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# Equilibrium Nonexistence in Spatial Competition with Quadratic Transportation Costs\*

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## Abstract

Under quadratic transportation costs, the existence of the sequential first-locate-then-price equilibrium in spatial competition is well known in the literature. In this paper, we find that the equilibrium may fail to exist under certain restrictions with respect to the location of firms and consumers in the market. This result is valid for both the linear and the circular models.

**Keywords:** Product differentiation, circular model, linear model, quadratic transportation costs, sequential equilibrium.

**JEL Classification:** C72, D43.

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## 1. Introduction

Rather frequently, commercial areas locate outside the cities. When consumers decide the store where to buy a given product, they then consider both the price and the transportation cost. Stores must account for this aspect when they first decide their location along the commercial area and the price to be charged for the product.

Although normally observed in practise, surprisingly the literature on spatial competition has not considered the existence of restrictions with respect to the locations of firms and consumers in cities, in particular, their effects on the selected locations and prices. One exception is the linear model of vertical differentiation proposed by Gabszewicz and Thisse (1986), which considers a uniform mass of consumers located along a linear city and two firms located outside one of the extremes of the city. It is well known that a first-location-then-price equilibrium does exist in this model, independently of transportation costs being convex or concave in distance.<sup>1</sup>

However, the vertical differentiation model is a rather hard simplification of real life situations. Quite often, we observe that commercial areas attract not only consumers of a particular neighbouring city or suburb, but consumers from several surrounding locations.

An illustrative example is the usual case of two linear cities with a commercial area between the two.<sup>2</sup> However, this case has not been analyzed in the literature, which, starting with Hotelling (1929), studies the model of horizontal differentiation where no location restrictions apply. The existence of the sequential equilibrium in this model is questioned in the literature, although it is well known that such an equilibrium exists in the case where the transportation cost are quadratic in distance, see D'Aspremont et al.

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<sup>1</sup> Gabszewicz and Thisse (1986) show the existence of equilibrium in the case where the transportation cost is linear-quadratic in distance, i.e.,  $C(d) = ad + bd^2$ ,  $a \geq 0$ ,  $b \geq 0$ , where  $d$  is the distance between the consumer and the firm. Very recently, Arguedas et al. (2005) have confirmed this existence result under the concave specification  $C(d) = ad - bd^2$ ,  $a \geq 0$ ,  $b \geq 0$ .

<sup>2</sup> This would correspond to consider two overlapped models of vertical differentiation.

(1979).<sup>3</sup> A common result in the literature is that, under existence, the sequential equilibrium involves maximal differentiation.

But location restrictions may also apply to circular cities, a case which has not been analyzed in the literature either. Till now, conclusions with respect to the circle model with no location restrictions summarize in the existence of the sequential equilibrium at least in the quadratic case and under certain concave and convex specifications of the transportation costs, see Anderson (1986) or De Frutos et al. (1999, 2002).

In this paper, we analyze whether the sequential equilibrium would persist under the mentioned restrictions on the locations of firms and consumers. In particular, we analyze the case of quadratic transportation costs, which, as explained above, is unquestioned in both the linear and the circular models.

Our results are rather negative. We find that there exists no sequential equilibrium in any model (linear or circular) when there is a separation between the residential and the commercial areas.

In the linear model, we find that there exist no locations of the firms in the commercial area for which Nash price equilibrium exists, other than locating exactly in the same place and charging a zero price (the Bertrand solution). The result is quite surprising, since intuition would suggest that firms would be tempted to differentiate as much as possible within the commercial area to avoid competition. However, if firms differentiated, stability in prices would not be possible. Given the discontinuity of the region where consumers live, firms would be tempted to either increase prices when the consumers of their own hinterland strictly prefer their products to the ones of their competitors or decrease prices when their natural consumers start thinking of travelling to the other firm to save total costs.

In the circle model, results are not that harsh. As before, there exists no Nash price equilibrium for all the possible locations of the firms in the commercial area. However,

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<sup>3</sup> However, under the linear-quadratic specification of Gabszewicz and Thisse (1986) and symmetric locations, the sequential equilibrium does not exist, a result which is after extended by Anderson (1988) to the case of asymmetric locations, and to functions of the type  $C(d) = d^n$ ,  $1 \leq n \leq 1.26$ , by Economides (1986). Arguedas et al. (2005) confirm the inexistence of the sequential equilibrium under concave transportation costs.

at least we find a region where such price equilibrium exists, which is positively related to the length of the commercial area.

