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Paulo Klinger Monteiro

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Optimal auctions in a general model of identical goods.

Paulo Klinger Monteiro*†
EPGE

Abstract

In this paper I study optimal auctions of identical goods. There is synergy in the number of goods and independent bidder's signals.

1 Introduction.

In this paper I study optimal simultaneous auctions of identical goods in an asymmetric bidders setting. My approach follows Myerson(1981) paper. The papers of Maskin and Riley(1989) and Branco(1996a-b) also study optimal auctions of identical goods. Indeed, Maskin and Riley suppose each bidder has a utility function $U(q,x)$ where x is the bidder signal and $q \in \mathbb{R}_+$ is the quantity received. Thus $U(1,x)$ is the utility of one unity of the good, $U(2,x)$ is the utility of two units and so on. If the good is indivisible[1] the meaning of (say) $U(5/4,x)$, is the expected utility of receiving one unit for sure and another unity with probability $5/4 - 1 = 1/4$. However Maskin and Riley's hypotheses preclude synergy: the topic of this paper.[2] The difference between the utility of several objects together and the sum of each objects' utility is called synergy.[3] For example in Krishna and Rosenthal (1996) and in Branco(1996a) the synergy is a positive constant. I allow a much more general form of synergy and even negative synergy up to a certain amount.

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The general existence of optimal auctions was studied by Page(1992). However I emphasize here an explicit characterization of the auction. This is of course possible only in very restrictive conditions. The main restriction in this paper, besides identical goods are independent bidders' signals. The general case seems very difficult. Myerson paper has still not been generalized to include dependent bidders' signals.

The hypothesis of identical goods has been made in I. Gale(1996). However his bidders' utility is dependent on the utility level of the other bidder.[4] He obtains that it is always better to sell in bundles. I prove that if the synergy is sufficiently high and doesn't increase too fast then it is best to sell in bundles.

2 Notation and basic definitions.

The set of bidders is $\mathbb{I} = \{1, 2, \ldots, I\}$. The number of objects to be sold at auction is $K \geq 1$. Bidder's i signal of each object, $x_1^i = \ldots = x_K^i$ is the realization of the integrable random variable $X^i : \Omega \to \mathbb{R}_+$, defined on the probability space $(\Omega, \mathcal{A}, P)$. The valuation of bidder i is a function $U^i : \mathbb{R}_+ \times \{0, 1, \ldots, K\} \to \mathbb{R}$. I suppose $U^i(x, 0) = 0$, $U^i(x, 1) = x$ and that $U_k^i(x) = U^i(x, k)$ is increasing.[5] Therefore if the realization of X^i is x and the number of objects received is k, bidder i has valuation $U_k^i(x)$. I suppose $U_k^i(\cdot)$ differentiable[6] with a bounded derivative. Therefore there exists an $L > 0$ such that $|U_k^i(x) - U_k^i(y)| \leq L|x - y|$ for every k, i and $x, y \geq 0$. It is convenient also to include the auctioneer as bidder number 0 and define $\mathbb{B} = \mathbb{I} \cup \{0\}$. The distribution of X^i is F_i with density $f_i(x)$. I suppose that $\{x \geq 0; f_i(x) > 0\}$ is an interval, Y_i, of the form $Y_i = [m_i, n_i)$ where $m_i \geq 0$ and $n_i \leq \infty$. The random variables $X^1, \ldots X^I$ are independent. The joint distribution of $X = (X^1, \ldots, X^I)$ is therefore $F = \Pi_{j \in \mathbb{I}} F^j$ with density $f(x) = \Pi_{j \in \mathbb{I}} f_j(x_j)$. The distribution of $(X^1, \ldots, X^{i-1}, X^{i+1}, \ldots, X^I)$ is $F_{-i} = \Pi_{j \in \mathbb{I} \setminus \{i\}} F^j$ with density f_{-i}. The set $Y = \Pi_{j \in \mathbb{I}} Y_j$ is the set of all possible bidders' signals. For each vector $y = (y_1, \ldots, y_I)$ I define the vector $y_{-i} = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_I)$ obtained by the removal of the i^{th} coordinate of y. Define also $y = (y_i, y_{-i})$. The set of allocations of objects among bidders is $\mathbf{A} = \{a \in \{0, 1, \ldots, K\}^{\mathbb{B}}; \sum_{b \in \mathbb{B}} a_b = K\}$. If $a \in \mathbf{A}$ the Bidder b gets a_b objects. For each $k = 0, 1, \ldots, K$ define the set of allocations that gives k objects to b, $\mathbf{A}_{bk} = \{a \in \mathbf{A}; a_b = k\}$. Define $S(\mathbf{A}) = \{(q_a)_{a \in \mathbf{A}}; q_a \geq$

$0, \sum_{a \in \mathbf{A}} q_a = 1\}$.

