a $N \times 1$ vector of ones, and $\mathbf{R}_{t+1}$ is a $N \times 1$ vector stacking all asset returns. Although they do not discuss it at any length in their paper, equation (30) shows that it is possible to identify M_{t+1}^* in the Hansen and Jagannathan framework. As in our case, (30) delivers an estimate of the SDF that is solely a function of asset returns and can therefore be used to verify whether preference-parameter values are admissible or not.

If one regards (30) as a means to identify M^*, there are some limitations that must be pointed out. First, it is obvious from (30) that a conditional econometric model is needed to implement an estimate for M_{t+1}^*, since one has to compute the conditional moment $\mathbb{E}_t \left(\mathbf{R}_{t+1} \mathbf{R}_{t+1}' \right)$. Second, as the number of assets increases ($N \to \infty$) the use

of (30) will suffer numerical problems in computing an estimate of $\left[\mathbb{E}_t\left(\mathbf{R}_{t+1}\mathbf{R}'_{t+1}\right)\right]^{-1}$. In the limit, the matrix $\mathbb{E}_t\left(\mathbf{R}_{t+1}\mathbf{R}'_{t+1}\right)$ will be of infinite order. Even for finite but large N there will be possible singularities in it, as the correlation between some assets may be very close to unity.

Our approach is very close to Mulligan's (2002), where return data is super-aggregated to compute the return to aggregate capital. For algebraic convenience, we use the log-utility assumption for preferences – where $M_{t+j} = \beta\frac{c_t}{c_{t+j}}$ – as well as the assumption of no production in the economy in illustrating their similarities.

Since asset prices are the expected present value of the dividend flows, and since with no production dividends are equal to consumption in every period, the price of the portfolio representing aggregate capital $\bar{p}_t$ is:

$$\bar{p}_t = \mathbb{E}_t\left\{\sum_{i=1}^{\infty}\beta^i\frac{c_t}{c_{t+i}}c_{t+i}\right\} = \frac{\beta}{1-\beta}c_t.$$

Hence, the return on aggregate capital $\overline{R}_{t+1}$ is given by:

$$\overline{R}_{t+1} = \frac{\bar{p}_{t+1} + c_{t+1}}{\bar{p}_t} = \frac{\beta c_{t+1} + (1-\beta)c_{t+1}}{\beta c_t} = \frac{c_{t+1}}{\beta c_t} = \frac{1}{M_{t+1}}, \qquad (31)$$

which is the reciprocal of the SDF. Therefore there is a duality between the approach in Mulligan and ours in the context above.

Taking logs of both sides of (31), using $\bar{r}_{t+1} = \ln\overline{R}_{t+1}$, yields:

$$\bar{r}_{t+1} = -m_{t+1},$$

which shows that the common feature in (11) is indeed the return to aggregate capital. Of course, it may not be so simple to derive this duality result under more general conditions but it can still be thought of as an approximation. Although similar in spirit, Mulligan's work and ours follow very different paths in empirical implementation. Our goal is to extract $-m_{t+1}$ from a large data set of asset returns, whereas

Mulligan uses national-account data to construct the return to aggregate capital. Because national-account data are prone to be measured with error, which will increase as the level of aggregation increases, the approach taken by Mulligan is likely to generate measurement error in the estimate of $\overline{R}_t$. However, our approach can avoid these problems for two reasons. First, we work with asset-return data, which is more reliable than national-account data. Second, averaging returns in the way we propose factors out idiosyncratic measurement error in $\widehat{M}_t$.

Factor models within the CCAPM framework have a long tradition in finance and in financial econometrics; see, for example, Fama and French (1993) and Lettau and Ludvigson (2001), who propose, respectively, a three- and a two-factor model where, in the former, factors are related to firm size, book-to-market equity and the aggregate stock market, and, in the latter, to a time-varying risk premium and deviations from the long-run consumption-wealth ratio. Compared to these papers, our focus is to consider only the first-order factor, m_t, in a novel way, i.e., a parsimonious representation. The discussion in Cochrane (2001, ch. 7) shows that increasing the number of factors in finance models does not necessarily generate a better model, since the risk of overfitting and instability across different samples is always present. Our effort was to find a parsimonious factor model with reasonable explanatory power for the behavior of asset returns, where the factor has a straightforward macroeconomic interpretation derived from theory – it is the stochastic discount factor pricing all assets, or the return to aggregate capital. The econometric technique itself allows for out-of-sample assessment of our estimator in pricing alternative assets not included in the computation of $\widehat{M}_t$. In our view, this is how we can impose discipline on our empirical model of M_t.