The remainder of the paper is organized as follows. In Section 2, we present the model. In Section 3, we analyze the equilibrium existence under location restrictions in the linear model. In Section 4, we study the circular model. We conclude in Section 5. All the proofs are in the Appendix.

## 2. The model

We consider the two traditional models of spatial competition with a slight variation with respect to the allowed locations of both consumers and firms in the market.

Our **linear model** is a market of length 1 composed of two parts (Figure 1): a commercial area of length  $0 \leq v \leq 1$  where firms locate, and a residential area of length  $1 - v$  where consumers live. For convenience, we assume that the commercial area is centrally situated between locations  $1/2 - v/2$  and  $1/2 + v/2$ . This area is occupied by two firms

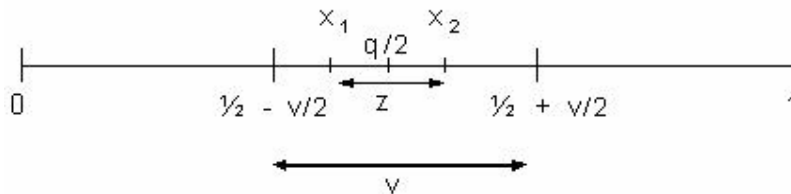


Figure 1: The linear model

which sell a homogeneous commodity with zero production costs. We denote by  $x_i$  the location of firm  $i$  in this area, such that  $1/2 - v/2 \leq x_1 \leq x_2 \leq 1/2 + v/2$ . Let  $z$  denote the distance between the two firms, i.e.,  $z = x_2 - x_1$ . Also let  $q$  be the sum of the two firms locations,  $q = x_1 + x_2$ . Given firms' locations,  $q/2$  represents the equidistant point between the two firms and constitutes a useful symmetry measure, as we will see later on. For given locations, firm  $i$  chooses the mill price which maximizes profits, i.e., the price times the number of units sold.

Consumers uniformly locate along the residential area. Let denote the consumer location in this area as  $x \in [0, \frac{1}{2} - v/2] \cup [0, \frac{1}{2} - v/2]$ . Each consumer buys only one unit of the goods at the firm with the lowest total cost, that is, the mill price plus the transportation cost.<sup>4</sup> The distance between the consumer and firm  $i$  is defined by  $d_i = |x - x_i|$ ,  $i = 1, 2$ . We assume that transportation costs are quadratic, as follows:

$$C(d_i) = d_i^2 \quad (1)$$

As noted in the introduction, this model integrates two cases of vertical differentiation: in the first one, consumers locate in  $[0, \frac{1}{2} - v/2]$  and firms locate in  $[\frac{1}{2} - v/2, \frac{1}{2} + v/2]$ ; in the second one, firms locate in the same region  $[\frac{1}{2} - v/2, \frac{1}{2} + v/2]$  and consumers live in  $[\frac{1}{2} + v/2, 1]$ .

For instance, this situation may reflect the case where two cities share a commercial area located between the two.<sup>5</sup>

Alternatively, our **circular model** consists of a market of perimeter 1 composed of a commercial area of length  $0 \leq v \leq \frac{1}{2}$  and a residential area of length  $1 - v$  (*Figure 2*). Now, the locations of the two firms satisfy that  $0 \leq x_1 \leq x_2 \leq v$  and the location of a typical consumer in the residential area satisfies  $x \in [1 - v]$ . The remaining ingredients are exactly the same as those of the linear model described above.

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<sup>4</sup> For simplicity, we assume that all consumers have enough willingness to pay. This assumption is common in all the literature on spatial differentiation.

<sup>5</sup> Alternatively, this model refers to the case where, for equal prices, a clearly identified subset of consumers prefer the product of firm 1 while the remaining consumers prefer the product of firm 2.