3 The model.

The auction will proceed in the following way:

1. The auctioneer publicly announces the functions[7] $q : \mathbb{R}_+^I \to S(\mathbf{A})$ and $P = (P^1, \ldots, P^I), P^i : \mathbf{A} \times \mathbb{R}_+^I \to \mathbb{R}, 1 \le i \le I$;
2. Each bidder knowing his signal X^i announces a number $y_i \ge 0$ privately to the auctioneer. The auctioneer forms the vector $y = (y_1, \ldots, y_I)$;
3. The allocation $a \in \mathbf{A}$ is drawn with probability $q_a(y) = q(y)(a), a \in \mathbf{A}$;
4. Bidder i receives a_i objects and pays $P^i(a, y)$.

As in Myerson(1981), pages 62-63, the direct mechanism (q, P) will be chosen among the direct mechanisms that satisfy individual rationality and incentive compatibility constraints.

To shorten several inequalities below I define for each (q, P) the function

$$T_i(c) = \int \sum_{a \in \mathbf{A}} q_a(c, y_{-i}) \left(U^i_{a_i}(c) - P^i_a(c, y_{-i})\right) dF_{-i}(y_{-i}), c \in \mathbb{R}_+ \quad (1)$$

and

$$Q^i_a(c) = \int q_a(c, y_{-i}) dF_{-i}(y_{-i}). \quad (2)$$

Definition 1 *1. The direct mechanism (q, P) is individually rational if*

$$T_i(y_i) \ge 0 \textit{ for all } y_i \ge 0. \quad (3)$$

2. The direct mechanism (q, P) is incentive compatible if for every $y_i, y_i' \ge 0$,

$$T_i(y_i) \ge \int \sum_{a \in \mathbf{A}} q_a(y_i', y_{-i})(U^i_{a_i}(y_i) - P^i_a(y_i', y_{-i}))dF_{-i}(y_{-i}). \quad (4)$$

The right hand side of the inequality above is the expected utility of bidder i if his signal is y_i and y_i' is announced to the auctioneer. The auctioneer problem is to choose (q, P) satisfying items (3) and (4) that maximizes:

$$\int \sum_{a \in \mathbf{A}} q_a(y) \sum_{i \in \mathbb{I}} P_a^i(y) dF(y).$$

To begin the study of inequalities (3) and (4) note that, given $(q, \hat{P})$, by defining $P^i(y) = \sum_{a \in \mathbf{A}} q_a(y) \hat{P}_a^i(y)$ a new pair of direct mechanisms, (q, P), is obtained with P^i independent of a that gives the same profit.

Remark 1 *The function Q_a^i may not be monotonic. The reason is simple: Suppose there are two objects and a allocates one object to i. If the bid of i, c, increases, i will eventually receives two objects. Therefore Q_a^i eventually decreases.*

The following lemma is similar to a lemma in Myerson. However since the functions Q_a^i may not be monotonic my proof is by necessity different.

Lemma 1 *If T_i is defined by (1) then*

1. *T_i is a primitive: For almost every $c \geq 0$, $T_i(c) = T_i(0) + \int_0^c T_i'(z) dz$ where T_i' is defined almost surely;*
2. *$T_i'(c) = \sum_{a \in \mathbf{A}} (U_{a_i}^i)'(c) Q_a^i(c)$ for almost every $c \geq 0$;*
3. *$\int T_i(x_i) dF_i(x_i) = T_i(0) + \int (1 - F^i(z)) \sum_{a \in \mathbf{A}} (U_{a_i}^i)'(z) Q_a^i(z) dz$.*

