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Exploratory semiparametric analysis of two-dimensional diffusions in finance

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ABSTRACT

We examine bivariate extensions of Aït-Sahalia's approach to the estimation of univariate diffusions. Our message is that extending his idea to a bivariate setting is not straightforward. In higher dimensions, as opposed to the univariate case, the elements of the Itô and Fokker-Planck representations do not coincide; and, even imposing sensible assumptions on the marginal drifts and volatilities is not sufficient to obtain direct generalisations. We develop exploratory estimation and testing procedures, by parametrizing the drifts of both component processes and setting restrictions on the terms of either the Itô or the Fokker-Planck covariance matrices. This may lead to highly non-linear ordinary differential equations, where the definition of boundary conditions is crucial. For the methods developed, the Fokker-Planck representation seems more tractable than the Itô's. Questions for further research include the design of regularity conditions on the time series dependence in the data, the kernels actually used and the bandwidths, to obtain asymptotic properties for the estimators proposed. A particular case seems promising: "causal bivariate models" in which only one of the diffusions contributes to the volatility of the other. Hedging strategies which estimate separately the univariate diffusions at stake may thus be improved.

(Preliminary version: February 2000)

1. INTRODUCTION

In spite of the landmarks in continuous-time derivatives pricing by Black and Scholes (1973) and Merton (1973), which opened a path followed by Vasicek (1977), Cox, Ingersoll and Ross (1985a, b) and Hull and White (1990), among others, the empirical literature has not followed this generally much more tractable and elegant alternative to discrete-time modelling. Indeed, the estimation of such pricing models usually abandons the continuous time environment, restricting itself to the discrete character of the data available.

The most commonly used estimation method for univariate diffusions in finance consists in parametrizing the drift and volatility functions and then discretize the model before estimating it. Lo (1988)'s pioneering proposal, based on the method of maximum-likelihood, suffered the drawback of requiring, except for very particular cases, the numerical solution of a partial differential equation for each optimising iteration. Nelson (1990) analysed the behaviour of discrete approximations when the interval between the observations goes to zero. Duffie and Singleton (1993) and Gourieroux, Monfort and Renault (1993) proposed the estimation of diffusions by simulation – given parameter values, sample paths are simulated, and their moments should be rendered as close as possible to the sample moments.

Aït-Sahalia (1996) sought to reconcile both the theoretical and empirical literature in option pricing. Though working with discrete data, he did not resort to discretizations of the model. Firstly, one parametrizes the drift, for instance, which guarantees the identification of the model and makes possible not to restrict the volatility specification. Next, one proceeds to estimate non-parametrically the marginal density of the process. Given the estimated (coefficients of the) drift and marginal density, a semiparametric estimator of the volatility is obtained through the Kolmogorov forward equation. The process can be improved by using the volatility estimates now as input to re-estimate (i) the drift parameters using Feasible Generalised Least Squares and (ii) the volatility itself.

In the case of interest-rate derivatives, parametrizing the drift makes sense given the importance of the instantaneous volatility in derivatives pricing and the difficulty in forming an *a priori* idea of its functional form. Moreover, as Pritsker (1998) pointed out, the availability of long time-series of daily data of spot interest rates is crucial for the good performance of the method.

The purpose of this paper is to explore the possibilities of bivariate extensions of Aït-Sahalia (1996)'s semiparametric framework. Since Brennan and Schwartz (1979), bivariate diffusions have appeared in several two-factor models and are many times treated independently. It would certainly be interesting to have a powerful estimating method for investigating, and testing, different relationships among the two univariate processes. However, when moving to a multivariate framework things become much more complicated. Actually, our main message is that extending Aït-Sahalia (1996)'s idea to a bivariate setting is by no means straightforward. First, the functions in the Itô's and Fokker-Planck's representations do not coincide in higher dimensions, as opposed to the univariate case. As a matter of fact, in the bivariate case, the Fokker-Planck volatilities turn out to be more adequate than those in the Itô representation. Second, even when imposing sensible assumptions on the drift and volatility functions, one can not obtain a direct generalisation of the univariate method.

In spite of these issues, the method might be interesting as an exploratory technique for uncovering certain relationships between the two processes, without imposing a fully parametric structure. Further work is however needed for rigorously establishing the asymptotic properties of the various possible estimators, as well as to acquire a better grasp of relevant potential applications.

The paper is organised as follows. Section 2 briefly recovers Aït-Sahalia's univariate approach, while the following section analyses the bivariate case. Section 4 proposes several semiparametric estimators for the covariance between the processes. The next section applies the bivariate approach to a pair of assets: the main stock indexes of Brazil and Argentina. Section 6 concludes.

2. UNIVARIATE SEMIPARAMETRIC ESTIMATION OF DIFFUSIONS

Aït-Sahalia (1996) considers the univariate Fokker-Planck (FP) equation (or the Kolmogorov forward equation, Karlin and Taylor (1981), p. 219), which describes the transition densities of continuous-time Markov processes without jump:

$$\frac{\partial}{\partial t} f(x,t;y,t') = -\frac{\partial}{\partial x}(\mu(x) f(x,t;y,t')) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2(x) f(x,t;y,t')) \qquad (1)$$

where:

$f(x,t;y,t')$:= transition density from point *(y,t')* to *(x,t)*;

$\mu(x)$:= drift of the process;

$\sigma^2(x)$:= volatility of the process.

The drift is parametrised as in Vasicek (1977) - himself inspired in the Ornstein-Uhlenbeck process -, with the mean-reverting property. Parametrization of the drift is fundamental to the identification of the pair (μ, σ^2): imposing no restriction on the pair makes it impossible to distinguish it from the pair ($a\mu, a\sigma^2$), where a is a constant, when considering a discrete sample with fixed time-intervals.

Supposing a general parametrization $\mu(x,\theta)$, under the assumption that the process is stationary – or rather, that it has converged to a steady state – we can write $\pi(x,t) = \pi(x)$, for its marginal density. Multiplying both sides of (1) by $\pi(y)$ and integrating with respect to y, one obtains,

$$\frac{\partial}{\partial x}(\mu(x,\theta)\int f(x,t;y,t')\pi(y)dy) = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2(x)\int f(x,t;y,t')\pi(y)dy) \qquad (2)$$

as $\frac{\partial}{\partial t}\int \pi(y) f(x,t;y,t')dy = \frac{\partial}{\partial t}\pi(x,t) = 0$. Using again the assumption of stationarity,

$$\frac{d}{dx}(\mu(x,\theta)\pi(x)) = \frac{1}{2}\frac{d^2}{dx^2}(\sigma^2(x)\pi(x)) \quad , \qquad (3)$$

integrating twice (3) with respect to x and using the boundary condition $\pi(0) = 0$:

$$\sigma^2(x) = \frac{2}{\pi(x)}\int_0^x \mu(u,\theta)\pi(u)du \quad . \qquad (4)$$

It is then possible to write the volatility as an explicit function $\varphi(\theta\,;\pi(x))$ of the marginal density and the parameter vector characterising the drift. If these two objects are estimated, a semiparametric estimate of the volatility function can be obtained as:

$$\hat{\sigma}^2(x) = \varphi(\hat{\theta};\hat{\pi}(x)) \quad . \qquad (5)$$

Aït-Sahalia (1996), with the help of a functional version of the delta method, shows that this estimator is point-wise consistent and asymptotically normal.

