Network security

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Abstract

In a variety of settings, some payoff-relevant item spreads along a network of connected individuals. In some cases, the item will benefit those who receive it (for example, a music download, a stock tip, news about a new research funding source, etc.) while in other cases the impact may be negative (for example, viruses, both biological and electronic, financial contagion, and so on). Often, good and bad items may propagate along the same networks, so individuals must weigh the costs and benefits of being more or less connected to the network. The situation becomes more complicated (and more interesting) if individuals can also put effort into security, where security can be thought of as a screening technology that allows an individual to keep getting the benefits of network connectivity while blocking out the bad items. Drawing on the network literatures in economics, epidemiology, and applied math, we formulate a model of network security that can be used to study individual incentives to expand and secure networks and characterize properties of a symmetric equilibrium.

1 Introduction

Being connected can be both a blessing and a curse. In this paper we build a model of network formation and security in which there are both benefits and risks to being highly connected to other agents. Our focus is on the role of the network in spreading non-rival goods (or bads) that are essentially costless to reproduce and transmit, such as news or viruses. While an agent joins the network to try to expose herself to any goods – denoted "tips" – that arise, as a consequence of joining she may also be exposed to bads, which we generically call "viruses." Because tips and viruses exploit the same network structure to spread, both positive and negative network externalities will arise from the fact that individual decisions about connections affect others' exposure. We assume that individuals can take costly actions that limit the spread of viruses without hindering tips; we call this security provision. This introduces a third externality, since an individual who blocks a virus benefits all of the downstream agents that she would have otherwise infected. We study the micro (individual connection and security choices) and macro (penetration rates of tips and viruses) properties of equilibrium in a large anonymous economy. To model "large and anonymous," we adopt a random graph approach to network formation.

To illustrate the type of settings where these issues arise, we consider a few potential applications. In epidemiology, the virus metaphor is literal: while people interact face-to-face in order to benefit from word of mouth information, these physical connections between people also form a vector for the spread of disease. People can mitigate disease spread through individual effort, such as hand-washing and vaccinations. Alternatively, the virus may be electronic. Computer users benefit from information spread by various types of networks (such as email, file-sharing, or social media), but undesirable items such as viruses or spam can piggyback on these networks as well. A user can protect herself (and others) by screening email, installing anti-virus software, and so forth. In both cases, it is most natural to think of the tip as information, which by its nature is easily reproduced and transmitted, and often non-rival. Both biological and computer viruses share these characteristics.

In some other settings, our model may provide useful intuition about tradeoffs between openness and security, even when the application does not precisely match the model assumptions. For example, in financial markets, both valuable and worthless information circulate, and acting on worthless information can be costly. In this case, one might think of security as undertaking due diligence before acting on a rumor and passing it on to other people. Here, the connection to our model is not tight for various reasons, but some of the incentives and externalities will be similar.[1] At an even broader, more heuristic level, there is a tension between the general benefits of an open, interconnected society (such as ease of communication, electronic banking, and so forth) and the threat that those very interconnections could make the society more vulnerable to attack, by terrorists or other malefactors.

As a general template for thinking about issues of connectivity and security in a large anonymous economy, suppose that we give an agent two choice variables: z, which will be some measure of his connection intensity or how actively he participates in the network, and f (for firewall), which will be some measure of his efforts toward avoiding the threats it poses. Suppose that a is some aggregate variable that summarizes the "good connectivity" of the network – that is, how well it spreads tips – and b is an aggregate variable that summarizes "bad connectivity" – that is, how prevalent viruses are. To make this the barest of reduced form models, we would need (*inter alia*) two elements: a payoff function for agents, and a general equilibrium mechanism specifying how individual choices (z, f) aggregate, *via* some network structure, to get the summary parameters a and b. A simple payoff function could have the form

$$Benefit\,(z, f\,;\, a) - Harm\,(z, f\,; b) - Cost\,(z, f) \tag{1}$$

where $C\,(z, f)$ reflects some direct costs of connections and security, and the first two terms capture the benefits earned and harms suffered from exposure to the network. At this stage, a modeler might impose some desired form of network externalities, perhaps in the form of constraints on the cross partial derivatives of the benefit and harm functions. One motive of the paper is to provide primitive foundations for this reduced form model. We will introduce an explicit network formation process, derive aggregate parameters a and b, and show that individual payoffs take the form of (1). Furthermore, by micro-founding the benefit and harm functions, we will be able to derive the structure of network externalities rather than impose it. In linking what might be called the structural and reduced form models, we aim to address the relatively common complaint that

[1] Two important distinctions are that financial information is usually not non-rival (although it spreads quickly nonetheless) and that the value of information is sensitive to when it is received. A third issue is that the harm suffered by acting on bad information due to insufficient due diligence begins to sound like a herding story. It would be more satisfying to see this modeled rather than to assume it in reduced form.

network theory is impractical for applied work because it relies too delicately on fine details of network structure that are rarely observed in data. In our model, the properties of the network that matter for incentives can be characterized very compactly.

The model is simplest to describe by beginning in the middle. Suppose that agents' connectivity and security choices have generated a network, modeled by a graph in which agents are nodes and connections between them are links. Certain links are *firewalled* in one or perhaps both directions; this means that a virus cannot travel along this link in the firewalled direction. The following completely exogenous events happen next. First, Nature creates a single tip or virus and endows it on a randomly chosen agent (the "Origin"). Then, if it is a tip, it spreads to every agent who is linked either directly or indirectly to the Origin. Alternatively, if Nature endows a virus, it spreads in the same way, but only along links that have not been firewalled. Finally, all agents who have been exposed receive a common payoff (positive for a tip, negative for a virus); there is no decay or discounting. The exogenous generation and transmission of tips and viruses is intentionally quite stylized, as the focus of this paper is elsewhere.

All of the strategic decisions happen one stage earlier. Although agents will act simultaneously, in spirit we wish to capture the idea that an agent faces a large, anonymous population of potential contacts, with only weak information about how they are interconnected or about who is particularly vigilant. To capture this, we use a random graph model of network formation. Agents simultaneously choose link intensities and security efforts (z, f). The link intensity could be thought of as undirected effort devoted toward forming contacts, which pays off with a stochastic number of links. A higher security level means that more of these links will be firewalled, although this too has a random component. Although agents are constrained by the technology of network formation, they are fully rational: in a Nash equilibrium, z and f are chosen with respect to correct expectations about others' strategies, and what they imply about the aggregate properties of the network.

While the model is formulated with a finite number of agents, all of the analysis will focus on the large population limit where we can exploit sharp limiting results for random graphs. There is one other feature of the model that serves a technical purpose in this paper but could be interesting to study in its own right. In addition to the strategic agents, there is one large, nonstrategic agent called the hub. At the network formation stage, separately from the strategic choice z, every agent also forms a link to the hub with a small exogenous probability. Tips and viruses pass through the hub just as they do through other agents. All of our analysis will focus on the decentralized limit in which the "size" of the hub (that is, the probability of linking to it) is taken to zero. The purpose of introducing this central institution only to take it away is somewhat analogous to seeding a rain cloud or providing a dust particle as a nucleus for a snowflake. The relevant limiting properties of the network as the hub vanishes are no different than they would have been had it never existed at all, but they are substantially easier to characterize rigorously. For an economic interpretation, it makes the most sense to think of an institution, such as a government or a healthcare system, or a forum where clusters of people can gather, such as a website or a club. However, since the role of the hub will be vanishing and mechanical in this paper, we will not push these interpretations too far.

A tip or a virus spreads widely if it reaches a positive fraction of the population. In a large population, a tip need not spread widely – for example, the Origin may be isolated – but if it does, the fraction of the population reached will be a deterministic function of individual decisions. The same is true for the fraction of the population infected by a virus. These fractions, which we call a and b, turn out to be the key aggregate variables for understanding individual incentives. The mathematical form of a is well known in the graph theory literature for the case when all link intensities are symmetric, but equilibrium analysis demands that we consider asymmetric cases as well (either because the equilibrium is asymmetric or to evaluate deviations from a symmetric equilibrium). Similar demands arise for b. A modest technical contribution of the paper is to provide rigorous derivations for these cases (Propositions 1 and 2).

We confine our equilibrium analysis to symmetric equilibria. The interaction of three externalities – positive and negative network externalities, and the public good problem of security provision – generates a rich and nuanced spectrum of outcomes. One feature that contributes to this richness is the fact that there are increasing returns to security. This is due to a type of finger-in-the-dyke effect: the marginal benefit from protecting a single link is small since one is likely to be infected anyway through other channels. By favoring all-or-nothing security investments, this works against symmetry, as it is not stable for *everyone* in the network to be fully secure or completely lax.[2] However, symmetric equilibria do exist when the cost of security is sufficiently convex, and we study comparative statics for cases in which those costs are also independent of the number of links protected.[3] When security is relatively expensive, equilibria fall into two general classes. If the *severity* of threats is low (that is, if viruses are rare or the payoff penalty from exposure is mild), then a Wild West scenario prevails: people form links profusely, pay lip service to security, and both the tip exposure rate a and the infection rate b are high. This reverses if threat severity is high: people cut links drastically (as a substitute for securing them), the network shrinks, and both a and b are small. Moreover, in the latter case, downward shifts in security costs (due perhaps to new technology or subsidy policies) will unambiguously make infection rates worse (higher b) before they eventually improve. The reason is that the lower security costs make it attractive to form more links, and the aggregate effect of a denser network on virus transmission swamps the increase in security. If such a change were to result from a policy initiative like a subsidy, it would be easy to jump to a judgement the policy had failed. We show that if total welfare is the yardstick (and not infection rates *per se*), then this judgement would be mistaken, as total welfare, including the benefits from a denser network, unambiguously rises with such a change.

Next, we put equilibrium networks through a stress test by taking the severity of threats to infinity. A network is *resilient* if it does not collapse under this process – that is, if a and individual payoffs remain strictly positive. We show that loosely speaking, a network will be resilient if and only if agents can obtain at least a small amount of security for free. (More precisely, the first three derivatives of the cost function must vanish at $f = 0$.)

The issues addressed in our model have received attention across a broad spectrum of academic fields and in various applied settings, as one might expect, given the diverse area of topics to which network flows are germane. In the epidemiology literature, models of disease spread traditionally have employed strong assumptions about how populations mix that allow the analyst to avoid modeling networks explicitly. Instead, these assumptions permit disease transmission to be characterized in terms of systems of differential equations. However, as more recent work by Mark

[2] We study the case in which equilibria must be asymmetric in companion work.

[3] Thus these cases are closer to a flu shot model of protection, where a single investment provides a certain level of protection for every interaction, than to a hand-washing model where the costs may be proportional to the number of interactions.

Newman and various coauthors (notably Newman et al. 2001) has illustrated, the consequences of assuming away the network structure of disease transmission are not innocuous. This more recent work takes a graph theoretic approach similar to ours. It is worth highlighting three important differences between these papers and ours. First, in considering disease epidemics, Newman (2002) employs a more detailed model that adheres more closely than we do to the specific features of a disease outbreak. In contrast, to make the novel features of our model as clear as possible, we have kept it simple and avoided wedding it too closely to any specific application.

More importantly, the methodology of Newman et al. (2001) and most of the closely related literature involves using ingenious but non-rigorous generating function arguments establish claims. To apply this approach, one must be willing to assume properties for the random graph, such as zero probabilities for some rare events, or independence for weakly correlated events, that are not strictly true for a finite graph but that one plausibly expects to hold in the limit. Vega-Redondo (2007) and Jackson (2008) provide accessible treatments of this approach in an economics context. On the other hand, when link intensities are asymmetric, rigorous proofs of aggregate network properties by probabilists, as in Molloy and Reed (1995) and (1998) and Chung and Lu (2002), can be quite lengthy. Since for an economist, these aggregate network properties are in the service of an economic insight but are not the end in themselves, a modeling strategy that admits parsimonious but rigorous proofs is attractive. Our device of an infinitesimal hub accomplishes this (although the proofs are still not as short as one might like).[4]

Finally, and most importantly for economists, existing network models based on a random graph generating function approach almost universally treat individuals' behavior as exogenous. Our main innovation is to take individual incentives seriously in the context of an equilibrium model.

Within economics, there has been a steadily growing interest in interactions on networks, with seminal treatments of network formation by Jackson and Wolinsky (1996) and Bala and Goyal (2000), among others; for book length treatments, see Goyal (2007), Jackson (2008), and Vega-Redondo (2007). A number of recent papers including Goyal and Vigier (2011), Hoyer and De Jaegher (2010) and Kovenock and Roberson (2010) study issues of security and reliability of a network under threat, but the focus in these papers is almost perfectly orthogonal to ours. They look at situations in which a centralized network designer (or designers) chooses a detailed network structure with an eye toward foiling the attack of a hostile strategic agent who is well informed about the structure of the network. Explicitly modeling the behavior of threats is an important element that we set aside to focus on other issues. The nature of threats differs somewhat in our setting, as our viruses operate not by disrupting the network but by hijacking it. But more importantly, our focus on the individual incentives of small, decentralized agents who are not able to act on the basis of detailed information about the network structure represents the polar opposite of a command and control design problem.

In the next section, we introduce the model for a finite number of agents and a positive hub size; then we discuss the limits as the number of agents grows and the hub shrinks. Section 3 derives the aggregate network properties of interest and uses them to derive expressions for agents' payoffs. An impatient reader who is willing to take the results of Propositions 1 and 2 on faith may want to skip ahead to Section 3.2, or even Section 4. Section 4 presents equilibrium conditions and

[4] For the general angle of attack on Propositions 1 and 2, we also owe a debt to Durrett (2007).

studies the properties of symmetric equilibria. Section 5 concludes with a discussion of limitations and extensions.

2 Model

2.1 Networks and Items

An $nh-$economy consists of n small strategic agents, indexed $\{0, 1, 2, ..., n-1\}$, and one large, nonstrategic player called the *hub* (H). The term h refers to the size of the hub and will be discussed later. We call the limit of an $nh-$economy as $n \to \infty$ a *large* $h-$*economy*, or just an $h-$economy. If we additionally let the influence of the hub vanish (in a way to be specified later), we call the result a *large decentralized economy*. Our results will focus mainly on large decentralized economies.

The small agents make strategic choices that influence the creation of a network, which we model as an undirected random graph in which agents are vertices and links between agents are edges. Formally, letting $\mathcal{I}_n = H \cup \{0, 1, 2, ..., n-1\}$, a network for an $n-$economy is a function $\mathcal{G} : \mathcal{I}_n^2 \to \{0, 1\}$, where $\mathcal{G}(i, j) = 1$ if and only if there is a link between agents i and j. Because the graph is undirected, we have $\mathcal{G}(i, j) = \mathcal{G}(j, i)$ for all i and j. Self-links ($\mathcal{G}(i, i) = 1$) are not ruled out *per se*, but they will not of any particular interest in what follows. Several notions of connectivity in $\mathcal{G}$ will be important. Define the t^{th} order neighbors of i by $\mathcal{N}_{\mathcal{G}}^1(i) = \{j \in \mathcal{I}_n \,|\, \mathcal{G}(i, j) = 1\}$ and $\mathcal{N}_{\mathcal{G}}^t(i) = \left\{j \in \mathcal{I}_n \,|\, \mathcal{G}(i', j) = 1 \text{ for some } i' \in \mathcal{N}_{\mathcal{G}}^{t-1}(i)\right\}$. We say that i and j communicate in $\mathcal{G}$ if there is a sequence of links that connects them. Mathematically, we write $j \in \mathcal{N}_{\mathcal{G}}(i)$, where $\mathcal{N}_{\mathcal{G}}(i) = \{i\} \cup \left(\bigcup_{t=1}^{\infty} \mathcal{N}_{\mathcal{G}}^t(i)\right)$; note that $j \in \mathcal{N}_{\mathcal{G}}(i)$ and $i \in \mathcal{N}_{\mathcal{G}}(j)$ are equivalent. From time to time, if there is no ambiguity about $\mathcal{G}$, we may use the notation $i \longleftrightarrow_1 j$ to mean that i and j are first order neighbors, and $i \longleftrightarrow j$ to mean that they communicate. A *component* of the network is a maximal subset $C \subseteq \mathcal{I}_n$ such that all of the agents in C communicate with each other; that is, $i \longleftrightarrow j$ for all $i, j \in C$, and $i \not\leftrightarrow k$ for all $i \in C$ and $k \in \mathcal{I}_n \backslash C$.

Any undirected graph can be reinterpreted as a directed graph in which each undirected link $i \longleftrightarrow_1 j$ is represented by a pair of directed edges from i to j, written $i \mapsto_1 j$, and from j to i. Given a network $\mathcal{G}$, an *unsecured subnetwork* $\mathcal{G}^f$ is a directed subgraph of $\mathcal{G}$, formed from $\mathcal{G}$'s directed counterpart by deleting a subset of the edges. Formally, let $\mathcal{G}^f$ be a function $\mathcal{G}^f : \mathcal{I}_n^2 \to \{0, 1\}$, where we do not require $\mathcal{G}^f(i, j) = \mathcal{G}^f(j, i)$, and $\mathcal{G}^f(i, j) = 1$ (equivalently, $i \longmapsto_1 j$) is interpreted to mean that there is a directed edge from i to j. Then $\mathcal{G}^f$ is an unsecured subnetwork of $\mathcal{G}$ if $\mathcal{G}^f \leq \mathcal{G}$. For reasons that will be clear shortly, we say that i directly infects j if $i \longmapsto_1 j$, and i infects j (written $i \longmapsto j$) if there is some chain of agents $\{k_1, k_2, ...\}$ and edges $i \longmapsto_1 k_1 \longmapsto_1 k_2 \longmapsto_1 ... \longmapsto_1 j$ leading from i to j. Define an agent's upstream and downstream neighbors: $\mathcal{N}_{\mathcal{G}^f}^{Up}(i) = \{j \in \mathcal{I}_n \,|\, j \longmapsto i\} \cup \{i\}$ and $\mathcal{N}_{\mathcal{G}^f}^{Down}(i) = \{j \in \mathcal{I}_n \,|\, i \longmapsto j\} \cup \{i\}$.[5] In general, these sets will not coincide, but note that $j \in \mathcal{N}_{\mathcal{G}^f}^{Up}(i)$ if and only if $i \in \mathcal{N}_{\mathcal{G}^f}^{Down}(j)$.

The role of the network in our model is to transmit *items*. Items come in two flavors, desirable and undesirable, which we label as *tips* and *viruses* respectively. Items are generated and spread in a mechanical way: after the network is formed, one randomly chosen agent is endowed with one item, which is a tip with probability λ and a virus with probability $1 - \lambda$. This agent is called the

[5] We could equivalently define these as unions of sets of order t upstream and downstream neighbors, in a similar manner to $\mathcal{N}_{\mathcal{G}}(i)$.

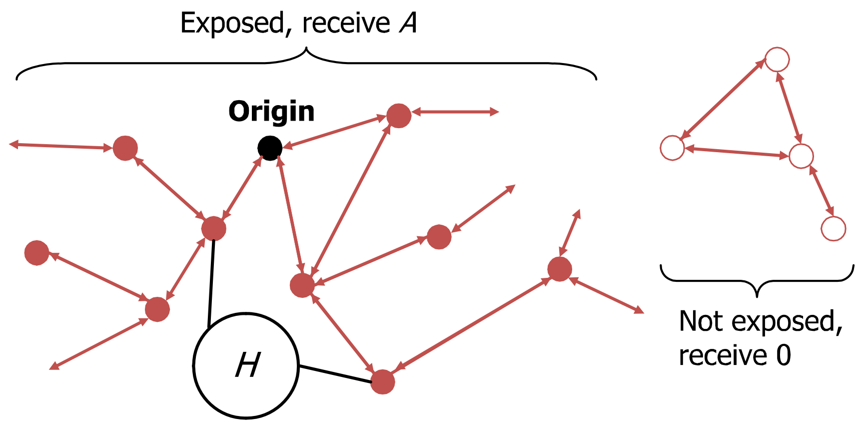


Figure 1: The spread of a tip

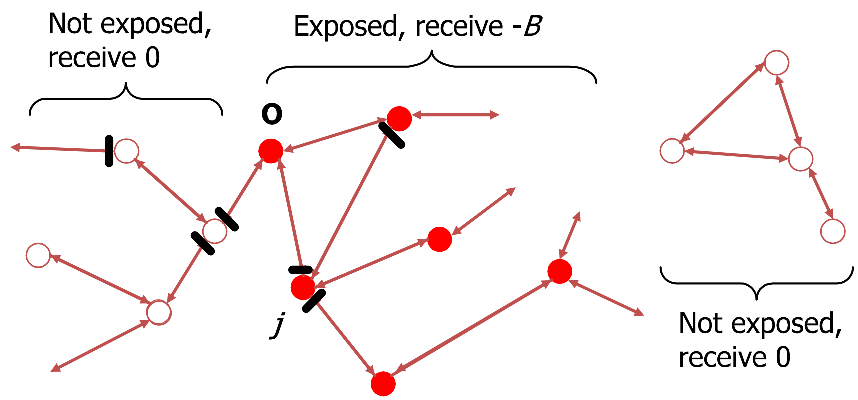


Figure 2: The spread of a virus. Firewalls indicated by black bars. While agent j firewalls her direct link to O, she is still infected indirectly.

origin, and without loss of generality, we take him to be agent 0.[6] If the item is a tip, it spreads (costlessly and with no degradation in quality) along the links in $\mathcal{G}$. On the other hand, a virus spreads only to agents who are downstream of the origin. An agent who is exposed to the item (including the origin) receives a payoff of $A > 0$ if it is a tip, or $-B$ (where $B > 0$) if it is a virus, while an agent who does not receive the item gets a payoff of zero. Multiple exposure has no additional effect; the payoff of A or $-B$ is realized at most once by each agent. Thus, agent i receives A if there is a tip and $i \in \mathcal{N}_{\mathcal{G}}(0)$, $-B$ if there is a virus and $i \in \mathcal{N}_{\mathcal{G}^f}^{Down}(0)$, and zero otherwise.

2.2 Network Formation and Security

An agent i makes two strategic decisions, denoted $z_i \in [0, \infty)$ and $f_i \in [0, 1]$. The first one is her link intensity z_i, this determines the mean number of links she will form to other small agents. The network $\mathcal{G}$ is a random graph in which a link between i and j appears with probability proportional to $z_i z_j$. The exception is the hub: each agent forms a link to the hub with exogenous probability

[6] The agents' labels are completely arbitrary, so we can always reorder them to make the origin agent 0.

h. We will sometimes refer to h as the size of the hub. The other decision is a security level f_i: this reflects effort to screen out incoming viruses. If $i \longleftrightarrow_1 j$ in $\mathcal{G}$, then i blocks an infection from j ($j \not\longmapsto_1 i$ in $\mathcal{G}^f$) with probability f_i.Thus, for a given network $\mathcal{G}$, positive security levels $f_i > 0$ create a wedge ensuring that viruses do not spread as widely as tips. Timing is as follows: all agents simultaneously choose z_i and f_i; then graphs $\mathcal{G}$ and $\mathcal{G}^f$ are generated; and finally, a single item is generated, endowed on agent 0, and spreads over the network as described above. Remember that agent 0 does not know that he is the origin; labeling him this way is just for our convenience. Now we turn to a more precise statement of how the random graphs $\mathcal{G}$ and $\mathcal{G}^f$ are formed.

