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Debt ceiling and fiscal sustainability in Brazil: a quantile autoregression approach

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Abstract

In this paper we investigate fiscal sustainability by using a quantile autoregression (QAR) model. We propose a novel methodology to separate periods of nonstationarity from stationary ones, which allows us to identify various trajectories of public debt that are compatible with fiscal sustainability. We use such trajectories to construct a debt ceiling, that is, the largest value of public debt that does not jeopardize long-run fiscal sustainability. We make out-of-sample forecast of such a ceiling and show how it could be used by Policy makers interested in keeping the public debt on a sustainable path. We illustrate the applicability of our results using Brazilian data.

JEL Classification: C22, E60, H60.
Keywords: Fiscal Policy, Debt Ceiling, Quantile Autoregression.

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1 Introduction

For decades, a lot of effort has been devoted to investigate whether long-lasting budget deficits represent a threat to public debt sustainability. Hamilton and Flavin (1986) was one of the first studies to address this question testing for the non-existence of Ponzi scheme in public debt. They conducted a battery of tests using data from the period 1962-84 and assuming a fixed interest rate. Their results indicate that the government intertemporal budget constraint holds. In a posterior work, Wilcox (1989) extends Hamilton and Flavin's work by allowing for stochastic variation in the real interest rate. His focus was on testing for the validity of the present-value borrowing constraint, which means that the public debt will be sustainable in a dynamically efficient economy[1] if the discounted public debt is stationary with unconditional mean equal to zero.

An important and common feature in the aforementioned studies is the underlying assumption that economic time series possess symmetric dynamics. In recent years, considerable research effort has been devoted to study the effect of different fiscal regimes on long-run sustainability of the public debt. When the public debt possesses a nonlinear dynamic, it may be sustainable in the long-run but can present episodes of unsustainability in the short-run. Indeed, some studies have reported the existence of short-run fiscal imbalances. For instance, Sarno (2001) uses a smooth transition autoregressive (STAR) model to investigate the U.S. debt-GDP ratio and states that the difficulty encountered in the literature to detect mean reversion of the debt process may be due to the linear hypothesis commonly adopted in the testing procedures. According to the author, the U.S. debt-GDP ratio is well characterized by a nonlinearly mean reverting process, and governments respond more to primary deficits (surpluses) when public debt is particularly high (low).

More recently, Davig (2005) uses a Markov-switching time series model to analyze the behavior of the discounted U.S. federal debt. The author uses an extended version of Hamilton and Flavin (1986) and Wilcox (1989) data and identifies two fiscal regimes: in the first one, the discounted federal debt is expanding, whereas, it is collapsing in the second one. He concludes that although the expanding regime is not sustainable, it does not pose a threat to the long-run sustainability of the discounted U.S. federal debt. Arestis et al. (2004) consider a threshold autoregressive model and, by using quarterly deficit data from the period 1947:2 to 2002:1, they find evidence that the U.S. budget deficit is sustainable in the long-run, but fiscal authorities only intervene to reduce budget deficit when it reaches a certain threshold, deemed to be unsustainable.

A common finding in the studies of Sarno (2001), Arestis et al. (2004), and Davig (2005) is that the presence of nonlinear dynamic in the public debt permits that there exist short episodes in which the public debt exhibits a nonsustainable behavior. Such a short-run behavior, however, does not pose a threat to the long-run sustainability. Therefore, there would be three possible paths for the public debt: (i) long-run sustainable paths with episodes of fiscal imbalances; (ii) long-run sustainable paths without episodes of fiscal imbalances and; (iii) long-run unsustainable paths. How could we identify and separate each of the aforementioned paths? This paper addresses this question by proposing a novel measure of debt ceiling that can be used to guide fiscal-policy managers in their task of keeping public debt sustainable in the long run.

The methodology developed in this paper is based on the so called quantile autoregressive (QAR) model,

[1] Abel et al. (1996) provides evidence that the U.S. economy is dynamically efficient.

introduced by Koenker and Xiao (2002, 2004a, 2004b). The QAR approach provides a way to directly examine how past information affects the conditional distribution of a time series. This feature of the QAR model is fundamental to the methodology proposed in this paper since our measure of debt ceiling ($\widetilde{D}_t$) will be nothing else than the upper conditional quantile of the public debt that satisfies the transversality condition of no-Ponzi game. Compared to the QAR approach, other non-linear methods such as the smooth transition autoregressive (STAR), threshold autoregressive (TAR) or Markov switching are not able to estimate conditional quantiles since they were originally proposed to estimate nonlinear models for conditional means (or variance).

The proposed measure of debt ceiling has the following main feature: if public debt y_t has a nonstationary behavior at time $t = t_A$, then $y_t > \widetilde{D}_t$ at $t = t_A$, otherwise $y_t \leq \widetilde{D}_t$. We also estimate $H \equiv \frac{1}{T}\sum_t I_t(y_t > \widetilde{D}_t)$, where $I(.)$ is an indicator function and T is the sample size, representing the percentage of periods in which public debt had an (local) unsustainable behavior. There are, therefore, two important issues we want to address in this paper. Firstly, how to identify $\widetilde{D}_t$ and, consequently, H? With this information in hand, the policy maker can evaluate whether a given fiscal policy is at risk, that is, if y_t is above $\widetilde{D}_t$, or whether it is sustainable but too austere, in the sense that y_t is too far below from $\widetilde{D}_t$. Secondly, how to make multi-step-ahead forecasts of the debt ceiling? A decision maker (fiscal authority) can use such a forecast to decide whether or not to take some action against long-run unsustainable paths of public debt.

The methodology developed in this paper complements the study of Garcia and Rigobon (2004), which proposed a very attractive technique to study debt sustainability from a risk management view by using a Value at Risk (VaR) approach based on Monte Carlo simulations. However, in their article, the choice of the quantile needed to compute the "risky" threshold of sustainability for public debt was somehow arbitrary. The methodology proposed in this paper goes beyond their approach by computing the exact quantile, the so-called critical quantile, that is used to separate sustainable fiscal policies from unsustainable ones. Therefore, our measure of debt ceiling can be viewed as a more elaborated concept of VaR in the sense that it appropriately uses economic theory to identify the quantile needed to compute the "risky" threshold, rather than choosing it arbitrarily.

We illustrate the applicability of our debt ceiling measurement by using Brazilian public debt data. Fiscal stabilization in Latin American countries, and specially in Brazil, has received a lot of attention over the last decade. In effect, Issler and Lima (2000) showed that public debt sustainability in Brazil from 1947 to 1992 was reached mostly because of the usage of revenue from seigniorage. However, after the Brazilian stabilization plan in 1994, this source of revenue disappeared, leading fiscal authorities to propose tax increases in order to run high primary surpluses needed to guarantee fiscal sustainability. The need of obtaining high primary surpluses possibly implied a shift to fiscal austerity and probably a cost in terms of foregone output and higher unemployment. Has the fiscal policy in Brazil been too austere or it has been just restrictive enough to avoid an excessive build up of debt? We answer these questions by using the measurement of debt ceiling developed in this paper.

This study is organized as follows: Section 2 describes the theoretical model to investigate public debt and the respective transversality condition to be tested, Section 3 presents the quantile autoregression model and a novel methodology to separate nonstationary observations from stationary ones. Section 4 describes debt ceiling on a QAR approach, Section 5 provides the empirical results for Brazilian public debt, and Section 6 summarizes the main conclusions.

2 Methodology

2.1 Theoretical Model

There is a large literature on the government intertemporal budget constraint. The general conclusion is that fiscal policy is sustainable if the government budget constraint holds in present value terms. In other words, the current debt should be offset by the sum of expected future discounted primary budget surpluses. The approaches used to analyze sustainability of fiscal policy consist in testing if the public debt and/or budget deficit is a stationary process.

The theoretical framework used here to investigate the sustainability of the Brazilian federal debt follows Uctum and Wickens (2000), which extends the results of Wilcox (1989) to a stochastic and time-varying discount rate, considering a discounted primary deficit that can be either strongly or weakly exogenous. According to the authors, a necessary and sufficient condition for sustainability is that the discounted debt-GDP ratio should be a stationary zero-mean process. As a starting point of the analysis, Uctum and Wickens (2000) investigate the one-period government intertemporal budget constraint, which can be written in nominal terms as

$$G_t - T_t + i_t B_{t-1} = \Delta B_t + \Delta M_t = -S_t, \tag{1}$$

where G = government expenditure, T = tax revenue, B = government debt at the end of period t, M = monetary base, S = total budget surplus, i = interest rate on government debt. Dividing each term of (1) by nominal GDP, one could obtain the budget constraint in terms of proportion of GDP

$$g_t - \tau_t + (i_t - \pi_t - \eta_t)b_{t-1} = \Delta b_t + \Delta m_t + (\pi_t + \eta_t)m_{t-1} = -s_t. \tag{2}$$

The variables g, τ, b, m, and s denote the ratio of the respective variables to nominal GDP, $\pi_t = (P_t - P_{t-1})/P_{t-1}$ and $\eta_t = (Y_t - Y_{t-1})/Y_{t-1}$, with P and Y standing for the price level and real GDP. This way, equation (2) can be rewritten as

$$d_t + \rho_t b_{t-1} = \Delta b_t, \tag{3}$$

where $d_t = g_t - \tau_t - \Delta m_t - (\pi_t + \eta_t)m_{t-1}$ is the primary government deficit expressed as a ratio to nominal GDP, and $\rho_t = i_t - \pi_t - \eta_t$ is the real ex-post interest rate adjusted for real output growth. According to the authors, if $\rho_t < 0$ for all t then equation (3) is a stable difference equation, which can therefore be solved backwards, implying that the debt-GDP ratio b_t will remain finite for any sequence of finite primary deficits d_t. It should be noted that for constant ρ and d, the steady-state value of b is given by $-d/\rho$.

On the other hand, if $\rho_t > 0$ for all t, then the debt-GDP ratio will eventually explode for $d_t > 0$. Thus, primary surpluses are required to avoid this case (i.e. $d_t < 0$), and equation (3) must be solved forwards, in order to determine whether the sum of expected future discounted surpluses is sufficient to meet the current level of debt-GDP ratio. In addition, the authors rewrite (in ex-ante terms) the budget constraint for period $t+1$ as

$$b_t = E_t[(1+\rho_{t+1})^{-1}(b_{t+1} - d_{t+1})], \tag{4}$$

where b_t is known in period t, and expectations are taken conditional on information at time t. Equation (4) is solved forwards, resulting on the n-period intertemporal budget constraint

$$b_t = E_t\delta_{t,n}b_{t+n} - E_t\sum_{i=1}^{n}\delta_{t,i}d_{t+i}, \tag{5}$$

where $\delta_{t,n} = \prod_{s=1}^{n}(1+\rho_{t+s})^{-1}$ is the time-varying real discount factor n periods ahead, adjusted for real GDP growth rate. The discount factor $\delta_{t,n}$ can also be written as $\delta_{t,n} = a_{t+n}/a_t$, where $a_t = \prod_{i=1}^{t}(1+\rho_i)^{-1}$.

The authors normalize $a_0 = 1$ and define $X_t = a_t b_t$ and $Z_t = a_t d_t$ as the discounted debt-GDP and primary deficit-GDP ratios respectively. This way, equation (5), representing the present-value borrowing constraint (PVBC), can be rewritten as

$$a_t b_t = E_t a_{t+n} b_{t+n} - E_t\sum_{i=1}^{n}a_{t+i}d_{t+i}, \tag{6}$$

or as

$$X_t = E_t X_{t+n} - E_t\sum_{i=1}^{n}Z_{t+i}. \tag{7}$$

The one-period budget constraint given by expression (3) can also be written in discounted terms, in the following way

$$b_{t-1} = (1+\rho_t)^{-1}(b_t - d_t) = (a_t/a_{t-1})(b_t - d_t), \tag{8}$$

$$\therefore X_{t-1} = a_{t-1}b_{t-1} = a_t b_t - a_t d_t = X_t - Z_t \therefore Z_t = \Delta X_t. \tag{9}$$

Hence, equation (4) can be expressed by

$$X_t = E_t(X_{t+1} - Z_{t+1}). \tag{10}$$

2.2 Sustainability for infinite horizon

According to Uctum and Wickens (2000), a necessary and sufficient condition for sustainability is that as n goes to infinity, the expected value of the discounted debt-GDP ratio converges to zero. This condition is usually known in the literature as the transversality condition (or no-Ponzi-scheme condition), and can be summarized by

$$\lim_{n\to\infty} E_t X_{t+n} = 0. \tag{11}$$

This way, the current debt-GDP ratio is counterbalanced by the sum of current and expected future discounted surpluses, also expressed as a proportion of GDP, implying that the government budget constraint is given (in present value terms) by

$$b_t = -\lim_{n\to\infty} E_t\sum_{i=1}^{n}\delta_{t,i}d_{t+i}, \tag{12}$$

or

$$X_t = -\lim_{n\to\infty} E_t \sum_{i=1}^{n} Z_{t+i}, \tag{13}$$

Uctum and Wickens (2000) show that the necessary and sufficient condition for the intertemporal budget constraint (13) to hold is that the discounted debt-GDP ratio (X_t) be a stationary zero-mean process. This way, if fiscal policy is currently (locally) unsustainable, then it will need to change in the future to guarantee (global) sustainability. In addition, the transversality condition requires discounted debt-GDP ratio to converge to zero.