In recent years there has been a trend to build less restrictive estimates of the SDF compared to the early functions of consumption growth; see, among others, Chapman (1998), Aït-Sahalia and Lo (1998, 2000), Rosenberg and Engle (2002), and Chen and Ludvigson (2004). In some of these papers a parametric function is still used to repre-

sent the SDF, although the latter does not depend on consumption; see Rosenberg and Engle, who project the SDF onto the payoffs of a single traded asset, and Aït-Sahalia and Lo (1998, 2000), who rely on equity-index option prices to nonparametrically estimate the projection of the average stochastic discount factor onto equity-return states. Sometimes non-parametric or semi-parametric methods are used, but the SDF is still a function of current or lagged values of consumption; see Chen and Ludvigson, who propose a semiparametric estimate of the functional form of the habit level given contemporaneous and lagged consumption, and Chapman, who approximates the pricing kernel using orthonormal Legendre polynomials in state variables that are functions of aggregate consumption. Although these efforts represent a step forward in terms of reducing the degree of stringent assumptions made to estimate the SDF, they still impose restrictions on preferences to achieve identification. In our framework, "preferences are revealed" in the common component of asset returns after using the Pricing Equation.

3 Using our Estimator to Evaluate the CCAPM

3.1 Testing Preference Specifications within the CCAPM

An important question that can be addressed with our estimator of M_t is how to test and validate specific preference representations. Here we focus on three different preference specifications: the CRRA specification, which has a long tradition in the finance and macroeconomic literatures, the external-habit specification of Abel (1990), and the Kreps and Porteus (1978) specification used in Epstein and Zin (1991), which are respectively:

$$M_{t+1}^{CRRA} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \tag{32}$$

$$M_{t+1}^{EH} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \left(\frac{c_t}{c_{t-1}} \right)^{\kappa(\gamma-1)} \tag{33}$$

$$M_{t+1}^{KP} = \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \right]^{\frac{1-\gamma}{\rho}} \left(\frac{1}{B_t} \right)^{1-\frac{1-\gamma}{\rho}}, \tag{34}$$

where c_t denotes consumption, B_t is the return on the optimal portfolio, β is the discount factor, γ is the relative risk-aversion coefficient, and κ is the time-separation parameter in the habit-formation specification. Notice that M_{t+1}^{EH} is a weighted average of M_{t+1}^{CRRA} and $\left(\frac{c_t}{c_{t-1}} \right)$. In the Kreps-Porteus specification the intertemporal elasticity of substitution in consumption is given by $1/(1-\rho)$ and $\alpha = 1-\gamma$ determines the agent's behavior towards risk. If we denote $\theta = \frac{1-\gamma}{\rho}$, it is clear that M_{t+1}^{KP} is a weighted average of M_{t+1}^{CRRA} and $\left(\frac{1}{B_t} \right)$, with weights θ and $1-\theta$, respectively.

For consistent estimates, we can always write:

$$m_{t+1} = \widehat{m_{t+1}} + \eta_{t+1}, \tag{35}$$

where η_{t+1} is the approximation error between m_{t+1} and its estimate $\widehat{m_{t+1}}$.

The properties of η_{t+1} will depend on the properties of M_{t+1} and $R_{i,t+1}$, and, in general, it will be serially dependent and heterogeneous. Using (35) and the expressions in (32), (33) and (34), we arrive at:

$$\widehat{m_{t+1}} = \ln \beta - \gamma \Delta \ln c_{t+1} - \eta_{t+1}^{CRRA}, \tag{36}$$

$$\widehat{m_{t+1}} = \ln \beta - \gamma \Delta \ln c_{t+1} + \kappa (\gamma - 1) \Delta \ln c_t - \eta_{t+1}^{EH}, \tag{37}$$

$$\widehat{m_{t+1}} = \theta \ln \beta - \theta \gamma \Delta \ln c_{t+1} - (1 - \theta) \ln B_{t+1} - \eta_{t+1}^{KP}, \tag{38}$$

Perhaps the most appealing way of estimating (36), (37) and (38), simultaneously testing for over-identifying restrictions, is to use the generalized method of moments (GMM) proposed by Hansen (1982). Lagged values of returns, consumption and income growth, and also of the logged consumption-to-income ratio can be used as

instruments in this case. Since (36) is nested into (37), we can also perform a redundancy test for $\Delta \ln c_t$ in (36). The same applies regarding (36) and (38), since the latter collapses to the former when $\ln B_{t+1}$ is redundant.