Let $(\mathbf{z}, \mathbf{f})$ be a strategy profile for the strategic agents, where $\mathbf{z} = (z_0, z_1, ..., z_{n-1})$ and $\mathbf{f} = (f_0, f_1, ..., f_{n-1})$. On occasion we may write $(\mathbf{z}^n, \mathbf{f}^n)$ where it is important to emphasize the size of the economy. Conditional on $(\mathbf{z}, \mathbf{f})$, generate the following collections of independent, binary zero-one Bernoulli random variables:

$$\begin{aligned} \zeta_{ij} \text{ for all } i, j &\in \mathcal{I}_n \backslash H \text{ such that } i \leq j \\ \eta_i \text{ for all } i &\in \mathcal{I}_n \backslash H \\ \phi_{ij} \text{ for all } i &\in \mathcal{I}_n \text{ and } j \in \mathcal{I}_n \backslash H \end{aligned}$$

The first set of $\frac{n(n+1)}{2}$ random variables determines whether there is a link between small agents i and j in $\mathcal{G}$. Whenever $i < j$, define a mirror image variable ζ_{ji} equal to the realization of ζ_{ij}. Then set $\mathcal{G}(i, j) = \zeta_{ij}$ for all i and j in $\mathcal{I}_n \backslash H$. The probability that a link exists between i and j is assumed to be $\Pr(\zeta_{ij} = 1) = p_{ij} = \frac{z_i z_j}{n \bar{z}}$, where $\bar{z} = \frac{1}{n} \sum_{m=0}^{n-1} z_m$ is the average link intensity.[7] Notice that agent i's expected number of links to other small agents is

$$E\left(\sum_{j \in \mathcal{I}_n \backslash H} \zeta_{ij} \right) = z_i \frac{\sum_{j \in \mathcal{I}_n \backslash H} z_j}{n \bar{z}} = z_i$$

so it is reasonable to think of z_i as agent i's choice of an expected number of connections.

The next set of n random variables determines whether there is a link between each small agent and the hub: we set $\mathcal{G}(i, H) = \mathcal{G}(H, i) = \eta_i$ for all $i \in \mathcal{I}_n \backslash H$. The probability of linking to the hub is $\Pr(\eta_i = 1) = h$, independently across agents, and regardless of the strategy profile.

The final set of $n(n+1)$ random variables determines whether there is a *firewall* that blocks the transmission of a virus from i to j. These firewalls determine the construction of the unsecured subnetwork $\mathcal{G}^f$ relative to $\mathcal{G}$:

$$\mathcal{G}^f(i, j) = \begin{cases} (1 - \phi_{ij}) \zeta_{ij} = (1 - \phi_{ij}) \mathcal{G}(i, j) & \text{if } i, j \in \mathcal{I}_n \backslash H \\ (1 - \phi_{Hj}) \eta_j = (1 - \phi_{Hj}) \mathcal{G}(H, j) & \text{if } i = H, j \in \mathcal{I}_n \backslash H \\ \eta_i = \mathcal{G}(i, H) & \text{if } j = H \end{cases}$$

As the definition above indicates, the firewall variable ϕ_{ij} only becomes relevant in the event that $\mathcal{G}$ has a link between i and j: in this case we construct the subnetwork $\mathcal{G}^f$ by removing the directed edge $i \longmapsto_1 j$ if $\phi_{ij} = 1$, and retaining it if $\phi_{ij} = 0$. Note that ϕ_{ij} and ϕ_{ji} are two different, independent random variables: j being protected against infection by i is unrelated to whether j

[7] To ensure that $\Pr(\zeta_{ij} = 1) \leq 1$, we do need to impose $\max_{\mathcal{I}_n \backslash H} (z_i)^2 \leq n \bar{z}$. For large n, in the equilibria that we look at, this will not be a binding constraint.

can infect i. Small agents protect their links with the hub in the same way that they protect all of their other links. By assumption, the hub puts up no defenses; any virus that reaches it passes through unhindered. The probability of blocking a virus depends entirely on the security level of the receiving agent: $\Pr\left(\phi_{ij} = 1\right) = f_j$, independently across $i \in \mathcal{I}_n$, for all $j \in \mathcal{I}_n \backslash H$.

A few remarks about this security technology may be helpful at this point. First, by choosing $f_j = 1$, an agent can guarantee that she screens items perfectly. For any $f_j < 1$, her security will make mistakes with positive probability, and by assumption, these mistakes are on the side of excessive laxness (viruses sneaking through) rather than excessive vigilance (tips being accidentally blocked). We make no defense for this assumption, other than that it simplifies the analysis – expanding the analysis to incorporate both types of mistake would be a natural extension. It is expositionally convenient but inessential to assume that the firewall variables are realized simultaneously before the item is generated. We could alternatively assume that ϕ_{ij} is not realized unless a virus actually arrives at j from i, in which case the security technology succeeds in blocking it with probability f_i – nothing in the model would change under this reinterpretation.[8]

To illustrate what it means for the random variables ϕ_{ij} to be independent across i, consider a concrete example of a computer virus. In one scenario, suppose there is a universe $\mathcal{V}$ consisting of $V = |\mathcal{V}|$ different viruses, one of which is dropped randomly on agent 0. Suppose a security technology of quality f is a software package that can recognize and block fV of the viruses in $\mathcal{V}$. In this case, we might think of ϕ_{ij} and $\phi_{i'j}$ as perfectly correlated: if (with probability f_j) the virus is one that her antivirus software recognizes, then j is protected along all of her links; otherwise she is not protected at all. In an alternative scenario, suppose that blocking an incoming virus requires different techniques depending on whether the connection is an email attachment, a transfer of packets with a web site, an instant message conversation, a Skype phone call, or some other communication channel. If j tends to communicate on different channels with different neighbors and antivirus software does not protect all of these channels equally well, then we would not expect ϕ_{ij} and $\phi_{i'j}$ to be perfectly correlated. For now, we focus on the latter type of situation.

We reiterate that an agent i commits to (and pays for, as we shall see momentarily) a security level f_i at the same time that she chooses how connected z_i to be. Thus she chooses 'under a veil of ignorance' in two senses. First, she does not see her realized number of links before choosing f_i; if she did, she would presumably want to adjust the diligence of her protection depending on whether she is more or less connected than she expected to be. Second, she cannot focus her security efforts toward some neighbors more than others – all of her links have the same chance f_i of being protected. While this is surely too extreme, we are constrained by tractability, and in keeping with the ethos of the paper, we err in the direction of making agents too ignorant rather than too informed.

2.3 Payoffs and Equilibrium Definition

We are now in a position to formally define payoffs and equilibrium. As above, let $(\mathbf{z}, \mathbf{f})$ be a strategy profile. A strategy profile in which agent i plays according to some (z_i, f_i) and the other agents play according to $(\mathbf{z}, \mathbf{f})$ is written $\left((z_i, f_i)\,;(\mathbf{z}, \mathbf{f})_{-i}\right)$. Agent i's payoff is

[8] This is because the strategic choices z_i and f_i precede the realization of any of the random variables, so it is immaterial when the agents become aware of those realizations.

$$\begin{aligned}\pi_i\left((z_i,f_i);(\mathbf{z},\mathbf{f})_{-i}\right) &= \lambda A \Pr_{\zeta,\eta,\phi}\left(i\in\mathcal{N}_{\mathcal{G}}(0)\mid z_i,\mathbf{z}_{-i}\right)\\ &\quad -(1-\lambda)\,B\Pr_{\zeta,\eta,\phi}\left(i\in\mathcal{N}_{\mathcal{G}f}^{Down}(0)\mid (z_i,f_i),(\mathbf{z},\mathbf{f})_{-i}\right)-C(z_i,f_i)\end{aligned}$$

The subscript (ζ,η,ϕ), which we will suppress from now on, is a reminder that these probabilities are taken over realizations of the link and firewall random variables. Because the frequency λ of tips never appears separately from their magnitude A, we will absorb λ into A. Similarly, we absorb $1-\lambda$ into B, rewriting i's payoff as

$$\pi_i\left((z_i,f_i);(\mathbf{z},\mathbf{f})_{-i}\right)=A\Pr\left(i\in\mathcal{N}_{\mathcal{G}}(0)\mid z_i,\mathbf{z}_{-i}\right)-B\Pr\left(i\in\mathcal{N}_{\mathcal{G}f}^{Down}(0)\mid (z_i,f_i),(\mathbf{z},\mathbf{f})_{-i}\right)-C(z_i,f_i)$$

The final term represents the explicit costs of choosing connectivity z_i and security f_i. For the most part, we will assume that forming links is not costly *per se*, but z_i still appears in the cost function because the cost of security may rise with the number of links secured. If the cost of attaining security level f rises linearly with the number of links, then we have $C(z,f)=zc(f)$; we refer to this as the *constant* MC_{link} case. At the other extreme, if $C(z,f)=c(f)$, then the cost of protecting z links at level f does not depend on z. This is referred to as the *zero* MC_{link} case. Which version of costs is appropriate (if either) will depend on the specific application. For example, the nuisance cost of handwashing might be expected to scale up with z, whereas a single flu shot could provide protection against any number of exposures to a particular strain of influenza. We will often impose the following set of regularity conditions on $c(f)$.

A1 The function $c:[0,1]\to\mathfrak{R}^+$ is continuously differentiable, strictly increasing and strictly convex on $(0,1]$, and satisfies $c(0)=0$, $c'(0)=0$, and $\lim_{f\to 1}c'(f)\geq B$.

While convexity and the other conditions in **A1** are introduced with an eye toward giving the individual's security decision a unique, interior optimum, they do not guarantee this – there are parameter regions, and cost functions satisfying **A1**, for which equilibria with symmetric interior security levels do not exist. The reason is that the returns to security will also turn out to be convex.

These payoff functions extend naturally to large $h-$economies and large decentralized economies. Now let $\mathbf{z}$ and $\mathbf{f}$ be infinite sequences, with $\mathbf{z}^n$ ($\mathbf{f}^n$) the truncation of $\mathbf{z}$ ($\mathbf{f}$) to its first n terms. We interpret the pair of truncated sequences $(\mathbf{z}^n,\mathbf{f}^n)$ as a strategy profile in a nh-economy. In the sequel, when we need to separate out the strategic decision (z_i,f_i) of one agent i with respect to anticipated behavior $(\mathbf{z},\mathbf{f})_{-i}$ by the other agents, we will often take $i=1$ without loss of generality. In a large decentralized economy, agent 1's payoff from playing (z,f) when the other agents play according to $(\mathbf{z},\mathbf{f})_{-1}$ is defined to be

$$\begin{aligned}\pi_1\left((z,f);(\mathbf{z},\mathbf{f})_{-1}\right) &= \lim_{h\to 0}\lim_{n\to\infty}\pi_1\left((z,f);(\mathbf{z}^n,\mathbf{f}^n)_{-1}\right)\\ &= A\mathcal{A}(z;\mathbf{z}_{-1})-B\mathcal{B}\left((z,f);(\mathbf{z},\mathbf{f})_{-1}\right)-C(z,f)\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A}(z;\mathbf{z}_{-1}) &= \lim_{h\to 0}\lim_{n\to\infty}\Pr\left(1\in\mathcal{N}_{\mathcal{G}}(0)\mid z,\mathbf{z}_{-1}^{n}\right)\\
\mathcal{B}\left((z,f);(\mathbf{z},\mathbf{f})_{-1}\right) &= \lim_{h\to 0}\lim_{n\to\infty}\Pr\left(1\in\mathcal{N}_{\mathcal{G}^f}^{Down}(0)\mid (z,f),(\mathbf{z}^n,\mathbf{f}^n)_{-1}\right)
\end{aligned}$$

Of course, this payoff is only well defined if the relevant limits exist; later we will show that they do for the cases that are of interest.

The reason for the order of limits, although technical, is important enough to mention. Our modeling owes a heavy debt to the random graph literature, where the typical approach is to study the connectivity of a finite hubless network as $n\to\infty$, in effect reversing our order of limits. In that literature, there is a substantial gap between what is 'known' to be true based on clever, semi-rigorous arguments, and what has actually been formally proved. One well established result is that a sufficiently connected large random graph has (with probability tending to one as $n\to\infty$) a single large $O(n)$ component, called the giant component, while all other components are small ($O(\log n)$ or smaller). In our model, the hub acts as a seed for this giant component. This simplifies proofs tremendously by allowing us to break down compound events involving the connectivity of i and j into simpler events about how each of them is connected to H. As far as we are able to tell, our results would be identical if the order of limits were reversed, but the proofs would be much more cumbersome.

Now we can define an equilibrium for a large decentralized economy.

Definition 1 *A large decentralized economy (LDE) equilibrium is a strategy profile* $(\mathbf{z}^*,\mathbf{f}^*)$ *such that*

i) $\pi_i\left((z_i,f_i);(\mathbf{z}^*,\mathbf{f}^*)_{-i}\right)$ *is well defined for all* $(z_i,f_i)\in[0,\infty)\times[0,1]$, *for all* $i\in\{0,1,2,...\}$, *and*

ii) (z_i^*,f_i^*) *maximizes* $\pi_i\left((z_i,f_i);(\mathbf{z}^*,\mathbf{f}^*)_{-i}\right)$ *for all* $i\in\{0,1,2,...\}$.

Notice that we define our equilibrium directly in terms of the limiting payoffs, not as the limit of nh-economy equilibria – one should keep this in mind when thinking about the LDE results as an approximation of behavior in a large finite economy.[9] One can think of a symmetric strategy profile, in the natural way, as a profile in which every agent uses the same strategy (z,f). Asymmetric profiles and mixed profiles over a finite set of supporting strategies can be thought of informally in terms of the limiting population shares of each strategy as n goes to infinity; we will be more precise about this later. For strategy profiles like these, existence of the limiting payoff functions is not an issue. For most of the paper, we will focus on symmetric equilibria; the next section is devoted to characterizing the limit payoffs for this case.

3 Limiting Properties of $\mathcal{G}$ and $\mathcal{G}^f$: the Symmetric Case

In this section, we derive explicit expressions for the limiting probabilities $\mathcal{A}(\tilde{z};\mathbf{z}_{-1})$ and $\mathcal{B}\left(\left(\tilde{z},\tilde{f}\right);(\mathbf{z},\mathbf{f})_{-1}\right)$ when the strategy profile $(\mathbf{z},\mathbf{f})_{-1}$ specifies that all agents other than agent 1 play the same strategy

[9] To be precise, the finite truncation of an LDE equilibrium strategy profile should be a (suitably defined) ε-equilibrium for the finite economy.

(z, f). As a side note, one could imagine a different modeling approach in which $\mathcal{A}$ and $\mathcal{B}$ are taken as the primitives that describe the flows of benefits and harms on a network. One secondary goal of the analysis in this section is to shed light on some of the features one might want to impose on $\mathcal{A}$ and $\mathcal{B}$ if they were taken as primitives of a network model. The analysis starts with the simpler case of tips ($\mathcal{A}$). In the course of informally deriving $\mathcal{A}$ (proofs are in the appendix), we provide a thumbnail sketch of some of the limiting properties of a large network. In what follows, the phrase "with probability tending to 1 as $n \to \infty$" is a common modifier, so it is useful to have shorthand. We will write this as (pt1).

3.1 Characterization of the Spread of Tips and Viruses

Proposition 1 *Suppose that* $\mathbf{z}$ *is the symmetric limit strategy profile with every term equal to* z. *Then for all* $\tilde{z}$,

$$\mathcal{A}(\tilde{z}; \mathbf{z}_{-1}) = \begin{cases} 0 & \text{if } z \leq 1 \\ a\left(1 - e^{-a\tilde{z}}\right) & \text{if } z > 1 \end{cases}$$

where a *is the unique positive solution to*

$$1 - a = e^{-az} \ . \tag{2}$$

We start by developing some intuition for Proposition 1 along with some of the machinery used in the proof; the formal proof is in the appendix. The basic idea is the following. First we show that it is without loss of generality (pt1) to assume that if agents 0 and 1 communicate, they do so through the hub. Then to compute the probability that, say, agent 0 communicates with H, we couple the true enumeration of extended neighbors of 0 (by first order neighbors, then second order neighbors, and so forth) to a related branching process. The branching process overestimates the number of neighbors by not treating a doubling back to a previously visited agent as a dead end. The probability that the branching process dies out before any of its members connect to H is easy to compute. While this underestimates the probability that the true extended neighborhood fails to connect to H, we can show that the difference vanishes as $n \to \infty$. The reason is that conditional on failing to connect to H, the branching process must have died out at a small size relative to n (pt1), in which case it is very likely (pt1) that the branching process and the true enumeration never diverged (due to no double-backs). Below, we provide a more complete discussion of some of these steps.

It helps to have a term for the set of small agents who communicate with i (including i herself), without using any links to or from H. Define

$$S_i = i \cup \{j \in \mathcal{I}_n \backslash H \mid i \text{ and } j \text{ communicate along at least one path that does not include } H\}$$

Next, notice that the event $i \nleftrightarrow H$ (i fails to connect to the hub) is equivalent to the event that for all agents $j \in S_i$, the random variables η_j have realization zero. We will get considerable mileage out of the fact that (for fixed h) this event becomes very unlikely if we condition on $|S_i|$ being large: an agent who has many small neighbors is very likely to be connected to the hub as well. In what follows, S_i large will often be taken to mean $|S_i| > \log n$, while we may say that S_i is small if $|S_i| \leq \log n$.

Next observe that the event that agents 0 and 1 communicate can happen in one of two disjoint ways. First, they may connect through the hub: $0 \leftrightarrow H$ and $1 \leftrightarrow H$ implies $0 \leftrightarrow 1$.[10] Or, they might be connected with each other despite the fact that neither is connected to H; this is the event $1 \leftrightarrow 0 \cap 1 \not\leftrightarrow H \cap 0 \not\leftrightarrow H$. The first step of the proof is to show that this second case is very unlikely. The logic is that if agent 0 fails to connect to H, then by the argument above, S_0 must be small. The same is true for agent 1 and S_1. But absent a connection to the hub, S_0 and S_1 must intersect if we are to have $0 \leftrightarrow 1$, and the chance of this intersection vanishes if S_0 and S_1 are both small.

Thus, to determine the chance that 0 and 1 communicate, it suffices to focus on the first case: the probability that agents 0 and 1 both communicate with the hub. The next step is to show that the events $0 \leftrightarrow H$ and $1 \leftrightarrow H$ are approximately independent, so $\mathcal{A}(\tilde{z}; \mathbf{z}_{-1})$ reduces to the product of their (limiting) probabilities. It is easier to grasp the intuition of this independence for the complementary events $0 \not\leftrightarrow H$ and $1 \not\leftrightarrow H$. These events can only occur if S_0 and S_1 are both small, but in this case, they cannot have much influence on each other.

In summary, it suffices to characterize the probabilities $\Pr^{nh}\left(0 \not\leftrightarrow H \mid \tilde{z}, \mathbf{z}_{-1}^{n}\right)$ and $\Pr^{nh}\left(1 \not\leftrightarrow H \mid \tilde{z}, \mathbf{z}_{-1}^{n}\right)$. To characterize the probability of $0 \not\leftrightarrow H$, we introduce a thought experiment. Let each small agent in $\{0, ..., n-1\}$ represent a 'type,' and suppose that we can generate arbitrarily many distinct individuals of each type. We grow a population of these individuals according to the following branching process. Let $\hat{W}_t$ be the set of individuals alive at generation t, and let $\hat{R}_t$ be the set of individuals who were previously alive. The individuals in $\hat{W}_t$ have children, according to probabilities described below, and then die; thus we have $\hat{R}_{t+1} = \hat{R}_t \cup \hat{W}_t$. An individual of type i has either zero or one child of each type, where the probability that she has a child of type j is p_{ij}. (Thus the total number of progeny for this individual is the sum of n Bernoulli random variables.) These random variables are independent across individuals. The set $\hat{W}_{t+1}$ is defined to be the set of all children of individuals in $\hat{W}_t$. In general, $\hat{W}_t$ and $\hat{R}_t$ may contain many distinct individuals of the same type. The process itself dies out if $\hat{W}_T$ is empty, for some T, in which case $\hat{R}_t = \hat{R}_T$ for all $t \geq T$.

If we initialize this process with $\hat{R}_0 = \emptyset$ and $\hat{W}_0$ containing exactly one individual of type 0, then we can associate $\hat{W}_1$ with agent 0's first order small neighbors. Furthermore, as long as all of the individuals in $\hat{R}_t$ have different types, we can associate $\hat{R}_t$ with the subset of agents in S_0 who are reachable within t links of agent 0. However at some point $\hat{W}_t$ may include a child whose type already appears in $\hat{R}_t$. From the standpoint of enumerating the agents in S_0, we should ignore this child, and her progeny, to avoid overcounting. Instead, the process counts her and treats her as an independent source of population growth. For this reason, $\sum_{t=0}^{\infty} \left|\hat{W}_t\right|$ is larger than $|S_0|$. However, in the event that S_0 turns out to be small, it is very likely that the branching process does not generate any of these duplicates, and therefore enumerates S_0 exactly.

Next, endow each individual in $\hat{W}_t$ (for all $t \in \{0, 1, 2, ...\}$) with an i.i.d. binary random variable that succeeds or fails with probability h or $1-h$. By analogy with the true model, call a success a connection to the hub. Let $\hat{Y}_0^{nh}$ be the probability that these random variables fail for every individual in $\bigcup_{t=0}^{\infty} \hat{W}_t$; since this comprises weakly more individuals than there are agents in S_0, we have $\Pr^{nh}\left(0 \not\leftrightarrow H \mid \tilde{z}, \mathbf{z}_{-1}^{n}\right) \geq \hat{Y}_0^{nh}$. Furthermore, the difference between these two probabilities is small because $0 \not\leftrightarrow H$ can only occur when S_0 is small, in which case the branching process is very likely to enumerate S_0 correctly. Fortunately $\hat{Y}_0^{nh}$ is straightforward to compute.

[10] Of course 0 and 1 might also be connected by additional paths that do not involve H, but this is immaterial.

To simplify the computation, pool individuals of types $\mathbf{N}\backslash\{1\}$ together as "normal" types, since their reproduction is governed by exactly the same probabilities, and refer to type 1 as the "deviant" type. A normal individual produces $Bin\,(n-1,p_{00})$ normal children and $Bin\,(1,p_{01})$ deviant ones, whereas a deviant individual has $Bin\,(n-1,p_{01})$ normal children and $Bin\,(1,p_{11})$ deviant ones. Let $\hat{E}_0$ be the event that no individual in the branching process beginning with a single normal individual connects to H, so $\hat{Y}_0^{nh} = \Pr^{nh}\left(\hat{E}_0\right)$. Let $\hat{E}_1$ be the corresponding event for a branching process beginning with a single deviant individual, with $\hat{Y}_1^{nh} = \Pr^{nh}\left(\hat{E}_1\right)$. Observe that $\hat{E}_0$ requires that (i) the initial individual does not connect to H, (ii) for each potential normal child, either this child does not exist, or she does exist, and event $\hat{E}_0$ holds for the independent process beginning with her, and (iii) for each potential deviant child, either the child does not exist, or she does exist, and event $\hat{E}_1$ holds for the independent process beginning with her. These conditions are all independent; (i) occurs with probability $1-h$, (ii) occurs with probability $\left(1-p_{00}+p_{00}\hat{Y}_0^{nh}\right)$ for each potential normal child, and (iii) occurs with probability $\left(1-p_{01}+p_{01}\hat{Y}_1^{nh}\right)$ for the single potential deviant child. Thus we have

$$\hat{Y}_0^{nh} = (1-h)\left(1-p_{00}+p_{00}\hat{Y}_0^{nh}\right)^{n-1}\left(1-p_{01}+p_{01}\hat{Y}_1^{nh}\right) \tag{3}$$

By the same type of argument, $\hat{Y}_1^{nh}$ must satisfy

$$\tilde{Y}_1^{nh} = (1-h)\left(1-p_{01}+p_{01}\hat{Y}_0^{nh}\right)^{n-1}\left(1-p_{11}+p_{11}\hat{Y}_1^{nh}\right) \tag{4}$$

Notice that we can write the expression for $\tilde{Y}_0^{nh}$ as $(1-h)\left(1-\frac{z}{n+d}\left(1-\hat{Y}_0^{nh}\right)\right)^{n-1}\left(1-\frac{\tilde{z}}{n+d}\left(1-\hat{Y}_1^{nh}\right)\right)$, where $d=\frac{\tilde{z}-z}{z}$. Taking limits, we have

$$\begin{aligned}\hat{Y}_0^{h} &= \lim_{n\to\infty}\hat{Y}_0^{nh} = (1-h)\lim_{n\to\infty}\left(1-\frac{z}{n+d}\left(1-\hat{Y}_0^{nh}\right)\right)^{n-1}\lim_{n\to\infty}\left(1-\frac{\tilde{z}}{n+d}\left(1-\hat{Y}_1^{nh}\right)\right)\\ &= (1-h)\left(e^{-z\left(1-\hat{Y}_0^{h}\right)}\right)(1)\end{aligned}$$

which has a unique positive solution that lies in the interval $(0,1)$. *Mutatis mutandis*, the result for $\hat{Y}_1^h$ is similar; taking limits, we have

$$\hat{Y}_1^{h} = (1-h)\,e^{-\tilde{z}\left(1-\hat{Y}_0^{h}\right)}$$

The expression in Proposition 1 follows by setting $a=\lim_{h\to 0}\left(1-\hat{Y}_0^h\right)$.