Proof. From (4) it is true for every $b, c \geq 0$ that:

$$T_i(c) - T_i(b) \geq \sum_{a \in \mathbf{A}} (U_{a_i}^i(c) - U_{a_i}^i(b)) Q_a^i(b) \geq -L|c - b| \sum_{a \in \mathbf{A}} Q_a^i(b). \quad (5)$$

Since $\sum_{a \in \mathbf{A}} Q_a^i(b) = \int \sum_{a \in \mathbf{A}} q_a(b, y_{-i}) dF_{-i}(y_{-i}) = \int dF_{-i}(y_{-i}) = 1$ we conclude that $T_i(c) - T_i(b) \geq -L|c - b|$. Changing places between b and c we obtain $T_i(b) - T_i(c) \geq -L|c - b|$. Therefore $|T_i(c) - T_i(b)| \leq L|c - b|$. So T_i is Lipschitzian and by Saks(1964) theorem 15.7 page 155, T_i is a primitive.

To prove the second part suppose $x > c$, c a point of differentiability of T_i. Then

$$\frac{T_i(x) - T_i(c)}{x - c} \geq \sum_{a \in \mathbf{A}} \frac{U^i_{a_i}(x) - U^i_{a_i}(c)}{x - c} Q^i_a(c).$$

Therefore passing to the limit $x \to c$ we get $T_i'(c) \geq \sum_{a \in \mathbf{A}} (U^i_{a_i})'(c) Q^i_a(c)$. Analogously we prove the other inequality. To prove item (3) we apply Tonellis' theorem in the second integral below:

$$\int T_i(z) dF_i(z) = T_i(0) + \int \int_0^z T_i'(c) dc\, dF_i(z) = T_i(0) + \int (1 - F^i(c)) T_i'(c) dc.$$

Now using item 2 we finish the proof. QED

Lemma 2 *The auctioneer problem is equivalent to maximize*

$$-\sum_{i \in \mathbb{I}} \alpha_i + \int_Y \sum_{a \in \mathbf{A}} q_a(y) \left\{ \sum_{i \in \mathbb{I}} \left(U^i_{a_i}(y_i) - \frac{1 - F_i(y_i)}{f_i(y_i)} (U^i_{a_i})'(y_i) \right) \right\} f(y) dy \quad (6)$$

subject to:

$$q(y) \in S(\mathbf{A}) \text{ for every } y \in \mathbb{R}^I_+;$$

$$\alpha_i \geq 0;$$

$$\int_b^c \sum_{a \in \mathbf{A}} (U^i_{a_i})'(z) Q^i_a(z) dz \geq \sum_{a \in \mathbf{A}} (U^i_{a_i}(c) - U^i_{a_i}(b)) Q^i_a(b) \text{ for all } b, c \geq 0. \quad (7)$$

Proof. Suppose (q, P) is a direct mechanism satisfying the incentive compatibility and individual rationality constraints. If T_i is defined as in (1) then from lemma 1, item 2, and the incentive compatibility constraints we obtain (7). The individual rationality constraints imply $T_i(0) \geq 0$. From lemma 1, item 3 we have

$$\int P^i(y) dF(y) = \int \left(\int P^i(y) dF_{-i}(y_{-i}) \right) dF_i(y_i) =$$

$$\int \sum_{a \in \mathbf{A}} \left(U^i_{a_i}(y_i) Q^i_a(y_i) - T_i(y_i) \right) dF_i(y_i) =$$

$$\int \sum_{a\in\mathbf{A}} U^i_{a_i}(y_i)Q^i_a(y_i)dF_i(y_i) - T_i(0) - \int (1-F_i(z)) \sum_{a\in\mathbf{A}} (U^i_{a_i})'(z)Q^i_a(z)dz.$$

Therefore the auctioneer profit is $\int \sum_{i\in\mathbb{I}} P^i(y)dF(y) =$

$$-\sum_{i\in\mathbb{I}} T_i(0) + \sum_{a\in\mathbf{A}}\sum_{i\in\mathbb{I}} \left\{ \int U^i_{a_i}(y_i)q_a(y)dF(y) - \int (1-F_i(z))(U^i_{a_i})'(z)Q^i_a(z)dz \right\}.$$