4 Empirical Application

4.1 Data

We apply our techniques to returns available to the average U.S. investor, who has increasingly become more interested in global assets over time. Real returns were computed using the consumer price index in the U.S. Our data base covers U.S.\$ real returns on G7-country stock indices and short-term government bonds, where exchange-rate data was used to transform returns denominated in foreign currency into U.S.\$. In addition to G7 returns on stocks and bonds, we also use U.S.\$ real returns on gold, U.S. real estate, bonds on AAA U.S. corporations, and on the SP 500. The U.S. government bond is chosen to be the 90-day T-Bill, considered by many to be a "riskless asset." All data were extracted from the DRI database, with the exception of real returns on real-estate trusts, which are computed by the National Association of Real-Estate Investment Trusts in the U.S.[5] Our sample period starts in 1972:1 and ends in 2001:4. Overall, we averaged the real U.S.\$ returns on these 18 portfolios or assets[6], which are, in turn, a function of thousands of assets. These are predominantly U.S. based, but we also cover a wide spectrum of investment opportunities across the globe. This is important element of our choice of assets,

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[5] Data on the return on real estate are measured using the return of all publicly traded REITs – Real-Estate Investment Trusts.

[6] The complete list of the 18 portfolio- or asset-returns, all measured in U.S.\$ real terms, is: returns on the NYSE, Canadian Stock market, French Stock market, West Germany Stock market, Italian Stock market, Japanese Stock market, U.K. Stock market, 90-day T-Bill, Short-Term Canadian Government Bond, Short-Term French Government Bond, Short-Term West Germany Government Bond, Short-Term Italian Government Bond, Short-Term Japanese Government Bond, Short-Term U.K. Government Bond. As well as on the return of all publicly traded REITs – Real-Estate Investment Trusts in the U.S., on Bonds of AAA U.S. Corporations, Gold, and on the SP 500.

since diversification allows reducing the degree of correlation of returns across assets, whereas too much correlation may generate no convergence in probability for sample means such as those appearing in Proposition 1.

In estimating equations (36) and (37), we must use additional series. Real per-capita consumption growth was computed using private consumption of non-durable goods and services in constant U.S.\$. We also used real per-capita GNP as a measure of income – an instrument in running some of these regressions. Consumption and income series were seasonally adjusted.

4.2 Estimating the SDF M_t and the Risk-Free Rate R_t^f

Figure 1 below shows our estimator of the SDF – $\widehat{M_t}$ – for the period 1972:1 to 2001:4. It is close to unity most of the time and bounded by the interval $[0.85, 1.15]$. The sample mean of $\widehat{M_t}$ is 0.9939, implying an annual discount factor of 0.9760, or an annual discount rate of 2.46%, a very reasonable estimate.
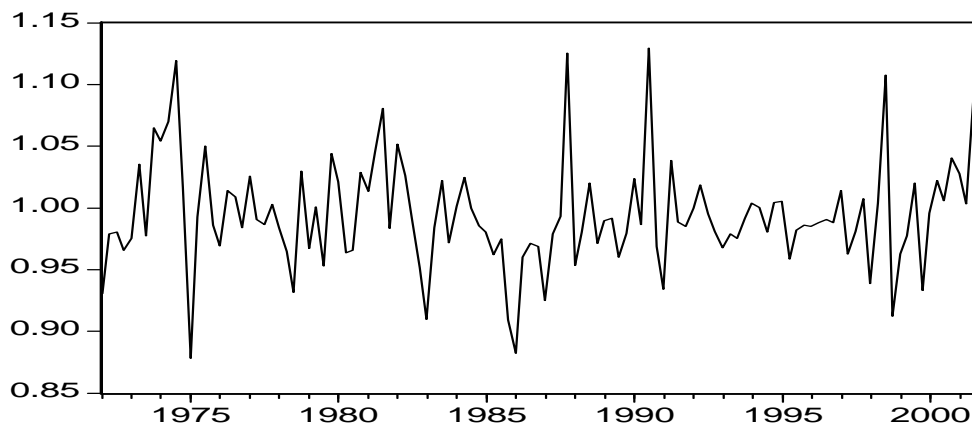