The formal proof follows the lines of the argument above. Most of the effort involves proving the various claims about events that occur with vanishing probability as n grows large. Also, in order to compare the branching process to S_0 rigorously, we must construct them on the same probability space; this involves a fair amount of notation but no major conceptual difficulties.

Readers familiar with the graph theory literature will recognize the condition defining a as a standard expression for the size of the giant component in a random graph. Thus, our device of "seeding" the network with a large agent H and then reducing its size to $h=0$ appears to generate the same payoff-relevant network properties that would arise in a model without H. With or without the hub, any two agents have a vanishing chance of communicating unless they both happen to belong to the unique giant component. Since its size implies that H belongs to the giant component (pt1), statements like $0 \leftrightarrow H$ turn out to be a tractable shorthand for agent 0's membership in the unique giant component.

Notice that if all agents (including agent 1) play z, the expected fraction of the population that benefits from a tip is a^2 : with probability a the originating agent communicates with a non-negligible set of agents, in which case fraction a of the population is exposed to her tip. In a large decentralized economy, a remains at zero unless connectivity rises above the threshold level $z = 1$. This is a particularly acute form of network effect: for an individual, there is no point in forming connections unless aggregate connectivity is great enough. Notice that this result smooths out in a large h-economy. If we write $a^h = 1 - \hat{Y}_0^h$, then it is not hard to see that a^h is positive and strictly increasing for all z.

Next we turn to the spread of viruses.

Proposition 2 *Let* $\left(\left(\tilde{z}, \tilde{f}\right); (\mathbf{z}, \mathbf{f})_{-1}\right)$ *be the strategy profile in which agent 1 plays* $\left(\tilde{z}, \tilde{f}\right)$ *and every other agent plays* (z, f)*. Then for all* $\left(\tilde{z}, \tilde{f}\right)$,

$$\mathcal{B}\left(\left(\tilde{z}, \tilde{f}\right); (\mathbf{z}, \mathbf{f})_{-1}\right) = \begin{cases} 0 & \text{if } z(1-f) \leq 1 \\ b\left(1 - e^{-b\left(1-\tilde{f}\right)\tilde{z}}\right) & \text{if } z(1-f) > 1 \end{cases}$$

where b *is the unique positive solution to*

$$1 - b = e^{-b(1-f)z} \ .$$

Notice that the probability $\mathcal{B}\left(\left(\tilde{z}, \tilde{f}\right); (\mathbf{z}, \mathbf{f})_{-1}\right)$ is exactly the expression that would result from substituting 'effective' link intensity $\left(1 - \tilde{f}\right)\tilde{z}$ for agent 1, or $(1 - f)\,z$ for the other agents, in place of $\tilde{z}$ or z in Proposition 1. The simplicity of this correspondence between $\mathcal{G}$ and $\mathcal{G}^f$ depends somewhat on the symmetry of this strategy profile; for general strategy profiles, one cannot simply substitute $(1 - f)\,z$ for z to get appropriate expressions for $\mathcal{G}^f$.

The logic follows the previous proposition quite closely. By analogy with S_i, define S_i^{up} to be the set of agents who infect i by some path that does not include H. That is $j \in S_i^{up}$ if and only if $j \longmapsto_1 i$ or $j \longmapsto_1 k_1 \longmapsto_1 \ldots \longmapsto_1 k_m \longmapsto_1 i$ for some path that does not include H. Define S_i^{down} to be the set of agents whom i infects along some path that does not include H. We claim that if there is no infecting path from agent 0 to agent 1 through the hub ($0 \longmapsto H \longmapsto 1$), then agent 0 does not infect agent 1 through any path (pt1). To see why this is true, suppose that $0 \longmapsto H \longmapsto 1$ fails because 0 does not infect H. This is only possible if 0 infects relatively few agents ($S_0^{down} \leq \log n$), in which case the chance that agent 1 is among those infectees is small. A similar argument applies if $0 \longmapsto H \longmapsto 1$ fails because agent 1 is not infected by H. This means that we can focus on infections that go through H. Next, by an independence argument similar to Proposition 1 it suffices to study the probability $0 \longmapsto H$ and $H \longmapsto 1$ one at a time. Finally, the complementary probabilities of $0 \not\longmapsto H$ and $H \not\longmapsto 1$ can be computed by a branching process approximation that is exact in the large n limit.

The branching process used here comes in two flavors, depending on whether we are trying to enumerate an agent's downstream infectees or her upstream infectors. For the downstream infectees of agent 0, define $\left(\hat{W}_t^{down}, \hat{R}_t^{down}\right)$ just as $\left(\hat{W}_t, \hat{R}_t\right)$ except for one key change. As above, an individual of type i has either zero or one child of each type, but the probability that she has a child of type j is now given by $p_{ij}(1-f_j)$, where earlier it was simply p_{ij}. The interpretation is that we only count progeny when a link is formed (with probability p_{ij}) and that link is not blocked by the downstream child (with probability $1-f_j$). As earlier, designate individuals of types $\mathbf{N}\backslash\{1\}$ as normal and individuals of type 1 as deviants. Under $\left(\hat{W}_t^{down}, \hat{R}_t^{down}\right)$, a normal individual produces $Bin(n-1, p_{00}(1-f))$ normal children and $Bin\left(1, p_{01}\left(1-\tilde{f}\right)\right)$ deviant ones, while a deviant individual produces $Bin(n-1, p_{01}(1-f))$ normal children and $Bin\left(1, p_{11}\left(1-\tilde{f}\right)\right)$ deviant ones. Each of these individuals is endowed with a probability h chance of infecting H. Let $\hat{Y}_i^{down,nh}$ be the probability that, in a branching process $\left(\hat{W}_t^{down}, \hat{R}_t^{down}\right)$ starting with a single individual of type i, no individual in the process infects H. By substituting the new reproduction rates into equations (3) and (4), we immediately have the following:

$$\hat{Y}_0^{down,nh} = (1-h)\left(1-p_{00}(1-f)\left(1-\hat{Y}_0^{down,nh}\right)\right)^{n-1}\left(1-p_{01}\left(1-\tilde{f}\right)\left(1-\hat{Y}_1^{down,nh}\right)\right)$$

Taking limits as before (and implicitly relying on the boundedness of $\hat{Y}_1^{down,nh}$), we arrive at[11]

$$\hat{Y}_0^{down,h} = \lim_{n\to\infty} \hat{Y}_0^{down,nh} = (1-h)\,e^{-z(1-f)\left(1-\hat{Y}_0^{down,h}\right)}$$

Recall that we are only interested in this branching process to the extent that it provides a good approximation of the probability that $0 \longmapsto H$ occurs in the true graph $\mathcal{G}^f$. Let $b^h = \lim_{n\to\infty} \Pr^{nh}(0 \longmapsto H)$ be this probability for a large $h-$economy; the final part of the proof links the branching process to the true graph by showing that $b^h = 1-\hat{Y}_0^{down,h}$.

Enumerating agent 1's upstream infectors requires a slight twist. Reverse the interpretation of the parent-child relationship, so that an individual's children are those individuals who directly infect her. To go with this interpretation, define the branching process $\left(\hat{W}_t^{up}, \hat{R}_t^{up}\right)$ with the following reproduction rates: A normal parent produces $Bin(n-1, p_{00}(1-f))$ normal children and $Bin(1, p_{01}(1-f))$ deviant ones. A deviant parent produces $Bin\left(n-1, p_{01}\left(1-\tilde{f}\right)\right)$ normal children and $Bin\left(1, p_{11}\left(1-\tilde{f}\right)\right)$ deviant ones. The only difference between these downstream and upstream branching processes is that in the latter case, it is the parent (as the potential infectee) whose firewall probability applies to the connection. In the downstream case, the child is the the potential infectee, so it is her type that determines whether a firewall blocks the connection. For the upstream case, define $\hat{Y}_i^{up,nh}$ to be the probability that no individual in a branching process $\left(\hat{W}_t^{up}, \hat{R}_t^{up}\right)$ is infected by H, when the process is started with one individual of type i. Making the necessary changes in the recursion relationship above, we have

$$\begin{aligned}
\hat{Y}_0^{up,nh} &= (1-h)\left(1-p_{00}(1-f)\left(1-\hat{Y}_0^{up,nh}\right)\right)^{n-1}\left(1-p_{01}(1-f)\left(1-\hat{Y}_1^{up,nh}\right)\right) \\
\hat{Y}_1^{up,nh} &= (1-h)\left(1-p_{01}\left(1-\tilde{f}\right)\left(1-\hat{Y}_0^{up,nh}\right)\right)^{n-1}\left(1-p_{11}\left(1-\tilde{f}\right)\left(1-\hat{Y}_1^{up,nh}\right)\right)
\end{aligned}$$

[11] $\hat{Y}_1^{down,h}$ can be calculated in exactly the same way, but it is of no particular interest here.

Taking limits as $n \to \infty$, gives us

$$\begin{aligned}
\hat{Y}_0^{up,h} &= \lim_{n\to\infty} \hat{Y}_0^{up,nh} = (1-h)\, e^{-z(1-f)\left(1-\hat{Y}_0^{up,h}\right)} \\
\hat{Y}_1^{up,h} &= \lim_{n\to\infty} \hat{Y}_1^{up,nh} = (1-h)\, e^{-\tilde{z}\left(1-\tilde{f}\right)\left(1-\hat{Y}_0^{up,h}\right)}
\end{aligned}$$

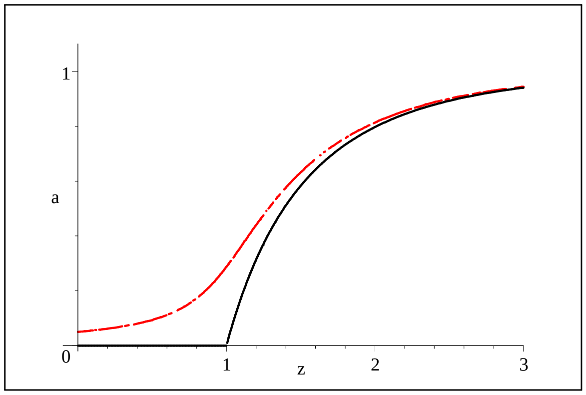
Notice that the expressions defining $\hat{Y}_0^{up,h}$ and $\hat{Y}_0^{down,h}$ are identical. Since this expression has a unique solution, we have $\hat{Y}_0^{up,h} = \hat{Y}_0^{down,h} = 1 - b^h$.[12] In other words, in a large $h-$ economy, with a symmetric strategy profile (except for one possible deviator), the chance that a normal agent infects the hub and the chance that he is infected by the hub are the same. This equality would not typically hold if agents were to choose different levels of protection, as agents who choose relatively higher f will be less likely than others to get infected, but no less likely to pass on an infection that they originate. To complete the argument, let $\beta_1^h = \lim_{n\to\infty} \Pr^{nh}(H \longmapsto 1)$ be the probability that H infects agent 1 in $\mathcal{G}^f$, for a large $h-$economy. The proof establishes that $\beta_1^h = 1 - \hat{Y}_1^{up,h}$.

Putting both results together, and taking the large decentralized economy limit, the chance that agent 0 infects agent 1 is $\lim_{h\to 0} b^h \beta_1^h = b\beta_1$, where $\beta_1 = 1 - e^{-\tilde{z}\left(1-\tilde{f}\right)b}$ and b solves $b = 1 - e^{-z(1-f)b}$.

3.2 Interpreting $\mathcal{A}$ and $\mathcal{B}$

The derivation of the probabilities $\mathcal{A}(\tilde{z}; \mathbf{z}_{-1})$ and $\mathcal{B}\left(\left(\tilde{z}, \tilde{f}\right); (\mathbf{z}, \mathbf{f})_{-1}\right)$ suggests that each one can be understood in two pieces. For a tip to be transmitted from agent 0 to agent 1, two things must happen. First agent 0 must belong to the giant component of $\mathcal{G}$; otherwise his tip will not get broad exposure.[13] The giant component has size a (as a fraction of the population), so given the symmetric strategy profile, agent 0 belongs to it with probability a. Second, agent 1 must have at least one link into the giant component. Given link intensity $\tilde{z}$, the chance that she fails to link into the giant component is $e^{-a\tilde{z}}$, or conversely, her chance of succeeding is $1 - e^{-a\tilde{z}}$. For viruses, the story is a bit more subtle. There is a set of agents (a fraction b of the population), each of whom is capable of infecting a positive fraction of the population. A virus cannot spread from 0 to 1 unless (with probability b) 0 is one of these agents. In this case, we also need agent 1 to form an unprotected link to someone downstream of 0; this happens with probability $1 - e^{-\left(1-\tilde{f}\right)\tilde{z}b}$. With a certain amount of imprecision, we will refer to a as total connectivity and to b as unsecured connectivity for the network.

[12] To be mathematically precise, there is a second, negative solution, but it has no relevance.
[13] And this turns out to be equivalent to connecting to H.



a vs. z (assuming all agents play z) Solid line: $h = 0$ limit. Dashed line: $h = 0.05$.

It is not difficult to confirm that a and b coincide if agents choose no security ($f = 0$) and that $b < a$ otherwise. Figure **??** illustrates how a varies with z. For fixed f, the relationship of b to z is similar, but the threshold is shifted right to $z = 1/(1-f)$. For comparison, the dashed curve shows a^h for a small positive hub size h. In this second curve, one can see a familiar S-shaped curve: initially there are increasing aggregate returns to additional individual links z, followed by a region of decreasing aggregate returns where the network is relatively saturated with connections. In this context, the sharp threshold behavior of a and b in the large decentralized economy can be understood as a limit of this pattern of first increasing, then decreasing returns.

Next, consider how the shape of agent 1's payoff function affects her incentives in choosing links and security. Increasing link intensity $\tilde{z}$ leads to greater benefits from tips through the $A\mathcal{A}$ term, but also to higher costs: both the explicit cost $C\left(\tilde{z}, \tilde{f}\right)$ and the "virus cost" $B\mathcal{B}$ (reflecting exposure to unblocked viruses) also rise. Notice that both the benefit and the virus cost terms are concave in $\tilde{z}$: each additional link yields a smaller marginal benefit from tips than the previous link, but also a smaller marginal harm from viruses. The logic is the same in both cases – the marginal effect of an additional link depends on the chance that the new link exposes an agent to a tip or virus, while all of her other links do not. An agent with many links is likely to already be exposed to the tip or virus, so the effect of the additional link is small. It might appear that in the absence of an explicit cost $C\left(\tilde{z}, \tilde{f}\right)$, and with $B\mathcal{B}$ concave, the optimal choice of $\tilde{z}$ could run off toward infinity. In fact, one can show that that as long as some security is being provided (either by agent 1 or by the other agents), the marginal benefit from tips shrinks faster with $\tilde{z}$ than the marginal harm from viruses does, and this would limit link intensity even if there were no explicit costs associated with $\tilde{z}$. Notice also that $A\mathcal{A}-B\mathcal{B}$ is convex in agent 1's security level $\tilde{f}$. This makes sense after considering that an increase in $\tilde{f}$ affects harm from viruses similarly to a reduction in $\tilde{z}$ – both changes reduce her effective link intensity $\tilde{z}_{eff} = \left(1-\tilde{f}\right)\tilde{z}$. In both cases, removing an effective link at the margin has a smaller impact for a highly connected agent ($\tilde{z}$ large or $\tilde{f}$ small) because she is still likely to catch the virus *via* one of her other effective links. The convexity of the benefits from security tends to stack the deck against moderate levels of $\tilde{f}$; unless $C\left(\tilde{z}, \tilde{f}\right)$ is also sufficiently convex, agents may find all or nothing security decisions ($\tilde{f} = 1$ or $\tilde{f} = 0$) optimal.

Finally, consider the interaction between aggregate properties of the network, like a and b, and individual incentives. First, unless other agents are sufficiently connected ($z > 1$), aggregate connectivity will be nil ($a = 0$), and so there will be no benefit for agent 1 in forming links. When $a > 0$, we have $\frac{\partial^2}{\partial a \partial \tilde{z}} A\mathcal{A}\left(\tilde{z}; \mathbf{z}_{-1}\right) = \left(2 - a\tilde{z}\right) A a e^{-a\tilde{z}}$, so whenever a and $\tilde{z}$ are not too large

($a\tilde{z} < 2$) a rise in total connectivity will tend to make marginal links more attractive for agent 1. Together, these two points constitute a positive network effect. However, if both the network and agent 1 individually are fairly saturated with connections ($a\tilde{z} > 2$), then a further rise in a reduces the return to $\tilde{z}$. Thus we anticipate the positive network effect to be self-limiting.[14] There is a similar relationship between unsecured connectivity and an individual agent's effective links; we have $\frac{\partial^2}{\partial b \partial \tilde{z}_{eff}} B\mathcal{B} = (2 - b\tilde{z}_{eff}) Bbe^{-b\tilde{z}_{eff}}$. Thus, when the network is relatively safe ($b\tilde{z}_{eff} < 2$), a rise in b increases the marginal harm to agent 1 from increasing links $\tilde{z}$ or reducing security $\tilde{f}$. This tends to act against the positive network effect from tips, because when other agents add links, b will rise, encouraging agent 1 to reduce $\tilde{z}$ or increase $\tilde{f}$. However, in a relatively unsafe network ($b\tilde{z}_{eff} > 2$), the incentives flip – in this case, a shift by other agents toward more connected or less secure behavior (and thus higher b) encourages agent 1 to become more connected and less cautious herself. Consequently, there may be a tendency for cautious or risky behavior in the network to be self-reinforcing.

4 Symmetric Equilibrium

In this section, we characterize the symmetric equilibria of a large decentralized economy under different cost structures. The sharpest results are obtained when security costs are extremely convex in f; we lead with the degenerate case in which security is free below a threshold $\bar{f}$ and infinitely costly above it. If we relax this to a smooth convex security cost, the results are similar, but it becomes harder to ensure that the necessary first order conditions for equilibrium are also sufficient. One complicating factor, as discussed earlier, is the convexity of benefits from security, which tempts agents toward all or nothing security decisions. Since this issue will be discussed in more depth when we consider elastic security costs, its treatment here will be brief.

Before turning to the case of step function security costs, consider the case in which $C(z, f)$ is differentiable. If a symmetric equilibrium exists at the interior strategy (z, f), then agent 1's payoff must satisfy the following first order conditions:

$$\begin{aligned}
\frac{\partial}{\partial \tilde{z}} \pi_1 \left(\left(\tilde{z}, \tilde{f} \right); (\mathbf{z}, \mathbf{f})_{-1} \right) \Big|_{(\tilde{z},\tilde{f})=(z,f)} &= Aa^2 e^{-az} - Bb^2 (1-f) e^{-(1-f)zb} - C_z(z,f) = 0 \\
\frac{\partial}{\partial \tilde{f}} \pi_1 \left(\left(\tilde{z}, \tilde{f} \right); (\mathbf{z}, \mathbf{f})_{-1} \right) \Big|_{(\tilde{z},\tilde{f})=(z,f)} &= Bb^2 z e^{-(1-f)zb} - C_f(z,f) = 0
\end{aligned}$$

and total and unsecured connectivity must be generated from individual choices according to $a = 1 - e^{-az}$ and $b = 1 - e^{-(1-f)zb}$ respectively. We can use these aggregation equations to write the first order conditions as

$$\begin{aligned}
Aa^2(1-a) &= C_z(z,f) + \frac{1-f}{z} C_f(z,f) \\
Bb^2(1-b) &= \frac{1}{z} C_f(z,f)
\end{aligned}$$

[14] Recall that $1 - e^{-a\tilde{z}}$ can be interpreted as the probability that agent 1 connects to a non-negligible fraction of the population. Notice that the effect of a on the marginal benefit from $\tilde{z}$ changes sign precisely when this probability is $1 - e^{-2} \approx 0.86$.

This highlights two facts. First, at equilibrium, the cost of an increase in connections can be represented by the direct cost $C_z(z,f)$ plus the indirect cost of 'sterilizing' the increase in z with an increase in f that is sufficient to hold $(1-f)z$ constant. Second, the lefthand side of both equations makes it clear that an agent's choices have the greatest impact on her connectivity when the network itself is moderately well connected (a and b near $\frac{2}{3}$).

4.1 Case 1: Inelastic Security Costs

As an idealization of a situation in which modest security measures are cheap, but additional measures are progressively more expensive, define $C(z,f)$ as

$$C(z,f) = \begin{cases} 0 & \text{if } f \leq \bar{f} \\ \infty & \text{if } f > \bar{f} \end{cases}$$

To avoid special cases, we will assume that $\bar{f} \in (0,1)$. Under these costs, an agent will always choose as much free security as possible, so we replace the first order condition for security with the condition $f = \bar{f}$.[15] Since it is correct but confusing to refer to an increase in $\bar{f}$ as a decline in security costs, we will describe this equivalently as a rise in 'security capacity.' Given a strategy profile of $(\mathbf{z}, \bar{\mathbf{f}})_{-1}$ for the other agents, agent 1's marginal payoff from $\tilde{z}$ can be written as

$$\frac{\partial}{\partial \tilde{z}} \pi_1 \left(\left(\tilde{z}, \bar{f}\right); (\mathbf{z}, \bar{\mathbf{f}})_{-1} \right) = e^{-a\tilde{z}} \left(Aa^2 - Bb^2 \left(1-\bar{f}\right) e^{\left(a - \left(1-\bar{f}\right)b\right)\tilde{z}} \right)$$

It is clear that there always exists a trivial zero connection equilibrium: if $z = 0$, then $a = b = 0$, and $\tilde{z} = 0$ is a weak best response for agent 1. We focus on the more interesting case in which $z > 1$, so the network has strictly positive total connectivity. Note that because $\left(1-\bar{f}\right)b < a$, the derivative is either negative for all $\tilde{z} \geq 0$ (if $Aa^2 \leq Bb^2\left(1-\bar{f}\right)$) or switches from positive to negative exactly once. Thus, any strictly positive solution that we find to agent 1's first order condition will be an optimal decision for her. This permits us to characterize equilibria as follows.

Proposition 3 *Suppose a large decentralized economy has costs as described above, with security capacity $\bar{f}$. There exists a unique symmetric equilibrium with positive total connectivity. The equilibrium strategy profile satisfies $z = -\frac{\ln(1-a)}{a}$ and $f = \bar{f}$, with a and b determined by the unique (on $(a,b) \in (0,1)^2, b \leq a$) solution to the following pair of equations:*

$$Aa^2(1-a) = Bb^2(1-b)\left(1-\bar{f}\right) \tag{5}$$

$$1-\bar{f} = \frac{a}{b}\frac{\ln(1-b)}{\ln(1-a)} \tag{6}$$

Equation (5) represents the first order condition for links, interpreted in terms of the aggregate network variables a and b, and evaluated when link choices are symmetric. One can see again

[15] To be more precise, an agent strictly prefers higher security unless $b = 0$, in which case she is indifferent. To streamline the discussion, we break this indifference by assuming that $f = \bar{f}$ is chosen even if $b = 0$. This is without loss of generality – if an equilibrium exists with $b = 0$ and $f < \bar{f}$, then there is an outcome equivalent equilibrium with $f = \bar{f}$.