3. A BIVARIATE GENERALIZATION

3.1 The bivariate Fokker-Planck equation

Consider now the bivariate Fokker-Planck equation, already with a parametrization on the drift:

$$\frac{\partial}{\partial t} f(x,t;y,t') = -\sum_{i=1}^{2} \frac{\partial}{\partial x_i}(\mu_i(x,\theta) f(x,t;y,t')) + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j}(b_{ij}(x) f(x,t;y,t')) \tag{6}$$

where

$f(x,t;y,t')$:= transition density from point *(y,t')* to *(x,t)*;

$\mu_i(x,\theta)$, i = 1,2 := drifts of the two processes;

$b_{ij}(x)$, i, j = 1,2 := volatilities of the Fokker-Planck representation.

It is worth to stress the correspondence, at least locally, between the Itô representation and the Fokker-Planck equation (Gardiner (1990), chapter 3). The bivariate version of the Itô stochastic differential equation is:

$$\begin{bmatrix} dx_{1t} \\ dx_{2t} \end{bmatrix} = \begin{bmatrix} \mu_1(x_t) \\ \mu_2(x_t) \end{bmatrix} dt + \begin{bmatrix} \sigma_{11}(x_t) & \sigma_{12}(x_t) \\ \sigma_{21}(x_t) & \sigma_{22}(x_t) \end{bmatrix} \begin{bmatrix} dB_{1t} \\ dB_{2t} \end{bmatrix} \tag{7}$$

where $x_t = (x_{1t}, x_{2t})$, and $\{(B_{it})_{i=1,2}, t \geq 0\}$ is a standard two-dimensional Brownian motion. The functions $\mu_i(.)$ and $\sigma_{ii}^2(.)$, i = 1,2 , are, respectively, the drift and the "volatility" of each process, and $\sigma_{ij}(.)$, i, j = 1,2, i≠j are instantaneous "covariances" between them.

The relation between the volatilities in both representations, i.e. in (6) and (7), is:

$$\begin{aligned} b_{11}(x) &= \sigma_{11}^2(x) + \sigma_{12}^2(x) \\ b_{12}(x) &= b_{21}(x) = \sigma_{11}(x)\sigma_{21}(x) + \sigma_{12}(x)\sigma_{22}(x) \\ b_{22}(x) &= \sigma_{21}^2(x) + \sigma_{22}^2(x) \end{aligned} \tag{8}$$

To obtain an analytical solution for the bivariate version of the FP equation in the spirit of the previous section, one needs assumptions that make the analysis more than a simple extension of the univariate method. First of all, we assume again stationarity of the process, and notice that $b_{12} = b_{21}$, which means that the "covariances" of the FP representation are equal. We then introduce the hypothesis that each drift depends only on its underlying process, which means,

$$\mu_i(x,\theta) = \mu_i(x_i,\theta),\ \mathrm{i} = 1, 2 \qquad . \tag{9}$$

Multiplying both sides of (6) by the joint density $\pi(y_1, y_2) \equiv \pi(y)$, integrating with respect to y and recalling that stationarity makes the left side equal to zero and one obtains:

$$\sum_{i=1}^{2}\frac{\partial}{\partial x_i}(\mu_i(x_i;\theta)\pi(x)) = \frac{1}{2}\sum_{i=1}^{2}\frac{\partial^2}{\partial x_i^2}(b_{ii}(x)\pi(x)) + \frac{\partial^2}{\partial x_1 \partial x_2}(b_{12}(x)\pi(x)) \qquad .$$

Integrating now with respect to x_1 and x_2:

$$\sum_{i,k=1;i\neq k}^{2}\mu_i(x_i;\theta)\int_{x_k}\pi(x)dx_k = b_{12}(\mathrm{x})\pi(\mathrm{x}) + \frac{1}{2}\sum_{i,k=1;i\neq k}^{2}\frac{\partial}{\partial x_i}\int_{x_k} b_{ii}(x)\pi(x)dx_k \qquad . \tag{10}$$

where all integration constants were set to zero.

Calling $\pi_i(x_i)$, i = 1,2, the marginal densities, in fixed-income analysis it is quite natural to assume $\pi_1(0) = \pi_2(0) = \pi(0,0) = 0$. Intuitively, this means assigning a probability zero to the event [nominal interest rate = 0]. When considering stock returns, for example, the integration interval is $[x_i^{min}, x_i^{max}]$ and an analogous hypothesis is $\pi_1(x_1^{min}) = \pi_2(x_2^{min}) = \pi(x_1^{min}, x_2^{min}) = 0$, and one obtains integration constants equal to zero once again.

Now, if in addition, each “variance” of the FP representation depends only on its own process,

$$b_{ii}(x) = b_{ii}(x_i),\ \mathrm{i} = 1,2 \qquad , \tag{11}$$

it is possible to write the FP “covariance” explicitly:

$$b_{12}(x) = \frac{1}{\pi(x)}\left[\sum_{i,k=1}^{2}\mu_i(x_i,\theta)\int_{x_k}\pi(x)dx_k - \frac{1}{2}\sum_{i,k=1}^{2}\frac{\partial}{\partial x_i}b_{ii}(x_i)\int_{x_k}\pi(x)dx_k\right] \qquad . \tag{12}$$

However, it is still necessary to identify the system (8), relating the FP and the Itô volatilities.

3.2 Parametrizing Itô volatilities

3.2.1 A general setting

The correspondence between the volatilities in the FP and Itô representations is much simpler in the univariate case than in the bivariate one. While the FP equation is more convenient in operational terms for developing our estimation procedure, the Itô representation has an intuitive appeal, especially when considering a continuous-time counterpart of a covariance matrix. The aim here is to analyse (10) and (12), according to various assumptions imposed on the (perhaps more natural) Itô representation (7).