Figure 1: Stochastic Discount Factor

In order to get a conditional model for the SDF, we project our SDF estimator on the first two powers of lag one and two of the returns used in computing it, of lag one and two of consumption and income growth, and of the log of the lagged

consumption–income ratio. In constructing the estimate of the risk-free rate R_t^f we take the reciprocal of the predicted value of this conditional model. The mean real return of our risk-free rate estimator is 3.12% on an annual basis, slightly higher than the mean real return of the T-Bill, 2.39% a year. Next, $\widehat{R_t^f}$ was then used to generate an estimator of the equity premium relative to the real return of the NYSE. The usual equity premium using the 90-day T-Bill was also computed, and a plot of both is presented in Figure 2. The correlation coefficient between these two series is 0.9431, and, as it is obvious from the picture, there is little difference in their behavior.
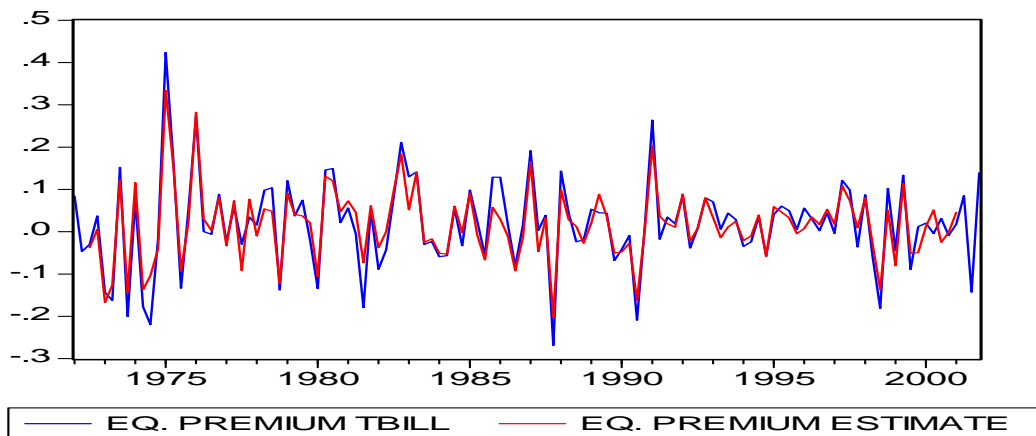

Figure 2: Equity Premia

4.3 Estimating Preference Specifications within the CCAPM

Tables 1, 2, and 3 present GMM estimation of equations (36), (37) and (38), respectively. We used as a basic instrument list two lags of all real returns employed in computing $\widehat{M_t}$, two lags of $\ln\left(\frac{c_t}{c_{t-1}}\right)$, two lags of $\ln\left(\frac{y_t}{y_{t-1}}\right)$, and one lag of $\ln\left(\frac{c_t}{y_t}\right)$. This basic list was altered in order to verify the robustness of empirical results. We also include OLS estimates to serve as benchmarks in all three tables.

Table 1 Power-Utility Function Estimates $\widehat{m_t} = \ln \beta - \gamma \Delta \ln c_t - \eta_t^{CRRA}$			
Instrument Set	β (SE)	γ (SE)	OIR Test (P-Value)
OLS Estimate	1.002 (0.006)	1.979 (0.884)	–
$r_{i,t-1}, r_{i,t-2}, \forall i = 1, 2, \cdots N$, constant	0.999 (0.003)	1.125 (0.517)	(0.9953)
$r_{i,t-1}, r_{i,t-2}, \forall i = 1, 2, \cdots N,$ $\Delta \ln c_{t-1}, \Delta \ln c_{t-2}$, constant	1.001 (0.003)	1.370 (0.511)	(0.9964)
$r_{i,t-1}, r_{i,t-2}, \forall i = 1, 2, \cdots N,$ $\Delta \ln y_{t-1}, \Delta \ln y_{t-2}$, constant	1.000 (0.003)	1.189 (0.523)	(0.9958)
$r_{i,t-1}, r_{i,t-2}, \forall i = 1, 2, \cdots N, \Delta \ln c_{t-1},$ $\Delta \ln c_{t-2}, \Delta \ln y_{t-1}, \Delta \ln y_{t-2}, \ln \frac{c_{t-1}}{y_{t-1}}$, const.	0.999 (0.003)	1.204 (0.514)	(0.9985)

Notes: (1) Except when noted, all estimates are obtained using the generalized method of moments (GMM) of Hansen (1982), with robust Newey and West (1987) estimates for the variance-covariance matrix of estimated parameters. (2) OIR Test denotes the over-identifying restrictions test discussed in Hansen (1982).