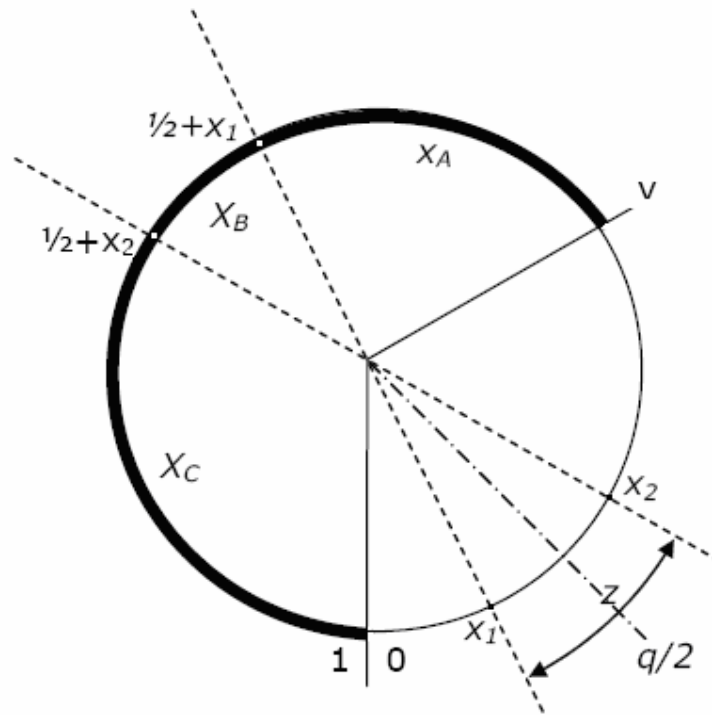


Figure 2: The circular model

The location of the indifferent consumer in any of the two described models is determined as follows:

$$p_1 + C(d_1) = p_2 + C(d_2) \quad (2)$$

We consider a sequential game where firms first decide their locations in the commercial area and then, they choose prices which maximize profits given the selected locations. We concentrate on the concept of sub-game perfect equilibrium. Thus, we solve backwards, first finding the Nash equilibrium prices for given locations.

In the next section, we analyze the existence of the sequential equilibrium in the linear model of spatial competition.

### 3. Equilibrium existence in the linear model

Consider the linear model as described in the previous section (*Figure 1*). We first derive firms' demands and then we analyze the existence of the sequential equilibrium.

### 3.1. Demand functions

To obtain the demand functions, we first obtain the location of the indifferent consumer, described by (2). Considering (1) and  $d_i = |x - x_i|$ ,  $i = 1, 2$ , the expression of the indifferent consumer is the following:

$$X = \frac{p_2 - p_1}{2z} + \frac{q}{2} \quad (3)$$

By the construction of the model, all consumers located to the left of  $X$  select firm 1, while the remaining consumers choose firm 2. Therefore, depending on the location of the indifferent consumer on the line, the demand of firm 1 is the following:<sup>6</sup>

$$D_1 = \begin{cases} 1 - v, & p_1 - p_2 \in (-\infty, -z(2 - q)] \\ X - v, & p_1 - p_2 \in [-z(2 - q), z(q - 1 - v)] \\ \frac{1-v}{2}, & p_1 - p_2 \in [z(q - 1 - v), z(q - 1 + v)] \\ X, & p_1 - p_2 \in [z(q - 1 + v), zq] \\ 0, & p_1 - p_2 \in [zq, +\infty) \end{cases} \quad (4)$$

The first part of (4) refers to the case where  $X \geq 1$  and, consequently, firm 1 attracts all the demand. In part 2,  $X$  belongs to the interval  $[\frac{1}{2} + v/2, 1]$ . In part 3, there exists a fictitious indifferent consumer situated in the commercial area  $[\frac{1}{2} - v/2, \frac{1}{2} + v/2]$  and, consequently, each firm attracts its own hinterland only. Part 4 refers to the case where the indifferent consumer lives in the region  $[0, \frac{1}{2} - v/2]$ . Finally, part 5 reflects the case where  $X \leq 0$  and consequently, no consumer buys at firm 1.

Since there are no production costs, the benefit functions are  $B_i = p_i D_i$  for all  $i$ . Given (3), (4) and the fact that  $D_2 = 1 - v - D_1$ , the corresponding expressions for the firms' benefit functions are as follows:

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<sup>6</sup> The demand of firm 2 is simply  $D_2 = 1 - v - D_1$



$$B_1 = \begin{cases} p_1(1-v), & p_1 - p_2 \in (-\infty, -z(2-q)] \\ p_1 \left( \frac{p_1 - p_2}{2z} + \frac{q-2v}{2} \right), & p_1 - p_2 \in [-z(2-q), z(q-1-v)] \\ p_1 \frac{1-v}{2}, & p_1 - p_2 \in [z(q-1-v), z(q-1+v)] \\ p_1 \left( \frac{p_1 - p_2}{2z} + \frac{q}{2} \right), & p_1 - p_2 \in [z(q-1+v), zq] \\ 0, & p_1 - p_2 \in [zq, +\infty) \end{cases}$$