A starting point to investigate this condition arises from a graphical analysis of the discounted debt time series, which should be declining over the sample period. In this paper, we perform a formal test of sustainability of the Brazilian federal debt, investigating the validity of the (necessary and sufficient) condition of stationarity with zero mean for the discounted debt-GDP ratio process. We will do so by using the quantile autoregression model which is briefly described in the next section.

3 The Quantile Autoregression Model

In a sequence of recent papers Koenker and Xiao (2002, 2004a, 2004b) introduced the so-called quantile autoregression (QAR) model. In this paper, we will show how one can separate nonstationary observations from stationary ones by using the QAR model. This result will have important implications on the literature of public-debt sustainability as shown in the next sections. For now, consider the following assumptions

Assumption 1 let $\{U_t\}$ be a sequence of iid standard uniform random variables;

Assumption 2 Let $\alpha_i(U_t)$, $i = 0, ..., p$ be comonotonic random variables.[2]

We define the pth order autoregressive process as follows,

$$y_t = \alpha_0(U_t) + \alpha_1(U_t)\, y_{t-1} + ... + \alpha_p(U_t)\, y_{t-p}, \tag{14}$$

where α_j's are unknown functions $[0,1] \to \mathbb{R}$ that we will want to estimate. We will refer to this model as the QAR(p) model. Given assumptions 1 and 2, the conditional quantile of y_t is given by

$$Q_{y_t}(\tau \mid \mathcal{F}_{t-1}) = \alpha_0(\tau) + \alpha_1(\tau)\, y_{t-1} + ... + \alpha_p(\tau)\, y_{t-p},$$

where $\mathcal{F}_{t-1} = (y_{t-1}, ..., y_{t-p})$ and τ is the quantile of U_t.

In order to develop an intuition about the QAR model, let us consider the following simple example

$$y_t = \alpha_0(U_t) + \alpha_1(U_t)\, y_{t-1}, \tag{15}$$

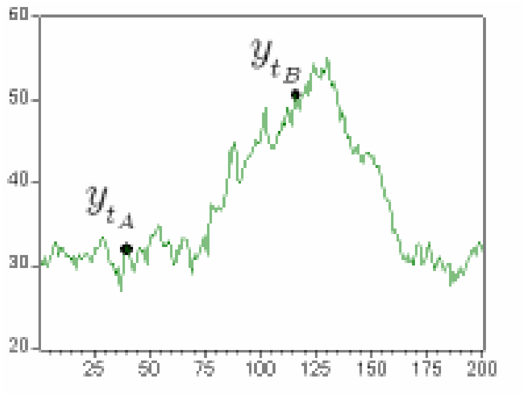
[2] According to Koenker (2006), two random variables $X, Y : \Omega \to \mathbb{R}$ are said to be comonotonic if there exist a third random variable $Z : \Omega \to \mathbb{R}$ and increasing functions f and g such that $X = f(Z)$ and $Y = g(Z)$. In our paper, $\beta_{i,t} = \alpha_i(U_t)$, $i = 0, 1, ..., p$ are comonotonic and $\alpha_i(\cdot)$ are, by definition, increasing functions. See our proofs in appendix to understand the crucial usefulness of this assumption.

which is simply a QAR(1) model. It should be noted that the QAR model can play a useful role in expanding the territory between classical stationary linear time series and their unit root alternatives. To see this, suppose in our QAR(1) example that $\alpha_1(U_t) = U_t + 0.5$. In this case, if $0.5 \leq U_t < 1$ then the model generates y_t according to the nonstationary model, but for smaller realizations of U_t, we have mean reversion tendency. Thus, the model exhibits a form of asymmetric persistence in the sense that sequences of strongly positive innovations of the iid standard uniform random variable U_t tend to reinforce its nonstationary like behavior, while occasional smaller realizations induce mean reversion and thus undermine the persistency of the process. Therefore, it is possible to have locally nonstationary time series being globally stationary.[3]

3.1 Identifying Nonstationary Observations

We continue our motivation by considering again the QAR(1) model (15) with the same autoregressive coefficient $\alpha_1(U_t) = U_t + 0.5$. If at a given period $t = t_A$, $U_{t_A} = 0.2$, then $\alpha_1(U_{t_A}) = 0.7$ and the model will present a mean reversion tendency at $t = t_A$. However, if at $t = t_B$, $U_{t_B} = 0.5$, then $\alpha_1(U_{t_B}) = 1$, and y_t will have a local unit-root behavior. Suppose, for illustrative purpose, that this model can be represented by the stochastic process depicted in Figure 1, in which y_t has a mean reversion tendency around the period t_A. Now assume that for periods $t > t_A$, there is a sequence of strong realizations of U_t inducing the model to a nonstationary behavior at period t_B.[4]

Figure 1 - Example of a QAR(1) model

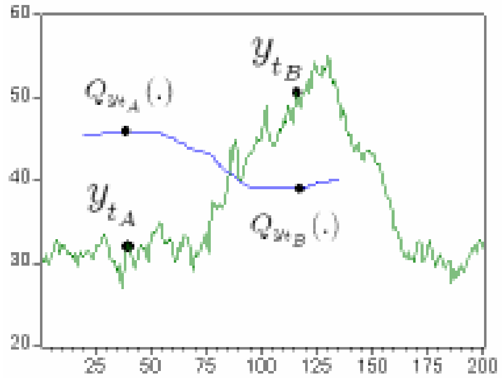


A natural question that arises in this context is how to separate periods of stationarity from periods where y_t exhibits a nonstationary behavior? In other words, is it possible to construct a function $Q_{y_t}(.)$ such that if y_t has a mean reversion tendency at time $t = t_A$ then $Q_{y_{t_A}}(.) \geq y_{t_A}$, but if y_t presents a nonstationary behavior at time $t = t_B$ then $Q_{y_{t_B}}(.) < y_{t_B}$?

[3] See Appendix for further details regarding the QAR model, including alternative representations, stationarity condtions, central limit theorem, estimation, autoregressive order choice, global stationarity, unconditional mean tests, and local analysis through the Koenker & Xiao (2004b) test.

[4] The DGP used to construct this example is represented by the QAR(1) model $y_t = \alpha_1(U_t)\, y_{t-1}$ where $\{U_t\}$ is a sequence of iid standard uniform random variables, and the coefficients α_1 is a function on $[0, 1]$, given by $\alpha_1(U_t) = \min\{1; \gamma_1 * U_t\}$, where $F : \mathbb{R} \rightarrow [0, 1]$ is the standard normal cumulative distribution function. We set the parameter $\gamma_1 = 0.8$ for $t = (1, ..., 65)$; 10 for $t = (66..., 90)$; 5 for $t = (91, ..., 152)$ and 0.8 for $t = (153, ..., 200)$.

Figure 2 - Separating periods of nonstationarity



This is a theoretical question that we aim to answer in this paper by using the QAR approach. In order to separate observations of y_t that exhibit a unit-root behavior from other observations with stationary behavior, we will need the following definitions:

Definition 1 *Critical Quantile ($\tau_{crit.}$) is the largest quantile $\tau \in \Gamma = (0,1)$ such that $\alpha_{1,t}(\tau) = \sum_{i=1}^{p} \alpha_i(\tau) < 1$, where τ is the quantile of U_t*

Definition 2 *Critical Conditional Quantile of y_t: $Q_{y_t}(\tau_{crit.} \mid \mathcal{F}_{t-1}) = \alpha_0(\tau_{crit.}) + \alpha_1(\tau_{crit.})\, y_{t-1} + ... + \alpha_p(\tau_{crit.})\, y_{t-p}$, where $\mathcal{F}_{t-1} = (y_{t-1}, ..., y_{t-p})$.*

The critical quantile $\tau_{crit.}$ can easily be identified by using the Koenker & Xiao (2004b) test for H_0 : $\alpha_{1,t}(\tau) = 1$ for selected quantiles $\tau \in \Gamma = (0,1)$, presented in Appendix. The critical conditional quantile $Q_{y_t}(\tau_{crit.} \mid \mathcal{F}_{t-1})$, is merely the τth conditional quantile function of y_t evaluated at $\tau = \tau_{crit.}$. Consider the additional assumption

Assumption 3 Let $\Omega = (t_1, t_2, ...t_T)$ be the set of all observations T. Assume that for the subset of time periods $\Upsilon \subset \Omega$, the time series y_t exhibits a nonstationary behavior, i.e., unit root model. Now we can state proposition 1[5].

Proposition 1 *Consider the QAR(p) model (14) and Assumptions 1, 2 and 3. The critical conditional quantile of y_t will always be lower than y_t for all periods in which y_t exhibits a unit-root behavior, that is, $Q_{y_t}(\tau_{crit.} \mid \mathcal{F}_{t-1}) < y_t \;\; ; \;\; \forall t \in \Upsilon$.*

Proof. See Appendix. ■

In order to clarify this result, suppose that all the observations of y_t, $t = 1, ...T$, exhibits a unit-root (stationary) behavior. In this case, the path of y_t would always be above (below) the path generated by $Q_{y_t}(\tau_{crit.} \mid \mathcal{F}_{t-1})$. There may exist an intermediate case in which some observations of y_t exhibits a unit-root

[5] A Monte Carlo experiment is presented in Appendix to verify the result of Proposition 1 in finite samples. The simulation reveals that the critical conditional quantile indeed exhibits a good behavior in finite samples, by correctly separating the nonstationary periods from the stationary ones.

behavior. In this case, the path of y_t would be above the path generated by $Q_{y_t}(\tau_{crit.} \mid \mathcal{F}_{t-1})$ only at the periods where y_t has a unit root.

In addition, by just comparing both time series y_t and $Q_{y_t}(\tau_{crit.} \mid \mathcal{F}_{t-1})$, one can compute the statistic H, which represents the percentage of periods in which y_t exhibits a (local) nonstationary behavior.

Definition 3 *Let H be the relative frequency of nonstationary periods, that is, $H \equiv \frac{1}{T}\sum_{t=1}^{T} I_t \left\{y_t > Q_{y_t}(\tau_{crit.} | \mathcal{F}_{t-1})\right\}$, where T is the sample size and I_t is an indicator function such that $I_t = \begin{cases} 1 \;\; ; \;\; if \;\; y_t > Q_{y_t}(\tau_{crit.} \mid \mathcal{F}_{t-1}) \\ 0 \;\; ; \;\; otherwise \end{cases}$*

In order to link the statistic H with the critical quantile, we can also state Proposition 2 :

Proposition 2 *If Assumptions 1 and 2 hold, then $H = (1 - \tau_{crit.})$.*

Proof. See Appendix. ■

These two propositions enable us to identify periods in which the time series y_t exhibits nonstationary (stationary) behavior. This methodology will be crucial to our analysis of fiscal sustainability as further described in section 4.

3.2 Out-of-sample forecast

In the previous sections we showed how to identify the critical conditional quantile. Now, we show how to make multi-step-ahead forecasts of the critical conditional quantile. In order to do so, we first forecast y_t based on the simple idea of recursive generation of its conditional density, which is quite a novelty approach introduced by Koenker and Xiao (2006)[6].

Recall that T is the sample size and let s be the forecast horizon. Given an estimated QAR model $\widehat{Q}_{y_t}(\tau \mid \mathcal{F}_{t-1}) = x_t'\widehat{\alpha}(\tau)$ based on data $t = 1, ..., T$ we can forecast

$$\widehat{y}_{T+s} = \widetilde{x}_{T+s}'\widehat{\alpha}(U_{T+s}) \quad ; \text{ for } s = 1, ..., s_{\max} \tag{16}$$

where U_{T+s} ∼iid Uniform $(0,1)$; $\widetilde{x}_{T+s}' = [1, \widetilde{y}_{T+s-1}, ..., \widetilde{y}_{T+s-p}]'$ and $\widetilde{y}_t = \begin{cases} y_t \text{ if } t \leq T \\ \widehat{y}_t \text{ if } t > T \end{cases}$

Conditional density forecasts can be made based on an "ensemble" of such forecast paths, i.e., a great number (k) of future trajectories of y_t enables us to construct the conditional density of y_t at each future period $T + s$.