This proves (6) with $\alpha_i = T_i(0)$. Reciprocally suppose q and $\alpha_i \geq 0$ satisfy the restrictions (6) and (7). Let us define P^i by

$$P^i(y) = \sum_{a\in\mathbf{A}} U^i_{a_i}(y_i)q_a(y) - \int_0^{y_i} \sum_{a\in\mathbf{A}} q_a(z, y_{-i})(U^i_{a_i})'(z)dz - \alpha_i.$$

Now note that

$$\int \left(\sum_{a\in\mathbf{A}} q_a(y)U^i_{a_i}(y_i) - P^i(y) \right) dF_{-i}(y_{-i}) = \sum_{a\in\mathbf{A}} U^i_{a_i}(y_i)Q^i_a(y_i) - \int P^i(y)dF_{-i}(y_{-i}) =$$

$$\alpha_i + \int \int_0^{y_i} \sum_{a\in\mathbf{A}} q_a(z, y_{-i})(U^i_{a_i})'(z)dz\, dF_{-i}(y_{-i}) = \alpha_i + \int_0^{y_i} \sum_{a\in\mathbf{A}} Q^i_a(z)(U^i_{a_i})'(z)dz.$$

Therefore the individual rationality constraint follows from $\alpha_i \geq 0$. To verify the incentive compatibility constraints fix $i \in \mathbb{I}$ and define $y' = (y'_i, y_{-i})$:

$$\int \left(\sum_{a\in\mathbf{A}} q_a(y')U^i_{a_i}(y_i) - P^i(y') \right) dF_{-i}(y_{-i}) = \sum_{a\in\mathbf{A}} Q^i_a(y'_i)U^i_{a_i}(y_i) - \int P^i(y')dF_{-i}(y_{-i}) =$$

$$\sum_{a\in\mathbf{A}} Q^i_a(y'_i)\left(U^i_{a_i}(y_i) - U^i_{a_i}(y'_i)\right) + \left(\sum_{a\in\mathbf{A}} Q^i_a(y'_i)U^i_{a_i}(y'_i) - \int P^i(y')dF_{-i}(y_{-i}) \right) \leq$$

$$\int_{y'_i}^{y_i} \sum_{a\in\mathbf{A}} Q^i_a(z)(U^i_{a_i})'(z)dz + \int_0^{y'_i} \sum_{a\in\mathbf{A}} Q^i_a(z)(U^i_{a_i})'(z)dz + \alpha_i =$$

$$\int_0^{y_i} \sum_{a\in\mathbf{A}} Q^i_a(z)(U^i_{a_i})'(z)dz + \alpha_i = \int \left(\sum_{a\in\mathbf{A}} q_a(y)U^i_{a_i}(y_i) - P^i(y) \right) dF_{-i}(y_{-i}).$$

The profit can be verified as in the proof's beginning. QED

Remark 2 *The lemma 2 generalizes Myerson's revenue equivalence theorem. The theorem of Engelbrecht-Wiggans(1988) is not applicable here since its differentiability hypothesis is not true. For example the price formula for the one bidder case (see equation 13 below) is not recoverable from its derivative as his proof needs.*

A few definitions are necessary for the main theorem enunciate. Whenever $y \in Y_i$ define $r_i(y) = \frac{1-F_i(y)}{f_i(y)}$. For each k and i define $h^i_k(y) = U^i_k(y_i) - r_i(y_i)(U^i_k)'(y_i)$. If $a \in \mathbf{A}$ the auctioneer revenue given y is $h_a(y) = \sum_{i \in \mathbb{I}} h^i_{a_i}(y)$. The set of maximizers of $h(\cdot, y)$ is

$$\mathbf{A}(y) = \{a \in \mathbf{A}; h_a(y) \geq h_v(y) \text{ for all } v \in \mathbf{A}\}.$$

The set of maximizers $a \in \mathbf{A}(y)$ which minimizes the number of goods delivered by the auctioneer is

$$\tilde{\mathbf{A}}(y) = \{a \in \mathbf{A}(y); \sum_{j \in \mathbb{I}} a_j \leq \sum_{j \in \mathbb{I}} v_j \text{ for all } v \in \mathbf{A}(y)\}.$$

The probability that bidder i receives k objects is $q_{ik}(y) = \sum_{a \in \mathbf{A}_{ik}} q_a(y)$.