The key to pass from one equation to the other seems to be identities in (8). A simple way to identify this system is to assume that $\sigma_{21} \equiv 0$, what gives:

$$
\begin{aligned}
&b_{11}(x) = \sigma_{11}^2(x) + \sigma_{12}^2(x) \\
&b_{12}(x) = b_{21}(x) = \sigma_{12}(x)\sigma_{22}(x) \\
&b_{22}(x) = \sigma_{22}^2(x)
\end{aligned}
\tag{13}
$$

If, for instance, (11) is also imposed, this would additionally imply that

$$
\begin{aligned}
&\sigma_{22}(x) = \sigma_{22}(x_2) \\
&\sigma_{11}^2(x) + \sigma_{12}^2(x) \text{ is independent of } x_2.
\end{aligned}
\tag{14}
$$

One way of fulfilling the second condition above is to make:

$$\sigma_{11}(x) = \sigma_{11}(x_1) \quad \text{and} \quad \sigma_{12}(x) = \sigma_{12}(x_1). \tag{15}$$

As it will be shown below, (14) and (15) are somewhat stringent conditions and make the procedure more useful for testing rather than estimation purposes.

Another idea would be to solve system (13) for the σ's, obtaining:

$$
\begin{aligned}
&\sigma_{11}^2(x) = b_{11}(x) - \frac{b_{12}^2(x)}{b_{22}(x)} = \frac{b_{11}(x)\, b_{22}(x) - b_{12}^2(x)}{b_{22}(x)} \\
&\sigma_{12}(x) = \frac{b_{12}(x)}{\sqrt{b_{22}(x)}} \\
&\sigma_{22}^2(x) = b_{22}(x)
\end{aligned}
\tag{16}
$$

Now, again, (8) may be imposed but clearly, in principle, both $\sigma_{11}(x)$ and $\sigma_{12}(x)$ will depend on the two components of vector x.

All the above assumptions are not sufficient to obtain an analytical solution for either (10) or (12). As a consequence, compared to the univariate case, one needs additional parametric assumptions. One interesting parametrization, which has also an intuitive appeal, consists in imposing functional forms on the Itô variances, i.e. on the diagonal terms of the instantaneous covariance matrix of the Itô representation. In particular, consider those variances taking the form of the volatilities in the Vasicek and Cox-Ingersoll-Ross (CIR) models; this will allow to write explicitly the covariance of the Itô representation as a function of the drifts and variances of both univariate components, and of the joint and marginal densities of the process. We shall now explore these specifications.

3.2.2 The double Vasicek model

Consider the Itô volatilities and assume that, besides the identification condition $\sigma_{21} = 0$, they are parametrized as constants, such as in the Vasicek (univariate) model:

$$\sigma_{11}(x_1) = C_1 \quad , \quad \sigma_{22}(x_2) = C_2 \quad . \tag{17}$$

These assumptions, together with (14), transform (13) into

$$\begin{aligned} b_{11}(x_1) &= \sigma_{11}^2(x_1) + \sigma_{12}^2(x_1) = C_1^2 + \sigma_{12}^2(x_1) \\ b_{12}(x) &= \sigma_{12}(x_1)\sigma_{22}(x_2) = C_2\sigma_{12}(x_1) \\ b_{22}(x_2) &= \sigma_{22}^2(x_2) = C_2^2 \end{aligned} \tag{18}$$

Inserting (18) in (12), one obtains a nonlinear ordinary differential equation (NODE) with variable coefficients

$$A_1\sigma_{12}(x_1) + A_2\sigma_{12}(x_1)\sigma_{12}'(x_1) + A_3\sigma_{12}^2(x_1) + A_4 = 0 \tag{19}$$

where

$$\begin{aligned}
A_1 &= C_2\pi(x_1, x_2) \\
A_2 &= \pi_1(x_1) \\
A_3 &= \frac{1}{2}\frac{d\pi_1(x_1)}{dx_1} \\
A_4 &= \frac{1}{2}\sum_{i=1}^{2} C_i^2 \frac{d\pi_i(x_i)}{dx_i} - \sum_{i=1}^{2}\mu_i(x_i,\theta)\int_{x_j}\pi(x)dx_j
\end{aligned} \tag{20}$$

In spite that the equation above shows that - once obtained the vector parameter θ, and constants C_1 and C_2 - it is possible to identify the covariance between the processes from the joint density $\pi(.,.)$ and the marginal densities $\pi_1(.)$ and $\pi_2(.)$, by hypothesis, the solution to (19) should be a function of x_1 only. Nevertheless, inspection of (20) shows that A_1 and A_4 are functions of the whole vector x, nothing a priori guaranteeing that the solution, in a given case, will be independent of the x_2 values. This fact makes the “double Vasicek specification”, within the context of our proposal, more suitable for a testing procedure rather than for estimation purposes.

3.2.3 The double Cox-Ingersoll-Ross model

As known, the Vasicek model has some undesirable features, like the occurrence of processes with negative interest rates. The CIR model overcomes this problem by the convenient specification of the volatility function. Consider then once again equations (13) and assume that the volatilities are parametrized as in the CIR model:

$$\sigma_{11}(x_1) = C_1\sqrt{x_1} \qquad , \qquad \sigma_{22}(x_2) = C_2\sqrt{x_2} \qquad . \tag{21}$$

Using again (14), (13) now becomes

$$\begin{aligned} b_{11}(x_1) &= \sigma_{11}^2(x_1) + \sigma_{12}^2(x_1) = C_1^2 x_1 + \sigma_{12}^2(x_1) \\ b_{12}(x) &= \sigma_{12}(x_1)\sigma_{22}(x_2) = C_2\sqrt{x_2}\,\sigma_{12}(x_1) \\ b_{22}(x_2) &= \sigma_{22}^2(x_2) = C_2^2 x_2 \end{aligned} \tag{22}$$

Inserting (22) into (12), one obtains a NODE with variable coefficients formally similar to (19):

$$B_1\sigma_{12}(x_1) + B_2\sigma_{12}(x_1)\sigma_{12}{}'(x_1) + B_3\sigma_{12}^2(x_1) + B_4 = 0 \tag{23}$$

where

$$\begin{aligned} B_1 &= C_2\sqrt{x_2}\,\pi(x_1, x_2) \\ B_2 &= \pi_1(x_1) \\ B_3 &= \frac{1}{2}\frac{d\pi_1(x_1)}{dx_1} \\ B_4 &= \frac{1}{2}\sum_{i=1}^{2} C_i^2 x_i \frac{d\pi_i(x_i)}{dx_i} + \frac{1}{2}\sum_{i=1}^{2} C_i^2 \pi_i(x_i) - \sum_{i=1}^{2}\mu_i(x_i,\theta)\int_{x_j}\pi(x)dx_j \end{aligned} \tag{24}$$

The equations above bear the same attributes and the same problem of those from the previous specification, so that the same comment applies.