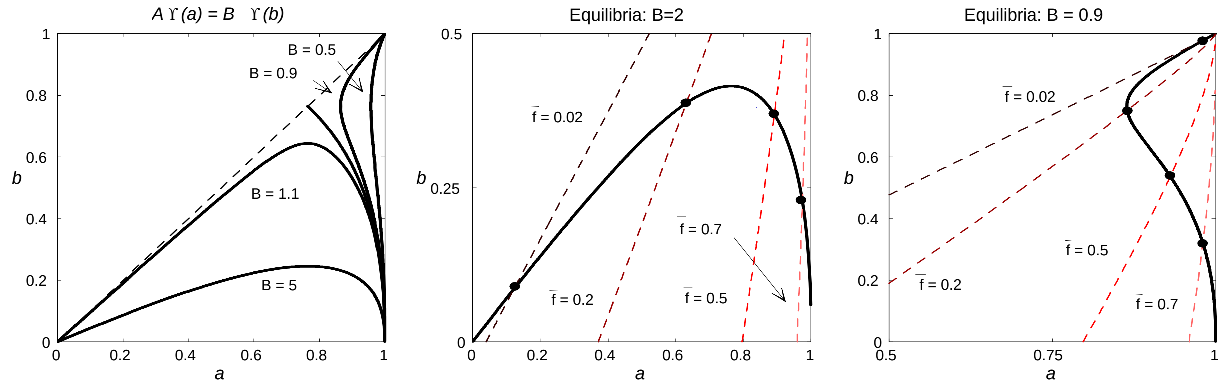


Figure 3: Equilibrium with security capacity $\bar{f}$. Left: eqn. (7) at assorted values of B. ($A = 1$.) Middle and right: equilibrium condition (6) – dashed – at increasing values of $\bar{f}$, vs. (7).

that the marginal benefit from links interacts in a non-monotonic way with total connectivity a, first rising, then falling. Likewise, the marginal harm from links is first rising and then falling in unsecured connectivity b. Equation (6) reflects the technological constraint specifying how total and unsecured connectivity must differ if agents choose security level $\bar{f}$.

With inelastic security costs, the model has only two free parameters: the security capacity $\bar{f}$, and the relative threat level B/A. In order to illustrate their comparative static effects, it is useful to transform the equilibrium conditions so that the effects of $\bar{f}$ and B are separated. Substitute (6) into (5) to get

$$A\Upsilon(a) = B\Upsilon(b), \tag{7}$$

where $\Upsilon(x) = -x(1-x)\ln(1-x)$. Together, (7) and (6) are an equivalent representation of the equilibrium conditions, but now $\bar{f}$ and B/A appear in separate equations. Examples of the curves defined by (7) and (6) are plotted in Figure 3.[16] The function Υ has a natural interpretation. Set $Benefit(\tilde{z}) = Aa\left(1 - e^{-a\tilde{z}}\right)$ and $Harm(\tilde{z}_{eff}) = Bb\left(1 - e^{-b\tilde{z}_{eff}}\right)$. We can recast the individual's decision problem as a maximization of $Benefit(\tilde{z}) - Harm(\tilde{z}_{eff})$ over total and effective connectivity, $\tilde{z}$ and $\tilde{z}_{eff}$, subject to the constraint that $\tilde{z}_{eff}/\tilde{z} = 1 - \bar{f}$. Because an individual can make a proportional increase or decrease in both types of connectivity without changing the value of the constraint, at an optimal decision the marginal benefit induced by a percentage increase in $\tilde{z}$ must balance the marginal harm induced by a percentage increase in $\tilde{z}_{eff}$. That is

$$\frac{dBenefit(\tilde{z})}{d\tilde{z}/\tilde{z}} = \frac{dHarm(\tilde{z}_{eff})}{d\tilde{z}_{eff}/\tilde{z}_{eff}}$$

Evaluated at a symmetric profile $\mathbf{z}$, we have

$$\left.\frac{dBenefit(\tilde{z})}{d\tilde{z}/\tilde{z}}\right|_{\tilde{z}=z} = A\tilde{z}a^2e^{-a\tilde{z}}\Big|_{\tilde{z}=z=-\frac{\ln(1-a)}{a},\, e^{-az}=1-a} = A\Upsilon(a).$$

Similarly, $B\Upsilon(b)$ is the marginal harm, or $-B\Upsilon(b)$ the marginal payoff change, induced by a

[16] As the right panel of Figure 3 suggests, the most delicate part of the proof of Proposition 3 is to show existence and uniqueness of equilibrium when $\bar{f}$ is near zero.

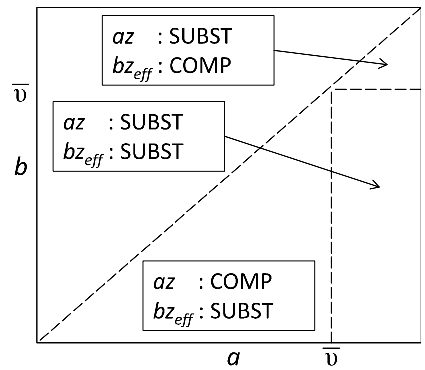


Figure 4: Regions where individual and aggregate connectivity are complements or substitutes at equilibrium. (az : total, bz_{eff} : unsecured)

percentage increase in $\tilde{z}_{eff}$, evaluated at a symmetric strategy profile.

The sign of the derivative $A\Upsilon'(a)$ tells us something about how an individual's incentive to form more links rises or falls with an increase in aggregate connectivity; we will say that individual and aggregate total connectivity are complements (substitutes) at equilibrium if $A\Upsilon'(a) > 0$ ($A\Upsilon'(a) < 0$). Similarly, let individual and aggregate unsecured connectivity be complements (substitutes) at equilibrium if $B\Upsilon'(b) < 0$ ($B\Upsilon'(b) > 0$); the sign change reflects the fact that threats enter payoffs with a negative sign. Figure 4 illustrates the regions in a-b space where each case applies. It is not hard to show that $\Upsilon(x)$ is quasiconcave on $(0,1]$, with $\Upsilon(1) = 0$, $\lim_{x\to 0}\Upsilon(x) = 0$, and a peak at $\bar{\upsilon} \approx 0.764$, so the feedback between individual and aggregate actions will tend to change around $a = \bar{\upsilon}$ or $b = \bar{\upsilon}$.

Comparative statics of a change in security capacity $\bar{f}$

Regardless of $\bar{f}$, any equilibrium must lie along the curve $A\Upsilon(a) = B\Upsilon(b)$, which is depicted in the left panel of Figure 3 for various values of B/A. One end of this curve is anchored at the social bliss point $(a,b) = (1,0)$. However, the other end of the curve depends on the relative threat level. If $B < A$, the curve terminates at a dense, unsecure network $(1,1)$, while if $B > A$ it ends at a sparse but safe network $(0,0)$.[17] The curve defined by (6) is upward sloping (in a-b space) and within the a-b unit square, it ranges between a positive intercept at $b = 0$ axis and a limit at $(1,1)$, as illustrated in the middle and right panels of Figure 3. As $\bar{f}$ increases, this curve shifts to the right. If we carry out the thought experiment of fixing A and B and tracing out the unique equilibrium as the security capacity rises from nil to full security, the path that is traced out is precisely the curve $A\Upsilon(a) = B\Upsilon(b)$. Examples are shown in Figure 3 for the case of a high ($B = 2$) or low ($B = 0.9$) threat level; we note as a remark that these examples are representative.

Remark 1 *Fix B, and trace out the sequence of equilibria as security capacity rises from $\bar{f} = 0$ to $\bar{f} = 1$. If the threat level is mild ($B < A$), then a and b both start at 1. Exposure to viruses b declines uniformly toward 0 as $\bar{f}$ rises, while exposure to tips a first declines, then eventually rises toward 1. If the threat level is severe ($B > A$), then a and b both start at 0. Exposure to tips rises uniformly toward 1 as $\bar{f}$ rises, while exposure to viruses first rises then eventually falls toward 0.*

[17] In the knife edge case $B = 1$, the other end of the curve $\Upsilon(a) = B\Upsilon(b)$ terminates at $a = b = \bar{\upsilon}$.

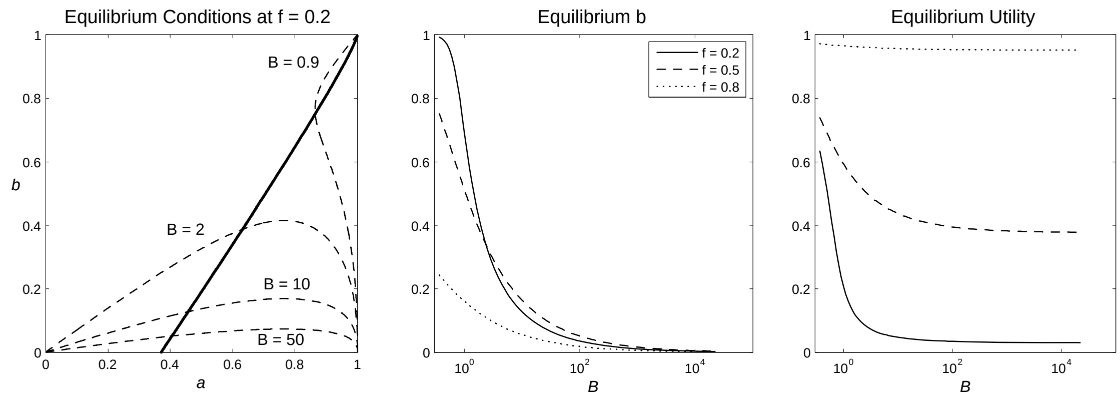


Figure 5: Comparative statics of a rising threat level B. Left: equilibrium condition (6) at $\bar{f} = 0.2$ (in bold) vs. condition (7) for various levels of B. ($A = 1$.) Middle and right: equilibrium b and utility vs. B.

In lieu of a proof, we offer an informal intuitive argument. The logic is simplest to convey in the other direction, as $\bar{f}$ declines. When perfect security is free, the economy must be at $(a, b) = (1, 0)$. From this point, agents initially absorb reductions in $\bar{f}$ by both reducing z and by accepting higher z_{eff}. At this stage, both $A\Upsilon(a)$ and $B\Upsilon(b)$ are rising, as each additional percentage reduction in good links, or increase in bad ones becomes more painful. As the security capacity continues to decline, eventually one of these two terms hits its maximum value $A\Upsilon(\bar{v})$ or $B\Upsilon(\bar{v})$ – the former if $A < B$, or the latter if $A > B$. Consider the first case, where $A < B$. At this point, accepting increases in z_{eff} is still increasingly painful as $\bar{f}$ falls, but accepting decreases in z becomes *less* painful. The reason is that z and a have become complements at equilibrium, so the fact that all other agents are reducing total links as $\bar{f}$ falls makes an individual's marginal links less valuable. Rather than continue to absorb reduced security by raising z_{eff} and reducing z, now it makes sense for an individual to begin to "buy back" reductions in z_{eff} with even larger reductions in z. Consequently, a and b begin to fall. To put the same point a different way, as a falls below $\bar{v}$, positive network effects begin to take over for total connectivity, so further declines in a tend to be reinforced.

Alternatively, if $A > B$, then as security capacity declines, b rises to $\bar{v}$ before a falls to $\bar{v}$. As $\bar{f}$ continues to decline, b and z_{eff} become complements, so further increases in b tend to make it *less* painful to accept additional unsecured links. The logic amounts to fatalism: for larger and larger b, one is so likely to catch the virus in any case that a few more links make little difference. In this case, it begins to make sense for an individual to buy back more total links z by accepting higher z_{eff}. As a result, both a and b begin to rise.

Put more simply, the economy responds to limited security differently depending on the threat level B. If B is low, agents opt to be highly connected in order to be sure to receive tips, implicitly giving up on trying to avoid viruses. However, if the threat level is high, agents emphasize avoiding viruses by remaining fairly isolated, implicitly accepting that this means they will rarely benefit from tips.

Comparative statics of a change in the threat level B

Next, we examine how equilibrium behavior changes as the threat level B rises, holding the security capacity $\bar{f}$ fixed. In this case, equation (6), which does not contain A or B, defines the set of states (a,b) that the economy traces out as B varies; examples are plotted in the left panel of Figure 5. For a threat level near zero, the economy tends toward a dense network in which all tips and viruses are received by all agents.[18] As the threat level rises, both aggregate exposure to tips and to viruses decline, driven by a decrease in individual links z. When the security capacity $\bar{f}$ is low, a and b decline more or less in tandem as B rises. However, when $\bar{f}$ is larger, equation (6) is more bowed away from the 45° line, so reductions in b are bought with less of a sacrifice in good connectivity a.

One interesting experiment is to study how the economy responds to the 'stress test' of taking the threat level B to infinity, holding A and $\bar{f}$ fixed; the middle and right panels of Figure 5 illustrate this. It is immediate from (5) that in equilibrium, b must tend to zero if $B \to \infty$. This implies that the number of unsecured links per agent, $z_{eff} = -\frac{\ln(1-b)}{b}$, tends to one as B grows. This in turn implies that $z \to \frac{1}{1-\bar{f}}$. In other words, the network survives the threat, with total connectivity falling just far enough to drive the infection rate b to zero. We are also concerned about how the individual equilibrium payoff of $\pi^{eq} = Aa^2 - Bb^2$ fares in this high threat limit. Let us write $\hat{z}$ and $\hat{a}$ for the limiting individual and aggregate connectivity, so $\hat{z} = -\frac{\ln(1-\hat{a})}{\hat{a}} = \frac{1}{1-\bar{f}}$. Although $B \to \infty$ and $b \to 0$, the expected harm term Bb^2 remains positive and finite. To evaluate it, use (5) to get

$$\lim_{B\to\infty} Bb^2 = \frac{A\hat{a}^2(1-\hat{a})}{1-\bar{f}} = A\hat{a}^2(1-\hat{a})\hat{z} = A\Upsilon(\hat{a}) \ .$$

An agent's limiting payoff is then

$$\hat{\pi} = A\left(\hat{a}^2 - \Upsilon(\hat{a})\right) = A\hat{a}\left(\hat{a} + (1-\hat{a})\ln(1-\hat{a})\right)$$

This payoff is strictly positive and increasing in $\hat{a}$.[19] In this sense, both the network and equilibrium payoffs are robust: no matter how severe the threat level, payoffs will never be driven below $\hat{\pi}$. Furthermore, $\hat{\pi}$ is increasing in $\bar{f}$ (because $\hat{z}$ and $\hat{a}$ increase in $\bar{f}$), so increasing the security capacity raises the lower bound on payoffs. To put this bound in perspective, note that in a threat-free world ($B \to 0$), a tends to one and payoffs tend to A. Thus, $1 - \frac{\hat{\pi}}{A} = 1 - \hat{a}^2 + \Upsilon(\hat{a})$ represents an upper bound on the fraction of these threat-free payoffs that a virus can destroy. For example, if agents' success rate at screening out viruses is $\bar{f} = \frac{1}{3}$, then viruses cannot erode more than $1 - \frac{\hat{\pi}}{A} \approx 87.3\%$ of the network's threat-free value. If screening improves to $\bar{f} = \frac{2}{3}$, this upper bound payoff improves to $1 - \frac{\hat{\pi}}{A} \approx 27.3\%$ of the threat-free value.

4.1.1 Welfare and Policy Implications

Our measure of welfare in a symmetric equilibrium will be an agent's payoff. Since the terms $\left(1 - e^{-a\tilde{z}}\right)$ and $\left(1 - e^{-b\left(1-\tilde{f}\right)\tilde{z}}\right)$ reduce to a and b at the equilibrium strategy $\left(\tilde{z}, \tilde{f}\right) = (z, f)$, welfare reduces to

$$\pi^{eq} = Aa^2 - Bb^2$$

[18] Of course an explicit cost to forming links would temper this result.

[19] One can see this by checking the derivative, or also by expanding $\ln(1-\hat{a})$ to get: $\hat{\pi} = A\hat{a}\left(\frac{\hat{a}^2}{2} + \frac{\hat{a}^3}{6} + \frac{\hat{a}^4}{12} + ... + \frac{\hat{a}^t}{t(t-1)} + ...\right)$.

We open the discussion of welfare with a simple policy question: if a social planner has the ability to improve the security technology available to individuals, should it do so? More precisely, will an increase in $\bar{f}$ improve equilibrium welfare?[20] Based on Remark 1, the answer is unambiguously yes if the initial level of security is sufficiently good. For $\bar{f}$ large enough, further increases in $\bar{f}$ will raise total connectivity a and reduce unsecured connectivity b (regardless of the relative size of A and B). If security is initially poor, the answer is less obvious – improving $\bar{f}$ will have at least one effect, either reducing exposure to tips a or increasing exposure to viruses b, that is deleterious to welfare. The latter case could be particularly embarrassing for a policymaker, as its pro-security intervention would appear to have an effect opposite to the one intended. The next result indicates that these concerns are of secondary importance.

Proposition 4 *The sign of $\frac{d\pi^{eq}}{d\bar{f}}$ is unambiguously positive. Thus, improvements in security technology always improve equilibrium welfare.*

Next, we consider a social planner who can enforce any symmetric strategy profile (z, f), subject to the technological constraint imposed by $\bar{f}$. One can think of the restriction to symmetric profiles as capturing some notion of equal treatment.[21] Clearly, the social planner will set each agent's security at $\bar{f}$, so the only question is the socially optimal link intensity z. The social planner's first order condition is

$$\frac{d\pi^{eq}}{dz} = MSB(z) - MSH(z) = 0$$

where $MSB(z) = 2Aa\frac{da}{dz}$ and $MSH(z) = 2Bb\frac{db}{dz}$ stand for marginal social benefit and harm respectively. For the purpose of comparison, write $MPB(z) = Aa^2(1-a)$ and $MPH = Bb^2(1-b)\left(1-\bar{f}\right)$ for the marginal private benefit or harm (evaluated at a symmetric strategy profile) that are the basis for individuals' decisions about z. Using the relationships $z = -\frac{\ln(1-a)}{a}$ and $z\left(1-\bar{f}\right) = -\frac{\ln(1-b)}{b}$, we can write

$$\begin{aligned} MSB(z) &= 2M(a) \cdot MPB(z) \\ MSH(z) &= 2M(b) \cdot MPH(z) \end{aligned}$$

where $M(x) = \frac{1}{1-\frac{1-x}{x}|\ln(1-x)|}$. The externality that separates the private and social benefits to additional links can be split into two pieces. First, there is an extra factor of two in the marginal social benefit; this reflects the fact that agents i and j each receive an identical expected benefit if a link forms between the two of them, but i only accounts for her own share of this gain. The second term $M(a)$ arises from the fact that an individual agent ignores the impact of her link decision on aggregate connectivity. The difference between the private and social harm terms has exactly the same structure. It may be instructive to note that the social benefit multiplier may also be written $M(a) = \frac{1}{1-z(1-a)} = 1 + z(1-a) + z^2(1-a)^2 + \ldots$, which would be the expected number of agents

[20] Of course, improving individuals' security technology may be costly for the government; those costs are not considered here.

[21] This is somewhat imprecise, as an asymmetric strategy profile could still give all agents the same payoff.

One naturally wonders how a social planner would allocate links and security if the symmetry restriction were lifted. (Of course the social welfare function would need to be extended.) This question appears to be substantially more difficult, and we do not have results on it at this time.

in agent i's neighborhood (including herself) if each agent were to form (in expectation) $z(1-a)$ onward links, with no overlaps. Standard results in branching theory suggest that $M(a)$ can be interpreted as agent i's expected neighborhood size, conditional on not linking to the hub.[22] In this interpretation, the multiplier reflects the number of agents who would benefit if a marginal link by i turns out to be successful. The social harm multiplier, which may be written $M(b) = \frac{1}{1-z(1-\bar{f})(1-b)}$, can be given a similar interpretation.

It is not difficult to confirm that $M(x)$ is decreasing on $(0,1)$, so we have $M(a) < M(b)$: the negative externality exceeds the positive one. There are two, complementary ways to think about this. One is to appeal again to the relationship between individual and aggregate connectivity illustrated in Figure **??**. Additional individual connections z have the greatest aggregate impact when a and b are positive but small. Because of security, b always lies further to the left on this curve than a does, where the marginal impact of z is larger. A second rationale, based on the neighborhood size interpretation, is that the expected size of a neighborhood that fails to connect to the hub is larger in the virus sub-network than in the tip network. This is true, although perhaps counterintuitive, and has to do with the conditioning: in the better-connected tip network, isolated neighborhoods tend to pick up a connection into the hub before they ever grow too large. As aggregate connectivity is just emerging, the externality is particularly acute: $M(x)$ tends to infinity as x goes to zero. In contrast, $M(x)$ tends to 1 as x goes to 1, so in a densely connected network, this component of the externality disappears.

If we write z^{eq} for the equilibrium level of links, we have $MPB(z^{eq}) = MPH(z^{eq})$, and therefore, $MSB(z^{eq}) < MSH(z^{eq})$. The next proposition follows immediately.

Proposition 5 *Fix security capacity $\bar{f}$. Suppose that $(z^{opt}, \bar{f})$ maximizes welfare over all symmetric strategy profiles, while z^{eq} is the symmetric equilibrium link intensity. Then $z^{opt} < z^{eq}$. That is, agents are more connected in equilibrium than would be socially optimal.*

Before moving on, it is worth making one final remark about the expression for the lower bound equilibrium payoff $\hat{\pi}$ derived in the last section. We argued that as B rises, individual incentives tend to drive b to zero roughly at rate $\frac{1}{\sqrt{B}}$. It is not hard to show that the social optimum involves driving b to zero more quickly – at rate $\frac{1}{B}$ – so that $Bb^2 \to 0$. This is accomplished by taking individual and total connectivity to the same limits as before as B rises, $\hat{z}$ and $\hat{a}$, but more quickly. In the $B \to \infty$ limit, socially optimal welfare is therefore $A\hat{a}^2$. Meanwhile, notice that the limiting equilibrium payoff may be written $\hat{\pi} = \frac{A\hat{a}^2}{M(\hat{a})}$. Thus, even though equilibrium payoffs can withstand high threat levels fairly successfully, a social planner could do substantially better by tweaking behavior only slightly (since $z^{opt} \to z^{eq} \to \hat{z}$ as $B \to \infty$). Moreover, the factor by which payoffs could be improved, $M(\hat{a})$, is directly related to the uninternalized effect of an agent's links on aggregate connectivity.

4.2 Case 2: Elastic, Link-Independent Security Costs

The assumption that agents cannot improve security above $\bar{f}$ at any price is unattractive, and in this section we relax it. Throughout the section, security costs are assumed to take the following form

$$C(z, f) = c(f) \tag{8}$$

[22] See, e.g., Durrett (2007).

where c is an increasing, convex function on $[0, \infty)$ with $c(0) = 0$. Since this cost function does not include z, it belongs to the class of zero MC_{link} functions discussed earlier. The natural examples of zero MC_{link} costs are investments like antivirus software or a flu shot, where a screening technology of quality f can screen arbitrarily many contacts at no additional cost. The necessary conditions for an interior equilibrium constrain the relationship between a and b just as earlier, but there is an additional constraint involving the marginal cost of security. Now an individual has two instruments for avoiding viruses – decreasing z or increasing f – and in equilibrium their marginal utility costs must be equalized.