Theorem 3 (Optimal auction of identical goods.) *Suppose for every $i \in \mathbb{I}$ and $y \in Y_i$ that*

1. *For every $l > k$, $h^i_l(y) - h^i_k(y)$ is either strictly increasing or constant.*
2. *$y_i - r_i(y_i)$ is strictly increasing;*
3. *$(U^i_K)'(y) \geq (U^i_{K-1})'(y) \geq \ldots \geq (U^i_1)'(y) = 1$ for every $y \geq 0$.*

Then the optimal auction (q, P) is the following:

$$q_a(y) = \begin{cases} \frac{1}{\#\tilde{\mathbf{A}}(y)} & \text{if } a \in \tilde{\mathbf{A}}(y) \text{ and } y \in Y; \\ 1 & \text{if } a = 0 \text{ and } y \in \mathbb{R}^I_+ \setminus Y; \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

and P^i is given by

$$P^i(y) = \sum_{k=1}^{K} \left(U^i_k(y_i) q_{ik}(y) - \int_0^{y_i} (U^i_k)'(z) q_{ik}(z, y_{-i}) dz \right). \quad (9)$$

Remark 3 *Since* $h_1^i(y) = y_i - r_i(y)$, $h_0(y) = 0$ *the hypothesis (2) in the theorem is really a strengthening of hypothesis (1) for* $l = 1$, $k = 0$. *Hypothesis (1) will imply that the number of goods delivered to a bidder does not decreases with his bid.*

Proof. Choose $\alpha_i = 0$ for every i. It is clear from (6) that (8) gives the greatest expected profit possible. To verify (7) define for each k:

$$t_{ik}(y) = \frac{\#(\tilde{\mathbf{A}}(y) \cap \mathbf{A}_{ik})}{\#\tilde{\mathbf{A}}(y)} \text{ for } 1 \leq k \leq K.$$

It follows from (8) that $q_{ik}(y) = t_{ik}(y)$. I will prove for every y_{-i} and $b, c \geq 0$ that $J \geq H$ where

$$J = \int_b^c \sum_{k=1}^{K} (U_k^i)'(z) t_{ik}(z, y_{-i}) dz \text{ and } H = \sum_{k=1}^{K} (U_k^i(c) - U_k^i(b)) t_{ik}(b, y_{-i}).$$

From $J \geq H$, we obtain (7) by integration in dF_{-i}. Define $I_k = \{z \in Y_i; t_{ik}(z, y_{-i}) > 0\}$. Then from lemma 4 below there exists $x_0 = 0 \leq x_1 \leq \ldots \leq x_K \leq x_{K+1} = n_i$ such that $(x_k, x_{k+1}) \subset I_k \subset [x_k, x_{k+1}]$ for $k = 0, 1, \ldots, K$. To prove that $J \geq H$ suppose first that $b < c$. There exists a unique l such that $b \in [x_l, x_{l+1})$. We have that

$$J \geq \int_b^c \sum_{k=l}^{K} (U_k^i)'(z) \chi_{I_k}(z) dz \geq \int_b^c \sum_{k=l}^{K} (U_l^i)'(z) \chi_{I_k}(z) dz = U_l^i(c) - U_l^i(b).$$

Now

$$H \leq \sum_{k=1}^{l} (U_l^i(c) - U_l^i(b)) t_{ik}(b, y_{-i}) = U_l^i(c) - U_l^i(b) \leq J.$$

The case $b > c$ is treated analogously. QED

Remark 4 *The mechanisms (8,9) may be interpreted as follows: The auctioneer keeps the objects if* $y \notin Y$. *If* $y \in Y$ *the auctioneer finds the allocation* $a \in \mathbf{A}$ *that maximizes* $\sum_{i \in \mathbb{I}} h_a^i(y)$. *Among those allocations he chooses the allocations that deliver the fewest numbers of objects. He chooses among those with equal likelihood. This is (8). From the characterization in lemma 4 we see that the price paid by the Bidder is his valuation for the smallest possible bid that will still give him the same number of objects.*