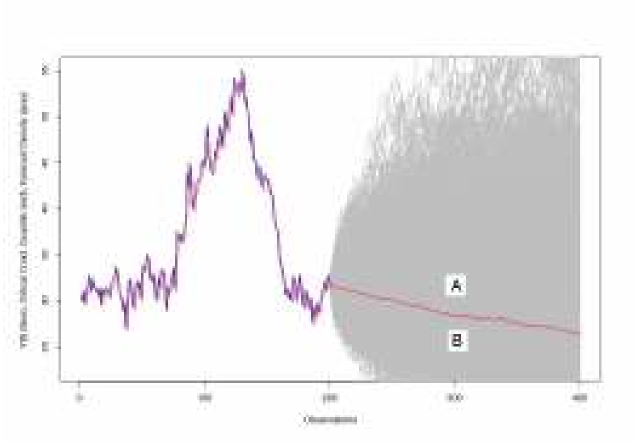
To better understand this idea, notice that $U_{T+s} \sim$ iid Uniform $(0, 1)$. Hence, it is always possible to establish a 1:1 relationship between τ and a realization, u_{T+s}, of this iid standard uniform random variable U_{T+s}. Thus, for each realization of U_{T+s}, there is a 1:1 corresponding quantile $\tau = u_{T+s}$. Moreover, in estimating the conditional quantile function of y_t, $Q_{y_t}(\tau \mid \mathcal{F}_{t-1})$, one can find the estimated coefficients $\widehat{\alpha}_i(\tau)$ for each τ and, therefore, we can find $\widehat{\alpha}(U_{T+s})$ for any realization of U_{T+s}. We proceed by generating

[6] Koenker and Xiao presented this forecasting approach in the Econometrics in Rio conference, which took place in the economic department of the Getulio Vargas Foundation, Rio de Janeiro, Brazil.

a sequence of realizations of U_{T+s} of size $s_{\max}$, that is, $\{u_{T+s}\}$, $s = 1, 2, ...s_{\max}$. This way, we can make an out-of-sample trajectory of y_t through equation (16). If we repeat the above steps k times, then we will end up with an ensemble of forecast paths. We can now forecast the critical conditional quantile based on this ensemble of forecasts. In order words, for a given period $T + s$, $\widehat{Q}_{y_{T+s}}(\tau_{crit.} \mid \mathcal{F}_T) = \widehat{y}^{k^*}_{T+s}$ so that $\Pr\left(\widehat{y}^{k}_{T+s} \leq \widehat{y}^{k^*}_{T+s} \mid \mathcal{F}_T\right) = \tau_{crit}$. This way, we are capable to generate the sequence $\{\widehat{Q}_{y_{T+s}}(\tau_{crit.} \mid \mathcal{F}_T)\}$ for $s = 1, .., s_{\max}$, which is nothing else than the forecast path of the critical conditional quantile. This methodology allows us to classify the future observations of the time series y_t into stationary and nonstationary ones.[7]

In order to clarify the idea of multi-step-ahead forecast, consider again the QAR(1) model discussed in section 3.1. Thus, based on the estimated coefficients $\widehat{\alpha}_i(\tau)$ and the generation of k sequences of $U_{T+s} \sim$ iid Uniform of size $s_{\max}$, we can compute (see Figure 3) the conditional densities of y_{T+s} for the forecast horizons $s = 1, ..., s_{\max}$. In our example, we considered $k = 1,000$ trajectories and $s_{\max} = 200$ periods.

Figure 3 - Out-of-sample forecast of y_t



Notes: (a) The picture shows the forecast conditional densities of the mentioned QAR(1) model for k=1,000 trajectories.

(b) The red line represents the in-sample and out-of-sample forecasts of the critical conditional quantile.

The Figure 3 summarizes the above discussion. The red line represents the forecast of the critical conditional quantile. We can see that the out-of-sample forecast of the critical conditional quantile splits the ensemble of forecasts into two regions, A and B. All the paths in region A are nonstationary whereas they are stationary in region B. As we will show in our empirical exercise on section 5, this separation has strong economic implications. Furthermore, the out-of-sample forecast of the critical conditional quantile apparently tends to zero as long as the forecast horizon increases. In fact, as we will formally show in Proposition 3 , if the time series process y_t is a zero-mean stationary process, then its critical conditional quantile will converge to zero at an infinite horizon.

[7] See Appendix B for further details regarding the numerical procedure.

4 Debt Ceiling and Fiscal Sustainability.

Hereafter let y_t be the discounted debt-GDP ratio process (X_t) presented in section 2. Before introducing a "sustainability" concept, lets consider the following (testable) additional assumptions:

Assumption 4: The time series y_t is covariance stationary.

Assumption 5: The unconditional mean of y_t is zero, i.e., $\mu_y = 0$.

Notice that assumption 5 holds if we set $\alpha_0(U_t) = 0$ in Eq. (14). In this case, the out-of-sample forecast $\widehat{y}_{T+s} = \widetilde{x}'_{T+s}\widehat{\alpha}\,(U_{T+s})$ would be computed from a vector without intercept $\widetilde{x}'_{T+s} = [\widetilde{y}_{T+s-1}, ..., \widetilde{y}_{T+s-p}]$.

Hence, based on the study of Uctum&Wickens (2000), we adopt the following concept of public debt sustainability:

Definition 4 *A fiscal policy is "globally sustainable" if and only if the discounted debt-GDP ratio y_t is a stationary zero-mean process, that is, it satisfies assumptions 4 and 5.*

The previous assumptions denote that y_t is a stationary zero-mean process, which is a necessary and sufficient condition for global sustainability. If a fiscal policy is sustainable in the long run, there can still be local episodes of fiscal imbalances. How can we identify such local episodes and separate sustainable fiscal policies from unsustainable ones? In order to answer these questions, we define the concept of debt ceiling.

Definition 5 *Debt ceiling ($\widetilde{D}_t$) is equal to the critical conditional quantile when assumptions 1-5 hold.*

The above definition establishes that the debt ceiling is nothing else than the critical conditional quantile of the discounted debt-GDP ratio, $\widetilde{D}_t \equiv Q_{y_t}(\tau_{crit.} \mid \mathcal{F}_{t-1})$. In order to clarify the concept of debt ceiling, suppose that all the observations of y_t, $t = 1, ...T$, exhibit an sustainable behavior. In this case, they would always be below or on the path generated by $\widetilde{D}_t$. There may exist an intermediate case in which the public debt is still globally sustainable despite some episodes of local unsustainability. In this case, the path of y_t would be above the path generated by $\widetilde{D}_t$ only at the periods where y_t takes on an unsustainable behavior. The proposed debt ceiling is a simple way to separate paths of public debt (fiscal policies) that are not sustainable from the ones that satisfy the long-run transversality condition. This discussion is summarized in the following corollary.

Corollary 1 *Consider the QAR(p) model (14), where now y_t represents the discounted debt-GDP ratio process. If Assumptions 1 until 5 hold, then the respective Debt Ceiling ($\widetilde{D}_t$) will always be lower than y_t in all periods where y_t is nonsustainable, that is, $\widetilde{D}_t < y_t$, $\forall t \in \Upsilon$.*

Proof. See Appendix. ■

Corollary 1 is an immediate consequence of Proposition 1 when the definitions 4-5 and the assumptions 4-5 are also considered. Based on Corollary 1, we have the nice result that $y_t > \widetilde{D}_t$ for all periods in which the public debt takes on an unsustainable dynamic. Moreover, given that y_t is, by (testable) assumptions, a stationary zero-mean process, by just comparing y_t and $\widetilde{D}_t$ one can also compute what we call "debt tolerance", that is, the percentage of episodes of local unsustainability that does not jeopardize long-run sustainability, that is:

$$H \equiv \frac{1}{T}\sum_{t=1}^{T} I_t \left\{y_t > \widetilde{D}_t\right\} \tag{17}$$

where $I(.)$ is an indicator function and T is the sample size. Therefore, given a globally sustainable fiscal policy, H represents the percentage of violations of the transversality condition still compatible with long-run fiscal sustainability[8].

Regarding the out-of-sample forecast, the following Proposition guarantees that the forecast path of debt ceiling will go to zero as the forecast horizon goes to infinity. This is an expected result from the literature of public debt sustainability, since the transversality condition (or no-Ponzi-game condition) states that the forecast value of a sustainable (discounted) debt-GDP ratio must converge to zero.

Proposition 3 *If assumptions 1 until 5 hold, then the forecast path of the Debt Ceiling ($\widehat{D}_{T+s}$) will go to zero as the forecast horizon s goes to infinity, i.e.,* $\lim_{s\to\infty} \widehat{D}_{T+s} = 0$.

Proof. See Appendix. ■

In the next section, we show that the debt ceiling concept can also be seen as a more elaborated concept of Value at Risk.

4.1 Debt Ceiling and Value at Risk

In the finance literature, Value at Risk (VaR) is a measurement representing the worst expected loss of an asset or portfolio over a specific time interval, at a given confidence level. It is typically used by securities houses or investment banks to measure the market risk of their asset portfolios.[9] The VaR_t can be defined as

$$\Pr\left(r_t \leq \mathrm{VaR}_t \mid \mathcal{F}_{t-1}\right) = \tau, \tag{18}$$

where r_t is the return of some financial asset, $\mathcal{F}_{t-1}$ is the information set available at time $t-1$, and $\tau \in (0,1)$ is the confidence level. From this definition, it is clear that finding a VaR_t is basically the same as finding a conditional quantile.

[8] Reinhart et al. (2003) developed the concept of "debt intolerance" based on a historical analysis about external debt. They divided the countries into debtors' clubs and vulnerability regions, depending principally on a country's own history of default and high inflation.

[9] For instance, if a given portfolio has a 1 day VaR of \$5 million (at the 95% confidence level), this implies that is expected that, with a probability of 95%, the value of its portfolio will decrease by 5 million or less during 1 day.

Following Hafner & Linton (2006), it is straightforward to show that the estimation of a VaR_t is a natural application of the QAR model, that is

$$\Pr\left(y_t \leq Q_{y_t}(.) \mid \mathcal{F}_{t-1}\right) = \tau_{crit.}. \tag{19}$$

In our application of the QAR model, we estimate the exact conditional quantile that represents the limit of stationarity (our critical conditional quantile), which is used to define the debt ceiling, in accordance to the government intertemporal budget constraint. Thus, our proposed debt ceiling is nothing else than a "qualified" Value at Risk, that is

$$\widetilde{D}_t \equiv Q_{y_t}(\tau_{crit.} \mid \mathcal{F}_{t-1}) = \mathrm{VaR}_t$$

It is important to note, however, that the proposed "qualified" VaR concept goes far beyond the finance applications, in which an "ad-hoc" value for $\tau_{crit.}$ is adopted (usually 1% or 5%).[10] In our approach, we identify the exact critical quantile that represents a threshold, $\tau_{crit.}$, according to a given theoretical economic model.

This is a novel approach in the literature of public debt sustainability, but it may have other applications in finance and macroeconomics. Garcia and Rigobon (2004) studied debt sustainability from a risk management perspective by using a Value at Risk (VaR) approach. The authors proposed a very attractive technique, based on Monte Carlo simulations, to compute "risk probabilities", i.e., probabilities that the simulated debt-GDP ratio exceeds a given threshold deemed "risky". However, their choice of the quantile needed to compute the "risky" threshold of sustainability was somehow arbitrary (see figure 4 of Garcia and Rigobon, 2004). The methodology proposed in this paper complements their approach by computing the exact "risky" quantile, the so-called $\tau_{crit.}$, which enable us to properly separate nonsustainable paths of public debt from sustainable ones, instead of choosing an "ad-hoc" threshold of sustainability.[11]

5 Empirical Results

5.1 The Database

The methodology presented in this paper is applied to analyze the discounted Brazilian federal debt. All data are quarterly and are obtained from the Central Bank of Brazil (BCB), Institute of Applied Economic Research (IPEA), and Brazilian Institute of Geography and Statistics (IBGE). Our sample covers the period 1976.I to 2005.I (117 observations). The undiscounted debt represents the series "Dívida Mobiliária Interna Federal fora do Banco Central", or federal domestic debt held by the public, in percentage of GDP.[12] The

[10] For instance, a bank capital requirements analysis usually fixes the critical quantile at 1%, whereas risk management models typically impose the confidence level at 5%.

[11] Moreover, this paper presents a distribution-free approach to make out-of-sample forecasts of the debt ceiling. The same does not happen in Garcia and Rigobon (2004) since their simulations are based on the assumption of normal distribution innovations.

[12] Following Rocha (1997), we focused the analysis on the domestic debt, since the sustainability of external debt is guaranteed by current account surpluses, and not by fiscal surpluses or seigniorage. Despite the debt-GDP ratio not be high in comparison to other nations, its sharp increase in the last decade is very concerning.

discounted debt is given by the undiscounted debt series multiplied by the stochastic discount factor. Bohn (2004) mentions that the debt-GDP ratio suggests a "more benign view" of fiscal policy than the nominal and real series.