3.3 Parametrizing FP volatilities

We explore now the combination of (16) with (11). The diffusion coefficients $b_{ii}(.)$, i=1,2, could, for instance, be specified in a "CIR fashion" as:

$$b_{ii}(y_i) = k_i y_i \text{, i = 1,2} \quad . \tag{25}$$

Alternatively, one could specify them in a "Vasicek fashion" as:

$$b_{ii} = k_i \text{, i = 1,2} \quad . \tag{26}$$

After imposing these parametrized volatilities, one may obtain a semiparametric estimate of the FP covariance $b_{12}(.)$ from (12). Constants k_i, i=1,2, must then be obtained beforehand. The fact that $\sigma_{21} = 0$ implies that the second process will be a true CIR or Vasicek one, so that k_2 may be obtained via standard methods. As for $k_{1,}$ it may be obtained in an iterative way. Other ideas to obtain an estimator for k_1 are discussed in the Appendix.

4. THE SEMIPARAMETRIC PROCEDURE

In order to implement the procedure developed in section 3.2, concerning parametrizations of the diagonal terms of the Itô volatility matrix, the drift parameters $\theta_i = (\alpha_{i,}\beta_i)$, the densities $\pi(.,.)$, $\pi_i(.)$ and $d\pi_i / dx_i$, and the parameters C_i, i=1,2 , should be replaced by consistent estimators. The densities are estimated using kernel smoothers (see Silverman (1986) for an introduction and Scott (1992) for an advanced treatment), while GMM estimation after discretization of each component process yields θ_i and C_i, i=1,2. The only parameter remaining to be estimated is $\sigma_{12}(.)$, the solution of either (19) or (23) depending on the assumptions concerning the diagonal terms of the Itô volatility matrix.

If instead one considers the implementation of the procedure suggested in section 3.3, concerning parametrizations of the FP volatilities, we propose to estimate the drift parameter vector θ and the parameter k_2 of the FP volatility of the second process using GMM. The densities $f(.|.)$, $f(.,.)$, $\pi(.)$ and $f_1(.|.)$ are estimated using kernel smoothers. The first approach suggested in Appendix 1, concerning the parameter k_1, could be accomplished through OLS. The other alternative, based on the density (A9), deserves more study but, as mentioned before, one possibility could be to choose a k_1 which minimises the Kullback-Leibler discrepancy measure between its associated normal density and the one estimated nonparametrically. Once k_1 is obtained, computation of $b_{12}(y)$ is straightforward.

5. AN APPLICATION TO THE BRAZILIAN AND ARGENTINIAN STOCK MARKETS

5.1 The data

To illustrate our approach, we use daily (logarithmic) returns of the Ibovespa and the Merval, which are, respectively, the main Brazilian and Argentinian stock indexes. The sample is from October 19, 1989 to March 16, 1999, and contains the market closure values of the index. It was assumed that Fridays are followed by Mondays, with no adjustment for weekend effects.

Although the series are non-stationary in levels, the returns seem to be stationary (see Figures 1 and 2). An interesting feature of the returns is the occurrence of outliers, especially in the Brazilian series, a characteristic of emerging markets. The stationarity assumption was tested for both series (see Table 1), being clearly satisfied; Table 2 shows some summary statistics. One should note that the null hypothesis of normality of the returns is clearly rejected by the Jarque-Bera test (see Davidson and MacKinnon (1993), p. 567), mostly because of kurtosis – this feature will be mentioned again, when considering the density estimates. As a consequence, estimation methods based on maximum likelihood under the normality assumption are expected to be inefficient.

5.2 GMM estimation

The GMM estimates for univariate Vasicek models $(\mu(r), \sigma(r)) = (\beta(\alpha\text{-}r), \sigma)$ were obtained from the following four moment conditions (Δ = 1 day) (see Karlin and Taylor (1981), p. 218, and Aït-Sahalia (1996) for details):

$$E\, f_t\,(\theta)' \equiv E[\ \varepsilon_{t+\Delta}\ ,\ r_t\ \varepsilon_{t+\Delta}\ ,\ \varepsilon_{t+\Delta}{}^2 - E[\varepsilon_{t+\Delta}{}^2|\ r_t]\ ,\ r_t\ (\varepsilon_{t+\Delta}{}^2 - E[\varepsilon_{t+\Delta}{}^2|\ r_t])\ ] = 0$$

where r_t are the observations of the process, and

$$\varepsilon_{t+\Delta} \equiv (r_{t+\Delta} - r_t) - E[(r_{t+\Delta} - r_t)|\ r_t]$$

$$E[(r_{t+\Delta} - r_t)|\ r_t] = (1 - e^{-\beta\Delta})\ (\alpha - r_t)$$

$$E[\varepsilon_{t+\Delta}{}^2|\ r_t] = (\sigma^2/2\beta)\ (1 - e^{-2\beta\Delta})$$

One should recall that this problem does not reduce to OLS, as we have an over-identified system; moreover, these moments correspond to transitions of length Δ, and are not subject to discretization bias. Notwithstanding, the moment conditions above are a first

Table 1. Unit root tests.

	ADF Test Statistics	10% Critical Value
Ibovespa Index	-2.41	-3.13
Merval Index	-1.93	-3.13

Insert Figures 1 and 2 by here

Table 2. Basic statistics of the returns.

	Ibovespa Returns	Merval Returns
Mean	0.0067	0.0013
Median	0.0051	0.0009
Std. Dev	0.2968	0.0398
Skewness	0.2732	-1.7508
Kurtosis	540.7741	72.9395
	Normality Test	
Jarque – Bera	27.08×10^6	0.46×10^6

approximation to the problem. As a matter of fact, under system (18), ideally, when estimating the parameters of the first component process, one should also consider the off-diagonal term σ_{12} of the Itô volatility matrix; what, as discussed in section 3, could be done iteratively.

Table 3. GMM estimation for the Vasicek model.

	Ibovespa Returns	Merval Returns
α	3.89×10^{-3}	1.23×10^{-3}
	(4.10)**	(1.39)*
β	2.410	2.794
	(6.51)**	(3.87)**
σ	7.83×10^{-3}	3.917×10^{-2}
	(10.81)**	(2.81)**

Notes: (i) the estimates reported are for daily sampling of the returns ; (ii) heteroskedasticity-robust t statistics are in parentheses.
* null rejected at 10 percent ; ** null rejected at 1 percent.