Lemma 4 *The set* $I_k = \{z \in Y_i; t_{ik}(z, y_{-i}) > 0\}$ *is an interval, possibly degenerated such that* $t_{ik}(z, y_{-i}) = 1$ *for all* $z \in \operatorname{int}(I_k)$*. Moreover the interval* I_k *is to the left of the interval* I_{k+1}*.*

Proof. Suppose $a < c$ are two points of I_k. I will prove that for every $b \in (a, c)$ $t_{ik}(b, y_{-i}) = 1$. There exists $u^a \in \tilde{\mathbf{A}}(a, y_{-i})$ and $u^c \in \tilde{\mathbf{A}}(c, y_{-i})$ such that $u_i^a = u_i^c = k$. Therefore we have that

$$h_k^i(a) + \sum_{j \in \mathbb{I} \setminus \{i\}} h_{u_j^a}^j(y_j) \geq h_{w_i}^i(a) + \sum_{j \in \mathbb{I} \setminus \{i\}} h_{w_j}^j(y_j) \text{ for every } w \in \mathbf{A} \tag{10}$$

and

$$h_k^i(c) + \sum_{j \in \mathbb{I} \setminus \{i\}} h_{u_j^c}^j(y_j) \geq h_{w_i}^i(c) + \sum_{j \in \mathbb{I} \setminus \{i\}} h_{w_j}^j(y_j) \text{ for every } w \in \mathbf{A}. \tag{11}$$

Choosing $w = u^c$ in (10) and $w = u^a$ in the inequality (11) we obtain that $\sum_{j \in \mathbb{I} \setminus \{i\}} h_{u_j^a}^j(y_j) = \sum_{j \in \mathbb{I} \setminus \{i\}} h_{u_j^c}^j(y_j)$. For any $w \in \mathbf{A}$ such that $w_i \leq k$ we have from (10) and hypothesis (1) of the theorem that

$$h_k^i(b) + \sum_{j \in \mathbb{I} \setminus \{i\}} h_{u_j^a}^j(y_j) \geq h_{w_i}^i(b) + \sum_{j \in \mathbb{I} \setminus \{i\}} h_{w_j}^j(y_j). \tag{12}$$

Analogously if $w_i > k$ we use (11) to conclude (12) for every $w \in \mathbf{A}$. Hence $u^a \in A(b, y_{-i})$. I omit the proof that $u^a \in \tilde{\mathbf{A}}(b, y_{-i})$. A reasoning analogous to the above using inequality (10) shows that for every $l < k$, $\mathbf{A}_{il} \cap A(b, y_{-i}) = \emptyset$. Also using (11), $\mathbf{A}_{il} \cap A(b, y_{-i}) = \emptyset$ if $l > k$. Hence $A(b, y_{-i}) = \mathbf{A}_{ik} \cap A(b, y_{-i})$ and $t_{ik}(b, y_{-i}) = 1$. From this also follows that I_k is to the left of I_{k+1}. This ends the proof. QED

Remark 5 *Item 2 is an usual regularity assumption. Item 1 is very general. It permits the possibility that a group of bidders demands fewer than* K *objects. For example a group wants at most one object, another two and so on.*

Remark 6 *The pricing formula (9) is not unique. What matters for the auctioneer revenue is* $\hat{P}(y_i) = \int P^i(y) dF_{-i}(y_{-i})$*. In particular it is easily verified that* $(q, \hat{P}^1, \ldots, \hat{P}^I)$ *is an optimal mechanism and is an all-pay auction. In other words: there is always an all-pay optimal auction. I chose (9) because it is analogous to Myerson's equation 4.8, page 64, and has the same interpretation he gives on page 67.*

Corollary 5 *Suppose $K = 2$. In the optimal auction the objects are sold in a bundle if*

$$U_2^i(x) - r_i(x)(U_2^i)'(x) \geq 2(x - r_i(x)) \text{ whenever } x - r_i(x) \geq 0.$$

Proof. If $i \neq j$, $h_1^i(y_i) \geq 0$ and $h_1^j(y_j) \geq 0$ it is true that $h_1^i(y_i) + h_1^j(y_j) \leq 2\max\{h_1^i(y_i), h_1^j(y_j)\} \leq \max\{h_2^i(y_i), h_2^j(y_j)\}$ it follows that no $a \in \mathbf{A}$ such that $a_i = 1 = a_j$ maximizes $h(v, y)$. QED

Corollary 6 *The auctioneer revenue is $\int \max_{a\in\mathbf{A}} \sum_{i\in\mathbb{I}} h_{a_i}^i(y_i)dF(y)$. It increases with I.*

Proof. The revenue formula is immediate from equation 6 and the definition of q. It is increasing since the maximum increases with I and $\int dF_{I+1}(y_{I+1}) = 1$.