The stochastic discount factor (a_t), as previously mentioned in the theoretical model, is generated from ρ_t (the real ex-post interest rate adjusted for real output growth), which depends on the inflation and nominal interest rates, and real output growth. The inflation rate (π_t) is measured in a standard approach by a general price index (IGP–DI), and the nominal interest rate (i_t) is measured by the over/selic interest rate (equivalent to the U.S. Fed funds rate). Regarding real output growth (η_t), we generate a quarterly series based on the quarterly GDP, which is released by IBGE, with seasonally adjustments made by the MA(12)[13] and X-11 methods.[14]

$$a_t = \prod_{i=0}^{t-1} \frac{1}{(1+\rho_i)} \quad ; \; a_0 = 1 \tag{20}$$

$$(1+\rho_t) = \frac{(1+i_t)}{(1+\pi_t)(1+\eta_t)} \tag{21}$$

According to Uctum and Wickens (2000), there are two major issues that must be addressed when using government debt data: whether to measure debt at market value or at face value (at par), and how to measure the discount rate.[15] The authors state that the correct implementation of the government intertemporal budget constraint requires the use of the discounted net market value of debt. However, the market value of debt is usually not available, and the debt is generally expressed at par. An estimate of the market value of debt is obtained by multiplying the face value by the implied market price $1/(1+p_t)$, where p_t is the yield on government debt. Some studies on the sustainability of the Brazilian public debt, such as Pastore (1995), Rocha (1997) and Giambiagi and Ronci (2004), used debt value at par, whereas Luporini (2000) uses market value. In our case, the analysis will only be conducted for the discounted debt at face value, since these two series, in our sample period, are very similar.

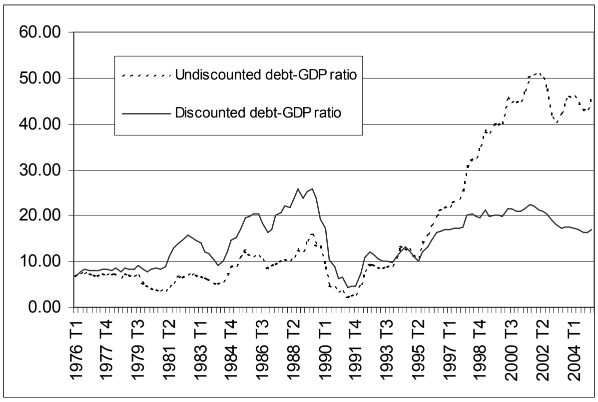
Figure 4 presents the undiscounted and discounted Brazilian federal debt-GDP ratio. A simply visual inspection in figure 4 suggests that the discounted debt seems to be stationary, despite the sharp increase path of the undiscounted series in the 1990s. The formal evidence on sustainability of the Brazilian public debt is investigated in the following sections.

[13] Following Garcia and Rigobon (2004).

[14] Since the results based on these two techniques are very similar, we only report the MA(12) results.

[15] According to Giambiagi and Ronci (2004), one should ideally use net-of-taxes real rate of interest. However, net-of-tax yield is a difficult task since tax rates vary according to security holder, and there is limited information on its identity.

Figure 4 - Brazilian federal debt (% GDP)



Note: Undiscounted debt corresponds to the federal domestic debt held by the public, in percentage of GDP.

5.2 Autoregressive Order Choice

We first determine the autoregressive order of the QAR(p) model (14) using the Kolmogorov-Smirnov test based on LR statistics, following Koenker & Machado (1999). We start estimating the quantile regression below with $p = p_{\max} = 3$, that is:

$$Q_{y_t}(\tau \mid y_{t-1}, ..., y_{t-p}) = \alpha_0(\tau) + \alpha_1(\tau) y_{t-1} + \alpha_2(\tau) y_{t-2} + \alpha_3(\tau) y_{t-3}.$$

The index set used for quantiles is $\tau \in \Gamma = [0.1, 0.9]$ with steps of 0.005. Next, we test if the third order covariate is relevant in our model, i.e., we considered the null hypothesis:

$$H_0 : \alpha_3(\tau) = 0, \quad for\ all\ \tau \in \Gamma.$$

The results are reported in Table 1. Using critical values obtained in Andrews (1993), we can infer that the autoregressive variable y_{t-3} can be excluded from our econometric model.

Table 1: Choice of the autoregressive order

excluded variable	$\sup_{\tau\in\Gamma} L_n(\tau)$ estimate	5% critical value	10% critical value	H_0	Result
y_{t-3}	3.989623	9.31	7.36	$\alpha_3(\tau) = 0$	*do not reject*
y_{t-2}	23.79831	9.31	7.36	$\alpha_2(\tau) = 0$	*reject*

Since the third order is not relevant, we proceed by analyzing if the second order covariate is relevant.[16] Thus, we considered the null hypothesis:

$$H_0 : \alpha_2(\tau) = 0, \quad for\ all\ \tau \in \Gamma$$

[16] As usual, we performed the test for exclusion of y_{t-2} with same sample size used to test the exclusion of y_{t-3}.

whose results are also presented in Table 1. Indeed, we verify that the second autoregressive variable cannot be excluded. Thus, the optimal choice of lag length in our model is $p = 2$ and this order will be used in the subsequent estimation and hypothesis tests presented in this paper. In summary, our econometric model will be:

$$y_t = \alpha_0 (U_t) + \alpha_1 (U_t) y_{t-1} + \alpha_2 (U_t) y_{t-2}, \tag{22}$$

and the associated ADF formulation is:[17]

$$y_t = \mu_0 + \alpha_{1,t} y_{t-1} + \alpha_{2,t} \Delta y_{t-1} + u_t, \tag{23}$$

where

$$\begin{aligned} \alpha_{1,t} &= \sum_{i=1}^{2} \alpha_i(U_t) \\ \alpha_{2,t} &= -\alpha_2(U_t), \\ u_t &= \alpha_0 (U_t) - \mu_0 \end{aligned}$$

5.3 Global sustainability

The concept of global sustainability used in this paper states that local episodes of fiscal imbalances must be offset by periods of fiscal responsibility, so that the PVBC condition holds in the long-run. Recall from section 2 that the necessary and sufficient condition for the intertemporal budget constraint (13) to hold is that the discounted debt-GDP ratio, represented by y_t, must be a stationary zero-mean process. If this happens, then the Brazilian federal debt will be globally sustainable.

In order to test for global stationarity, we need to test the null hypothesis H_0: $\alpha_{1,t}= 1$ in Eq. (23). If such a null hypothesis is rejected against the alternative H_1: $\alpha_{1,t}< 1$, then we say that the Brazilian federal debt is globally stationary. We test H_0: $\alpha_{1,t}= 1$ by using the so-called Quantile Komogorov-Smirnoff (QKS) test proposed by Konker and Xiao (2004). The computational details on the QKS test statistic are described in the appendix. The critical values used in the QKS test are computed by the residual-based block (RBB) bootstrap recently proposed by Paparoditis and Politis (2003). Therefore, the critical values will ultimately depend on the block length arbitrarily chosen by the user.[18] Table 2 reports the statistics and critical values

[17] For the sake of completion, we carried on the same tests in the ADF form. As expected, the Kolmogorov-Smirnov based on LR statistics estimates were exactly the same as the estimates reported in Table 2.

[18] The fundamental issue of the RBB bootstrap is its ability to simulate the weak dependence appearing in the original data series by separating the residuals in blocks. For more detais, see Lima and Sampaio (2005)

for eight different block lengths, b, arbitrarily chosen. We considered 10,000 bootstrap replications.

Table 2: Results for the global stationarity test

Block length b	QKS	5% critical value	10% critical value	H_0: $\alpha_{1,t}$= 1
12	13.2753601	14.0261732	11.9415994	reject at 10%
14	13.2753601	13.8578960	11.91955628	reject at 10%
16	13.2753601	14.6971102	12.18249493	reject at 10%
18	13.2753601	14.2410137	12.00697801	reject at 10%
20	13.2753601	15.4325526	12.76201548	reject at 10%
22	13.2753601	13.7142703	11.36398838	reject at 10%
24	13.2753601	12.8470731	11.12157297	reject at 5%
26	13.2753601	12.3618035	10.87127688	reject at 5%

There is evidence that the discounted debt is not a unit root process, with significance level of 10% for almost all values of b (except for $b = 24$ and 26, where we reject the unit root null at significance level of 5%). Overall, the results in Table 2 suggest that, at worst, the discounted Brazilian debt is globally stationary at 10% of significance.

We now test the null hypothesis that y_t has zero unconditional mean, i.e., $H_0 : \mu_y = 0$. We conduct a t-test for the unconditional mean and use the NBB resampling method with 10,000 replications to compute 5% critical values. Table 3 reports the t-statistic for the discounted public debt series. The reported results suggest that the unconditional mean of the autoregressive process is not statistically different from zero. The result of the test depends on the block length used to compute the bootstrap sample. The results in Table 3 showed to be robust to various values of the block length (b).

Table 3 : Results for the unconditional mean test

Block length b	t	2.5% critical value	97.5% critical value	Ho: intercept=0
12	28.9146968	17.8434187	32.8290331	do not reject at 5%
14	28.9146968	19.5784337	34.5324403	do not reject at 5%
16	28.9146968	21.241210	35.5579257	do not reject at 5%
18	28.9146968	22.7204141	36.7304168	do not reject at 5%
20	28.9146968	24.1544067	38.4237097	do not reject at 5%
22	28.9146968	25.4286859	39.7284730	do not reject at 5%
24	28.9146968	26.7328681	40.4919047	do not reject at 5%
26	28.9146968	28.0966684	41.7003689	do not reject at 5%

Putting all together, the discounted Brazilian federal debt is indeed globally sustainable. This result is in accordance with many previous studies, such as in Pastore (1995), Rocha (1997), and Issler and Lima (2000) , suggesting the sustainability of the Brazilian public debt.

5.4 Local Sustainability Test

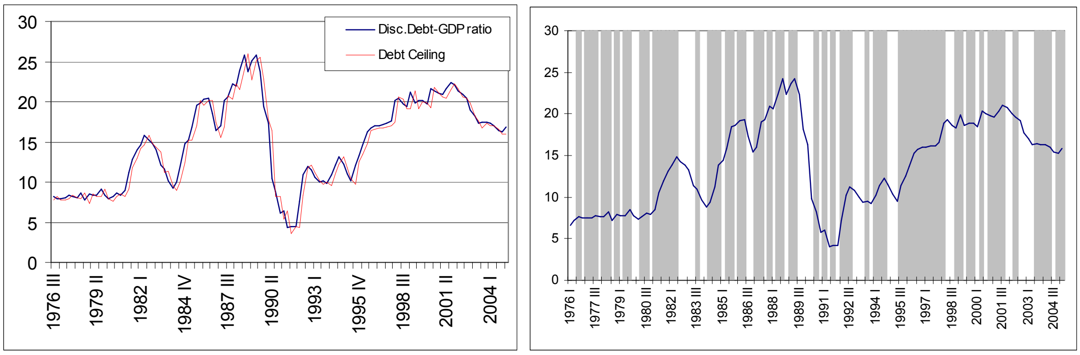
Provided that the Brazilian public debt is a stationary zero mean process, we can now proceed to the "local" analysis by using the Koenker & Xiao (2004b) test. In order to identify the debt ceiling of the Brazilian public debt, we need to test the null hypothesis $H_0 : \alpha_1(\tau) = 1$ at various quantiles by using the t-ratio test $t_n(\tau)$ proposed by Koenker and Xiao (2004.b), with the zero-mean restriction imposed in the ADF representation of Eq. (22) . Table 4 reports the results. The second column displays the estimate of the autoregressive term at each decile. Note that, in accordance with our theoretical model, $\widehat{\alpha}_1(\tau)$ is monotonic increasing in τ, and it is close to unity when we move towards upper quantiles. Table 4 shows that the null hypothesis $H_0 : \alpha_1(\tau) = 1$ is rejected against the alternative hypothesis $H_1 : \alpha_1(\tau) < 1$ for $\tau \in [0.1; 0.4]$. The critical values were obtained by interpolation of the critical values extracted from Hansen (1995, page 1155). The last column summarizes the local sustainability analysis.