5.3 Density estimation

The densities of both Ibovespa and Merval returns are characterised by heavy tails. Consider first the joint Ibovespa and Merval returns density, in Figure 3, and then the non-parametric marginal densities estimates of each return compared to the normal densities with same mean and variance in Figures 4 and 5. As the density estimates are inputs to the estimation procedure, it is taking into account the heavy tails which characterise the data at stake.

5.4 Itô covariance estimation: the double Vasicek and CIR models

Given the estimates for θ_i, C_i , i = 1,2, and the density estimates for $\pi(.,.)$ and $\pi_i(.)$, i=1,2, the covariance estimate $\hat{\sigma}_{12}$ solves (19) and (23), respectively, for the Vasicek and CIR assumptions about the diagonal of the Itô covariance matrix. We shall use in this exercise, somewhat improperly, the C_i estimates obtained above. This could be avoided by considering more complex moment conditions which take into account the off-diagonal terms of the Itô volatility matrix, as already mentioned in section 5.2.

Insert Figures 3, 4 and 5 by here

One important question concerning the boundary conditions remains. Consider first the Vasicek model in the case of fixed income. At the point $(x_1,x_2) = (0,0)$ one may rewrite (20) as:

$$A_1 = C_2\, \pi(0,0)$$

$$A_2 = \pi_1(0)$$

$$A_3 = \frac{1}{2}\left.\frac{d\pi_1(x_1)}{dx_1}\right|_{x_1=0}$$

$$A_4 = \frac{1}{2}\sum_{i=1}^{2} C_i^2 \left.\frac{d\pi_i(x_i)}{dx_i}\right|_{x_1=0} - \sum_{i=1}^{2}\mu_i(0,\theta)\pi_i(0) \qquad ,$$

if we impose $\pi_1(0) = \pi_2(0) = \pi(0,0) = 0$, which is the bivariate counterpart of Aït-Sahalia (1996)'s boundary condition, the result is

$$\sigma_{12}(0) = \left[-\left(\left.\frac{d\pi_1(x_1)}{dx_1}\right|_{x_1=0}\right)^{-1}\left(\sum_{i=1}^{2} C_i^2 \left.\frac{d\pi_i(x_i)}{dx_i}\right|_{x_i=0}\right)\right]^{1/2} \qquad . \tag{27}$$

It is straightforward to see that one may get a complex-valued boundary condition. Imposing a real $\sigma_{12}(0) > 0$ implies that

$$\frac{\left.\frac{d\pi_2(x_2)}{dx_2}\right|_{x_2=0}}{\left.\frac{d\pi_1(x_1)}{dx_1}\right|_{x_1=0}} < -\frac{C_1^2}{C_2^2} \qquad ,$$

as both C_1 and C_2 are assumed to be strictly positive and the derivatives are likely to have the same (positive) sign, the assumption is invalid.

Consider next the Vasicek model in the case of variable income. Making $\pi_1(x_1^{\min}) = \pi_2(x_2^{\min}) = \pi(x_1^{\min}, x_2^{\min}) = 0$, one gets, at $(x_1, x_2) = (x_1^{\min}, x_2^{\min})$, for (20):

$$A_1 = 0$$

$$A_2 = 0$$

$$A_3 = \frac{1}{2}\left.\frac{d\pi_1(x_1)}{dx_1}\right|_{x_1^{\min}}$$

$$A_4 = \frac{1}{2}\sum_{i=1}^{2} C_i^2 \left.\frac{d\pi_i(x_i)}{dx_i}\right|_{x_i^{\min}} - \sum_{i=1}^{2}\mu_i(x_i^{\min}, \theta)\pi_i(x_i^{\min})$$

leading to:

$$\sigma_{12}(x_1^{\min}) = \left[-\left(\frac{d\pi_1(x_1^{\min})}{dx_1}\right)^{-1}\left(\sum_{i=1}^{2} C_i^2 \frac{d\pi_i(x_{i1}^{\min})}{dx_i}\right)\right]^{1/2} \qquad . \qquad (28)$$

Once again one may get a complex-valued boundary condition, what in fact happened with the data at stake. Of course, there are other boundary conditions to be considered; given the results above, several were tried, and those actually used were obtained as follows.

For the Vasicek model, assume that $x_1^{\min}$ is such that A_1=0 and A_4=0, a reasonable assumption for our data set. The differential equation (19), related to the Vasicek model, may then be written, at $x_1^{\min}$, as

$$A_2 \frac{d\sigma_{12}(x_1^{\min})}{dx_1} \sigma_{12}^{-1}(x_1^{\min}) + A_3 = 0 \qquad (29)$$

and, by straightforward calculation, one gets:

$$\log(\sigma_{12}(x_1^{\min})) = -\frac{A_3}{A_2} x_1^{\min} + K_2 \qquad ;$$

so that, assuming that K_2 = 0:

$$\sigma_{12}(x_1^{\min}) = \exp\left\{-\frac{A_3}{A_2} x_1^{\min}\right\} \qquad . \qquad (30)$$

Alternatively, one may set $\sigma_{12}(x_1^{\min})$ = $\sigma_{12}'(x_1^{\min})$ = w and A_1 = 0. This allows to rewrite (19), at $x_1^{\min}$, as:

$$(A_2 + A_3)\, w^2 + A_4 = 0 \qquad . \qquad (31)$$

The solutions are given by:

$$\sigma_{12}(x_1^{\min}) = w = \pm\left(-\frac{A_4}{A_2 + A_3}\right)^{1/2} \qquad . \qquad (32)$$

The solution of (19) subject to the boundary conditions (30) and (32) is rather computer intensive. The negative root in (32) did not produce a sensible surface and, in all cases, the numerical iterations did not converge for values of x_1 (Ibovespa resturns) less than –0.05. Moreover, for similar reasons, the range of x_2 (Merval resturns) was restricted to (-0.2 , 0.2).

Although each boundary condition may originate a completely different behaviour for the covariance estimate, the shape in each of the two cases considered does not vary too much along the range of x_2, the returns of the Merval index. In fact, the two solutions shown are flat for a considerably large range of x_2 (say between –0.10 and 0.10). This may indicate that there is independence between the covariance estimates and the variable x_2, a question that deserves more study in the future.

6. CONCLUSIONS

We developed an exploratory procedure to estimate and test two-dimensional diffusions used in finance. The extension of Aït-Sahalia's univariate idea to the bivariate case is not immediate, since the solution of the corresponding Fokker-Planck equation can easily become very difficult, if not impossible in analytical terms. By parametrizing the drifts of both processes and imposing restrictions on the terms of the Itô and Fokker-Planck covariance matrices, it is sometimes possible to obtain a nonparametric estimate of the covariance between the processes. However, a delicate issue might still remain, regarding the definition of the boundary conditions for the partial differential equations to be actually solved.