4 The optimal auction in particular cases

In this section I consider some particular cases to make it easier to understand the optimal auction.

4.1 The one bidder case.

It is interesting to compare the single bidder optimal auction when there are several objects to sell. Two objects will give the general idea. Suppose therefore that $K = 2$, $\mathbb{I} = \{1\}$ and $Y_1 = [0, 1]$. Define $\delta(x) = U_2^1(x)$ and $r(x) = r_1(x)$. Suppose also that $k(x) = U_2(x) - r(x)(U_2)'(x) - (x - r(x))$ and $x - r(x)$ are strictly increasing. There exist $x_0, p \in [0, 1]$ such that $k(p) = 0$ and $x_0 = r_1(x_0)$. For definiteness I suppose $0 < x_0 < p < 1$. The optimum probabilities are:

$$q_u(x) = 1 \text{ if } \begin{cases} u = 0 & \text{and} & x \leq x_0; \\ u = 1 & \text{and} & x \in (x_0, p]; \\ u = 2 & \text{and} & x \in (p, 1]. \end{cases}$$

Let's calculate the bidder's payments. We have

$$P(x) = P_0(x) = P_1(x) = P_2(x) = xq_1(x) + \delta(x)q_2(x) - \int_0^x (q_1(z) + \delta'(z)q_2(z))dz$$

Therefore

$$P(x) = \begin{cases} 0 & \text{if } \ x < x_0 \\ x_0 & \text{if } \ x_0 \le x \le p; \\ x_0 + \delta(p) - p & \text{if } \ p < x \le 1. \end{cases} \tag{13}$$

And the auctioneer revenue is $x_0 P(x_0 \le X \le p) + (x_0 + \delta(p) - p)P(p < X)$.

4.2 An example.

In this section the optimal auction is analysed in a particular case in which the synergy is negative. I suppose $K = 2$, $I = 2$ and $U^i(x, 2) = \alpha x$, $1 < \alpha < 2$. The distribution of signals is uniform in $[0, 1]$ for each bidder $i = 1, 2$. Let us verify the hypotheses of theorem 3. Since $r_i(y) = 1 - y$ we have that $h_1^i(y) = 2y - 1$ and $h_2^i(y) = \alpha(2y - 1)$. Items 1 and 3 of theorem 3 are true if $\alpha \ge 1$. The set of allocations is $\mathbf{A} = \{(a_0, a_1, a_2); a_b \in \{0, 1, 2\}, a_0 + a_1 + a_2 = 2\}$. The auctioneer wants to maximize $h_a(y) = h_a(y_1, y_2)$ where

$$h_a(y) = \begin{cases} 0 & \text{if } \ a = (2, 0, 0), \\ 2y_1 - 1 & \text{if } \ a = (1, 1, 0), \\ 2y_2 - 1 & \text{if } \ a = (1, 0, 1), \\ 2(y_1 + y_2 - 1) & \text{if } \ a = (0, 1, 1), \\ \alpha(2y_1 - 1) & \text{if } \ a = (0, 2, 0), \\ \alpha(2y_2 - 1) & \text{if } \ a = (0, 0, 2). \end{cases}$$

So the auctioneer keeps the objects if $y_1 < 1/2$, $y_2 < 1/2$. If $y_1 = 1/2$ and $y_2 < 1/2$ the auctioneer keeps the objects with probability $1/3$, deliver one object to Bidder 1 with probability $1/3$ and deliver two objects to Bidder 1 with probability $1/3$. The auctioneer may not deliver the objects in a bundle: he delivers one object to Bidder 1 and one to Bidder 2 if

$$y_1 > 1/2 \text{ and } \frac{y_1}{\alpha - 1} - \frac{2 - \alpha}{2\alpha - 2} > y_2 > \frac{2 - \alpha}{2} + (\alpha - 1)y_1.$$