Table 4 : Koenker-Xiao test

τ	$\widehat{\alpha}_1(\tau)$	$t_n(\tau)$	δ^2	H_0: $\alpha_1(\tau) = 1$	Sustainability
0.10	0.8955590	-5.89314	0.12338216	reject	OK
0.20	0.9547887	-3.58005	0.09245611	reject	OK
0.30	0.9671567	-4.85814	0.28034278	reject	OK
0.40	0.9810935	-3.53123	0.16280786	reject	OK
0.50	0.9963589	-0.49040	0.13264691	do not reject	-
0.60	1.0093934	1.05544	0.19937313	do not reject	-
0.70	1.0339169	2.81045	0.18691883	do not reject	-
0.80	1.0694750	5.68227	0.04007214	do not reject	-
0.90	1.0948026	6.29170	0.02279413	do not reject	-

Table 4 shows that the critical quantile found using Brazilian public-debt data is equal to 0.40 ($\tau_{crit.} = 0.40$). Consequently, the debt ceiling of the Brazilian debt-GDP ratio corresponds to the path generated by the fourth conditional decile, that is, $\widetilde{D}_t = Q_{y_t}(0.40 \mid y_{t-1}, ..., y_{t-p})$. Hence, according to corollary 1 in this paper, if a given fiscal policy yielding a path of the (discounted) debt-GDP ratio above $Q_{y_t}(0.40 \mid y_{t-1}, ..., y_{t-p})$ were to persist forever, then such a fiscal policy would not be sustainable in the long run. Figure 5 displays the in-sample path of the debt-ceiling which is nothing else but the in-sample forecast of th 0.4th conditional decile function.

Figure 5 - Debt ceiling ($\widetilde{D}_t$) and discounted debt-GDP ratio (y_t)



Note: The debt ceiling series is constructed through in-sample forecast of the 0.4th conditional quantile, given by the ADF formulation: $\widetilde{D}_t = \widehat{\alpha}_1\,(0.4)\,y_{t-1} + \widehat{\alpha}_2\,(0.4)\,\Delta y_{t-1}$

The gray bar in Figure 5 indicates episodes in which the public debt presented an unsustainable behavior. Recall that Tables 2 and 3 show that the discounted debt-GDP ratio in Brazil is globally sustainable. It means that despite the many episodes of fiscal imbalances exhibited in Figure 5 by the gray bars, there were other episodes of fiscal adjustments (white bars) enough to guarantee global sustainability of the Brazilian debt. These episodes of fiscal imbalances were triggered by external shocks, such as oil price shocks in the 70s, and the sequence of financial crises in the 80s and 90s. In the domestic scenario, some recent macroeconomic shocks such as the exchange rate fluctuation in 1999, and the political uncertainty related to the presidential elections of 2002, are also related to periods of local unsustainability of Brazilian debt.

In sum, the results displayed by Figure 5 suggest that the Brazilian authorities are able to intervene through deficit cuts when debt has reached high levels. However, as suggested in Issler and Lima (2000), their mechanism of intervention is never based on spending cuts: it is either based on increases in the tax burden or on the usage of seigniorage revenue.

Table 5 gives us a historical perspective of the Brazilian public debt solvency. The overall result of Table 5 reveals that the debt tolerance $\widehat{H} = 0.60$, i.e., the percentage of episodes in our sample period in which the discounted debt-GDP ratio was above its debt ceiling ($y_t > \widetilde{D}_t$) was 60%, which is perfectly compatible with Proposition 2, since we have found $\tau_{crit.} = 0.40$.[19] Furthermore, due to the nonlinear dynamics of y_t, it is possible to identify different fiscal regimes by estimating, for each historical period, the respective statistic H. Indeed, our estimates for the fiscal policy by the end of the military regime suggest that for 59% of this period the public debt was above the debt ceiling, which is an amount slightly below the theoretical value for the debt tolerance H. As for the beginning of the new republic, in the Sarney's administration (1985.II-1990.I), the fiscal policy implemented in that period was not sustainable during 55% of the time, which is lower than the debt tolerance of 60%. However, we should remind that seigniorage revenue played a crucial role to balance public budget in that period.

[19] The Brazilian debt is globally sustainable despite 60% of its observations exhibiting an (local) unsustainable behavior. This finding results from the combination of the global stationarity and unconditional mean tests with the local investigation in a selected range of quantiles, based on the Koenker & Xiao (2004) test.

Table 5: Quarters during which the discounted public debt-GDP ratio is larger than the 0.4th conditional quantile forecast ($y_t > \widetilde{D}_t$)

	number of quarters (a)	total of quarters (b)	H = (a) / (b)
End of Military Regime (1976.I-1985.I)	22	37	0.59
Sarney's administration (1985.II-1990.I)	11	20	0.55
Collor and Franco's administration (1990.II-1994.IV)	10	19	0.53
Cardoso's first administration (1995.I-1998.IV)	12	16	0.75
Cardoso's second administration (1999.I-2002.IV)	9	16	0.56
Lula's administration (2003.I-2005.I)	6	9	0.67
Total sample (1976.I-2005.I)	70	117	0.60

Regarding the Collor and Franco's administration (1990.II-1994.IV), it is important to notice that the fiscal stabilization plan launched in the middle of March 1990 was responsible for the sharp decrease observed in the public debt stock, since around 80% of the money stock was "frozen" (M4=M1+all other financial assets).[20] As a result, the percentage of periods in which the public debt moved above its debt ceiling was only 53%. Notice, however, that such a number should be analyzed with some caution since the Brazilian Supreme Court decided that the majority part of this "unpaid" debt had to be repaid in the Cardoso's government under the recognition of hidden liabilities (skeletons). Indeed, regarding Cardoso's first term (1995.I-1998.IV), the episodes of fiscal unsustainability was equal to 75%, well above the 60% debt tolerance. The rising in the debt-GDP ratio was mainly due to the recognition of skeletons of around 10% of GDP. However, despite the sharp increasing of the debt, the recognition of skeletons improved the fiscal statistics, providing greater transparency and accuracy of the Brazilian fiscal stance.

Table 5 shows an improvement of the Brazilian fiscal stance in the second term of President Cardoso. Such a improvement occurred despite of the significant real exchange rate depreciation starting in 1999.I,[21] which provoked a considerable debt increase because most of the Brazilian bonds were at that time indexed to hard currencies. Since government spending did not stop rising in the Cardoso's second term, most of the fiscal effort was based on the fact that tax revenue increased much faster than government spending.

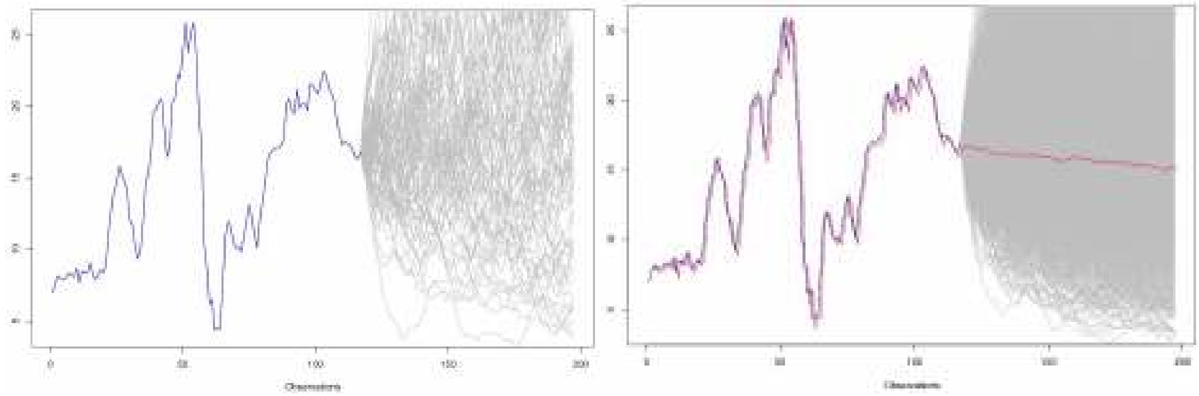
More recently, regarding President Lula's administration, it should be noticed that despite the fiscal effort to keep discounted debt in a sustainable path, the majority of the observations are beyond the debt ceiling. Therefore, we find that the fiscal policy in effect since the beginning of 2003 has not been austere enough to guarantee long-run sustainability.

Next, we present the out-of-sample forecasts of the Brazilian public debt, based on the methodology of recursive generation of conditional densities of y_t, previously described in section 3.2. The out-of-sample forecasts were constructed with a maximum forecast horizon $s_{\max} = 80$ periods (or 20 years), with 1,000 trajectories for the y_t process:

[20] See Rocha (1997).

[21] Real exchange rate adjustment has occurred under the new floating exchange regime.

Figure 6 - Out–of-sample forecast of Brazilian debt



Notes: (a) The pictures respectively show the out-of-sample forecasts for 100 and 1,000 trajectories.

(b) The right picture exhibits (in red line) the in-sample and out-of-sample forecast of the critical conditional quantile.

The red line, representing the forecast debt ceiling, which is the upper trajectory that satisfies the transversality condition of no Ponzi scheme. Notice that it is indeed decreasing, in accordance to Proposition 3, which states that it must converge to zero in the long run. A decision maker will use the forecast debt ceiling to decide whether or not to take some action. For example, if the future values of public debt are above its predicted ceiling, then the fiscal authorities may decide to cut expenditure or increase tax revenue to bring public debt back to its sustainable path. Conditional on the information available up to time T, one can consider the following additional statistic:

Definition 6 *Future percentage of violations* $H^* \equiv \frac{1}{s_{\max}} \sum_{t=T+1}^{T+s_{\max}} I_t^*$, *where* I_t^* *is an indicator function, for* $t = T + s$ *and* $s = 1, ..., s_{\max}$, *such that* $I_t^* = \begin{cases} 1 & ; \text{ if } \widehat{y}_t > \widetilde{D}_t \\ 0 & ; \text{ otherwise} \end{cases}$.

Based on the above definition,we could classify the future paths of the public debt into three different categories:

(i) Globally sustainable fiscal policies: those trajectories always below the "red line", i.e., $H^* = 0$;

(ii) Unsustainable fiscal policies: those paths always above the red line, or in a percentage of violations above 60%.i.e., $H^* > 0.6$.

(ii) Globally sustainable fiscal policies but with some local unsustainable episodes: those trajectories with percentage of violations below 60%, i.e., $H^* \leq 0.6$;

Therefore, a decision maker (fiscal authority) may decide to intervene in the path of public debt (by increasing budget surplus) if the percentage of violations (H^*) during, say, the next four quarters is larger than 60%. Since our sample ends in 2005.I, and (by now) new observations became available, we can compare them to the forecast debt ceiling. Notice that the actual undiscounted debt-GDP ratio for the periods 2005.II, 2005.III, 2005.IV, and 2006.I was respectively 47.21%,48.95%, 49.53% and 50.76%. However, the predicted debt ceiling for the same period was 42.83%, 42.78%, 42.75%, and 42.51%, respectively. Hence, for the 4 quarters considered, there were 100% of violations, that is, $H^* = 1$. Therefore, the out-of-sample forecast based analysis reveals that the more recent dynamic of the Brazilian public debt is not sustainable and additional fiscal efforts are needed to bring the debt-GDP ratio back to values below the debt ceiling.

It is important to mention that other decision rules might also be considered by the fiscal authority. For example, the government might have to decide today (at the time where the forecast is made) how much expenditure cuts or tax revenue increases should occur in the next four quarters in order to guarantee that the public debt would be lower than its forecast ceiling. Another interesting application is to define $z_t = 1$ if $y_t > \widetilde{D}_t$ and $z_t = 0$, otherwise. Hence, we could estimate $\widehat{\pi}_t^i = prob(y_t > \widetilde{D}_t)$ according to some economic model "i" and use the Kuipers Score to evaluate such probability forecasts (See Granger and Pesaran ,1999, for further details). We did not consider either of these techniques in this paper, but we recognize that they can easily be employed to study other aspects of the public debt sustainability, such as the determinants of local fiscal imbalances.'

Since additional fiscal effort is needed, it is relevant to understand how long-run fiscal sustainability has normally been reached in Brazil. Issler and Lima (2000) show that from 1947 to 1994, public budget in Brazil was balanced through seigniorage revenue and no reduction in government spending. After the Real plan, the seigniorage revenue disappeared, leading the Brazilian government to restore fiscal imbalances through tax increases. Indeed, the tax burden in Brazil is already 38% of GDP, meaning that Brazilians are now the most heavily taxed citizens in Latin America with almost no counterpart in public goods. Hence, it would be ideal that the aforementioned fiscal goal of raising the primary surplus were to be achieved through expenditure cuts. It turns out, however, that in the last four quarters of the Lula administration, the GDP growth rate has been very low and the government spending has increased by 14% (year to date), while tax revenue increased by only 11% (year to date).

If public expenditure keeps rising faster than tax revenue, then we might expect that the fiscal stance in Brazil would worsen in the near future. Notice, however, that a new president term will start in January, 2007. Based on the fact that popularity concerns[22] (political constraints) are partially eliminated at the beginning of a new term, we could expect that a fiscal policy based on expenditure cuts through the reduction of interest rate payments is perfectly viable in Brazil as long as the market believes that the new government is able to implement a reform agenda that would increase the productivity of the Brazilian economy in the long run. Such a agenda should include changes in the job-market legislation, social security system, education system, and simplification of the bureaucracy, among other changes needed to increase the productivity of the Brazilian economy.[23] Without such reforms, it will be hard for the Brazilian fiscal authorities to convince the market that they are able to bring the debt-GDP ratio back to its sustainable path, unless, of course, they decide to resort to seigniorage revenue.