Our main message is that extending in a general way Aït-Sahalia (1996)'s framework to a multivariate setting is by no means straightforward. The basic reason is perhaps because the correspondence between the Itô and Fokker-Planck representations in higher dimensions is not the same as in the univariate case. For the methods developed here, the Fokker-Planck is more tractable than the Itô representation, suggesting that parametrizations should be made on the former. Notwithstanding, even when imposing sensible assumptions on the drift and volatility functions, one cannot obtain a direct generalisation of the univariate method.

Questions for improvement and further research include (i) the development of the testing procedure concerning the differential equations resulting from the parametrizations on the Itô volatilities in 3.2; (ii) the implementation of the estimation and testing procedure sketched in section 3.3 and Appendix 1; (iii) the improvement of the GMM estimation procedure used in section 5.2; (iv) alternative boundary conditions for differential equations (19) and (22), as exemplified in 5.4; (v) the allowance of different bandwidths, taking also into account the heavy tails of returns distributions, and the use of bivariate kernels which would exploit better the dependence structure of the bivariate data. Moreover, by imposing suitable regularity conditions on (a) the time series dependence in the data, (b) the kernels actually used and (c) the bandwidths, asymptotic results must be developed for the various estimators proposed.

Finally, a particular case that seems promising is that of “causal bivariate models” in which one of the diffusions contributes to the volatility of the other. The appeal of this idea is immediate – investors could improve their hedging strategies when considering a flexible estimator of the covariance between two (groups of) assets of a given portfolio, instead of assuming independence among processes and estimating separately univariate diffusions.

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APPENDIX. ESTIMATING FP VOLATILITIES

A natural way to obtain an estimator of the constant in $b_{11}(.)$, either in (25) or (26), is to consider the kernel estimator of the conditional density using its definition:

$$\hat{f}(x,t \mid y,t') \equiv \frac{\hat{f}(x,y)}{\hat{\pi}(y)} \tag{A1}$$

with $x = (x_1, x_2)$ and $y = (y_1, y_2)$, so that the numerator of the right-hand side of (A1) is the joint density of two observations at the time distance (t' + Δt) – t' = Δt, which is the interval between observations, and the denominator is the corresponding marginal density. The (product) kernel estimators of the joint and marginal densities are given by, respectively,

$$\hat{f}(x,y) = \frac{1}{nh^4}\sum_{i=1}^{n} K\left(\frac{x_1 - x_{1i}}{h}\right)K\left(\frac{x_2 - x_{2i}}{h}\right)K\left(\frac{y_1 - y_{1i}}{h}\right)K\left(\frac{y_2 - y_{2ii}}{h}\right) \tag{A2}$$

and

$$\hat{\pi}(y) = \frac{1}{nh^2}\sum_{i=1}^{n} K\left(\frac{y_1 - y_{1i}}{h}\right)\mathrm{K}\left(\frac{\mathrm{y}_2 - y_{2i}}{h}\right) \tag{A3}$$

where the kernel $K(.)$ is a symmetric, finite variance, univariate density, and the parameter h, assumed to be the same for every kernel, is the bandwidth.

The marginal $\hat{f}_1(x_1 \mid y)$ of the bivariate conditional $\hat{f}\,(x \mid y)$ will be:

$$\hat{f}_1(x_1 \mid y) \equiv \int \frac{\hat{f}(x,y)}{\hat{\pi}(y)}\,dx_2 = \frac{1}{\hat{\pi}(y)}\int \hat{f}(x,y)\,dx_2 \quad . \tag{A4}$$

By defining $u = (x_2 - x_{2i})/h$, integrating (A4) with respect to u, and using the fact that the kernel function is a density, one obtains

$$\hat{f}_1(x_1 \mid y) = \frac{1}{h} \frac{\sum_{i=1}^{n} K\left(\frac{x_1 - x_{1i}}{h}\right) K\left(\frac{y_1 - y_{1i}}{h}\right) \mathrm{K}\left(\frac{\mathrm{y}_2 - y_{2i}}{h}\right)}{\sum_{i=1}^{n} K\left(\frac{y_1 - y_{1i}}{h}\right) \mathrm{K}\left(\frac{\mathrm{y}_2 - y_{2i}}{h}\right)} , \tag{A5}$$

which can be used to compute the marginal variances for selected $y_j = (y_{1j}, y_{2j})$.

By defining $v = (x_1 - x_{1i})/h$, integrating with respect to v, and using the properties of the kernel function, one obtains for the marginal mean:

$$\hat{\mu}_1 = \frac{\sum_{i=1}^{n} x_1\, K\left(\frac{y_1 - y_{1i}}{h}\right) K\left(\frac{y_2 - y_{2i}}{h}\right)}{\sum_{i=1}^{n} K\left(\frac{y_1 - y_{1i}}{h}\right) K\left(\frac{y_2 - y_{2i}}{h}\right)} \tag{A6}$$

and for the marginal variance

$$\hat{b}_{11} = \frac{\sum_{i=1}^{n} (x_1 - \hat{\mu}_1)^2\, K\left(\frac{y_1 - y_{1i}}{h}\right) \mathrm{K}\left(\frac{\mathrm{y}_2 - y_{2i}}{h}\right)}{\sum_{i=1}^{n} K\left(\frac{y_1 - y_{1i}}{h}\right) \mathrm{K}\left(\frac{\mathrm{y}_2 - y_{2i}}{h}\right)} . \tag{A7}$$

Fitting a line through the points $y_j = (y_{1j},\ \hat{b}_{11}(y_j))$ plus the origin will produce an estimate for k_1.