Let us calculate the Bidder 2 payment if $y_1 > 1/2$, $y_2 > 1/2$. We have that Bidder 2 receives one object for sure if(and only if) $\frac{y_1}{\alpha-1} - \frac{2-\alpha}{2\alpha-2} > y_2 > \frac{2-\alpha}{2} + (\alpha - 1)y_1$. He receives two objects if $y_2 > \frac{y_1}{\alpha-1} - \frac{2-\alpha}{2\alpha-2}$. Hence his payment is:

$$P_2(y) = \begin{cases} \frac{2-\alpha}{2} + (\alpha - 1)y_1 & \text{if } \ \frac{y_1}{\alpha-1} - \frac{2-\alpha}{2\alpha-2} > y_2 > \frac{2-\alpha}{2} + (\alpha - 1)y_1, \\ \\ \frac{y_1}{\alpha-1} - \frac{2-\alpha}{2\alpha-2} & \text{if } \ y_2 > \frac{y_1}{\alpha-1} - \frac{2-\alpha}{2\alpha-2}. \end{cases}$$

4.3 The optimal auction in the symmetric case.

It may help to understand the optimal auction to consider the case when there is (ex-ante) symmetry among bidders. Therefore let me suppose that $F^i = F^1$ and $U^i = U^1$ for every $i = 2, \ldots, I$. Define $r(x) = r_1(x)$, $h_k(x) = h_k^1(x)$ and $U_k(x) = U_k^1(x)$, $k = 1, \ldots, K$. Define also $X^{(1)}$ the greatest value of $\{X^1, \ldots, X^I\}$ and $X^{(2)}$ the second greatest and so on, $X^{(K)}$ being the K^{th} greatest.[8] Then the following is true:

Corollary 7 (The symmetric case) *Suppose that $h_l - h_k$ is strictly increasing if $l > k$ and that $x - r(x)$ is strictly increasing. Then the optimal auction allocates the objects accordingly to the greatest number below:*

$h_K(X^{(1)})$	*one bidder receives K objects,*
$h_{K-1}(X^{(1)}) + h_1(X^{(2)})$	*one bidder receives $K-1$ objects and one bidder one object,*
$h_{K-1}(X^{(1)}))$	*exactly one bidder receives $K-1$ objects,*
$h_{K-2}(X^{(1)}) + h_2(X^{(2)})$	*one bidder receives $K-2$ objects and one bidder 2 objects,*
$h_{K-2}(X^{(1)}) + h_1(X^{(2)}) + h_1(X^{(3)})$	*one bidder receives $K-2$ objects and two bidders 1 object,*

and so on...

For example if there are two objects($K = 2$) the auctioneer chooses the allocation that maximizes

$$U_2(X^{(1)}) - r(X^{(1)})(U_2)'(X^{(1)}); \tag{14}$$

$$X^{(1)} - r(X^{(1)}) + X^{(2)} - r(X^{(2)}); \tag{15}$$

$$X^{(1)} - r(X^{(1)}) \text{ and} \tag{16}$$

$$0. \tag{17}$$

The auctioneer keeps the objects if (17) is the greatest number. Deliver the objects separately if (15) is the greatest and so on.

Remark 7 (Efficiency) *We see from (14,15,16,17) that the optimal auction allocation does not maximize social welfare. For example the objects are not always delivered. Branco(1996a) shows[9] for every bounded distribution that the optimal auction is not (ex-post) efficient.*

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Footnotes

1. Most of the time I will call an indivisible good an object.

2. For example $U(2,x) < 2U(1,x)$ with Maskin and Riley's hypotheses.

3. I.e. $U(2,x) - 2U(1,x)$.

4. He studies the selling of licenses in a production game with two participants. Here I suppose the bidders are consumers.

5. The function $f : \mathbb{R} \to \mathbb{R}$ is increasing if $x \leq y \Rightarrow f(x) \leq f(y)$. It is strictly increasing if $x < y \Rightarrow f(x) < f(y)$.

6. It is enough to suppose $U_k^i(\cdot)$ absolutely continuous.
7. Henceforward called direct mechanisms.
8. It is possible that $X^{(k)} = X^{(k+1)}$.
9. He supposes additive valuations.