Proposition 6 *Given aggregate connectivity $(a, b) \in (0, 1)^2$, an interior best response $\left(\hat{z}, \hat{f}\right)$ for an individual agent must satisfy the first order conditions*

$$Aa^2 \hat{z} e^{-a\hat{z}} = \left(1 - \hat{f}\right) c'\left(\hat{f}\right) = Bb^2 \hat{z}_{eff} e^{-b\hat{z}_{eff}} \tag{9}$$

(where $\hat{z}_{eff} = \left(1 - \hat{f}\right) \hat{z}$). If (a, b) represents an interior symmetric equilibrium, then (9) implies that the following conditions hold (where $f = 1 - \frac{a}{b} \frac{\ln(1-b)}{\ln(1-a)}$)

$$A\Upsilon(a) = (1 - f) c'(f) = B\Upsilon(b) \ . \tag{10}$$

Once again, any equilibrium must lie along the curve defined by $A\Upsilon(a) = B\Upsilon(b)$. Equation (10) offers some insight about why existence and uniqueness of an interior equilibrium are less straightforward here than they were in the previous section. The middle term can be written $(1 - f) c'(f) = \frac{d}{d\delta} c(f + \delta(1 - f))\big|_{\delta=0}$; this is the marginal cost of increasing security so as to reduce one's exposure $(1 - f)$ by a small percentage. The outer terms reflect the marginal benefit of that security investment, in a symmetric strategy profile, depending on whether those benefits are taken in the form of lower exposure to viruses ($B\Upsilon(b)$) or by increasing exposure to tips while holding virus exposure constant ($A\Upsilon(a)$).

There are several issues to address. First, a conventional regularity assumption would be to set $c'(0) = 0$, but here that is not enough to guarantee a positive level of security in equilibrium. To see where the problem arises, recall from the previous section that (given $A\Upsilon(a) = B\Upsilon(b)$) a low security equilibrium must have a and b both near zero or both near one. If $c'(0) = 0$, the cost of a small security investment by all agents will be negligible for each agent, but the marginal benefit it induces will also be near zero. For a and b small, the positive network externality (which causes $A\Upsilon(a)$ to scale up or down with a, and $B\Upsilon(b)$ with b) is to blame. For a and b near 1, network saturation is to blame – a small change in f has little effect on connectivity. If $c'(f)$ does not tend to zero quickly enough with f, then there will be values of A and B for which no interior equilibrium can be sustained. The same basic factors (increasing marginal benefits on some range, paired with insufficiently convex marginal costs) can give rise to multiple interior equilibria.

At the other end of the spectrum, if $(1 - f) c'(f)$ is bounded above, then for A and B large enough agents will be attracted to full security: $f = 1$. However, this implies that any equilibrium would need to be asymmetric. (If all agents chose $f = 1$, we would have $b = 0$, obviating the need for security in the first place.) The simplest condition that ensures an interior equilibrium choice $f < 1$, is substantially stronger than convexity of c:

C $(1 - f) c'(f)$ is increasing (equivalently, $\frac{c''(f)}{c'(f)} > \frac{1}{1-f}$) and tends to ∞ as $f \to 1$.

With these issues in mind, we take a Second Welfare Theorem approach to equilibrium existence: we show that every outcome along the curve $A\Upsilon(a) = B\Upsilon(b)$ is supported as an equilibrium by some sufficiently convex cost function. To formalize 'sufficiently convex,' we imagine tilting the marginal cost function about some pivot point f^*.

Definition 2 *Fix a convex cost function $c(f)$. An alternative cost function $c_{m,f^*}(f)$, called the m-tilt of c about f^*, is defined by $c'_{m,f^*}(f) = r(f)\,c'(f)$, where $r(f) = \left(\frac{c'(f)}{c'(f^*)}\right)^m$.*

Then we have:

Proposition 7 *Suppose that $(a^*, b^*) \in \left\{(0,1)^2 : b^* < a^*\right\}$ satisfies $A\Upsilon(a^*) = B\Upsilon(b^*)$. Let $f^* = 1 - \frac{a^*}{b^*}\frac{\ln(1-b^*)}{\ln(1-a^*)}$. Then,*

(i) There exists some increasing cost function $c_1(f)$, satisfying $c_1(0) = c'_1(0) = 0$, c'_1 strictly convex, and Condition C, for which the first order condition (10) holds at (a^*, b^*, f^*).[23]

(ii) Let c_m be the m-tilt of c_1 about f^*. Then for m sufficiently large, (a^*, b^*, f^*) constitutes an interior symmetric equilibrium for the cost function c_m.

One can interpret this as a relaxation of Proposition 3 which guarantees existence of a unique interior equilibrium if security choices are perfectly inelastic. Proposition 7 shows that an equilibrium exists if security choices that are sufficiently inelastic, without guaranteeing uniqueness.

4.2.1 Comparative Statics and Network Resilience

Comparative statics of a change in security costs

In order to discuss changes in the cost of security, we temporarily write the cost with a scale factor k:

$$C(z, f) = kc(f) \ .$$

If we fix the shape of the cost function $c(f)$, then the model has only three parameters, A, B, and k. Moreover, equilibrium behavior is determined entirely by the two ratios A/k and B/k. In this section, we examine how equilibrium behavior, and agents' welfare, respond to changes in those parameters. To do this, we exploit the fact that the equilibrium conditions (10) permit us to treat a change in any one of these parameters as a shift along a curve defined by the other two parameters.

First consider a shift in the cost of security, holding A and B fixed. From (10), we can see that changes in k simply shift the economy along the curve $A\Upsilon(a) = B\Upsilon(b)$. It is not hard to show that the direction of this shift is unambiguous – an increase in k has the same qualitative effect that a reduction in the security capacity $\bar{f}$ had in the previous section. In both cases, the aggregate effect of limiting individual security is to move the economy away from the bliss point at $(a, b) = (1, 0)$ and toward the no-security line $b = a$. As earlier, a connected and secure economy ((a, b) near $(1, 0)$) absorbs higher security costs both by using reduced connectivity as a security substitute (a falls) and by accepting a higher infection rate (b rises). If costs continue to rise, the

[23] A suitable c_1 can always be found. One example is

$$c_1(f) = k\left(\frac{1}{1-f} + \ln(1-f) - 1\right) = k\left(\frac{1}{2}f^2 + \frac{2}{3}f^3 + \frac{3}{4}f^4 + \ldots\right)$$

with the constant k chosen to satisfy (10).

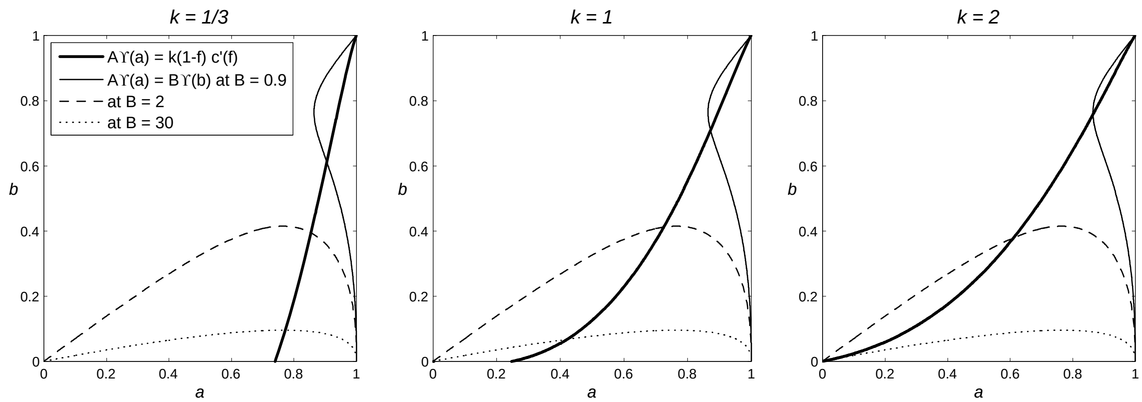


Figure 6: Equilibrium conditions with security cost $C(f) = kc(f)$, $c(f) = f^3/(1-f)$. (Condition (11) in bold. Condition $A\Upsilon(a) = B\Upsilon(b)$ shown for $B = .9, 2, 30$. In each case, $A = 1$.)

equilibrium response eventually depends on whether the harm from viruses exceeds the benefit from tips ($B \gtreqless A$). If they do ($B > A$), then as k rises, agents continue to shed links as a substitute for security (a falls), and eventually this effect dominates the reduction in f, pushing down the infection rate (b falls). However, if the harm from viruses is less severe ($B < A$), then increases in k continue to be absorbed as a rise in infections (b rises). Eventually b is large enough that shedding links becomes an ineffective security substitute; at this point, further increases in k begin to push total connectivity back up (a rises).

Comparative statics with respect to the threat level B and network resilience

Next, consider the exercise of raising the threat level B, while A and k are fixed. To conserve on notation, we absorb k back into the cost function, so we return to $C(z, f) = c(f)$. In the inelastic security case, we used the fixed level of security $1 - \bar{f} = \frac{a}{b}\frac{\ln(1-b)}{\ln(1-a)}$ to identify the (a, b) pairs that could represent an equilibrium for some value of B. Here, in place of (6), we have a relationship that balances marginal benefits and costs of security: any equilibrium must lie on the curve

$$A\Upsilon(a) = k(1-f)c'(f)\Big|_{1-f=\frac{a}{b}\frac{\ln(1-b)}{\ln(1-a)}} \tag{11}$$

regardless of B. Examples of the curve defined by (11) are sketched in bold in Figure 6. Because equilibrium condition (7) has not changed from the inelastic security case, the effect on equilibrium of making security more flexible can begin to be understood by looking at the change from (6) to (11). For comparison, let us write (6) (equivalently but less succinctly) as $k\left(1-\bar{f}\right)c'\left(\bar{f}\right) = k(1-f)c'(f)\big|_{1-f=\frac{a}{b}\frac{\ln(1-b)}{\ln(1-a)}}$. Both (6) and (11) tend to $(a, b) = (1, 1)$ as a approaches 1. Now consider some other arbitrary point $\left(\hat{a}, \hat{b}\right)$ lying on (6). Starting from $\left(\hat{a}, \hat{b}\right)$, do we need to shift to the right or to the left to reach curve (11)? The answer depends on the equilibrium returns to marginal connectivity $A\Upsilon(\hat{a})$. If $A\Upsilon(\hat{a})$ is too large and $\hat{a} > \bar{v}$, then individuals have the incentive to form more links and sterilize this increase with an increase in security. This tends to shift a to the right; because we are in the negative feedback region where $A\Upsilon'(a) < 0$, a sufficient shift to the right will bring the marginal benefits from links and the marginal cost of security into balance, satisfying (11).

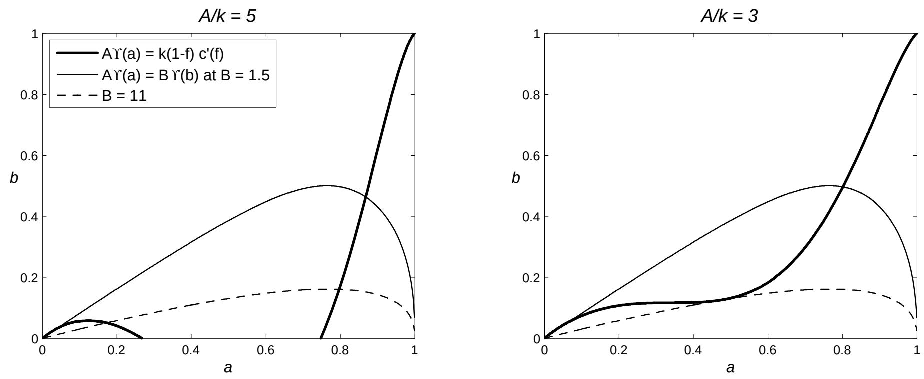


Figure 7: Non-resilience with moderately convex security costs. $C(f) = kf^2/(1-f)$.

Alternatively, suppose instead that the marginal returns to connectivity $A\Upsilon(\hat{a})$ are too low to satisfy (11) at $\left(\hat{a}, \hat{b}\right)$ and that $\hat{a} < \bar{v}$. Then individuals have the incentive to cut back on both links and security. (In essence, at $\left(\hat{a}, \hat{b}\right)$ it is less painful to cut virus exposure by cutting z than by maintaining f.) At the individual level this makes sense, since *holding a fixed*, the individual return to z, $A\hat{a}^2 e^{-\hat{a}z}$, rises as z is reduced. However, here we are in the region of positive network effects, so when *all* individuals cut links, thereby reducing a, the individual return to additional links actually falls further. In order to satisfy (11), a and f must fall sufficiently far that declining marginal security costs can catch up with the downward spiral of returns to connectivity. If the security cost function is not sufficiently convex, this may not be possible.

A complementary intuition for this last case is the following. When security is free up to a limit $\bar{f}$, individuals do not scale back on security even as the prevalence of the virus b declines. In contrast, when security costs are more elastic, individuals respond to a reduction in b by economizing on f, and this countervailing reaction can prevent b from falling too rapidly.

Figure 6 illustrates these possibilities for the cost function $c(f) = k\frac{f^3}{1-f}$.[24] For low k, equation (11) is close to linear, but for higher k, it becomes very shallow as a and b fall, reflecting the fact that total connectivity must fall quite a long way before marginal security costs are worth bearing. As the figure illustrates, the equilibrium comparative statics of an increase in the threat level B are loosely similar to what we saw with inelastic security: as B rises, holding other parameters fixed, both total and unsecured connectivity decline, with the latter tending to zero as $B \to \infty$. However, there is an important difference. In the first two panels with lower cost parameters, the network survives as $B \to \infty$, in the sense that a tends to a positive limit. In the last case, it does not: both a and b vanish as $B \to \infty$. This outcome was not possible under a positive and inelastic security capacity $\bar{f}$, but here agents' incentives to substitute isolation for security effort lead the network to dissolve. Motivated by Figure 6, we introduce the concept of resiliency.

Definition 3 *An economy is resilient (for particular A and cost function $C(f)$) if there is a sequence of symmetric equilibria in which total connectivity and payoffs remain positive and bounded away from zero as the threat level B goes to infinity.*

In a non-resilient economy the network may wither away smoothly with B, as in Figure 6,

[24] The denominator is chosen to ensure that $c'(f)$ and $(1-f)c'(f)$ grow unboundedly large as $f \to 1$.

or there may be a threshold threat level at which interior symmetric equilibria suddenly vanish. Figure 7 shows the latter possibility for costs $C(f) = k\frac{f^2}{1-f}$. If costs are low relative to network benefits ($A/k = 5$), then the economy is resilient: connectivity tends toward $a \approx 0.75$ and payoffs tend toward approximately $0.38A$ as B rises.[25] However, if benefits are smaller relative to costs ($A/k = 3$), then near $B \approx 11$, the interior equilibrium collapses: connectivity falls from $a \approx 0.49$ to zero, while payoffs drop from roughly $0.09A$ to zero.[26] Our last result shows that these are examples of a more general relationship between resilience and the cost of security.

Proposition 8 *Fix an increasing, convex, analytic cost function $C(f) = kc(f)$ satisfying $c(0) = 0$ and $(1-f)c'(f)$ increasing, and $k > 0$.*

i) If $c'(0) = c''(0) = c'''(0)$, then for any $A > 0$, the economy is resilient.

ii) If at least one of the derivatives $c'(0)$, $c''(0)$, and $c'''(0)$ is strictly positive, then there is some threshold $\bar{L} > 0$ such that such that the economy is not resilient if $A/k < \bar{L}$.

5 Concluding Remarks

We develop a model of network formation and security in large decentralized populations and characterize some of its important features. Among other results, we show conditions under which equilibrium networks are overconnected (because negative externalities dominate positive ones at equilibrium) and improvements in security technology induce equilibrium infection rates to get worse.

While the relative simplicity of the model – aside from the shape of costs, there are only two free parameters – makes it a useful testbed, there are many ways that it could be extended or made more realistic. One issue is the random graph assumption. While this is a convenient modeling metaphor for the idea that agents are too poorly informed about the network structure to fine-tune whom they connect to, it fails in two respects. First, it is too pessimistic about agents' information. A natural relaxation would be to assume that an agent can see (and use in his decisions) some information about a potential direct connection – such as the number of additional links he has – but cannot observe connections that are more than one stage removed. The second failure of the random graph assumption is that it cannot generate the high level of clustering that is a feature of most known human networks. One way to improve this mismatch might involve letting the hub play a real role in the model, rather than a merely instrumental one. If there were many small to moderate-sized hubs, each one could act as a type of local nucleus where people congregate, with natural interpretations in terms of clubs, internet forums, restaurants and so forth. Or, in the context of network security, if we interpret the hub as a large interested party such as Google or the Centers for Disease Control as suggested earlier, there may be interesting crowd-in or crowd-out interactions between initiatives taken by the large party and the incentives of the smaller ones.

There are also a number of interesting alternative assumptions about payoffs to explore. Besides the obvious fact that the creators of a virus could be promoted to full strategic status, one could look at notions of benefits or harm that decay with distance from the originating agent. A different

[25] The figure also illustrates the fact that for insufficiently convex costs, there may be multiple solutions to the necessary conditions for equilibrium. We anticipate that at least some of these intersections are also equilibria but have not confirmed this.

[26] One caveat is that we have not ruled out the possibility that the economy switches to an asymmetric equilibria and some degree of network connectivity survives; additional work is required to clarify this.

assumption that has some real-world resonance would be that benefits derive mainly from direct contacts, but threats and viruses can skip across the whole network as in our model.

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6 Proofs

Proposition 1

Proof. Write $\mathcal{A}^{nh}(\tilde{z};\mathbf{z}_{-1}) = \Pr^{nh}\left(1 \leftrightarrow 0 \,|\, \tilde{z}, \mathbf{z}^{n}_{-1}\right)$ for the probability that agents 0 and 1 communicate in a nh-economy. Henceforth we suppress conditioning on $(\tilde{z};\mathbf{z}_{-1})$ in the notation. We can write

$$\overset{nh}{\Pr}(1 \leftrightarrow 0) = \overset{nh}{\Pr}(E_{01+H}) + \overset{nh}{\Pr}(E_{01-H})$$

where E_{01+H} and E_{01-H} are the events

$$E_{01+H}:\ 1 \leftrightarrow H \cap 0 \leftrightarrow H \quad , \quad E_{01-H}:\ 1 \leftrightarrow 0 \cap 1 \not\leftrightarrow H \cap 0 \not\leftrightarrow H$$

That is, if 0 and 1 communicate, then they are either both connected to the hub, or neither of them is. The proof proceeds though a series of lemmas. First, Lemmas 1 and 2 establish that $\lim_{n\to\infty} \Pr^{nh}(E_{01-H}) = 0$. That is, in a large h-economy, agents 0 and 1 never communicate with each other unless each is also connected to the hub. Next, Lemma 3 establishes that the events $1 \leftrightarrow H$ and $0 \leftrightarrow H$ are asymptotically independent, so we have $\lim_{n\to\infty} \Pr^{nh}(E_{01+H}) = X_0^h X_1^h$, where $X_0^h = \lim_{n\to\infty} X_0^{nh}$ and $X_1^h = \lim_{n\to\infty} X_1^{nh}$ are the agents' limiting probabilities of connecting to H:

$$X_0^{nh} = \overset{nh}{\Pr}(0 \leftrightarrow H) \quad , \quad X_1^{nh} = \overset{nh}{\Pr}(1 \leftrightarrow H)$$

Lemma 4 characterizes X_0^h and X_1^h using the branching process approximation described in the text. We have

$$1 - X_0^h = (1-h)\, e^{-zX_0^h} \quad \text{and} \quad 1 - X_1^h = (1-h)\, e^{-\tilde{z}X_0^h}$$

For any hub size $h > 0$, the first equation has a unique solution with $X_0^h \in (0,1)$. (There is also an irrelevant strictly negative solution.) Then, X_1^h is determined by X_0^h (and $\tilde{z}$). The final step is to take $\mathcal{A}(\tilde{z};\mathbf{z}_{-1}) = \lim_{h\to 0} X_0^h X_1^h$. Showing that X_0^h converges to zero if $z \le 1$ and to a if $z > 1$ completes the proof. To show this, let $X_0 = \lim_{h\to 0} X_0^h$. By continuity, X_0 must be a solution of

$$1 - x = e^{-zx}\ .$$

One solution is obviously at $x = 0$. If $z = 1$, this is the unique solution; otherwise there is exactly one additional solution which is positive (negative) if $z > 1$ ($z < 1$). (If one plots the equation, this is obvious. To be a bit more careful, note that the function $\nu(x) = 1 - x - e^{-zx}$ is strictly concave, approaches $-\infty$ as $x \to \infty$ or $x \to -\infty$, and satisfies $\nu(0) = 0$ and $\nu'(0) = z - 1$. Thus if $z - 1 > 0$, $\nu(x) = 0$ has exactly one additional positive solution, and conversely if $z < 1$.) This immediately implies (since $X_0^h > 0$ for all $h > 0$) that $X_0 = 0$ if $z \le 1$.

Finally, suppose that $z > 1$. In this case, X_0^h must converge either to zero, or to the positive solution a. But Lemma 5 shows that X_0^h can be bounded away from zero uniformly for all h. Thus, we must have $X_0 = a$, as claimed. ■

Lemma 1 *Let S_i be the set of agents who are reachable from agent i, including i herself, without using any links to or from H. Let Lg_i (respectively Sm_i) be the event $|S_i| > \log n$ (respectively $|S_i| \leq \log n$). Then* $\lim_{n\to\infty} \Pr^{nh}\left(i \nleftrightarrow H \,\cap\, Lg_i \,|\, \tilde{z}, \mathbf{z}_{-1}^{n}\right) = 0.$

Proof. Here and elsewhere, we suppress the conditioning on $\tilde{z}, \mathbf{z}_{-1}^{n}$ in the notation.Write $\Pr^{nh}(i \nleftrightarrow H \,\cap\, Lg_i) = \Pr^{nh}(i \nleftrightarrow H \,|\, Lg_i)\Pr^{nh}(Lg_i) \leq \Pr^{nh}(i \nleftrightarrow H \,|\, Lg_i)$. The event $i \nleftrightarrow H$ is equivalent to $\eta_j = 0$ for all $j \in S_i$, so we have

$$\overset{nh}{\Pr}(i \nleftrightarrow H \,|\, Lg_i) \leq (1-h)^{\log n} = n^{\log(1-h)}$$

Since $\log(1-h)$ is negative, the final expression converges to 0 with n, which suffices to prove the lemma. ■

Lemma 2 *Let E_{01-H} be the event $1 \leftrightarrow 0 \cap 1 \nleftrightarrow H \cap 0 \nleftrightarrow H$. Then* $\lim_{n\to\infty} \Pr^{nh}\left(E_{01-H} \,|\, \tilde{z}, \mathbf{z}_{-1}^{n}\right) = 0$.