A second idea – likely to be more demanding and dependent on approximations – starts by recalling that, for small displacements $\tau = t - t'$, if the derivatives of the FP drifts and volatilities are negligible compared of those of the transition density, the equation to be solved is approximately:

$$\frac{\partial}{\partial t} f(x,t \mid y,t') = -\sum_{i=1}^{2} \mu_i(y,\theta) \frac{\partial f(x,t \mid y,t')}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} b_{ij}(y) \frac{\partial^2 f(x,t \mid y,t')}{\partial x_i \partial x_j} \quad . \qquad \text{(A8)}$$

Subject to the initial condition

$$f(x,t;y,t') = \delta(x-y) \qquad ,$$

the solution to this equation will be a Gaussian distribution, with mean $y + \mu(y,t';\theta)(t - t')$, whose covariance matrix coincides with that of the FP b's (see Gardiner (1997), section 3.5):

$$f(x,t \mid y,t') = (2\pi)^{-N/2} \{\det[B(y,t')]\}^{1/2} (t-t')^{-1/2} \cdot$$

$$\cdot \exp\left\{-\frac{1}{2}\frac{[x-y-\mu(y,t';\theta)(t-t')]^T [B(y,t')]^{-1} [x-y-\mu(y,t';\theta)(t-t')]}{(t-t')}\right\} \qquad \text{(A9)}$$

where

$$B(y,t') = \begin{bmatrix} b_{11}(y,t') & b_{12}(y,t') \\ b_{12}(y,t') & b_{22}(y,t') \end{bmatrix},\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},\ \mu(y,t';\theta) = \begin{bmatrix} \mu_1(y_1,t';\theta) \\ \mu_2(y_2,t';\theta) \end{bmatrix} \quad .$$

Recalling that – under stationarity – the left hand side of (A9) is estimable using (A1)-(A3), this opens a range of possibilities both for estimation and testing. With the parametrizations at stake - in a “CIR or Vasicek fashion” - the parameter k_1 is not known but, if one estimates parameter k_2 (e.g. by GMM) and takes (12) into account, inserting it into (A9), $b_{12}(y)$ is eliminated and there is only one unknown in this equation – k_1. One might then choose a k_1 which minimises the Kullback-Leibler discrepancy measure between the associated bivariate normal density and the one estimated non-parametrically. Again, once k_1 is obtained, $b_{12}(y)$ follows in a straightforward manner.

Finally, if the assumptions leading to (A9) are considered reasonable, it is also possible to test a variety of hypotheses comparing the densities $f(x,t \mid y,t')$ resulting from the two alternatives. For instance, one could compare different parametrizations concerning both the drifts and volatilities of the bivariate process.

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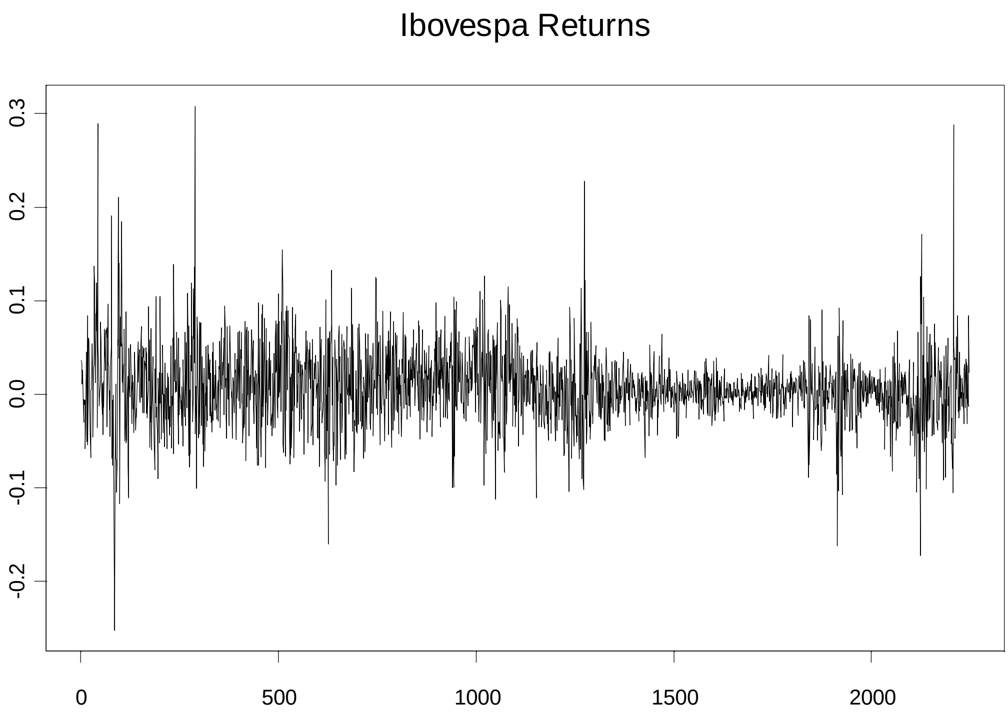