$$B_2 = \begin{cases} 0, & p_1 - p_2 \in (-\infty, -z(2-q)] \\ p_2 \left( \frac{p_1 - p_2}{2z} + \frac{2-q}{2} \right), & p_1 - p_2 \in [-z(2-q), z(q-1-v)] \\ p_2 \frac{1-v}{2}, & p_1 - p_2 \in [z(q-1-v), z(q-1+v)] \\ p_2 \left( \frac{p_1 - p_2}{2z} + \frac{2(1-v)-q}{2} \right), & p_1 - p_2 \in [z(q-1+v), zq] \\ p_2(1-v), & p_1 - p_2 \in [zq, +\infty) \end{cases}$$

### 3.2. Equilibrium existence

We now analyze the existence of a Nash-price equilibrium for given locations. Given  $(x_1, x_2)$ , a Nash-price equilibrium is a pair  $(p_1^*, p_2^*)$  such that each firm selects the price which maximizes profits, considering the other firm's equilibrium price as given. That is,  $(p_1^*, p_2^*)$  satisfies the following:

$$B_i(p_i^*, p_j^*) \geq B_i(p_i, p_j^*), \text{ for all } i \text{ and } j, i \neq j$$

It is well known that Nash price equilibrium exists under quadratic transportation costs when there are no restrictions on firms and consumers' locations, see D'Aspremont et al. (1979). However, we now show that the equilibrium fails to exist under those restrictions, i.e., when  $v \geq 0$ . The result is now summarized in the following:

**Proposition 1.** *Given  $v \geq 0$ , the only possible Nash-price equilibrium in the linear model implies that  $p_1^* = p_2^* = 0$  and  $z = 0$ .*

This result is quite surprising. In fact, one would expect prices' equilibria to exist if firms differentiated as much as they can to obtain maximum profits. Also, given the symmetry of the problem, one would also expect these prices to be equal. In fact, this is what the literature on spatial differentiation predicts, at least when there are no

restrictions on firms and consumers' locations. However, when these location restrictions apply, we find that there exist no possible firms' locations for which such a price equilibrium exists other than the minimum differentiation result, which yields to Bertrand's solution.

The explanation of this result is the following. Assume that, initially, the two firms charge the same price (including zero as a possibility) and each one locates at one extreme of the commercial area. Then, each firm attracts consumers of its own hinterland only. The reason is that there exists a separation between the firms and, therefore, there is a transportation cost associated with travelling from one firm to the other. Knowing this, one of the firms may decide to increase the price to increase its revenues, at least till consumers of its own hinterland are indifferent between this and the other firm. But then, the other firm may decide to decrease its price to start attracting consumers of the other hinterland. And the first firm may decrease its price as well to try to recover some of the lost consumers. Etc. In summary, no possible price equilibrium exists when firms differentiate as much as they can. Moreover, the same type of argument can be applied when there is some differentiation between the firms, not necessarily maximum. As a consequence, the only possible equilibrium is no separation between firms and competition *a la Bertrand*.

#### **4. Equilibrium existence in the circular model**

In this section, we analyze the equilibrium existence in the circular model. We present the results in the same way as those of the previous section, first determining the demand functions and then analyzing the existence of the sequential equilibrium.

##### **4.1. Demand functions**

To obtain the demand functions, we first calculate the location of the indifferent consumer(s) in the circle. Remember that the indifferent consumer satisfies (2). To do this, we first distinguish three regions in the area where consumers live, depending on the way they take to travel to the firms. These are regions A, B and C (*Figure 2*). Locations of consumers in these areas satisfy  $x_A \in [v, x_1 + 1/2]$ ,  $x_B \in [x_1 + 1/2, x_2 + 1/2]$ , and

$x_C \in [x_2+1/2, I]$ , respectively. Thus, all consumers in region A travel clockwise whereas all consumers in C travel counter clockwise, independently of the selected firm. In the case of region B, only consumers who choose firm 2 travel clockwise.