[22] The existence of delayed stabilization in Brazil was recently reported by Lima and Simonassi (2005) who investigated whether the Brazilian public debt is sustainable in the long run by considering threshold effects on the Brazilian budget deficit. They show that popularity concerns (political constraints) taking place in the end of the presidential term are the main reason for the existence of delays in the fiscal stabilization in Brazil.

[23] It is important to notice that the government intervention through deficit cuts might not necessarily be incompatible with the minimization of output and employment loss. Indeed, Giavazzi and Pagano (1990) found empirical envidence, for some European Countries, in favour of an "expansionary expectational effect" of a fiscal consolidation.

Table 6: Out-of-sample forecast of Brazilian debt (% GDP)

Periods	Debt ceiling (discounted debt)	Debt ceiling (undiscounted debt)	Observed debt
2005.II	15.71	42.83	47.21
2005.III	15.69	42.78	48.95
2005.IV	15.68	42.75	49.53
2006.I	15.59	42.51	50.76
2006.II	15.54	42.37	-
2006.III	15.43	42.06	-

Note: The stochastic discount factor used to transform the discounted Debt Ceiling (% GDP) into the undiscounted value is the same one used in the last sample point, that is, 2005.I. However, there are other ways to deal with future values of the stochastic discount factor. For example, one could use the market expectations on inflation, output growth and interest rate, published by the Central Bank of Brazil.

6 Conclusions

After the fiscal stabilization plan in 1994, the Brazilian government was no longer able to use seigniorage as a (major) source of revenue. In order to avoid an excessive build up of the debt and a consequent pressure on monetary policy, fiscal authorities had to adopt restrictive fiscal policies. The fiscal austerity led the Brazilian economy to grow at very low rates, with negative impacts on employment. Some politicians have constantly argued that primary budget surplus in Brazil is too large and, therefore, should be reduced to allow the increase of public spending on infrastructure, education and health services. They claim that fiscal policy would still be sustainable (without the necessity to use seigniorage) with lower budget surpluses.

Running lower budget surpluses without resorting to seigniorage revenue, would ultimately lead to an increase in public debt. In this paper, we attempted to answer the following question: how austere should fiscal policy be to guarantee long-run sustainability? By using a fresh econometric model, we showed that: (i) contrary to politicians' thought, the Brazilian public debt is not currently low enough to guarantee long-run sustainability and, therefore, budget surplus should rise rather than lower. In other words, we found that the debt-GDP ratio has moved beyond its ceiling during the majority of quarters in the last two years; (ii) in the absence of shocks, the Brazilian government would have to reduce the debt-GDP ratio during the next quarters to guarantee long-run fiscal sustainability and; (iii) despite the occasional periods in which the Brazilian public debt moved beyond its sustainability ceiling, our historical analysis reveals that the public debt in Brazil has been globally sustainable, suggesting that the Brazilian government authorities react to high levels of public debt, mainly through increases in the tax burden or seigniorage revenue.

Issler and Lima (2000) concluded their article with a brief reflection on the solvency of the Brazilian public debt. They suggested that, for exogenous expenditures, as verified by them in the sample 1947-1992, there would be just two polar forms of restoring long-run sustainability in Brazil: tax increases or increases of seigniorage revenue. Since tax burden have risen almost twofold and already reached 38% of GDP,[24] it seems that Brazilian fiscal authorities did opt to balance budget via tax increases. With such tax burden, Brazilians are now the most heavily taxed citizens in Latin America and, therefore, may start penalizing politicians who propose additional tax increases. Hence, the aforementioned fiscal goal of raising the primary surplus will probably have to be achieved through expenditure cuts or increase in seigniorage revenue. In the second case, inflation will increase again, a price Brazilians may be willing to pay for tax relief. As in Issler and Lima (2000), we all hope that expenditures will cease to be "exogenous" in Brazil.

Despite the process of institutional transformations and the recent austere fiscal policy adopted in Brazil, with the implementation of a target for the budget surplus, Brazil has an unfortunate history of serious difficulties to balance its public budget. Therefore, seems to be necessary the construction of indebtedness targets for Brazil, providing a benchmark to guide fiscal authorities in their task of keeping the public debt on a sustainable path. The measure of debt ceiling introduced in this paper aims to contribute along this direction, developing a "debt-warning system" that helps the macroeconomist to identify "dangerous" debt paths, deemed to be unsustainable.

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Appendix A. Inference Methods of the QAR model

6.1 A.1 Other Representations and Regularity Conditions of the QAR(p) Model.

We define the pth order autoregressive process as follows,

$$y_t = \alpha_0 (U_t) + \alpha_1 (U_t) y_{t-1} + ... + \alpha_p (U_t) y_{t-p}$$

where α_j's are unknown functions $[0, 1] \to \mathbb{R}$ that we will want to estimate. We will refer to this model as the QAR(p) model.[25]

[24] In the first semester of 2006.

[25] More on regularity conditions underlying model (14) are found in Koenker and Xiao (2004a) as well as in the appendix of this paper.

In order to investigate stationarity of the y_t process, we initially rewrite the QAR(p) model in a vector QAR(1) representation, as it follows

$$Y_t = \mu + A_t Y_{t-1} + V_t$$

where $Y_t = [y_t, ..., y_{t-p+1}]'$; $\mu = \begin{bmatrix} \mu_0 \\ 0_{p-1} \end{bmatrix}$; $A_t = \begin{bmatrix} a_t & \alpha_p(U_t) \\ I_{p-1} & 0_{p-1} \end{bmatrix}$; $V_t = \begin{bmatrix} u_t \\ 0_{p-1} \end{bmatrix}$; $a_t = [\alpha_1(U_t), ..., \alpha_{p-1}(U_t)]$ and $u_t = \alpha_0(U_t) - \mu_0$.

Then, lets assume the following conditions:

C.1 $\{u_t\}$ is *iid* with mean 0 and variance $\sigma^2 < \infty$. The CDF of u_t, F, has a continuous density f with $f(u) > 0$ on $U = \{u : 0 < F(u) < 1\}$.

C.2 Eigenvalues of $\Omega_A = E(A_t \otimes A_t\}$ have moduli less than unity.

Koenker & Xiao (2004b) state that under conditions C.1 and C.2, the QAR(p) process y_t is covariance stationary and satisfies a central limit theorem

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} (y_t - \mu_y) \Rightarrow N(0, \omega_y^2) \tag{24}$$

with

$$\begin{aligned} \mu_y &= \frac{\mu_0}{1 - \sum_{j=1}^{p} \beta_j} \\ \beta_j &= E(\alpha_j(U_t)) \quad , \ j = 1, ..., p \\ \omega_y^2 &= \lim \frac{1}{n} E\left[\sum_{t=1}^{n} (y_t - \mu_y)\right]^2 \end{aligned} \tag{25}$$

The QAR(p) model (14) can be reformulated in a more conventional random coefficient notation as

$$y_t = \mu_0 + \beta_{1,t} y_{t-1} + ... + \beta_{p,t} y_{t-p} + u_t, \tag{26}$$

where

$$\begin{aligned} \mu_0 &= E\alpha_0(U_t), \\ u_t &= \alpha_0(U_t) - \mu_0, \\ \beta_{j,t} &= \alpha_j(U_t), \quad j = 1, ..., p. \end{aligned}$$

Thus, $\{u_t\}$ is an iid sequence of random variables with distribution $F(\cdot) = \alpha_0^{-1}(\cdot + \mu_0)$, and the $\beta_{j,t}$ coefficients are functions of this u_t innovation random variable.

An alternative form of the model (26) widely used in economic applications is the ADF (augmented Dickey-Fuller) representation (27). According to Koenker & Xiao (2004b), in the ADF formulation the first order autoregressive coefficient plays an important role in measuring persistency in economic and financial time series, and in our case will be crucial to determine the sustainability of public debt.

$$y_t = \mu_0 + \alpha_{1,t} y_{t-1} + \sum_{j=1}^{p-1} \alpha_{j+1,t} \Delta y_{t-j} + u_t, \tag{27}$$

where, corresponding to (14),

$$\alpha_{1,t} = \sum_{i=1}^{p} \alpha_i(U_t),$$

$$\alpha_{j+1,t} = -\sum_{i=j}^{p} \alpha_i(U_t), \quad j = 1, ..., p,$$

Under regularity conditions, if $\alpha_{1,t} = 1$, y_t contains a unit root and is persistent; and if $\mid \alpha_{1,t} \mid < 1$, y_t is stationary. Notice that equations (14), (26) and (27) are equivalent representations of our econometric model. Each representation is convenient for the inference analysis.

Appendix B. Proofs of Propositions

Proof of Proposition 1. Consider the ADF representation of the QAR(p) model (27). The existence and uniqueness of the critical conditional quantile is proven by the following simple argument:

For $\forall t \in \Upsilon$, let u_t be a realization of the iid uniform random variable U_t such that $\alpha_{1,t} = \sum_{i=1}^{p} \alpha_i(u_t) = 1$, and $\alpha_{j+1,t} = -\sum_{i=j+1}^{p} \alpha_i(u_t) = \alpha_{j+1} \quad j = 1, ..., p$. By assumptions 1 and 2, $\alpha_i(u_t)$, $i = 0, ...p$, are increasing functions in u_t.Since the sum of monotone increasing functions is itself a monotone increasing function, it follows that $\alpha_{1,t}(u_t)$ and $\alpha_{j+1,t}(u_t)$ are monotone increasing. Assumptions 1 and 2 guarantee that $Q_{\alpha_i(U_t)} = \alpha_i(Q_{U_t}) = \alpha_i(\tau)$, which is an increasing function in τ. Moreover, comonotonicity guarantees that $Q_{\sum_{i=1}^{p} \alpha_i(U_t)} = \sum_{i=1}^{p} Q_{\alpha_i(U_t)} = \sum_{i=1}^{p} \alpha_i(Q_{U_t}) = \sum_{i=1}^{p} \alpha_i(\tau)$. This implies that $\alpha_{1,t}(\tau)$ and $\alpha_{j+1,t}(\tau)$ are monotone increasing in τ. Thus, assumptions 1 and 2 guarantee that the conditional quantile function of y_t is monotone increasing in τ.

Given assumption 1, we know that $u_t \in (0, 1)$. Based on the above argument, it is always possible to find an unique quantile τ^* such that $\alpha_{1,t}(\tau^*) = \sum_{i=1}^{p} \alpha_i(\tau^*) = 1$ and $\alpha_{j+1,t}(\tau^*) = -\sum_{i=j+1}^{p} \alpha_i(\tau^*) = \alpha_{j+1}$. This suggests the nice result that

$$Q_{y_t}(\tau^* \mid \mathcal{F}_{t-1}) = y_t, \ \forall t \in \Upsilon$$

that is, the trajectory of the conditional quantile function $Q_{y_t}(\tau^* \mid \mathcal{F}_{t-1})$ will hit the points in which the time series process y_t has a unit root behavior.