Table 1 reports results obtained using a power-utility specification for preferences. The first thing to notice is that there is no evidence of rejection in over-identifying restrictions tests in any GMM regression we have run. Moreover, all of them showed sensible estimates for the discount factor and the risk-aversion coefficient: $\widehat{\beta} \in [0.999, 1.001]$, where in all cases the discount factor is not statistically different from unity and $\widehat{\gamma} \in [1.125, 1.370]$, where in all cases the relative risk-aversion coefficient is likewise not statistically different from unity. Our preferred regression is the last one in Table 1, where all instruments are used in estimation. There, $\widehat{\beta} = 0.999$ and $\widehat{\gamma} = 1.204$. These numbers are close to what could be expected *a priori* when power utility is considered; see the discussion in Mehra and Prescott (1985). They are in line with several panel-data estimates of the relative risk-aversion coefficient, such as Runkle (1991), Attanasio and Weber (1985) and Blundell, Browning and Meghir (1994). For the latter, the intertemporal substitution elasticity at the sample mean ($-1/\gamma$ here) is found to be between -0.77 and -0.75. Here, it would be between -0.89 and -0.72. Our estimates are also in line with recent results using time-series

data obtained in Mulligan (2002), where the estimates of $1/\gamma$ are close to unity most of the time.

Our estimates $\widehat{\beta}$ and $\widehat{\gamma}$ in Table 1 are somewhat different from early estimates of Hansen and Singleton (1982, 1984). As is well known, the equity-premium puzzle emerged as a result of rejecting the over-identifying restrictions implied by the complete system involving real returns on equity and on the T-Bill: Hansen and Singleton's estimates of γ are between 0.09 and 0.16, with a median of 0.14, all statistically insignificant in testing. All of our estimates are statistically significant, and their median estimate is 1.20 – almost ten times higher.

Table 2				
External-Habit Utility-Function Estimates				
$\widehat{m_t} = \ln\beta - \gamma\Delta\ln c_t + \kappa\left(\gamma - 1\right)\Delta\ln c_{t-1} - \eta_t^{EH}$				
Instrument Set	β	γ	κ	OIR Test
	(SE)	(SE)	(SE)	(P-Value)
OLS Estimate	1.002	1.975	-0.008	–
	(0.006)	(0.972)	(0.997)	
$r_{i,t-1}, r_{i,t-2}, \forall i = 1, 2, \cdots N$, constant	1.005	1.263	-2.847	(0.9911)
	(0.003)	(0.618)	(8.333)	
$r_{i,t-1}, r_{i,t-2}, \forall i = 1, 2, \cdots N,$	0.9954	1.308	1.997	(0.9954)
$\Delta\ln c_{t-1}, \Delta\ln c_{t-2}$, constant	(0.003)	(0.562)	(3.272)	
$r_{i,t-1}, r_{i,t-2}, \forall i = 1, 2, \cdots N,$	0.987	1.592	3.588	(0.9951)
$\Delta\ln y_{t-1}, \Delta\ln y_{t-2}$, constant	(0.003)	(0.688)	(3.742)	
$r_{i,t-1}, r_{i,t-2}, \forall i = 1, 2, \cdots N, \Delta\ln c_{t-1},$	0.987	1.161	8.834	(0.9980)
$\Delta\ln c_{t-2}, \Delta\ln y_{t-1}, \Delta\ln y_{t-2}, \ln\frac{c_{t-1}}{y_{t-1}}$, const.	(0.002)	(0.621)	(32.769)	

Notes: Same as Table 1.

Table 2 reports results obtained when (external) habit formation is considered in preferences. Results are very similar to those obtained with power utility. A slight difference is the fact that, with one exception, all estimates of the discount factor are smaller than unity. We cannot reject time-separation for all regressions we have run – κ is statistically zero in testing everywhere. In this case, the external-habit specification collapses to that of power-utility, which should be preferred as a more parsimonious model.

Table 3 Kreps–Porteus Utility-Function Estimates $\widehat{m_t} = \theta \ln \beta - \theta \gamma \Delta \ln c_t - (1 - \theta) \ln B_t - \eta_t^{KP}$				
Instrument Set	β (SE)	γ (SE)	θ (SE)	OIR Test (P-Value)
OLS Estimate	1.007 (0.006)	3.141 (0.886)	0.831 (0.022)	–
$r_{i,t-1}, r_{i,t-2}, \forall i = 1, 2, \cdots N$, constant	1.001 (0.004)	1.343 (0.723)	0.933 (0.014)	(0.9963)
$r_{i,t-1}, r_{i,t-2}, \forall i = 1, 2, \cdots N$, $\Delta \ln c_{t-1}, \Delta \ln c_{t-2}$, constant	1.003 (0.004)	1.360 (0.768)	0.922 (0.012)	(0.9980)
$r_{i,t-1}, r_{i,t-2}, \forall i = 1, 2, \cdots N$, $\Delta \ln y_{t-1}, \Delta \ln y_{t-2}$, constant	1.000 (0.004)	0.926 (0.756)	0.927 (0.013)	(0.9969)
$r_{i,t-1}, r_{i,t-2}, \forall i = 1, 2, \cdots N, \Delta \ln c_{t-1},$ $\Delta \ln c_{t-2}, \Delta \ln y_{t-1}, \Delta \ln y_{t-2}, \ln \frac{c_{t-1}}{y_{t-1}}$, const.	0.997 (0.004)	0.362 (0.761)	0.901 (0.012)	(0.9996)