Proof. Write $\Pr^{nh}(E_{01-H}) = \Pr^{nh}(E_{01-H} \,\cap\, Sm_1) + \Pr^{nh}(E_{01-H} \,\cap\, Lg_1)$ and note that $\Pr^{nh}(E_{01-H} \,\cap\, Lg_1)$ tends to zero with n by Lemma 1. For the first term, observe that

$$\overset{nh}{\Pr}(E_{01-H} \,\cap\, Sm_1) = \overset{nh}{\Pr}(E_{01-H} \,|\, Sm_1)\overset{nh}{\Pr}(Sm_1) \leq \overset{nh}{\Pr}(E_{01-H} \,|\, Sm_1) \leq \overset{nh}{\Pr}(0 \in S_1 \,|\, Sm_1)$$

Given the symmetric strategy profile for all agents other than 1, agent 0 has the same chance as any other agent of belonging to S_1, so $\Pr^{nh}(0 \in S_1 \,|\, Sm_1) \leq \frac{\log n-1}{n-1}$. Thus, $\Pr^{nh}(E_{01-H} \,\cap\, Sm_1)$ also tends to zero with n, completing the proof. ■

Lemma 3 $\lim_{n\to\infty} \left|\Pr^{nh}\left(1 \leftrightarrow H \,\cap 0 \leftrightarrow H \,|\, \tilde{z}, \mathbf{z}_{-1}^{n}\right) - \Pr^{nh}\left(1 \leftrightarrow H \,|\, \tilde{z}, \mathbf{z}_{-1}^{n}\right)\Pr^{nh}\left(0 \leftrightarrow H \,|\, \tilde{z}, \mathbf{z}_{-1}^{n}\right)\right| = 0.$

Proof. We will prove the equivalent statement

$$\lim_{n\to\infty}\left|\overset{nh}{\Pr}(1 \nleftrightarrow H \,\cap 0 \nleftrightarrow H) - \overset{nh}{\Pr}(1 \nleftrightarrow H)\overset{nh}{\Pr}(0 \nleftrightarrow H)\right| = 0\ .$$

Let E_i be the event $i \nleftrightarrow H \cap Sm_i$ for $i = 0, 1$. By Lemma 1, it suffices to prove that

$$\left|\overset{nh}{\Pr}(E_0 \cap E_1) - \overset{nh}{\Pr}(E_0)\overset{nh}{\Pr}(E_1)\right|$$

tends to zero with n. Furthermore, since we have $\Pr^{nh}(E_0 \cap E_1) = \Pr^{nh}(E_0 \,|\, E_1)\Pr^{nh}(E_1)$, it is enough to show that $\Pr^{nh}(E_0 \,|\, E_1)$ converges to $\Pr^{nh}(E_0)$. To show this, construct independent copies $\left(\zeta'_{ij}, \eta'_j\right)$ of all of the link random variables, and let $\mathcal{G}'$ be the graph generated by these variables. Let $\mathcal{G}''$ be the graph generated by using:

$$\begin{aligned} &\zeta_{ij} \text{ if } i \ \in\ S_1 \text{ or } j \in S_1 \ ;\ \text{otherwise, use } \zeta'_{ij} \\ &\eta_j \text{ if } j \ \in\ S_1 \ ;\ \text{otherwise use } \eta'_j \end{aligned}$$

That is, $\mathcal{G}''$ coincides with the original graph $\mathcal{G}$ for links to and from agents who are in S_1 but follows the copies for other links. Define S'_0, S''_0, Sm'_0, and Sm''_0 for $\mathcal{G}'$ and $\mathcal{G}''$ just as S_0 and Sm_0

were defined for $\mathcal{G}$, and let E_0' and E_0'' be the events corresponding to E_0. By construction, graphs $\mathcal{G}$ and $\mathcal{G}'$ are independent, so we have

$$\overset{nh}{\Pr}\left(E_0' \,|\, E_1\right) = \overset{nh}{\Pr}\left(E_0'\right) = \overset{nh}{\Pr}\left(E_0\right) \ .$$

We also have $E_1'' = E_1$ and

$$\overset{nh}{\Pr}\left(E_0'' \,|\, E_1\right) = \overset{nh}{\Pr}\left(E_0 \,|\, E_1\right) \ .$$

so to establish the lemma, it suffices to show that $\Pr^{nh}\left(E_0' \,|\, E_1\right) - \Pr^{nh}\left(E_0'' \,|\, E_1\right)$ tends to zero with n. Let $\hat{S}_0'$ be the set of agents who are reachable in $\mathcal{G}'$ from agent 0, including herself, when links to or from H and links to or from agents in S_1 are excluded. Define $\hat{S}_0''$ similarly for $\mathcal{G}''$, and note that the sets $\hat{S}_0'$ and $\hat{S}_0''$ are identical; write $\hat{S}_0 = \hat{S}_0' = \hat{S}_0''$. Whenever there are no links between the sets $\hat{S}_0$ and S_1 in either $\mathcal{G}'$ or $\mathcal{G}''$, the events E_0' and E_0'' must coincide, as they are determined both determined by the same set of (copy) random variables. Furthermore, whenever $\left|\hat{S}_0\right| > \log n$, the chance of either E_0' or E_0'' must be each tending to zero with n by Lemma 1. Thus,

$$\begin{aligned}
\left|\overset{nh}{\Pr}\left(E_0' \,|\, E_1\right) - \overset{nh}{\Pr}\left(E_0'' \,|\, E_1\right)\right| &\leq \overset{nh}{\Pr}\left(\left(\bigcup_{(i,j)\in\hat{S}_0\times S_1}\left(\zeta_{ij}' = 1 \cup \zeta_{ij} = 1\right)\right) \cap \left|\hat{S}_0\right| \leq \log n \,|\, E_1\right) + o\left(1\right) \\
&\leq \overset{nh}{\Pr}\left(\bigcup_{(i,j)\in\hat{S}_0\times S_1}\left(\zeta_{ij}' = 1 \cup \zeta_{ij} = 1\right) \,|\, E_1 \cap \left|\hat{S}_0\right| \leq \log n\right) + o\left(1\right) \\
&\leq \overset{nh}{\Pr}\left(\bigcup_{(i,j)\in\hat{S}_0\times S_1}\left(\zeta_{ij}' = 1\right) \,|\, E_1 \cap \left|\hat{S}_0\right| \leq \log n\right) \\
&\quad + \overset{nh}{\Pr}\left(\bigcup_{(i,j)\in\hat{S}_0\times S_1}\left(\zeta_{ij} = 1\right) \,|\, E_1 \cap \left|\hat{S}_0\right| \leq \log n\right) + o\left(1\right) \\
&\leq 2\delta\frac{\left(\log n\right)^2}{n} + o\left(1\right)
\end{aligned}$$

where we define $\delta = \frac{\max(z,\tilde{z})^2}{\min(z,\tilde{z})}$. The first line simply defines the probability of a link between the sets $\hat{S}_0$ and S_1 in either $\mathcal{G}'$ or $\mathcal{G}''$. The second step applies the fact that $\Pr\left(x \cap y\right) = \Pr\left(x \,|\, y\right)\Pr\left(y\right) \leq \Pr\left(x \,|\, y\right)$. The third step uses the inclusion-exclusion principle. The last step uses the inclusion-exclusion principle again, along with the conditioning on both sets having at most $\log n$ elements and the fact that the probability of a direct link between any particular pair of agents is bounded above by $\frac{\delta}{n}$. The final expression tends to zero as $n \to \infty$, and this completes the proof. ■

Lemma 4 *Let $X_0^{nh} = \Pr^{nh}\left(0 \leftrightarrow H \,|\, \tilde{z}, \mathbf{z}_{-1}^n\right)$ and $X_1^{nh} = \Pr^{nh}\left(1 \leftrightarrow H \,|\, \tilde{z}, \mathbf{z}_{-1}^n\right)$ as in Proposition 1. Define $\hat{Y}_0^h$ as the positive solution to*

$$\hat{Y}_0^h = \left(1-h\right)e^{-z\left(1-\hat{Y}_0^h\right)} \ .$$

Then

$$\lim_{n\to\infty} X_0^{nh} = 1 - \hat{Y}_0^h \quad \textit{and}$$
$$\lim_{n\to\infty} X_1^{nh} = 1 - (1-h)\, e^{-\tilde{z}\left(1-\hat{Y}_0^h\right)} \; .$$

Proof. As discussed in the text, we proceed by deriving the complementary probability of $0 \nleftrightarrow H$. We omit a derivation of the probability of $1 \nleftrightarrow H$, as the arguments involved are identical to the $0 \nleftrightarrow H$ case.

Let $\left(\hat{R}_t, \hat{W}_t\right)$ be the branching process described in the text. We will couple $\left(\hat{R}_t, \hat{W}_t\right)$ to a second stochastic process (R_t, W_t) such that (i) the second process must eventually die out at some generation, say T, (ii) R_T correctly enumerates the members of S_0, and (iii) $\left|\hat{R}_t\right| \geq |R_t|$ and $\left|\hat{W}_t\right| \geq |W_t|$ for all $t \geq 0$.

Let ζ_{ij} refer to the true random variables that generate $\mathcal{G}$, and let $\hat{\zeta}_{ij}$ be an independent copy of ζ_{ij}, with the understanding that every reference to $\hat{\zeta}_{ij}$ implies that a new independent copy is drawn. When we need to refer to an arbitrary individual of type i, we write $i_\#$. Put $\hat{R}_0 = \emptyset$ and $\hat{W}_0 = \{0_\#\}$ as in the text. We generate $\hat{W}_{t+1}$ as described in the text, with the following provisos. Suppose that every individual in $\hat{R}_t \cup \hat{W}_t$ has been tagged as either legitimate or illegitimate. In particular, the unique member of $\hat{W}_0$ is legitimate. For each parent $i_\# \in \hat{W}_t$ and each type j of potential child that $i_\#$ could generate :

If $i_\#$ is illegitimate or if there is a legitimate type j individual in $\hat{R}_t \cup \hat{W}_t$, then
- Generate the type j child according to an independent copy $\hat{\zeta}_{ij}$.
- If the child is created, tag her as illegitimate.

If $i_\#$ is legitimate and there is no legitimate type j individual in $\hat{R}_t \cup \hat{W}_t$, then
- Generate the type j child according to ζ_{ij}.
- If the child is created, leave her untagged for the moment.

After this step, the members of set $\hat{W}_{t+1}$ have been created. Next return to the untagged members of $\hat{W}_{t+1}$. For each type j, rank the untagged type j individuals in $\hat{W}_{t+1}$ by parent's type, from lowest to highest. Declare the first individual in this list to be legitimate (if she exists) and any other individuals to be illegitimate.

Define $R_0 = \emptyset$, $R_{t+1} = R_t \cup W_t$, and $W_t = \left\{ i_\# \,|\, i_\# \in \hat{W}_t \text{ and } i_\# \text{ is legitimate} \right\}$. Clearly property (iii) holds by construction. Also by construction, at most one individual of each type is ever tagged as legitimate. This implies that R_t contains at most one individual of each type, and that (R_t, W_t) must die off within at most n generations (so property (i) holds). Finally, we claim that the types of the individuals in R_T enumerate the agents in S_0. By construction, W_1 contains the first order small neighbors of agent 0. At each subsequent step W_t generates (according to the probabilities ζ_{ij}) those t^{th} order small neighbors of 0 who have not been counted at an earlier step. If we reach agent j simultaneously along several separate length t paths, then we only add him to W_t once (arbitrarily keeping the copy with the lower ranking parent.)

Finally, endow each individual in $\hat{W}_t$ with an i.i.d. binary random variable with probability of success h. Specifically, for each individual $j_\# \in \hat{W}_t$, assign $\eta_{j_\#} = \eta_j$ if $j_\#$ is legitimate or use independent copy $\eta_{j_\#} = \hat{\eta}_j$ if $j_\#$ is illegitimate. Likewise, individual $j_\# \in W_t$ gets the random variable η_j. The event $E_{S_0} = 0 \nleftrightarrow H$ is equivalent to $\left(\eta_{j_\#} = 0 \text{ for all } j_\# \in R_T\right)$. By analogy, define the event for which the branching process starting with 0 fails to connect to H as $E_B = \left(\eta_{j_\#} = 0 \text{ for all } j_\# \in \bigcup_{t=0}^{\infty} \hat{W}_t\right)$. Clearly, by construction, we have $E_B \subseteq E_{S_0}$, so $\Pr^{nh}(E_{S_0}) = \Pr^{nh}(E_B) + \Pr^{nh}(E_{S_0} \backslash E_B)$. We intend to show that the probability of the event $E^* = E_{S_0} \backslash E_B$ (that is, the branching process connects to H, but the true process does not) vanishes as $n \to \infty$.

Toward this end, partition E^* as $E^* = (E^* \cap Sm_0) \cup (E^* \cap Lg_0)$, so we have

$$\overset{nh}{\Pr}(E^*) = \overset{nh}{\Pr}(E^* \cap Sm_0) + \overset{nh}{\Pr}(E^* \cap Lg_0)$$

Intuitively, the first term vanishes because (R_t, W_t) and $\left(\hat{R}_t, \hat{W}_t\right)$ are very likely to coincide if S_0 is small, in which case E^* is null. If S_0 is large, then it doesn't matter whether (R_t, W_t) and $\left(\hat{R}_t, \hat{W}_t\right)$ coincide because both processes are very likely to hit H, so the chance of event E^* is still small. We present these arguments as two claims.

Claim: $\lim_{n\to\infty} \Pr^{nh}(E^* \cap Sm_0) = 0$

A prerequisite for events E_{S_0} and E_B to differ is that process $\left(\hat{R}_t, \hat{W}_t\right)$ generates at least one illegitimate individual. A necessary condition for the first illegitimacy in $\left(\hat{R}_t, \hat{W}_t\right)$ to occur is that a legitimate parent (who is thus in S_0) generates a child whose tagged as illegitimate. A necessary condition for this child to be tagged as illegitimate is that his type appears in S_0. There are at most $|S_0|^2$ potential parent-child pairings that could create the first illegitimacy, and each of these potential pairings has probability no greater than $\frac{\delta}{n}$ of generating the child. Thus, conditional on $|S_0|$, the probability that E_{S_0} and E_B differ is bounded above by $\frac{\delta |S_0|^2}{n}$. It follows that $\Pr^{nh}(E^* \mid Sm_0) \leq \frac{\delta (\log n)^2}{n}$, which justifies the claim.

Claim: $\lim_{n\to\infty} \Pr^{nh}(E^* \cap Lg_0) = 0$

Because $E^* \subseteq E_{S_0}$, it suffices to show that $\lim_{n\to\infty} \Pr^{nh}(E_{S_0} \cap Lg_0) = 0$, but this follows from Lemma 1.

The last two claims give us $\lim_{n\to\infty} \Pr^{nh}(E^*) = 0$, so as a result, we have $\lim_{n\to\infty} \left(\Pr^{nh}(E_{S_0}) - \Pr^{nh}(E_B)\right) = 0$. Thus, in order to calculate the limiting probability of $0 \not\leftrightarrow H$, it suffices to calculate the limiting probability $\hat{Y}_0^h$ that the branching process starting at 0 fails to hit H; the derivation of $\hat{Y}_0^h$ is in the text. ■

Lemma 5 *For $h \geq 0$ and $z > 1$, let $x^*(h)$ be the unique positive solution to $\nu(x) = 0$ for $\nu(x) \equiv 1 - x - (1-h)e^{-zx}$. Then $x^*(h) > \delta_z > 0$, for all $h \geq 0$, where $\delta_z = \frac{\log z}{z}$ does not depend on h.*

Proof. Clearly $\nu(x) \to -\infty$ as $x \to \infty$, so it suffices to show that $\nu(\delta_z) > 0$ for all $h \geq 0$. We have

$$\nu(\delta_z) = \frac{z - 1 - \log z}{z} + \frac{h}{z} \geq \frac{z - 1 - \log z}{z} \text{ for all } h \geq 0$$

■

Proposition 2

Proof. The structure of the proof is similar to Proposition 1; we will focus on the novel features, occasionally omitting details that pass through unchanged. Write $\mathcal{B}^{nh}\left(\left(\tilde{z}, \tilde{f}\right); (\mathbf{z}, \mathbf{f})_{-1}\right) = \Pr^{nh}\left(0 \longmapsto 1 \mid \left(\tilde{z}, \tilde{f}\right); (\mathbf{z}, \mathbf{f})_{-1}\right)$. Henceforth conditioning on the strategy profile is suppressed.

Let E be the event $(0 \longmapsto H \cap H \longmapsto 1)$ and let E^c be its complement; we have $\Pr^{nh}(0 \longmapsto 1) = \Pr^{nh}(0 \longmapsto 1 \cap E) + \Pr^{nh}(0 \longmapsto 1 \cap E^c) = \Pr^{nh}(E) + \Pr^{nh}(0 \longmapsto 1 \cap E^c)$, where the last step follows because E implies $0 \longmapsto 1$. Note that $E^c = 0 \not\longmapsto H \cup H \not\longmapsto 1$, so by the inclusion-exclusion principle, we have

$$\begin{aligned}\overset{nh}{\Pr}(0 \longmapsto 1) &= \overset{nh}{\Pr}(E) + \overset{nh}{\Pr}(0 \longmapsto 1 \cap 0 \not\longmapsto H) + \overset{nh}{\Pr}(0 \longmapsto 1 \cap H \not\longmapsto 1) \\ &\quad - \overset{nh}{\Pr}(0 \longmapsto 1 \cap 0 \not\longmapsto H \cap H \not\longmapsto 1)\end{aligned}$$

The last three terms involve events in which agent 0 infects agent 1 via some path that does not include the hub; Lemma 6 shows that these terms vanish as $n \to \infty$. Lemma 7 shows that $0 \longmapsto H$ and $H \longmapsto 1$ are asymptotically independent, so we have $\lim_{n\to\infty} \Pr^{nh}(0 \longmapsto 1) = b^h \beta^h$, where $b^h = \lim_{n\to\infty} \Pr^{nh}(0 \longmapsto H)$ and $\beta^h = \lim_{n\to\infty} \Pr^{nh}(H \longmapsto 1)$. Lemma 8 adapts the branching process approximation from Proposition 1 to accomodate the firewall random variables, demonstrating that b^h and β^h solve

$$1 - b^h = (1-h)\, e^{-z(1-f)b^h} \text{ and } 1 - \beta^h = (1-h)\, e^{-\tilde{z}\left(1-\tilde{f}\right)b^h} \quad .$$

The final step of taking the $h \to 0$ limit proceeds just as in Proposition 1. ■

Lemma 6 *Define event* $E = (0 \longmapsto H \cap H \longmapsto 1)$ *as in Proposition 2. Then,*

$$\lim_{n\to\infty} \overset{nh}{\Pr}(0 \longmapsto 1) = \lim_{n\to\infty} \overset{nh}{\Pr}(E) \quad .$$

Proof. Following the argument in the proof of Proposition 1, we have

$$\overset{nh}{\Pr}(0 \longmapsto 1) = \overset{nh}{\Pr}(E) + \overset{nh}{\Pr}\left(E^{c1}\right) + \overset{nh}{\Pr}\left(E^{c2}\right) - \overset{nh}{\Pr}\left(E^{c3}\right)$$

where $E^{c1} = 0 \longmapsto 1 \cap H \not\longmapsto 1$, $E^{c2} = 0 \longmapsto 1 \cap 0 \not\longmapsto H$, and $E^{c3} = 0 \longmapsto 1 \cap 0 \not\longmapsto H \cap H \not\longmapsto 1$. Observe that $E^{c3} \subseteq E^{c1}$ and $E^{c3} \subseteq E^{c2}$, so it suffices to show that $\Pr^{nh}\left(E^{c1}\right)$ and $\Pr^{nh}\left(E^{c2}\right)$ each tend to zero as $n \to \infty$.

By analogy with S_i, define S_i^{up} to be the set of agents who infect i by some path that does not include H. That is $j \in S_i^{up}$ if and only if $j \longmapsto_1 i$ or $j \longmapsto_1 k_1 \longmapsto_1 \ldots \longmapsto_1 k_m \longmapsto_1 i$ for some path that does not include H. Define S_i^{down} to be the set of agents whom i infects along some path that does not include H.

Claim: $\lim_{n\to\infty} \Pr^{nh}\left(E^{c1}\right) = 0$

Event E^{c1} is equivalent to $0 \in S_1^{up} \cap H \not\longmapsto 1$. Partition E^{c1} based on the number of upstream neighbors that agent 1 has: $\Pr^{nh}\left(E^{c1}\right) = \Pr^{nh}(0 \in S_1^{up} \cap H \not\longmapsto 1 \cap |S_1^{up}| \leq \log n) + \Pr^{nh}(0 \in S_1^{up} \cap H \not\longmapsto 1 \cap |S_1^{up}| > \log n)$. The second term counts events in which at least $\log n$ agents fail to connect to H (since otherwise we would have $H \longmapsto 1$). This has probability no greater than $(1-h)^{\log n}$, which vanishes as $n \to \infty$. For the first term, note that by symmetry all agents in $\mathbf{N}\backslash 1$ are equally likely to appear in S_1^{up}, so conditional on $|S_1^{up}| \leq \log n$, the chance of $0 \in S_1^{up}$ is no greater than $\frac{\log n - 1}{n-1}$, which also tends to zero with n.

Claim: $\lim_{n\to\infty} \Pr^{nh}\left(E^{c2}\right) = 0$

In this case, we have $E^{c2} = 1 \in S_0^{down} \cap 0 \not\longmapsto H$. Split E^{c2} based on the size of S_0^{down}:

$$\Pr^{nh}\left(E^{c2}\right) = \Pr^{nh}\left(1 \in S_0^{down} \cap 0 \not\longmapsto H \cap \left|S_0^{down}\right| \leq \log n\right) + \Pr^{nh}\left(1 \in S_0^{down} \cap 0 \not\longmapsto H \cap \left|S_0^{down}\right| > \log n\right)$$

The second term vanishes by the same argument as the second term in the previous claim. (If 0 infects more than $\log n$ agents, then the chance that none of them connects to H is small.) For the first term, we adapt the symmetry argument from the previous claim. We have $1 \in S_0^{down}$ only if $j \longmapsto_1 1$ for some $j \in S_0^{down}$. For any arbitrary agent $j \in \mathbf{N}\backslash 1$, the probability that j directly infects 1 is no greater than $\frac{\delta}{n}$, where $\delta = \frac{\max(z,\tilde{z})^2}{\min(z,\tilde{z})}$ as defined earlier. Then, conditional on $\left|S_0^{down}\right| \leq \log n$, the probability that at least one of the agents in S_0^{down} directly infects 1 is thus no greater than $\frac{\delta \log n}{n}$. So the first term above is bounded above by $\frac{\delta \log n}{n}$, and therefore tends to zero with n.