Now recall that the critical quantile $\tau_{crit.}$ is the largest quantile τ such that $\alpha_{1,t}(\tau) < 1$. Define $\widetilde{\Gamma} \subseteq \Gamma = (0; 1)$ as the subset of quantiles so that $\alpha_{1,t}(\tau) < 1$. Hence, based on the fact that $\alpha_{1,t}(\tau)$ is monotone increasing in τ, it follows that the critical quantile is

$$\tau_{crit.} = \sup \widetilde{\Gamma}$$

Thus, $\tau_{crit.} < \tau^*$ by definition and, since $Q_{y_t}(\tau \mid \mathcal{F}_{t-1})$ is monotone increasing in τ, we must have that $Q_{y_t}(\tau_{crit.} \mid \mathcal{F}_{t-1})$ must lie below y_t in all periods where y_t is nonstationary, that is, $Q_{y_t}(\tau_{crit.} \mid \mathcal{F}_{t-1}) < y_t = Q_{y_t}(\tau^* \mid \mathcal{F}_{t-1})$, $\forall t \in \Upsilon$ ■

Proof of Corollary 1. Based assumptions 4 and 5, we have that the public debt process y_{t_i} for $\forall t_i \in \Upsilon$ is now represented by

$$y_{t_i} = y_{t_i-1} + \sum_{j=1}^{p-1} \alpha^*_{j+1,t_i} \Delta y_{t_i-j} \quad ; \quad i = 1, ..., N$$

which is the same y_{t_i} process discussed in Proposition 1, but without intercept. In the same manner, the conditional quantile function can be written, by using the ADF formulation, as

$$Q_{y_t}(\tau \mid \mathcal{F}_{t-1}) = \alpha_{1,t}(\tau) y_{t-1} + \sum_{j=1}^{p-1} \alpha_{j+1,t}(\tau) \Delta y_{t-j}$$

This way, the proof of Corollary 1 is straightforward to be achieved, by just following the proof of Proposition 1 considering no intercept in the stochastic process y_t, given that the local analysis of public debt depends on the zero-mean process assumption (or global sustainability for public debt). ■

Proof of Proposition 2. By definition, we have that $H \equiv \frac{1}{T} \sum_t I_t \left\{ y_t > Q_{y_t}(\tau_{crit.} | \mathcal{F}_{t-1}) \right\}$, where $I_t(.)$ is an indicator function and T is the sample size. By Assumption 3, we can rewrite this expression as $H = \frac{1}{T} \left(\sum_{t \in \Upsilon} I_t \left\{ y_t > Q_{y_t}(\tau_{crit.} | \mathcal{F}_{t-1}) \right\} + \sum_{t \in (\Omega/\Upsilon)} I_t \left\{ y_t > Q_{y_t}(\tau_{crit.} | \mathcal{F}_{t-1}) \right\} \right) = \frac{(N+0)}{T}$, based on Proposition 1. On the other hand, we can state the critical quantile as $\tau_{crit.} = \Pr\left(y_t < Q_{y_t}(\tau_{crit.} \mid \mathcal{F}_{t-1})\right) = \Pr\left(t \in [\Omega/\Upsilon] \mid \mathcal{F}_{t-1}\right)$ $= \Pr\left(t \in \Omega \mid \mathcal{F}_{t-1}\right) - \Pr\left(t \in \Upsilon \mid \mathcal{F}_{t-1}\right) = 1 - \frac{N}{T} = 1 - H$ ■

Proof of Proposition 3. Notice that for each realization of U_{T+s}, there is a 1:1 corresponding quantile $\tau = u_{T+s}$. Hence, let u_{crit} be the largest realization of U_{T+s} so that $\alpha_{1,t} = \sum_{i=1}^{p} \alpha_i(U_{T+s}) < 1$, which guarantees stationarity whenever the realization u_{crit} takes place. By proposition 1, there exist a critical quantile $\tau_{crit.} = u_{crit}$, and its corresponding conditional critical quantile $\widehat{D}_{T+s} = \widehat{Q}_{y_{T+s}}(\tau_{crit.} \mid \mathcal{F}_T)$, so that $\widehat{y}_{T+s} = \widehat{D}_{T+s}$ whenever the realization u_{crit} takes place. Given that the process y_t has zero mean (no intercept) and $\widehat{y}_{T+s}$ is a forecasted path of y_t, it follows that $\lim_{s \to \infty} \widehat{D}_{T+s} = 0$. ■

7 Appendix C. Estimation and Hypothesis Testing

Provided that the right hand side of (14) is monotone increasing in U_t, it follows that the τth conditional quantile function of y_t can be written as

$$Q_{y_t}(\tau \mid y_{t-1}, ..., y_{t-p}) = \alpha_0(\tau) + \alpha_1(\tau) y_{t-1} + ... + \alpha_p(\tau) y_{t-p}, \tag{28}$$

or somewhat more compactly as

$$Q_{y_t}(\tau \mid y_{t-1}, ..., y_{t-p}) = x_t' \alpha(\tau),$$

where $x_t' = (1, y_{t-1}, ..., y_{t-p})'$. The transition from (14) to (28) is an immediate consequence of the fact that for any monotone increasing function g and a standard uniform random variable, U, we have:

$$Q_{g(U)}(\tau) = g(Q_U(\tau)) = g(\tau),$$

where $Q_U(\tau) = \tau$ is the quantile function of U_t. Analogous to quantile estimation, quantile autoregression estimation involves the solution to the problem

$$\min_{\{\alpha \in R^{p+1}\}} \sum_{t=1}^{n} \rho_\tau (y_t - x_t'\alpha), \tag{29}$$

where ρ_τ is defined as in Koenker and Basset (1978):

$$\rho_\tau (u) = \begin{cases} \tau u, u \geq 0 \\ (\tau - 1) u, u < 0 \end{cases}.$$

The quantile regression method is robust to distributional assumptions, a property that is inherited from the robustness of the ordinary sample quantiles. Moreover, in quantile regression, it is not the magnitude of the dependent variable that matters but its position relative to the estimated hyperplane. As a result, the estimated coefficients are less sensitive to outlier observations than, for example, the OLS estimator. This superiority over OLS estimator is common to any M-estimator.[26]

Autoregressive Order Choice

Equation (14) gives our pth order quantile autoregression model. We now discuss how to choose the optimal lag length p. We follow Koenker and Machado (1999) in testing for the null hypothesis of exclusion for the pth control variable.τ

$$H_0 : \alpha_p (\tau) = 0, \quad \text{for all } \tau \in \Gamma, \tag{30}$$

and some index set $\Gamma \subset (0, 1)$. Let $\widehat{\alpha}(\tau)$ denote the minimizer of

$$\widehat{V}(\tau) = \min_{\{\alpha \in \mathbb{R}^{p+1}\}} \sum \rho_\tau (y_t - x_t'\alpha),$$

where $x_t' = (1, y_{t-1}, y_{t-2}, ..., y_{t-p})'$and $\widetilde{\alpha}(\tau)$ denotes the minimizer for the corresponding constrained problem without the pth autoregressive variable, with

$$\widetilde{V}(\tau) = \min_{\{\alpha \in \mathbb{R}^{p}\}} \sum \rho_\tau (y_t - x_{1t}'\alpha),$$

where $x_{1t}' = \left(1, y_{t-1}, y_{t-2}, ..., y_{t-(p-1)}\right)'$. Thus, $\widehat{\alpha}(\tau)$ and $\widetilde{\alpha}(\tau)$ denote the unrestricted and restricted quantile regression estimates. Koenker and Machado (1999) state that we can test the null hypothesis (30) using a related version of the Likelihood process for a quantile regression with respect to several quantiles. Suppose that the $\{u_t\}$ are iid but drawn from some distribution, say, F, and satisfying some regularity conditions. The LR statistics at a fixed quantile is derived as follows:

$$L_n(\tau) = \frac{2\left(\widetilde{V}(\tau) - \widehat{V}(\tau)\right)}{\tau(1-\tau)s(\tau)}, \tag{31}$$

where $s(\tau)$ is the sparsity function

$$s(\tau) = \frac{1}{f(F^{-1}(\tau))}.$$

[26] The quantile estimator is (in fact) a M-estimator.

The sparsity function, also termed the quantile-density function, plays role of a nuisance parameter. We want to carry out a joint test about the significance of the pth autoregressive coefficient with respect to a set of quantiles Γ (not only at fixed quantile). Koenker and Machado (1999) suggest using the Kolmogorov-Smirnov type statistics for the joint test:

$$\sup_{\tau\in\Gamma} L_n(\tau),$$

and show that under the null hypothesis (30):

$$\sup_{\tau\in\Gamma} L_n(\tau) \rightsquigarrow \sup_{\tau\in\Gamma} Q_1^2(\tau),$$

where $Q_1(\cdot)$ is a Bessel process of order 1. Critical values for $supQ_q^2(\cdot)$ are extensively tabled in Andrews (1993).

Global Stationarity

Given the choice of the optimal lag length p, one must check for global stationarity of the y_t process, in order to verify whether conditions C.1 and C.2 described in section 3 indeed hold, and y_t is covariance stationary in the sense of Koenker & Xiao (2004b). An approach to test the unit root property is to examine it over a range of quantiles $\tau \in \Gamma$, instead of focusing only on a selected quantile. We may, then, construct a Kolmogorov-Smirnov (KS) type test based on the regression quantile process for $\tau \in \Gamma$. We considered $\tau \in \Gamma = [0.1, 0.9]$ with steps of 0.005 Koenker and Xiao (2004b) proposed the following quantile regression based statistics for testing the null hypothesis of a unit root:

$$QKS = \sup_{\tau\in\Gamma} \mid U_n(\tau) \mid, \tag{32}$$

where $U_n(\tau)$ is the coefficient based statistics given by:

$$U_n(\tau) = n\left(\widehat{\alpha}_1(\tau) - 1\right).$$

Koenker and Xiao (2004b) suggest to approximate the limiting distribution of (32) under the null hypothesis by using the autoregressive bootstrap (ARB). In this paper, we approximate the distribution under the null using the residual based block bootstrap procedure (RBB). The advantages of the RBB over ARB are documented in Lima and Sampaio (2005).

Unconditional Mean Test

In order to test whether or not the unconditional mean of the process is zero, we recall that the following null hypotheses are equivalent:

$$\begin{aligned} H_0 &: \mu_y = 0 \\ H_0' &: \mu_0 = 0 \end{aligned}$$

Consider the pth order quantile autoregressive process given by

$$\begin{aligned} y_t &= \alpha_0(U_t) + \alpha_1(U_t)\, y_{t-1} + ... + \alpha_p(U_t)\, y_{t-p} \\ &= \mu_0 + \beta_{1,t} y_{t-1} + ... + \beta_{p,t} y_{t-p} + u_t \end{aligned}$$

where $u_t = \alpha_0(U_t) - \mu_0$. Now note that the τth conditional quantile function of y_t is given by

$$Q_{y_t}(\tau \mid y_{t-1}, ..., y_{t-p}) = \alpha_0(\tau) + \alpha_1(\tau)\, y_{t-1} + ... + \alpha_p(\tau)\, y_{t-p}$$

and it does not allow us to identify the intercept coefficient μ_0, since $Q_u(\tau) = \alpha_0(\tau) - \mu_0$, where $\tau = Q_U(\tau)$ is the quantile function of U. Thus, the next natural attempt would be to ignore the existence of asymmetric dynamic and estimate a symmetric regression (constant coefficient model)

$$y_t = \mu_0 + \beta_1 y_{t-1} + ... + \beta_p y_{t-p} + v_t \tag{33}$$

The null hypothesis H_0' could be tested using the conventional t-statistics

$$t = \frac{\widehat{\mu_0}}{\widehat{SE(\mu_0)}}$$

However, in omitting asymmetries, the new error term v_t is no longer an iid sequence, i.e.,

$$v_t = \left(\beta_{1,t} - \beta_1\right) y_{t-1} + ... + \left(\beta_{p,t} - \beta_p\right) y_{t-p} + u_t$$

which *invalids* the conventional t-statistics type test. Putting that aside, we decided to directly test the null hypothesis $H_0 : \mu_y = 0$ using a resampling method for dependent data due to Carlstein (1986), named Nonoverlapping Block Bootstrap (NBB). The key feature of this bootstrap method is that its blocking rule is based on nonoverlapped segments of the data, making it able to simulate the weak dependence in the original series, y_t. Further details regarding NBB bootstrap are available in Lahiri (2003).

The Koenker-Xiao Test

In this section we introduce the Koenker-Xiao test, which is used to test the null hypothesis $H_0 : \alpha_1(\tau) = 1$, for a given $\tau \in (0, 1)$. We express the null hypothesis in the ADF representation (16) as:

$$H_0 : \alpha_1(\tau) = 1, \text{ for selected quantiles } \tau \in (0, 1)\,.$$

In order to test such a hypothesis, Koenker and Xiao (2004b) proposed a statistic similar to the conventional augmented Dick-Fuller (ADF) t-ratio statistic. The t_n statistics is the quantile autoregression counterpart of the ADF t-ratio test for a unit root and is given by:

$$t_n(\tau) = \frac{f\left(\widehat{F^{-1}(\tau)}\right)}{\sqrt{\tau(1-\tau)}} \left(Y_{-1}^{\intercal} P_X Y_{-1}\right)^{\frac{1}{2}} \left(\widehat{\alpha}_1(\tau) - 1\right),$$

where, $f\left(\widehat{F^{-1}(\tau)}\right)$ is a consistent estimator of $f\left(F^{-1}(\tau)\right)$, Y_{-1} is a vector of lagged dependent variables (y_{t-1}) and P_X is the projection matrix onto the space orthogonal to $X = (1, \Delta y_{t-1}, ..., \Delta y_{t-p+1})$. Koenker and Xiao (2004b) show that the limiting distribution of $t_n(\tau)$ can be written as:

$$t_n(\tau) \Rightarrow \delta \left(\int_0^1 \underline{W}_1^2 \right)^{-\frac{1}{2}} \int_0^1 \underline{W}_1 dW_1 + \sqrt{1-\delta^2} N(0,1),$$

where, $\underline{W}_1(r) = W_1(r) - \int_0^1 W_1(s)\,ds$ and $W_1(r)$ is a standard Brownian Motion. Thus, the limiting distribution of $t_n(\tau)$ is nonstandard and depends on parameter δ given by:

$$\delta = \delta(\tau) = \frac{\sigma_{\omega\psi}(\tau)}{\sigma_\omega^2}.$$

and can be consistently estimated (see Koenker and Xiao, 2004b, for more details). Critical values for the statistic $t_n(\tau)$ are provided by Hansen (1995, page 1155) for values of δ^2 in steps of 0.1. For intermediate values of δ^2, Hansen suggests obtaining critical values by interpolation.