Notes: Same as Table 1.

Results using the Kreps-Porteus specification are reported in Table 3. To implement its estimation a first step is to find a proxy to the optimal portfolio. We followed Epstein and Zin (1991) in choosing the NYSE for that role, although we are aware of the limitations they raise for this choice. With that caveat, we find that the optimal portfolio term has a coefficient that is close to zero in value (θ close to unity), although $(1 - \theta)$ is not statically zero in any regressions we have run. If it were, then the Kreps-Porteus would collapse to the power-utility specification. The estimates of the relative risk-aversion coefficient are not very similar across regressions, ranging from 0.362 to 1.360. Moreover, they are not statistically different from zero at the 5% significance level, which differs from previous estimates in Tables 1 and 2. There is no evidence of rejection in over-identifying restrictions tests in any GMM regression we have run, which is in sharp contrast to the early results of Epstein and Zin using this same specification.

Since the Kreps-Porteus encompasses the power utility specification, the former should be preferred to the latter in principle because $(1 - \theta)$ is not statistically zero. A reason against it is the limitation in choosing a proxy for the optimal portfolio. Therefore, the picture that emerges from the analysis of Tables 1, 2 and 3 is that both

the power-utility and the Kreps-Porteus specifications fit the CCAPM reasonably well when our estimator of the SDF is employed in estimation. These results *resurrect the CCAPM* as Lettau and Ludvigson (2001) propose, and to which Mulligan (2002, 2004) implicitly agrees.

4.4   Investigating the Equity-Premium Puzzle

We now turn to investigating the existence of the Equity-Premium Puzzle (EPP) using our estimator of M_t. Signs of the EPP have been known since the early GMM estimates in Hansen and Singleton (1982, 1984), because of rejections in the over-identifying restrictions tests. Mehra and Prescott (1985) put forth the EPP in some-what different terms. To avoid estimation of CCAPM parameters, they use a "cali-bration" argument: if the discount-factor coefficient β is set at a "reasonable" level, then an absurd value for the risk-aversion coefficient ($\gamma > 10$, at least) is needed to explain the historical difference between the return on equity and on a riskless bond (the equity premium). This difference is 6.2% per year in real terms in their sample and 7.6% in ours. Therefore, the puzzle is the inability of the CCAPM to explain such a large equity premium once we consider reasonable values for β and γ with a CRRA specification.

In the previous Section we have shown that it is possible to obtain reasonable estimates for β and γ when our estimator of the SDF is used in the analysis, c.f. Tables 1, 2 and 3. To claim that there is not an EPP, we still have to show that, using our estimator of M_t, the observed behavior of the equity premium is consistent with economic theory. We look for some testable implications arising from theory that involve the equity premium and our SDF estimator. Recall the Pricing Equation:

$$\mathbb{E}_{t-1}\left\{M_t R_{i,t}\right\} = 1, \qquad i = 1, 2, \dots, N. \tag{39}$$

If we focus on the behavior of a risky asset, whose return is labelled R_t^e, and a risk-free

asset, whose return is labelled R_t^f, we can combine the two resulting equations and use the Law-of-Iterated Expectation to obtain,

$$\mathbb{E}\left\{M_t\left(R_t^e - R_t^f\right)\right\} = 0. \tag{40}$$

Equation (40) offers a testable implication of the equity premium using our estimator of M_t: $M_t\left(R_t^e - R_t^f\right)$ must have a zero mean. Following Hansen and Singleton and Mehra and Prescott, who considered the return on the risky asset to be the return on equity, and on the riskless asset to be the return on the T-Bill, $\left(R_t^e - R_t^f\right)$ is observable, and a direct test of the existence of the EPP using our estimator of M_t consists in verifying whether (40) holds. There is one interesting variant of this test that we also implement here, which is the use of consumption-based estimates of M_t in verifying (40).