Lemma 7 $\lim_{n\to\infty} \left|\Pr^{nh}\left(0 \not\longmapsto H \cap H \not\longmapsto 1\right) - \Pr^{nh}\left(0 \not\longmapsto H\right) \Pr^{nh}\left(H \not\longmapsto 1\right)\right| = 0.$

Proof. The procedure is similar to Lemma 3. By now standard arguments, it suffices to show that $\lim_{n\to\infty} \left|\Pr^{nh}\left(\tilde{E}_0 \cap \tilde{E}_1\right) - \Pr^{nh}\left(\tilde{E}_0\right) \Pr^{nh}\left(\tilde{E}_1\right)\right| = 0$, where $\tilde{E}_0 = 0 \not\longmapsto H \cap \left|S_0^{down}\right| \leq \log n$ and $\tilde{E}_1 = H \not\longmapsto 1 \cap |S_1^{up}| \leq \log n$. Let $\left(\zeta'_{ij}, \eta'_j, \phi'_{ij}\right)$ be independent copies of the variables that generate $\mathcal{G}^f$, and let $\mathcal{G}^{f\prime}$ be the secured network that they generate. Let $\tilde{E}'_0$ be the counterpart of $\tilde{E}'_0$ on graph $\mathcal{G}^{f\prime}$. Clearly events $\tilde{E}'_0$ and $\tilde{E}_1$ are independent, so $\Pr^{nh}\left(\tilde{E}'_0 \cap \tilde{E}_1\right) = \Pr^{nh}\left(\tilde{E}'_0\right) \Pr^{nh}\left(\tilde{E}_1\right)$. Let $\mathcal{G}^{f\prime\prime}$ be the graph generated by using:

$$\begin{aligned}
\zeta_{ij} \text{ if } i &\in S_1^{up} \text{ or } j \in S_1^{up} \;\;;\;\; \text{otherwise, use } \zeta'_{ij} \\
\eta_j \text{ if } j &\in S_1^{up} \;\;;\;\; \text{otherwise use } \eta'_j \\
\phi_{ij} \text{ if } j &\in S_1^{up} \;\;;\;\; \text{otherwise use } \phi'_{ij}
\end{aligned}$$

In other words, $\mathcal{G}^{f\prime\prime}$ uses the original random variables to determine which agents are upstream of agent 1 and then switches to the copy random variables for all other links and firewalls. Let $\tilde{E}''_0$ be the counterpart of $\tilde{E}_0$ on $\mathcal{G}^{f\prime\prime}$. Note that $\Pr^{nh}\left(\tilde{E}''_0 \cap \tilde{E}_1\right) = \Pr^{nh}\left(\tilde{E}_0 \cap \tilde{E}_1\right)$. Now compare event $\tilde{E}'_0$ on graph $\mathcal{G}^{f\prime}$ with event $\tilde{E}''_0$ on graph $\mathcal{G}^{f\prime\prime}$. Let $\hat{S}_0^{down}$ be the set of small agents who are infected by agent 0 (in either $\mathcal{G}^{f\prime}$ or $\mathcal{G}^{f\prime\prime}$ – it is the same set of agents for both graphs) if links to H and links to agents in S_1^{up} are excluded. If $\left|\hat{S}_0^{down}\right| > \log n$ or $|S_1^{up}|$ then events $\tilde{E}''_0 \cap \tilde{E}_1$ and $\tilde{E}'_0 \cap \tilde{E}_1$ coincide. (They both fail.) Furthermore, whenever neither $\mathcal{G}^{f\prime}$ nor $\mathcal{G}^{f\prime\prime}$ contains an unblocked link from $\hat{S}_0^{down}$ into S_1^{up}, then $\tilde{E}'_0$ and $\tilde{E}''_0$ coincide, since in this case they are both determined by the same set of copy variables $\left(\zeta'_{ij}, \eta'_j, \phi'_{ij}\right)$. Thus,

$$\begin{aligned}
\left|\overset{nh}{\Pr}\left(\tilde{E}_0''\cap\tilde{E}_1\right)-\overset{nh}{\Pr}\left(\tilde{E}_0'\cap\tilde{E}_1\right)\right| &\leq \overset{nh}{\Pr}\left(\begin{array}{c}\left(\begin{array}{c}\mathcal{G}^{f\prime}\text{ has an unblocked link from }\hat{S}_0^{down}\text{ into }S_1^{up}\cup\\ \mathcal{G}^{f\prime\prime}\text{ has an unblocked link from }\hat{S}_0^{down}\text{ into }S_1^{up}\end{array}\right)\\ \cap\left(\left|\hat{S}_0^{down}\right|\leq\log n\right)\cap\left(\left|S_1^{up}\right|\leq\log n\right)\end{array}\right)\\
&\leq 2\overset{nh}{\Pr}\left(\begin{array}{c}\left(\mathcal{G}^{f\prime}\text{ has an unblocked link from }\hat{S}_0^{down}\text{ into }S_1^{up}\right)\\ \cap\left(\left|\hat{S}_0^{down}\right|\leq\log n\right)\cap\left(\left|S_1^{up}\right|\leq\log n\right)\end{array}\right)\\
&\leq 2\delta\frac{(\log n)^2}{n}
\end{aligned}$$

Since the righthand side tends to zero with n, we have

$$\begin{aligned}
\lim_{n\to\infty}\overset{nh}{\Pr}\left(\tilde{E}_0\cap\tilde{E}_1\right) &= \lim_{n\to\infty}\overset{nh}{\Pr}\left(\tilde{E}_0''\cap\tilde{E}_1\right)\\
&= \lim_{n\to\infty}\overset{nh}{\Pr}\left(\tilde{E}_0'\cap\tilde{E}_1\right)\\
&= \lim_{n\to\infty}\overset{nh}{\Pr}\left(\tilde{E}_0'\right)\lim_{n\to\infty}\overset{nh}{\Pr}\left(\tilde{E}_1\right)\\
&= \lim_{n\to\infty}\overset{nh}{\Pr}\left(\tilde{E}_0\right)\lim_{n\to\infty}\overset{nh}{\Pr}\left(\tilde{E}_1\right)
\end{aligned}$$

as claimed. ■ ■

Lemma 8 *Given strategy profile $\left(\left(\tilde{z},\tilde{f}\right);(\mathbf{z},\mathbf{f})_{-1}\right)$ in which agent 1 plays $\left(\tilde{z},\tilde{f}\right)$ and every other agent plays (z,f), we have*

$$\begin{aligned}
\lim_{n\to\infty}\overset{nh}{\Pr}\left(0\not\longmapsto H\right) &= 1-b^h\\
\lim_{n\to\infty}\overset{nh}{\Pr}\left(H\not\longmapsto 1\right) &= 1-\beta_1^h
\end{aligned}$$

where b^h is the positive solution to

$$1-b^h=(1-h)\,e^{-z(1-f)b^h}$$

and

$$\beta_1^h=1-(1-h)\,e^{-\tilde{z}\left(1-\tilde{f}\right)b^h}$$

Proof. As the proof is quite similar to the proof of Lemma 4, we focus on highlighting the differences. To determine $\lim_{n\to\infty}\Pr^{nh}(0\not\longmapsto H)$, define $\left(\hat{R}_t^{down},\hat{W}_t^{down}\right)$ with reproduction rates as in the text. Here, ϕ_{ij} will refer to the true firewall random variables that determine $\mathcal{G}^f$, while $\hat{\phi}_{ij}$ will be independent copies of these random variables. As in the proof of Lemma 4, we couple $\left(\hat{R}_t^{down},\hat{W}_t^{down}\right)$ to $\mathcal{G}^f$ by using the true, unhatted variables when enumerating members of S_0^{down} for the first time and switching to independent copies when we begin to generate duplicates of those members. Put $\hat{R}_0^{down}=\emptyset$ and $\hat{W}_0^{down}=\{0_\#\}$, and tag this initial type 0 individual as legitimate. At stage $t+1$, assume that all individuals in $\hat{R}_t^{down}\cup\hat{W}_t^{down}$ have been tagged. For each parent $i_\#\in\hat{W}_t^{down}$ and each type j of potential child that $i_\#$ could generate :

If $i_\#$ is illegitimate or if there is a legitimate type j individual in $\hat{R}_t^{down} \cup \hat{W}_t^{down}$, then

- Generate the type j child if $\hat{\zeta}_{ij}\left(1-\hat{\phi}_{ij}\right)=1$.
- If the child is created, tag her as illegitimate.

If $i_\#$ is legitimate and there is no legitimate type j individual in $\hat{R}_t^{down} \cup \hat{W}_t^{down}$, then

- Generate the type j child if $\zeta_{ij}\left(1-\phi_{ij}\right)=1$.
- If the child is created, leave her untagged for the moment.

The children created at this stage are the members of $\hat{W}_{t+1}^{down}$. Return to the untagged members of this set and tag them as legitimate or illegitimate as in Lemma 4. Define $R_0^{down}=\emptyset$, $R_{t+1}^{down}=R_t^{down}\cup W_t^{down}$, and $W_t^{down}=\left\{i_\# \,|\, i_\# \in \hat{W}_t^{down} \text{ and } i_\# \text{ is legitimate}\right\}$. By the same logic as in Lemma 4, R_t^{down} must cease growing by some finite T, and the types of the individuals in R_T enumerate the agents in S_0^{down}. From here, one follows Lemma 4 more or less line for line. $\hat{Y}_0^{down,nh}$ (the probability that no individual created in the branching process $\left(\hat{R}_t^{down},\hat{W}_t^{down}\right)$ hits H) and $\Pr^{nh}(0 \not\longmapsto H)$ can differ only over events in which no agent in S_0^{down} hits H but $\hat{R}_t^{down}$ eventually contains some illegitimate agent who does hit H. As $n\to\infty$, these events almost never occur, either because the creation of an illegitimate individual is very unlikely (if S_0^{down} turns out to be small relative to n) or because some agent in S_0^{down} is almost certain to hit H (if S_0^{down} turns out to be large relative to n). Thus we have $1-b^h=\lim_{n\to\infty}\Pr^{nh}(0 \not\longmapsto H)=\hat{Y}_0^{down,h}$. As shown in the text, $\hat{Y}_0^{down,h}$ solves $\hat{Y}_0^{down,h}=(1-h)\,e^{-z(1-f)\left(1-\hat{Y}_0^{down,h}\right)}$, from which the first part of the lemma follows.

Determining $\lim_{n\to\infty}\Pr^{nh}(H \not\longmapsto 1)$, involves an essentially identical argument. In this case, we use the branching process $\left(\hat{R}_t^{up},\hat{W}_t^{up}\right)$, initialized with $\hat{R}_0^{down}=\emptyset$ and $\hat{W}_0^{down}=\{1_\#\}$, and we couple the process to S_1^{up} by assigning random variables as follows. For each parent $i_\# \in \hat{W}_t^{up}$ and each type j of potential child that $i_\#$ could generate :

If $i_\#$ is illegitimate or if there is a legitimate type j individual in $\hat{R}_t^{up} \cup \hat{W}_t^{up}$, then

- Generate the type j child if $\hat{\zeta}_{ij}\left(1-\hat{\phi}_{ji}\right)=1$.
- If the child is created, tag her as illegitimate.

If $i_\#$ is legitimate and there is no legitimate type j individual in $\hat{R}_t^{up} \cup \hat{W}_t^{up}$, then

- Generate the type j child if $\zeta_{ij}\left(1-\phi_{ji}\right)=1$.
- If the child is created, wait until all members of $\hat{W}_{t+1}^{up}$ have been generated, and then tag her as in Lemma 4.

The only difference from the $\left(\hat{R}_t^{down},\hat{W}_t^{down}\right)$ case is the reversal of subscripts on the firewall variables, indicating that we create the child j upstream of parent i only if the *parent's* firewall variable ϕ_{ji} (or $\hat{\phi}_{ji}$) equals 1. One proceeds just as for the downstream case to show that $1-\beta_1^h=\lim_{n\to\infty}\Pr^{nh}(H \not\longmapsto 1)=\hat{Y}_1^{up,h}$. The conclusion that $\beta_1^h=1-(1-h)\,e^{-\tilde{z}\left(1-\tilde{f}\right)b^h}$ follows from the argument in the text. ■

Proposition 3

Proof. The first condition follows from setting $\tilde{z}=z$ in agent 1's first order condition with respect to $\tilde{z}$ and using the identities $a=1-e^{-az}$ and $b=1-e^{-\left(1-\bar{f}\right)zb}$. The second condition follows from those two identities. The argument in the text establishes that these two conditions are necessary and sufficient for an equilibrium. It remains to show that this pair of equations has a unique solution on $(a,b)\in(0,1)^2, b\le a$.

For brevity, let D denote the triangular region $\left\{(a,b) \mid (a,b) \in (0,1)^2, b < a\right\}$ and $\bar{D}$ its closure. We start with a few remarks on the curves defined by the two conditions. Let $g(x) = -\frac{\ln(1-x)}{x}$, and note that g is a positive, strictly increasing, and continuous function on $(-\infty, 1)$ with $g(0) = 1$ and $\lim_{x\to\infty} g(x) = \infty$. Let $\underline{a} > 0$ be defined (uniquely) by $\left(1-\bar{f}\right) g(\underline{a}) = 1$. Since g is invertible, for the second equilibrium condition we can write b as a strictly increasing function of a: $b = \gamma(a)$, with $\gamma(x) = g^{-1}\left(\left(1-\bar{f}\right) g(x)\right)$. The pair $(a,b) = (a, \gamma(a))$ satisfies condition (6) and lies in D if and only if $a \in (\underline{a}, 1)$. (If $a \leq \underline{a}$, $\gamma(a) \leq 0$.) To the southwest end of D, $(a, \gamma(a))$ tends toward $(\underline{a}, 0)$, while to the northeast, $(a, \gamma(z))$ tends toward $(1,1)$. Next, we claim that $\gamma'(a)$ tends to infinity as $a \to 1$. For this, it suffices to show that $\lim_{a\to 1} \frac{1-\gamma(a)}{1-a} = \infty$. With a bit of algebra, we can write $\ln\left(\frac{1-\gamma(a)}{1-a}\right) = \left(\left(1-\bar{f}\right)\frac{\gamma(a)}{a} - 1\right)\ln(1-a)$. The term in parentheses tends to $-\bar{f}$ as $a \to 1$, while $\ln(1-a)$ tends to $-\infty$, so we have $\lim_{a\to 1} \ln\left(\frac{1-\gamma(a)}{1-a}\right) = \infty$, so *a fortiori*, $\frac{1-\gamma(a)}{1-a}$ tends to infinity also.

Next we turn to the first condition, $Aa^2(1-a) = Bb^2(1-b)\left(1-\bar{f}\right)$. We can write this as $A\chi(a) = B\left(1-\bar{f}\right)\chi(b)$, where $\chi(x) = x^2(1-x)$. Note that χ has an inverse-U shape on $x \in (0,1)$, with a range of $(0, \frac{4}{27}]$ and its single peak at $x = \frac{2}{3}$. For each $y \in \left(0, \frac{4}{27}\right)$, $y = \chi(x)$ has two solutions for $x \in (0,1)$; one to the left and one to the right of $x = \frac{2}{3}$. Let $\chi_-^{-1}(y)$ and $\chi_+^{-1}(y)$ denote these two solutions. (That is, $\chi\left(\chi_-^{-1}(y)\right) = \chi\left(\chi_+^{-1}(y)\right) = y$, with $\chi_-^{-1}(y) < \frac{2}{3} < \chi_+^{-1}(y)$.) There are three cases, depending on whether $A \lesseqqgtr B\left(1-\bar{f}\right)$.

$\underline{A < B\left(1-\bar{f}\right)}$ In this case, consider a solution to the condition for arbitrary $a' \in (0,1)$. Any satisfactory b' must satisfy $\chi(b') = y'$, where $y' = \frac{A}{B\left(1-\bar{f}\right)}\chi(a') \in (0, \frac{4}{27}\frac{A}{B\left(1-\bar{f}\right)}]$. There are two solutions, call them b_- and b_+. Notice that $y' < \chi(a')$, so b_- must lie to the left of $\chi_-^{-1}(\chi(a'))$ and b_+ must lie to the right of $\chi_+^{-1}(\chi(a'))$. But this implies that $b_- < a' < b_+$, so only the smaller solution (a', b_-) lies in D. Since this is true for arbitrary $a' \in (0,1)$, for this case, all solutions to condition (5) on D take the form $\left\{(a,b) \mid a \in (0,1), b = \chi_-^{-1}\left(\frac{A}{B\left(1-\bar{f}\right)}\chi(a)\right)\right\}$. Observe that $\chi_-^{-1}(y)$ is increasing in y and that $\chi_-^{-1}(0) = 0$. Thus, this solution approaches $(a,b) = (0,0)$ as $a \to 0$ or $(1,0)$ as $a \to 1$, and b is first increasing in a (since $\frac{A}{B\left(1-\bar{f}\right)}\chi(a)$ is increasing for $a < \frac{2}{3}$) and then decreasing (for $a > \frac{2}{3}$).

$\underline{A > B\left(1-\bar{f}\right)}$ For this case, the roles of a and b are reversed: a is an inverse-U shaped function of b. To see this, pick an arbitrary $b' \in (0,1)$ and consider candidate solutions a' that solve $\chi(a') = \frac{B\left(1-\bar{f}\right)}{A}\chi(b') < \frac{4}{27}$. Again, there are two solutions, $a_- = \chi_-^{-1}\left(\frac{B\left(1-\bar{f}\right)}{A}\chi(b')\right)$ and $a_+ = \chi_+^{-1}\left(\frac{B\left(1-\bar{f}\right)}{A}\chi(b')\right)$. In this case, because $\frac{B\left(1-\bar{f}\right)}{A} < 1$, we have $\chi(a_+) = \chi(a_-) < \chi(b')$, so $a_- < b' < a_+$. The only candidate solution lying in D is a_+, so solutions to condition (5) must take the form $\left\{(a,b) \mid b \in (0,1), a = \chi_+^{-1}\left(\frac{B\left(1-\bar{f}\right)}{A}\chi(b)\right)\right\}$. Because $\chi_+^{-1}(0) = 1$, this solution tends to $(a,b) = (1,0)$ or $(1,1)$ as $b \to 0$ or $b \to 1$. Furthermore, since $\chi_+^{-1}(y)$ is decreasing in y, the solution has a first decreasing (for $b < \frac{2}{3}$) and then increasing (for $b > \frac{2}{3}$) in b.

$\underline{A = B\left(1-\bar{f}\right)}$ For this boundary case, we have $a^2(1-a) = b^2(1-b)$. The solution $a = b$ does not lie in D. The only other solutions, parametrized by $y \in (0, \frac{4}{27}]$, are $(a,b) = \left(\chi_-^{-1}(y), \chi_+^{-1}(y)\right)$ or $(a,b) = \left(\chi_+^{-1}(y), \chi_-^{-1}(y)\right)$. Only the latter lies in D. Its endpoints tend toward $(a,b) = (1,0)$

as $y \to 0$ and $(a,b) = \left(\frac{2}{3}, \frac{2}{3}\right)$ as $y \to \frac{4}{27}$.

Now we can turn to first existence and then uniqueness, case by case.

Existence

$\underline{A < B\left(1-\bar{f}\right)}$ Each equilibrium condition determines b as a continuous function of a. Define the difference between these functions: $\nu(a) = \chi_{-}^{-1}\left(\frac{A}{B(1-\bar{f})}\chi(a)\right) - \gamma(a)$ for $a \in [\underline{a}, 1]$. We have $\nu(\underline{a}) > 0$ and $\nu(1) = -1$, so by continuity there exists some pair $(a,b) \in D$ (with $a \in (\underline{a}, 1)$) that satisfies both equilibrium conditions simultaneously.

$\underline{A = B\left(1-\bar{f}\right)}$ This is identical to the previous case, but define ν on $a \in \left[\max\left(\underline{a}, \frac{2}{3}\right), 1\right]$. If $\underline{a} > \frac{2}{3}$, then $\gamma(\underline{a}) = 0$ implies $\nu(\underline{a}) > 0$. If $\frac{2}{3} \geq \underline{a}$, then $\chi_{-}^{-1}\left(\chi\left(\frac{2}{3}\right)\right) = \frac{2}{3}$ and $\gamma(a) < a$ imply $\nu\left(\frac{2}{3}\right) > 0$. As above, we have $\lim_{a\to 1}\nu(a) = -1$, so some $(a,b) \in D$ satisfying both equilibrium conditions must exist.

$\underline{A > B\left(1-\bar{f}\right)}$ In this case, both conditions can be framed to express a as a continuous function of b. Define the difference $\mu(b) = \chi_{+}^{-1}\left(\frac{B(1-\bar{f})}{A}\chi(b)\right) - \gamma^{-1}(b)$, and note that $\lim_{b\to 0}\mu(b) = 1 - \underline{a} > 0$. Furthermore, we have $\lim_{b\to 1}\mu(b) = 0$ and we claim that $\lim_{b\to 1}\mu'(b) > 0$. This implies that $\mu(b)$ is strictly negative for b sufficiently close to 1, and therefore that $\mu(b)$ has a zero for some $b \in (0,1)$. This establishes existence of an equilibrium. For the claim on $\mu'(b)$, we showed earlier that $\gamma'(a)$ tends to ∞ (and so $\frac{d\gamma^{-1}(b)}{db} \to 0$) as a and b tend to 1, so it suffices to show that $\frac{da}{db}$ is strictly positive along condition (5) as a and b approach 1. It is straightforward to show (use the implicit function theorem) that in fact $\frac{da}{db} \to \frac{B(1-\bar{f})}{A} > 0$ as $(a,b) \to (1,1)$ along the solution for condition (5).

Uniqueness

$\underline{A < B\left(1-\bar{f}\right)}$ We will show that if $\nu(\hat{a}) = 0$, then $\nu'(\hat{a}) < 0$. Thus $\nu(a)$ has at most one zero on $a \in (0,1)$, and therefore the economy has at most one equilibrium. Suppose that we have an equilibrium $(\hat{a}, b^*)$ with $\hat{b} = \gamma(\hat{a})$ and $\nu(\hat{a}) = 0$. Direct computations show that

$$\nu'(\hat{a}) = \frac{1}{1-\bar{f}}\frac{A\hat{a}(2-3\hat{a})}{B\hat{b}\left(2-3\hat{b}\right)} - \left(1-\bar{f}\right)\frac{\hat{b}^2\left(1-\hat{b}\right)}{\hat{a}^2(1-\hat{a})}\frac{\hat{a}+(1-\hat{a})\ln(1-\hat{a})}{\hat{b}+\left(1-\hat{b}\right)\ln\left(1-\hat{b}\right)}$$

Using both equilibrium conditions, this may be rewritten as

$$\nu'(\hat{a}) = \frac{A}{B}\frac{\ln(1-\hat{a})}{\ln\left(1-\hat{b}\right)}\left(\frac{2-3\hat{a}}{2-3\hat{b}} - \frac{\omega(\hat{a})}{\omega\left(\hat{b}\right)}\right)$$

where $\omega(x) = 1 - x + \frac{x}{\ln(1-x)}$. Note that because $\hat{b} = \chi_{-}^{-1}\left(\frac{A}{B(1-\bar{f})}\chi(\hat{a})\right)$, we must have $2 - 3\hat{b} > 0$. One can show computationally that $\omega(x)$ is strictly negative on $x \in (0,1)$, decreasing on $x \in (0,\bar{x})$, and increasing on $x \in (\bar{x}, 1)$, where $\bar{x} \approx 0.83359$. Thus for $\hat{a} < \frac{2}{3}$, because $\hat{b} < \hat{a}$ we have $\frac{2-3\hat{a}}{2-3\hat{b}} < 1$ and $\frac{\omega(\hat{a})}{\omega(\hat{b})} > 1$, so the term in parentheses is strictly negative. For $\hat{a} \geq \frac{2}{3}$, we have $\frac{2-3\hat{a}}{2-3\hat{b}} \leq 0$ and $\frac{\omega(\hat{a})}{\omega(\hat{b})}$ strictly positive, so again, the term in parentheses is strictly negative. This shows that $\nu'(\hat{a}) < 0$.

$\underline{A = B\left(1-\bar{f}\right)}$ In this case, any equilibrium $\left(\hat{a},\hat{b}\right)$ must satisfy $\hat{a} > \frac{2}{3}$ and $\hat{b} < \frac{2}{3}$. The argument in the previous case for $\hat{a} > \frac{2}{3}$ applies directly to show that $\nu\left(\hat{a}\right) < 0$.
$\underline{A > B\left(1-\bar{f}\right)}$ The general idea remains the same, but here we need to show that $\mu'\left(\hat{a}\right) < 0$ at any equilibrium $\hat{a}$. Substituting equilibrium conditions into $\mu'\left(\hat{a}\right)$ as above, we have

$$\begin{aligned}
\mu'\left(\hat{a}\right) &= \left(1-\bar{f}\right)\frac{B\hat{b}\left(2-3\hat{b}\right)}{A\hat{a}\left(2-3\hat{a}\right)} - \left(\left(1-\bar{f}\right)\frac{\hat{b}^2\left(1-\hat{b}\right)}{\hat{a}^2\left(1-\hat{a}\right)}\frac{\hat{a}+\left(1-\hat{a}\right)\ln\left(1-\hat{a}\right)}{\hat{b}+\left(1-\hat{b}\right)\ln\left(1-\hat{b}\right)}\right)^{-1} \\
&= \frac{B}{A}\frac{\ln\left(1-\hat{b}\right)}{\ln\left(1-\hat{a}\right)}\left(\frac{2-3\hat{b}}{2-3\hat{a}} - \frac{\omega\left(\hat{b}\right)}{\omega\left(\hat{a}\right)}\right)
\end{aligned}$$

In this case, any equilibrium has $\hat{a} > \frac{2}{3}$, so if $\hat{b} \leq \frac{2}{3}$, we have $\frac{2-3\hat{b}}{2-3\hat{a}} \leq 0$, so we are done. If $\hat{b} > \frac{2}{3}$, then write

$$\mu'\left(\hat{a}\right) = \frac{B}{A}\frac{\ln\left(1-\hat{b}\right)}{\ln\left(1-\hat{a}\right)}\frac{2-3\hat{b}}{2-3\hat{a}}\left(1 - \frac{\omega\left(\hat{b}\right)}{2-3\hat{b}}\Big/\frac{\omega\left(\hat{a}\right)}{2-3\hat{a}}\right)$$

One can show computationally that the function $\frac{\omega(x)}{2-3x}$ is positive and strictly decreasing for $x \in (0,1)$, so since we have $\frac{2}{3} < \hat{b} < \hat{a}$, the term in parentheses is negative, and thus $\mu'\left(\hat{a}\right) < 0$. ■

Proposition 4

Proof. We have $\frac{d\pi^{eq}}{d\bar{f}} = 2Aa\frac{da}{d\bar{f}} - 2Bb\frac{db}{d\bar{f}}$. Throughout the proof, we will appeal to a number of facts about equilibrium that are summarized in Lemma 9. If $\bar{f}$ is sufficiently high, then (by Lemma 9) we have $\frac{da}{d\bar{f}}$ positive and $\frac{db}{d\bar{f}}$ negative, so we are done. Otherwise, differentiate the equilibrium identity $A\Upsilon\left(a\right) = B\Upsilon\left(b\right)$ to get $A\Upsilon'\left(a\right)\frac{da}{d\bar{f}} = B\Upsilon'\left(b\right)\frac{db}{d\bar{f}}$. Substitute to get

$$\begin{aligned}
\frac{d\pi^{eq}}{d\bar{f}} &= 2Aa\frac{da}{d\bar{f}}\left(1-\frac{b}{a}\frac{\Upsilon'\left(a\right)}{\Upsilon'\left(b\right)}\right) \\
&= 2Aa\frac{da}{d\bar{f}}\left(1-\frac{\tau\left(a\right)}{\tau\left(b\right)}\right)
\end{aligned}$$

where $\tau\left(x\right) = \Upsilon'\left(x\right)/x = 1 - \frac{1-2x}{x}\ln\left(1-x\right)$. The function $\tau\left(x\right)$ is strictly decreasing on $(0,1)$, with $\tau\left(\bar{\upsilon}\right) = 0$. From Lemma 9, there are only two cases in which $\frac{da}{d\bar{f}}$ and $-\frac{db}{d\bar{f}}$ are not both positive: either $B < A$ and $\bar{\upsilon} < b < a$, or $B > A$ and $b < a < \bar{\upsilon}$. For the first case, we have $\frac{da}{d\bar{f}} < 0$ and $0 > \tau\left(b\right) > \tau\left(a\right)$ (and thus $\frac{\tau(a)}{\tau(b)} > 1$), implying $\frac{d\pi^{eq}}{d\bar{f}} > 0$. For the second case, we have $\frac{da}{d\bar{f}} > 0$ and $\tau\left(b\right) > \tau\left(a\right) > 0$ (thus $\frac{\tau(a)}{\tau(b)} < 1$), again implying $\frac{d\pi^{eq}}{d\bar{f}} > 0$. ■

Lemma 9 *The symmetric equilibrium of the model with inelastic security costs has the following comparative statics with respect to* $\bar{f}$.