Figure 1

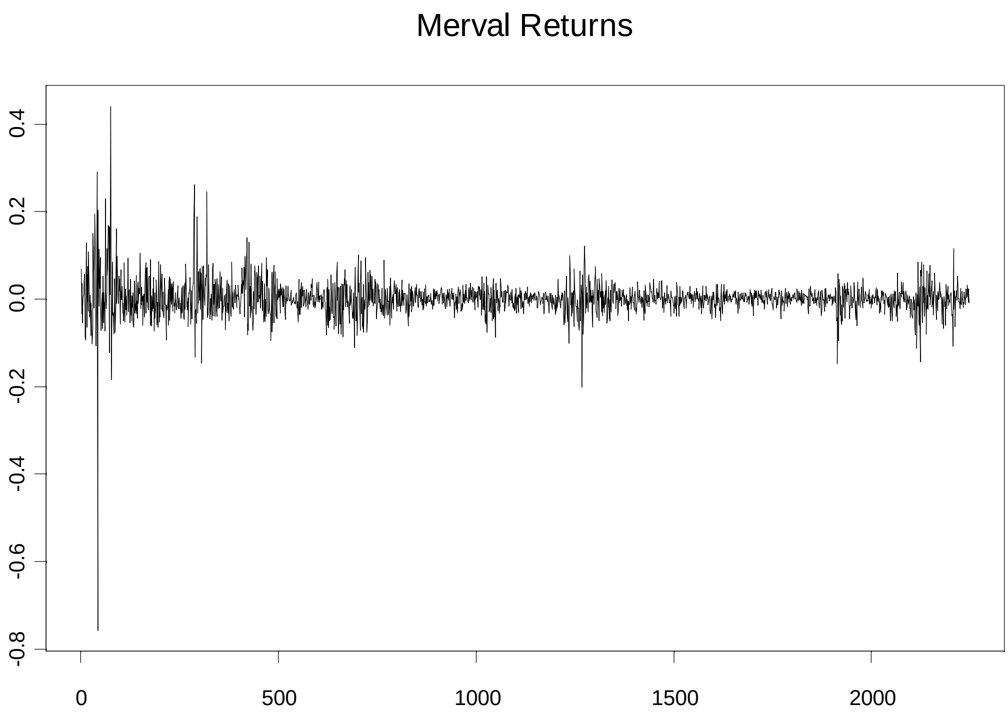

Figure 2

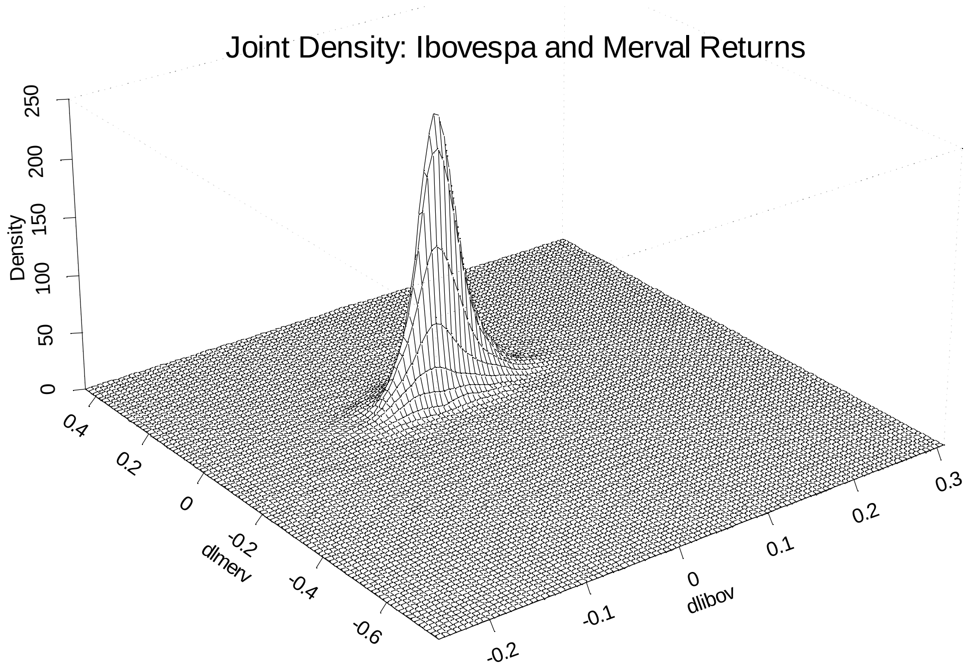

Figure 3

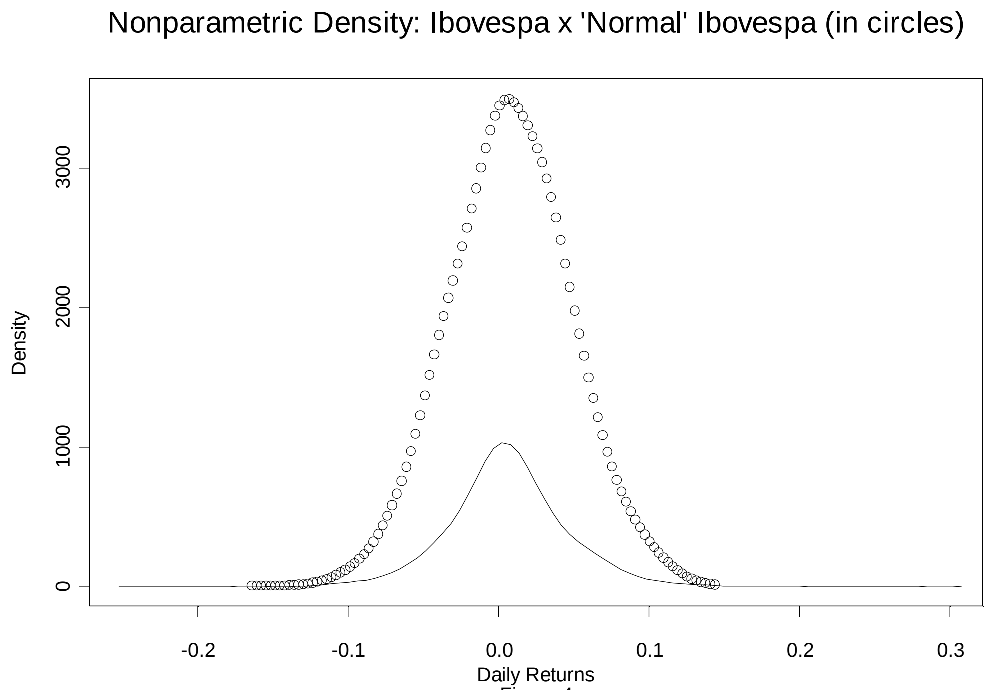

Figure 4

Nonparametric Density: Merval x 'Normal' Merval (in circles)

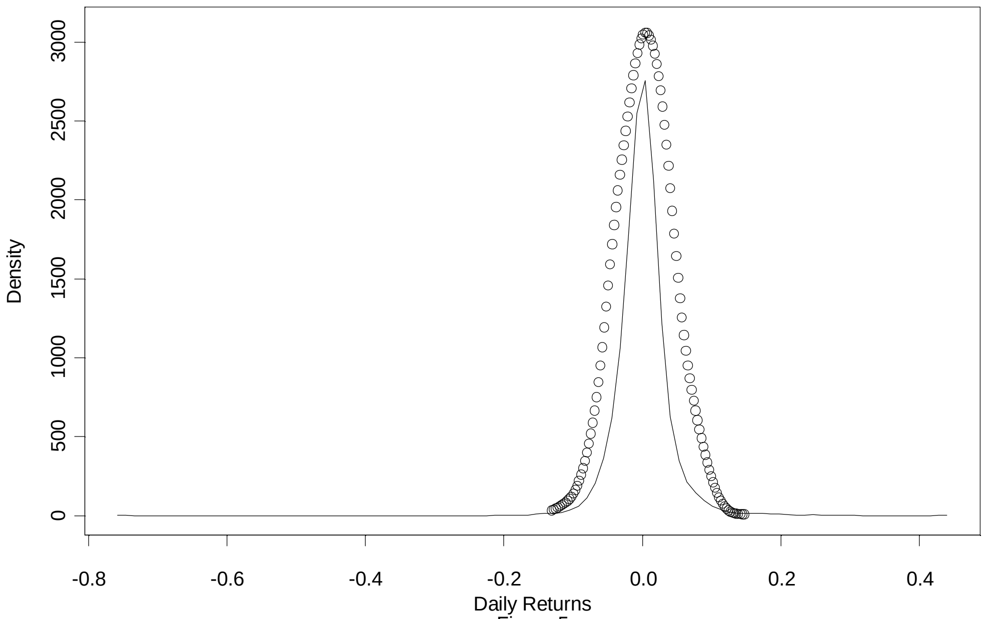

Figure 5