In addition to these tests, we go a step further investigating the existence of the EPP, asking whether or not there is any statistical difference between the mean real return on the T-Bill, labelled R_t^{T-Bill}, and that on our estimate of R_t^f. There are no theoretical grounds to implement such a test, because there is no basis to treat the T-Bill as measurable, given that its real return is not known in advance. Nevertheless, several authors believe that the T-Bill is a close enough approximation to a riskless asset. Therefore, we also test whether

$$\mathbb{E}\left(R_t^{T-Bill} - R_t^f\right) = 0 \tag{41}$$

holds.

Zero-mean tests such as those in (40) and (41) are straightforward to implement. The only issue is how to construct a robust estimate for the variance of sample-mean estimates, taking into account possible serial correlation and heteroskedasticity in their components. Here we employ the non-parametric estimate proposed by Newey and West (1987).

Table 4	
Equity-Premium Tests	
Version of M_t Used in Testing	Null Hypothesis in Two-Sided Tests $\mathbb{E}\left\{M_t\left(R_t^e - R_t^{T-Bill}\right)\right\} = 0$
	Sample Mean (Robust P-Value)
Estimator in Proposition 1	0.014064 (0.0859)
Consumption-Based CRRA Specification	0.018146 (0.0287)
Consumption-Based Habit-Formation Specification	0.017972 (0.0292)
Consumption-Based Kreps-Porteus Specification	0.017200 (0.0383)

Notes: (1) The return on equity is labelled R_t^e and the risk-free return is labelled R_t^{T-Bill}. (2) The return on equity used the NYSE index in real terms and the risk-free return used the real return on the 90-day T-Bill. (3) There are four possible series used to represent M_t – the estimator in Proposition 1, and consumption-based estimates using the CRRA, Habit-Formation and Kreps-Porteus specifications, with parameter estimates in the last rows of Tables 1, 2, and 3, respectively. (4) P-values come from the asymptotic distribution when the variance of the sample mean is computed using the non-parametric estimate proposed by Newey and West (1987).

Table 4 summarizes the results of equity-premium tests. When we use our estimator of M_t to investigate whether or not the zero-mean condition in (40) holds, we obtained a sample mean for $M_t\left(R_t^e - R_t^{T-Bill}\right)$ of 0.014, with a p-value of 0.086, not rejecting the null at the 5% level. A different picture emerges when we use consumption-based estimates of M_t. If the CRRA specification is employed, the sample mean for $M_t\left(R_t^e - R_t^{T-Bill}\right)$ is 0.018, with a p-value of 0.029, which rejects the null at this same level. The same happens when the Habit-Formation and Kreps-Porteus specifications are used: the sample means for $M_t\left(R_t^e - R_t^{T-Bill}\right)$ are 0.018 and 0.017, respectively, which yield the following respective p-values: 0.029 and 0.038, both rejecting the null at the usual levels.

It is important to reconcile our findings with previous investigations of the EPP, the literature on which spans two decades. In order to obtain $\mathbb{E}\left\{M_t\left(R_t^e - R_t^{T-Bill}\right)\right\} =$

0 one needs a large negative covariance between M_t and $\left(R_t^e - R_t^{T-Bill}\right)$. This happens because $\mathbb{E}(M_t) \cong 1$, therefore, $\mathbb{E}\left\{M_t\left(R_t^e - R_t^{T-Bill}\right)\right\} = 0$ implies:

$$-\text{Cov}\left[M_t, \left(R_t^e - R_t^{T-Bill}\right)\right] \cong \mathbb{E}\left(R_t^e - R_t^{T-Bill}\right).$$

Notice that $\mathbb{E}\left(R_t^e - R_t^{T-Bill}\right)$ is the mean equity premium, a large number by all accounts. Therefore, (40) requires first a negative covariance between M_t and $\left(R_t^e - R_t^{T-Bill}\right)$, which must be sufficiently large in absolute value.

Turning to sample estimates, the equity premium is 0.0186 per quarter in our sample. The CRRA specification yields the smallest covariance (in absolute value) among SDF estimates (-0.000072), followed by the Habit-Formation specification (-0.000153), by the Kreps-Porteus specification (-0.00181), and by our estimator in Proposition 1 (-0.003371), which is almost twice the largest absolute covariance of consumption-based estimates. Perhaps it is easier to look at correlation coefficients, which are, respectively, -0.1321, -0.2396, -0.7363, -0.7567. Therefore, rejections of $\mathbb{E}\left\{M_t\left(R_t^e - R_t^{T-Bill}\right)\right\} = 0$ are a consequence of a small correlation (in absolute value) between M_t and $\left(R_t^e - R_t^{T-Bill}\right)$: our estimator has just the appropriate covariance with $\left(R_t^e - R_t^{T-Bill}\right)$ to yield a mean zero for $M_t\left(R_t^e - R_t^{T-Bill}\right)$.