1. *If* $B < A$*, then* $a > \bar{\upsilon}$ *and* $\frac{db}{d\bar{f}} < 0$*. There is some threshold* f_0 *such that*
   if $\bar{f} < f_0$*, then* $\frac{da}{d\bar{f}} < 0$ *and* $b > \bar{\upsilon}$
   if $\bar{f} > f_0$*, then* $\frac{da}{d\bar{f}} > 0$ *and* $b < \bar{\upsilon}$.

2. If $B > A$, then $b < \bar{v}$ and $\frac{da}{df} > 0$. There is some threshold f_1 such that

if $\bar{f} < f_1$, then $\frac{db}{df} > 0$ and $a < \bar{v}$

if $\bar{f} > f_1$, then $\frac{db}{df} < 0$ and $a > \bar{v}$.

Proof. Some of the analysis will be similar to the proof of equilibrium existence and uniqueness. Equilibrium is defined by (7) and (6). As earlier, represent (6) as the condition $b = \gamma\left(a\,;\,\bar{f}\right)$, where $\gamma\left(x\,;\,\bar{f}\right) = g^{-1}\left(\left(1-\bar{f}\right)g\left(x\right)\right)$ as before, but now we emphasize the role of $\bar{f}$ by treating it as an argument. Equivalently, we can write $a = \gamma^{-1}\left(b\,;\,\bar{f}\right)$, with $\gamma^{-1}\left(x\,;\,\bar{f}\right) = g^{-1}\left(\frac{g(x)}{1-\bar{f}}\right)$. The function $\Upsilon\left(x\right)$ is positive on $(0,1)$. It is increasing (decreasing) on the interval $(0,\bar{v})$ (on $(\bar{v},1)$). Furthermore, its image satisfies $\Upsilon\left(0,\bar{v}\right) = \Upsilon\left(\bar{v},1\right) = \left(0,\Upsilon\left(\bar{v}\right)\right)$.

<u>Part 1 $B < A$</u>

If $B < A$, then for arbitrary $b \in (0,1)$, any a meeting (7) must satisfy $\Upsilon\left(a\right) = \frac{B}{A}\Upsilon\left(b\right) < \Upsilon\left(b\right)$. There are two such values of a, one to the left and one to the right of b, but only the latter is admissible. Thus we can write this equilibrium condition as $a = \kappa\left(b\right)$ where κ is a continuous function with $\kappa\left(0\right) = \kappa\left(1\right) = 1$. Define $\hat{\mu}\left(b\,;\,\bar{f}\right) = \kappa\left(b\right) - \gamma^{-1}\left(b\,;\,\bar{f}\right)$ and note that $\hat{\mu}\left(0\,;\,\bar{f}\right) > 0$. For each $\bar{f}$, by Proposition 3 there is a unique (equilibrium) b satisfying $\hat{\mu}\left(b\,;\,\bar{f}\right) = 0$. Fix f' and f'', with $f' < f''$. Let b' and b'' be the corresponding zeros of $\hat{\mu}$. Consider $\hat{\mu}\left(b'\,;\,f''\right)$. The function g^{-1} is increasing, so we have $\gamma^{-1}\left(b'\,;\,f''\right) > \gamma^{-1}\left(b'\,;\,f'\right)$. Since $\hat{\mu}\left(b'\,;\,f'\right) = 0$, we must have $\hat{\mu}\left(b'\,;\,f''\right) < 0$. Given $\hat{\mu}\left(0, f''\right) > 0$, we conclude that $b'' \in (0, b')$. This suffices to show $\frac{db}{df} < 0$. With the fact that equilibrium b tends toward 1 as $\bar{f} \to 0$ and toward 0 as $\bar{f} \to 1$, this ensures that there is some threshold f_0 such that in equilibrium, $b > \bar{v}$ if $\bar{f} < f_0$ and $b < \bar{v}$ if $\bar{f} > f_0$.

Next, recall that (7) does not depend on $\bar{f}$. Thus, once $\frac{db}{df}$ is determined, $\frac{da}{df}$ is pinned down by the co-movement of a and b along this curve: $\frac{da}{df} = \frac{db}{df}\frac{da}{db}$, where $\frac{da}{db} = \frac{B}{A}\frac{\Upsilon'(b)}{\Upsilon'(a)}$. Suppose that $\bar{f} < f_0$. The facts that $b > \bar{v}$ and $a > b$ imply $a > \bar{v}$. But this implies that $\Upsilon'\left(b\right)$ and $\Upsilon'\left(a\right)$ are both negative, and so $\frac{da}{df}$ has the same sign as $\frac{db}{df}$. Alternatively, suppose that $\bar{f} > f_0$. Since $B < A$ implies $\Upsilon\left(a\right) < \Upsilon\left(b\right)$ and Υ is increasing on $(b,\bar{v})$, we once again must have $a > \bar{v}$. Now we have $b < \bar{v} < a$, $\Upsilon'\left(b\right)$ and $\Upsilon'\left(a\right)$ have opposite signs, and therefore, $\frac{da}{df}$ and $\frac{db}{df}$ must have opposite signs.

<u>Part 2 $B > A$</u>

Most of the arguments from Part 1 reverse. In this case, we have $\Upsilon\left(b\right) = \frac{A}{B}\Upsilon\left(a\right) < \Upsilon\left(a\right)$ for the first condition. For any $a \in (0,1)$, there are two value of b that solve this equation, of which only the one lying below a is relevant. Write this condition as $b = \hat{\kappa}\left(a\right)$ where $\hat{\kappa}$ is a continuous function with $\kappa\left(0\right) = 0$ and $\kappa\left(1\right) = 0$. Define $\hat{\nu}\left(a\,;\,\bar{f}\right) = \hat{\kappa}\left(a\right) - \gamma\left(a\,;\,\bar{f}\right)$, and note that $\hat{\nu}\left(1;\,\bar{f}\right) = -1$. Fix $f' < f''$ and let a' and a'' be the corresponding zeros of $\hat{\nu}$. Observe that $\hat{\nu}\left(a', f'\right) = 0$ and $\gamma\left(a', f''\right) < \gamma\left(a', f'\right)$ imply that $\hat{\nu}\left(a', f''\right) > 0$. With $\hat{\nu}\left(1;\,\bar{f}\right) < 0$, this implies that $a'' \in (a', 1)$. This shows that $\frac{da}{df} > 0$. The existence of f_1 such that $a < \bar{v}$ for $\bar{f} < f_1$ and $a > \bar{v}$ for $\bar{f} > f_1$ follows as above.

Next observe that $b < \bar{v}$. If $a < \bar{v}$, this follows from $b < a$. If $a > \bar{v}$, this follows from $\Upsilon\left(b\right) < \Upsilon\left(a\right)$ and the fact that Υ is decreasing on $(\bar{v}, a)$. As in the previous part, $\frac{db}{df}$ has the same sign as $\frac{da}{df}$ if and only if $\frac{db}{da} = \frac{A}{B}\frac{\Upsilon'(a)}{\Upsilon'(b)}$ is positive. For $\bar{f} < f_1$, we have $b < a < \bar{v}$, $\Upsilon'\left(a\right)$ and $\Upsilon'\left(b\right)$ have the same sign, and so $\frac{db}{df} > 0$. For $\bar{f} > f_1$, we have $b < \bar{v} < a$, $\Upsilon'\left(a\right)$ and $\Upsilon'\left(b\right)$ have opposite signs, and $\frac{db}{df} < 0$. ■

Proposition 6

Proof. An individual's first order condition for z implies $Aa^2(1-a) = Bb^2(1-b)(1-f)$ in a symmetric equilibrium, just as in the previous section. Applying the aggregation relationship $1-f = \frac{a}{b}\frac{\ln(1-b)}{\ln(1-a)}$, this can once again be written as $A\Upsilon(a) = B\Upsilon(b)$. An individual choosing $\left(z, \tilde{f}\right)$ when everyone else plays (z, f) has the security first order condition

$$Bb^2 z e^{-\left(1-\tilde{f}\right)zb} - c'\left(\tilde{f}\right) = 0$$

so in a symmetric equilibrium we must have (using $e^{-(1-f)zb} = 1-b$ and $z_{eff} = (1-f)z$)

$$Bb^2(1-b)z_{eff} = (1-f)c'(f)$$

The result follows from $z_{eff} = -\frac{\ln(1-b)}{b}$. ■

Proposition 7

Proof. Fix (a^*, b^*) and let $z^* = -\frac{\ln(1-a^*)}{a^*}$ and $f^* = 1 - \frac{a^*}{b^*}\frac{\ln(1-b^*)}{\ln(1-a^*)}$ be the candidate equilibrium strategies. Let $\left(\hat{z}, \hat{f}\right)$ be the individual best response to (a^*, b^*). We require conditions under which $\left(\hat{z}, \hat{f}\right) = (z^*, f^*)$. Write $\pi(z, f\,; a, b) = Aa\left(1-e^{-az}\right) - Bb\left(1-e^{-b(1-f)z}\right) - c(f)$, and formulate an agent's choice problem in two stages:

$$\begin{aligned} \hat{z}(f) &= \arg\max_{z\in[0,\infty)} \pi(z, f\,; a^*, b^*) && (12) \\ \left(\hat{z}, \hat{f}\right) = \left(\hat{z}\left(\hat{f}\right), \hat{f}\right), \text{ where } \hat{f} &= \arg\max_{f\in[0,1]} \pi(\hat{z}(f), f\,; a^*, b^*) && (13) \end{aligned}$$

First we claim that $\hat{f} < 1$. This follows because Condition C implies $\lim_{f\to 1} c(f) = \infty$. The other payoff terms are bounded, so choosing $f = 1$ cannot be optimal. Next derive $\hat{z}(f)$. The first order condition for (12) is

$$\frac{\partial \pi(z, f\,; a^*, b^*)}{\partial z} = A(a^*)^2 e^{-a^* z} - B(b^*)^2(1-f)e^{-b^*(1-f)z} \tag{14}$$

We know that the first term decays faster than the second (because $a^* > b^*(1-f)$) and we claim that $A(a^*)^2 > B(b^*)^2$, so $\left.\frac{\partial\pi(z,f\,;a^*,b^*)}{\partial z}\right|_{z=0}$ is strictly positive. From these points, it follows that (for $f < 1$) the unique solution of $\left.\frac{\partial\pi(z,f\,;a^*,b^*)}{\partial z}\right|_{z=0}$ solves (12). Thus,

$$\hat{z}(f) = \frac{1}{a^* - b^*(1-f)}\ln\left(\frac{A(a^*)^2}{B(b^*)^2(1-f)}\right) \text{ for } f \in [0,1)$$

Note that $\hat{z}(f^*) = z^*$, so if f^* solves (13), we are done.

Turning to (13), By the envelope theorem, we have

$$
\begin{aligned}
\frac{d\pi\left(\hat{z}\left(f\right),f\,;a^{*},b^{*}\right)}{df} &= \frac{\partial\pi\left(\hat{z}\left(f\right),f\,;a^{*},b^{*}\right)}{\partial f} \\
&= B\left(b^{*}\right)^{2}\hat{z}\left(f\right)e^{-b^{*}(1-f)\hat{z}(f)}-c'\left(f\right)
\end{aligned}
$$

If $c = c_m$ (for any m), then f^* is a zero of this first order condition by construction. To show that f^* is optimal, it suffices to show that $\frac{d\pi(\hat{z}(f),f\,;a^*,b^*)}{df}$ is positive on $[0, f^*)$ and negative on $(f^*, 1)$. We will choose m large enough to ensure this this is true for c_m.

$\underline{f \in [0, f^*)}$ Define $\delta(f) = \frac{c_1'(f)}{B(b^*)^2\hat{z}(f)e^{-b^*(1-f)\hat{z}(f)}}$ and note that $\delta(f^*) = 1$. It suffices to show that we can choose m_l such that $r_m(f)\,\delta(f) < 1$ for all $m > m_l$ and all $f \in [0, f^*)$. The function $\delta(f)$ is weakly positive and continuously differentiable on $[0, 1)$, so let $L_1 = \max_{f\in[0,f^*]}\left|\delta'(f)\right|$. Let $L_2 = \min_{f\in[0,f^*]} c_1''(f) > 0$. Observe that

$$
r_1\left(f\right) = \frac{c_1'\left(f\right)}{c_1'\left(f^*\right)} \leq 1 - \frac{L_2}{c_1'\left(f^*\right)}\left|f^*-f\right| = 1 - L_3\left|f^*-f\right| \leq \frac{1}{1+L_3\left|f^*-f\right|}
$$

where $L_3 = \frac{L_2}{c_1'(f^*)}$. Then we have

$$
r_m\left(f\right) \leq \frac{1}{\left(1+L_3\left|f^*-f\right|\right)^m} \leq \frac{1}{1+mL_3\left|f^*-f\right|}
$$

Also note that $\delta(f) \leq 1 + L_1|f^* - f|$. Combining, we have

$$
r_m\left(f\right)\delta\left(f\right) \leq \frac{1+L_1\left|f^*-f\right|}{1+mL_3\left|f^*-f\right|}
$$

Choosing $m_l \geq L_1/L_3$ ensures that the righthand side is strictly less than one for all $m > m_l$, uniformly in $f \in [0, f^*)$.

$\underline{f \in (f^*, 1)}$ First, we claim that $\lim_{f\to 1}\delta(f) = \infty$. To see this, use (14) to write

$$
\delta\left(f\right) = \frac{\left(1-f\right)c_1'\left(f\right)}{A\left(a^*\right)^2\hat{z}\left(f\right)e^{-a^*\hat{z}(f)}}
$$

As $f \to 1$, $\hat{z}(f) \to \infty$, so the denominator tends to zero while the numerator tends to infinity by Condition C. Given this, there exists some $L_4 = \left|\min_{[f^*,1)}\frac{d\ln\delta(f)}{df}\right| < \infty$, and so we have $\ln\frac{\delta(f)}{\delta(f^*)} = \ln\delta(f) \geq -L_4(f - f^*)$. Meanwhile,

$$
r_1\left(f\right) \geq 1 + L_5\left(f-f^*\right) \quad \text{where } L_5 = \frac{c_1''\left(f^*\right)}{c_1'\left(f^*\right)} > 0
$$

so $\ln r_m(f) \geq m\ln(1 + L_5(f - f^*))$. Finally, $\ln(r_m(f)\,\delta(f)) \geq m\ln(1 + L_5(f - f^*)) - L_4(f - f^*)$. To guarantee $r_m(f)\,\delta(f) > 1$ for all $f \in (f^*, 1)$, we require

$$
m > \left.\frac{L_4 x}{\ln\left(1+L_5 x\right)}\right|_{x=f-f^*} \quad \text{for all } f \in \left(f^*,1\right)
$$

The function $\frac{L_4 x}{\ln(1+L_5 x)}$ is increasing, so set $m_h = \frac{L_4(1-f^*)}{\ln(1+L_5(1-f^*))}$. Then, for $m > m_h$, $r_m(f)\,\delta(f) > 1$ holds uniformly over $f \in (f^*, 1)$.

To complete the proof, set $\bar{m} = \max(m_l, m_h)$. It only remains to prove the claim that $A(a^*)^2 > B(b^*)^2$. From $A\Upsilon(a^*) = B\Upsilon(b^*)$ it follows that $\frac{A(a^*)^2}{B(b^*)^2} = \left(-\frac{(1-b^*)\ln(1-b^*)}{b^*}\right) / \left(-\frac{(1-a^*)\ln(1-a^*)}{a^*}\right)$. The function $-\frac{(1-x)\ln(1-x)}{x}$ is positive and decreasing on $(0,1)$, so the claim follows from $b^* < a^*$. ∎

Proposition 8

Proof. Because equilibria only depend on the ratio A/k, it suffices to fix $k = 1$ and show the result for a threshold $\underline{A}$ on A. First observe that as $B \to \infty$, the condition $A\Upsilon(a) = B\Upsilon(b)$ implies that any sequence of equilibria must have b tending to zero. With this in mind, define a function equal to the limiting value of the other equilibrium condition as $b \to 0$:

$$\begin{aligned} Q(a) &= \lim_{b\to 0}\left(A\Upsilon(a) - (1-f)\,c'(f)\Big|_{f=1-\frac{a}{b}\frac{\ln(1-b)}{\ln(1-a)}} \right) \\ &= A\Upsilon(a) - (1-f)\,c'(f)\Big|_{f=1+\frac{a}{\ln(1-a)}} \end{aligned}$$

If $Q(a) = 0$ has a positive solution $a^* > 0$, then there is a sequence of equilibria with total connectivity tending to a^* as $B \to \infty$. If furthermore individual payoffs are positive in this limit, then the economy is resilient. On the other hand, if $Q(a) = 0$ has no solution besides $a = 0$, then the economy is not resilient.

Part (i)

The function $Q(a)$ is negative for a large enough, so to establish the existence of some $\hat{a} > 0$ with $Q(\hat{a}) = 0$, it suffices to show that $Q(a)$ is strictly positive for a small. Observing that $f = 1 + \frac{a}{\ln(1-a)}$ tends to zero with a, expand the terms of $Q(a)$ around $a = 0$ and $f = 0$ to get:

$$\begin{aligned} A\Upsilon(a) &= Aa^2 - \frac{1}{2}a^3 + o\left(a^3\right) \\ (1-f)\,c'(f) &= (1-f)\left(c'(0) + fc''(0) + \frac{f^2}{2}c'''(0) + \frac{f^3}{6}c^{(4)}(0) + \ldots\right) \\ &= \frac{f^3}{6}c^{(4)}(0) + o\left(f^3\right) \quad \text{and} \\ f &= 1 - \frac{a}{a + \frac{a^2}{2} + \frac{a^3}{3} + \ldots} = \frac{a}{2} + o(a) \end{aligned}$$

In summary, $Q(a) = Aa^2 - \left(\frac{1}{2} + \frac{1}{48}c^{(4)}(0)\right)a^3 + o\left(a^3\right)$. Thus, for any $A > 0$, we can choose a small enough to make $Q(a)$ strictly positive.

Next, we show that limiting payoffs are strictly positive. Suppose that $a \to \hat{a}$ as $B \to \infty$. Write the limiting payoff as

$$\hat{\pi} = A\hat{a}^2 - \widehat{Bb^2} - \widehat{c(f)}$$

where $\widehat{Bb^2}$ and $\widehat{c(f)}$ are the limiting values of Bb^2 and $c(f)$ respectively. Using equilibrium conditions we have

$$\frac{A\Upsilon(\hat{a})}{\widehat{Bb^2}} = \lim_{\substack{B\to\infty,\\ b\to 0}} \frac{B\Upsilon(b)}{Bb^2} = \lim_{b\to 0}\left(-\frac{(1-b)\ln(1-b)}{b}\right) = 1$$

so $\widehat{Bb^2} = A\Upsilon(\hat{a})$. Convexity of c and $c(0) = 0$ gives us $c(f) \leq fc'(f)$, so by the other equilibrium condition we have $c(f) \leq \frac{f}{1-f}A\Upsilon(a)$. Thus,

$$\widehat{Bb^2} + \widehat{c(f)} \leq \frac{A\Upsilon(\hat{a})}{1-\hat{f}} \ ,$$

or using $1 - \hat{f} = -\frac{\hat{a}}{\ln(1-\hat{a})}$, we arrive at

$$\hat{\pi} \geq A\left(\hat{a}^2 - (1-\hat{a})\ln(1-\hat{a})^2\right).$$

The function $g(x) = x^2 - (1-x)\ln(1-x)^2$ satisfies $g(0) = g'(0) = 0$ and $g''(x) = -2\frac{x+\ln(1-x)}{1-x}$. Note that $x+\ln(1-x)$ is strictly negative on $(0,1)$, so g is strictly convex on $(0,1)$. This establishes that $g(x)$ is strictly positive on $(0,1)$, so the limiting payoff $\hat{\pi}$ is also strictly positive.

Part (ii)

Suppose that $c'(0) = c''(0)$ and $c'''(0) > 0$; the other cases are simple extensions. Following the same expansion as before, we have

$$(1-f)\,c'(f)\big|_{f=1+\frac{a}{\ln(1-a)}} = \frac{c'''(0)}{8}a^2 + o\left(a^2\right)$$

so $Q(a) = \left(A - \frac{c'''(0)}{8}\right)a^2 + o\left(a^2\right)$. The strategy is to show that we can pick A small enough that $Q(a) < 0$ for all $a > 0$. Let $A_1 = \frac{c'''(0)}{16}$, so $Q(a) = -A_1a^2 - (A_1 - A)\,a^2 + o\left(a^2\right)$. There exists $\bar{a}$ small enough that $Q(a) < 0$ whenever $A < A_1$ and $a < \bar{a}$. Now return to the original formulation $Q(a) = A\Upsilon(a) - (1-f)\,c'(f)\big|_{f=1+\frac{a}{\ln(1-a)}}$. Let $L = (1-f)\,c'(f)\big|_{f=1+\frac{a}{\ln(1-a)},a=\frac{\bar{a}}{2}}$. Because $\Upsilon(a)$ is bounded above, there exists $A_2 > 0$ such that $A\Upsilon(a) < L$ for all $A < A_2$ and all $a \in (0,1)$. Then, because $(1-f)\,c'(f)$ is increasing in f, and $f = 1 + \frac{a}{\ln(1-a)}$ is increasing in a, we have $Q(a) < 0$ whenever $A < A_2$ and $a > \frac{\bar{a}}{2}$. Choose $\underline{A} = \min(A_1, A_2)$. Then for all $A < \underline{A}$, we have $Q(a)$ strictly negative on both $(0, \bar{a})$ and $\left(\frac{\bar{a}}{2}, 1\right)$, so $Q(a)$ is negative on $(0,1)$ as claimed.

Alternatively, if $c'(0) > 0$ or $c''(0) > 0$, then $Q(a)$ has a negative term of order a^0 or a^1 respectively, so *a fortiori* the same method applies. ■