8 Appendix D. Monte Carlo Simulation

A Monte Carlo simulation is designed to investigate the finite sample performance of the result showed in Proposition 1, that is, the critical conditional quantile is able to separate nonstationary points from stationary ones. One of the critical issues regarding this experiment is the Data-Generating Process (DGP), which will be represented by the following QAR(1) model

$$y_t = \alpha_0(U_t) + \alpha_1(U_t)\, y_{t-1} \tag{34}$$

where $\{U_t\}$ is a sequence of iid standard uniform random variables, and the coefficients α_0 and α_1 are functions on $[0,1]$, given by $\alpha_0(U_t) = F^{-1}(U_t)$, where $F : \mathbb{R} \to [0,1]$ is the standard normal cumulative distribution function, and $\alpha_1(U_t) = \min\{1 \;\; ; \;\; \gamma_0 + \gamma_1 U_t\}$ with $\gamma_0 \in (0,1)$ and $\gamma_1 > 0$.

In our case, we initially assume $\gamma_0 = 0.7$ and $\gamma_1 = 0.4$ in order to limit the variance of α_1. If $U_t > \frac{(1-0.7)}{0.4} = 0.75$ then the model generates y_t according to the unit root model, but for smaller realizations of U_t we have mean reversion tendency. In other words, we expect that 25% of the U_t realizations will induce a unit root behavior. We also consider the case $\gamma_0 = 0.8$, which leads to 50% of the realizations of U_t generating a unit root model.

In our experiment, we construct 10,000 replications of $\{y_t\}$ with 100 or 300 observations. We adopt a hybrid solution for this experiment using R and Ox environments, since the proposed simulation is extremely computational intensive. Ox is much faster than R in large computations. On the other hand, R language is more interactive and user-friendly than Ox, and the QAR model must be estimated in R, since its package for quantile regressions (quantreg) is more complete and updated than the Ox package. The main steps of the algorithm used in the Monte Carlo simulation are as it follows:[27]

[27] Both R and Ox codes are available from the authors upon request.

a) Initialization of the R code (setting parameters γ_0, γ_1)

b) Generation of one DGP

b.1) R code calls Ox code informing the input parameters

b.2) Ox code generates one DGP y_t

b.3) R code imports the data generated by Ox code

c) Calculation of the optimal lag length (p) for the QAR(p) model

d) Estimation of the coefficients for the QAR(p) model

e) Testing for local unit root in all quantiles

f) Search for the critical quantile

g) Generation of the conditional quantiles

h) Computing the Debt Ceiling

i) Save the results for this DGP

j) Repeat the steps from (b) to (i) for 10,000 replications

Therefore, we proceed as follows: Ox code initially generates the time series y_t and, then, returns these data to R, which estimates the QAR(p) model, computes the descriptive statistics and saves the results in a text file. Once the Ox code generates the $\{y_t\}$ process, the optimal lag length of the QAR(p) model is chosen based on Koenker and Machado (1999) procedure. This way, the coefficients are estimated for all quantiles and a local unit root test is conducted in order to find the critical quantile $\tau_{crit.}$, i.e., the last quantile associated with an autoregressive coefficient, which still represents a mean reversion tendency (or in other words, where the null $H_0 : \alpha_1(\tau) = 1$ is still rejected, according to the Koenker-Xiao test for unit root). Furthermore, the R code generates the conditional quantiles, including the critical quantile, according to the following ADF formulation

$$\widehat{Q}_{y_t}(\tau_{crit.} \mid \mathcal{F}_{t-1}) = \widehat{\alpha}_0(\tau_{crit.}) + \widehat{\alpha}_1(\tau_{crit.})\, y_{t-1} + \sum_{j=1}^{p-1} \widehat{\alpha}_{j+1}(\tau_{crit.})\, \Delta y_{t-j} \tag{35}$$

Based on the critical conditional quantile, one can verify if the adopted QAR(p) model for a finite sample is able to correctly identify the stationarity limit, by comparing the $\{y_t\}$ process with $\widehat{Q}_{y_t}(\tau_{crit.} \mid \mathcal{F}_{t-1})$ for observations where the DGP imposes a unit root model. To investigate this issue carefully, lets initially define (for a given replication i) the following dummy variables W_t^i and Z_t^i

$$W_t^i = \left\{ \begin{array}{ll} 1 \ ; & \text{if } \alpha_1(U_t) = 1 \\ 0 \ ; & \text{otherwise} \end{array} \right\} \tag{36}$$

$$Z_t^i = \left\{ \begin{array}{ll} 1 \ ; & \text{if } y_t > \widehat{Q}_{y_t}(\tau_{crit.} \mid \mathcal{F}_{t-1}) \ \text{ and } \ \alpha_1(U_t) = 1 \\ 0 \ ; & \text{otherwise} \end{array} \right\} \tag{37}$$

Thus, the W_t^i variable indicates observations with an autoregressive coefficient equal to unity, according to the DGP, and Z_t^i reveals observations associated with a unit root behavior and, at the same time, where the generated y_t time series is above the critical conditional quantile. Note that $\frac{1}{T}\sum_{t=1}^{T} Z_t^i = H^i$, which is exactly the H statistic, presented in definition 3, computed for replication i. Therefore, one can compute the ratio R^i as it follows

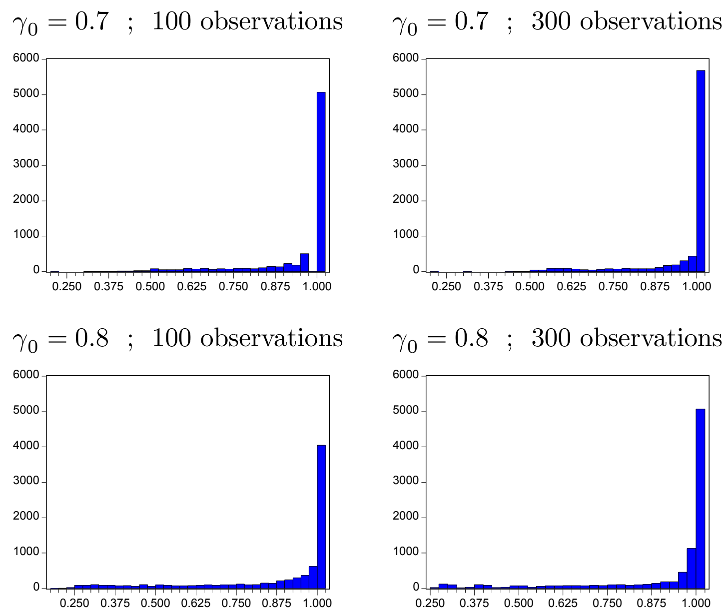
$$R^i \equiv \frac{\frac{1}{T}\sum_{t=1}^{T} Z_t^i}{\frac{1}{T}\sum_{t=1}^{T} W_t^i} \tag{38}$$

One should expect the ratio R^i to be as close to unity as possible, since in the QAR(p) model all observations of the y_t process associated with a unit root model must be above the critical quantile, according to Proposition 1.

Our simulation compute the R^i statistic for each replication i and summarizes the results in the following histograms, where the frequency of R^i is plotted for the set of 10,000 replications.[28] It is worth mentioning that only the replications in which the null hypothesis of a local unit root for the y_t process can not be rejected are displayed in the following histograms. In other words, we select among the 10,000 replications only those representing a stochastic process y_t containing at least one quantile with a local unit root, i.e., the null $H_0 : \alpha_1(\tau) = 1$ is not rejected for (at least) one quantile $\acute{\tau} \in (0,1)$.

Since U_t follows a standard uniform distribution and $\alpha_1(U_t) = \min\{1 \ ; \ \gamma_0 + \gamma_1 U_t\}$ it is possible that for a given replication j the stochastic process $\{y_t^j\}$ has no local unit root, i.e., $\alpha_1(\tau) < 1; \forall \tau \in (0,1)$. In fact, these cases occur for lower values of γ_0 and γ_1, and since they are not object of our investigation we decided to not consider them in our analysis.

Figure 8 - Histograms with the frequency of R^i



Note: Total of i=10,000 replications, excluding those with no local unit root.

[28] Each vertical bar graph represents the frequency distribution of R^i, in which the height of bars is proportional to the frequency within each class interval.

Table 7 - Summary of the Monte Carlo results

Parameter γ_0	T	T^*	mean (R_i)	median (R_i)	std.dev.(R_i)
$\gamma_0 = 0.7$	100	25	0.935	1.000	0.131
	300	75	0.951	1.000	0.112
$\gamma_0 = 0.8$	100	50	0.866	0.982	0.213
	300	150	0.909	1.000	0.182

Notes: (a) Total of i=10,000 replications for each simulation, excluding those with no local unit root;

(b) T is the total number of observations and T^* is the expected number of observations,

across the 10,000 replications, associated with a unit root model.

According to the Monte Carlo experiment, we found that the result of Proposition 1 indeed exhibits a good performance in the finite sample investigation. As long as the number of observations T increases (for a given parameter γ_0), the empirical distribution of R^i approaches the unity value, with a respective decreasing standard deviation, as we already expected. In our simulations, the distribution of R^i for $\gamma_0 = 0.7$ is more concentrated than the respective distribution for $\gamma_0 = 0.8$, since the DGP for $\gamma_0 = 0.8$ induces a larger expected number T^* of realizations of U_t generating a unit root model. In this case, for $\gamma_0 = 0.7$ and $T = 100$ observations, we found that (in average) the QAR model imposes 93,5% of observations of the y_t process associated with a unit root model (T^*) correctly above the estimated critical quantile.

Appendix E. Out-of-sample Forecast: generation of a discrete uniform random variable.

In practical terms, the numerical procedure described in the construction of the out-of-sample forecast of y_t must be implemented by an algorithm considering a perfect match between the discrete set of quantiles $\tau \in \Lambda = [0.1, ..., 0.9]$ and a discrete support of the U_t random variable. Firstly, we must choose the number of elements n for the grid Λ of quantiles and, then, estimate the QAR model to generate the set of coefficients $\widehat{\alpha}_i(\tau)$ for all $\tau \in \Lambda$. The discrete set of quantiles Λ, containing n elements, is defined by

$$\tau \in \Lambda \equiv [0.1, 0.1 + \tau_{step}, 0.1 + 2\tau_{step}, ..., 0.9 - \tau_{step}, 0.9] \tag{39}$$

where $\tau_{step} = (0.9 - 0.1)/(n-1)$. In addition, one must ensure that the dropping of the discrete version of the random variable U_t, defined as $\widetilde{U}_t$, is made based on the same set Λ, in order to guarantee that, for every realization of U_t, the algorithm correctly calculates the respective $\widetilde{U}_t$, in order to find an associated quantile τ and, therefore, an estimated coefficient $\widehat{\alpha}_i\left(\tau = \widetilde{U}_t\right)$.

This way, the perfect 1:1 mapping between τ and U_t depends on the random variable $\widetilde{U}_t$, which can be obtained from the realization of the continuous random variable U_t, in the following way: Assume that U_t

belongs to the continuous set $[0.1, 0.9]$. If we define $\widetilde{U}_t$ as it follows, we can guarantee that indeed $\widetilde{U}_t$ belongs to the same discrete set Λ of quantiles.

$$\widetilde{U}_t \equiv 0.1 + \tau_{step} * round\{\frac{(U_t - 0.1)}{\tau_{step}}\} \tag{40}$$

where the $round(.)$ function approximates its argument to the nearest integer value.[29]

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[29] Lets present a simple example for some grid Λ_1, with $n = 9 \therefore \tau_{step} = 0.1 \therefore \Lambda_1 = [0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9]$. Now, assume the realized value $U_t = 0.376815$. By definition, we have $\widetilde{U}_t = 0.1 + 0.1 * round\{\frac{(0.376815 - 0.1)}{0.1}\} = 0.4 \in \Lambda_1$. Therefore, for a given drop of U_t from a continuous range set $[0.1, 0.9]$ we constructed its respective discrete version $\widetilde{U}_t$ that indeed belongs to the considered grid Λ_1 of quantiles. Now consider another example, with a different grid Λ_2 by assuming $n = 3 \therefore \tau_{step} = 0.4$ and $\Lambda_2 = [0.1, 0.5, 0.9]$. If one sorts again $U_t = 0.376815$, then $\widetilde{U}_t = 0.1 + 0.4 * round\{\frac{(0.376815 - 0.1)}{0.4}\} = 0.5 \in \Lambda_2$. On the other hand, if $U_t = 0.68$, then $\widetilde{U}_t = 0.5 \in \Lambda_2$, but if $U_t = 0.72$, then $\widetilde{U}_t = 0.9 \in \Lambda_2$.

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