We now investigate whether or not the zero-mean condition in (41) holds. The mean difference between the return on $\widehat{R_t^f}$ and that of the T-Bill is -0.001633 on a quarterly basis. Testing whether the mean difference is zero yields a robust p-value of 0.6702 – showing that $\mathbb{E}\left\{\left(R_t^f - R_t^{T-Bill}\right)\right\} = 0$ when our estimator of the risk-free rate is used. This is not surprising, given that the sample average of the former is 2.39% a year and that of the latter is 3.12%; see also their implied risk-premia depicted in Figure 2.

5   Conclusions

In this paper, we propose a novel estimator for the stochastic discount factor (SDF), or pricing kernel, that exploits both the time-series and the cross-sectional dimensions of the data. It depends exclusively on appropriate averages of asset returns, which makes its computation a simple and direct exercise. In deriving it, instead of assuming a parametric function to characterize preferences, we treat the SDF as a random process that can be estimated consistently as the number of time periods and assets grow without bounds. Because our SDF estimator does not depend on any assumptions about preferences, or on consumption data, we are able to use it to test directly different preference specifications which are commonly used in finance and in macroeconomics. Moreover, our SDF estimator offers an immediate estimate of the *risk-free* rate, allowing us to discuss important issues in finance, such as the equity-premium puzzle.

A key feature of our approach is that it combines a general Taylor Expansion of the Pricing Equation with standard panel-data asymptotic theory to derive a novel consistent estimator for the SDF. In this context, we show that the econometric identification of the SDF only requires using the "serial-correlation common-feature property" of the logarithm of the SDF. We have followed two trends here: first, in financial econometrics, recent work avoids imposing stringent functional-form restrictions on preferences prior to estimation of the SDF; see Chapman (1998), Aït-Sahalia and Lo (1998, 2000), Rosenberg and Engle (2002), and Chen and Ludvigson (2004); second, in macroeconomics, early rejections of the optimal behavior for consumption using time-series data found by Hall(1978), Flavin(1981, 1993), Hansen and Singleton(1982, 1983, 1984), Mehra and Prescott(1985), Campbell (1987), Campbell and Deaton(1989), and Epstein and Zin(1991) were overruled by subsequent results using panel data by Runkle (1991), Blundell, Browning, and Meghir (1994) and Attanasio and Browning (1995), among others.

The techniques discussed above were applied to quarterly data of U.S.\$ real re-

turns from 1972:1 through 2002:4 representing investment opportunities available to the average U.S. investor. They cover thousands of assets worldwide, but are predominantly U.S.-based. Our SDF estimator – $\widehat{M_t}$ – is close to unity most of the time and bounded by the interval $[0.85, 1.15]$, with an equivalent average annual discount factor of 0.9760, or an annual discount rate of 2.46%. When we examined the appropriateness of different functional forms to represent preferences, we concluded that standard preference representations used in finance and in macroeconomics cannot be rejected by the data. Moreover, estimates of the relative risk-aversion coefficient are close to what can be expected *a priori* – between 1 and 2, statistically significant and not different from unity in statistical tests. These results can be reconciled with the recent literature using time-series data (Mulligan(2002)) and panel-data estimates (Blundell, Browning and Meghir (1994)).

A direct test of the equity-premium puzzle using our SDF estimator cannot reject that the discounted equity premium in the U.S. has mean zero. If one takes the equity-premium puzzle to mean the need to have incredible parameter values either for the discount factor β or the relative risk-aversion coefficient γ (or both) in order to achieve a mean-zero discounted equity premium in the U.S., then our results show little signs of the equity-premium puzzle. This happens mainly because our estimator of the SDF is highly negatively-correlated with the equity premium (-0.7567), whereas standard consumption-based estimates of the SDF have too small a correlation with it.

As we have argued above, the set of assumptions needed to derive our results is common to many papers in macroeconomics, finance, and financial econometrics. However, our estimator of the SDF and our empirical results are not. A striking characteristic of our approach is the combination of economic theory (Pricing Equation) with basic econometric tools (standard panel-data asymptotics) in deriving an estimator of M_t which is "preference free," delivers sensible overall empirical results, and bears little evidence of the equity-premium puzzle.

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