We first concentrate on region A to obtain the location of the indifferent consumer  $X$ . Considering (1) and (2),  $X_A$  satisfies the following:

$$p_1 + (X_A - x_1)^2 = p_2 + (X_A - x_2)^2,$$

since any consumer in region A travels clockwise. Thus, the location of the indifferent consumer in this region is:

$$X_A = \frac{p_2 - p_1}{2z} + \frac{q}{2} \quad (5)$$

Since  $X_A \in [v, x_1+1/2]$ , it is easy to see that there exists an indifferent consumer in region A characterized by (5) if and only if  $z(2v - q) \leq p_2 - p_1 \leq z(I - z)$ . Since, by definition,  $v \geq q/2$ , a necessary condition for  $X_A$  to exist is clearly  $p_2 \geq p_1$ . Else, all consumers in region A would prefer firm 2, since it is the nearest firm. Given region A and  $X_A$ , all consumers to the right of  $X_A$  travel clockwise to firm 1, and all consumers to the left of  $X_A$  travel clockwise to firm 2.

In region B, consumers selecting firm 2 travel clockwise while the remaining consumers travel counter clockwise. Therefore, the location of the indifferent consumer in this region,  $X_B$ , satisfies the following:

$$p_1 + [(I - X_B) + x_1]^2 = p_2 + (X_B - x_2)^2$$

Operating in this expression, we obtain:

$$X_B = \frac{p_1 - p_2}{2(1 - z)} + \frac{1 + q}{2} \quad (6)$$

Since  $X_B \in [x_1+1/2, x_2+1/2]$ , there exists an indifferent consumer in region B if and only if  $-z(I - z) \leq p_2 - p_1 \leq z(I - z)$ . Here, it is interesting to see that both  $p_1 \geq p_1$  and  $p_1 \leq p_2$  are valid for  $X_B$  to exist. In fact, in the particular case where  $p_1 = p_2$ , the indifferent consumer is located exactly opposite to the equidistant point between the firms,  $q/2$ .

Thus, consumers to the right of  $X_B$  select firm 2 travelling clockwise, whereas consumers to the left of  $X_B$  choose firm 1 travelling counter clockwise.

Finally, all consumers in region C travel counter clockwise. Therefore, the location of the indifferent consumer in this region,  $X_C$  satisfies the following:

$$p_1 + [(1 - X_C) + x_1]^2 = p_2 + [(1 - X_C) + x_2]^2,$$

from which we obtain:

$$X_C = \frac{p_2 - p_1}{2z} + 1 + \frac{q}{2} \quad (7)$$

Since  $X_C \in [x_2 + 1/2, 1]$ , we can conclude that there exists an indifferent consumer in region C if and only if  $-z(1-z) \leq p_2 - p_1 \leq -zq$ . In this case, a necessary condition for  $X_C$  to exist is  $p_2 \leq p_1$ . Else, all consumers in region C would select firm 1, their nearest firm. Thus, given region C and  $X_C$ , all consumers in  $[x_2 + 1/2, X_C]$  travel counter clockwise to firm 1 while the remaining consumers travel counter clockwise to firm 2.

Next, a relevant question to derive the demand functions is to determine the number of indifferent consumers along the residential area. This crucially depends on the prices difference,  $p_2 - p_1$ . By our assumptions, we have  $-z(1-z) < -zq < z(2v-q) < z(1-z)$ . Therefore, considering all the previous analysis, there exist two indifferent consumers when  $-z(1-z) < p_2 - p_1 < -zq$ , those given by expressions (6) and (7). Also, there exist two indifferent consumers when  $z(2v-q) < p_2 - p_1 < z(1-z)$ , those given by expressions (5) and (6). However, there is only one indifferent consumer given by (6) when  $-zq < p_2 - p_1 < z(2v-q)$ . In the remaining cases, only one firm attracts all the demand.

Consequently, the demand function of firm 1 can be expressed as follows:<sup>7</sup>

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<sup>7</sup> The corresponding demand of firm 2 is simply  $D_2 = 1 - v - D_1$

$$D_1 = \begin{cases} 1 - v, & p_1 - p_2 \in (-\infty, -z(1-z)] \\ (1-v) - (X_B - X_A), & p_1 - p_2 \in [-z(1-z), -z(2v-q)] \\ 1 - X_B, & p_1 - p_2 \in [-z(2v-q), zq] \\ X_C - X_B, & p_1 - p_2 \in [zq, z(1-z)] \\ 0, & p_1 - p_2 \in [z(1-z), \infty) \end{cases} \quad (8)$$

Firms' profits are  $B_i = D_i p_i$ , since production costs are zero. Therefore, considering (5), (6), (7), (8) and  $D_2 = 1 - v - D_1$  we have:

$$B_1 = \begin{cases} (1-v)p_1, & p_1 - p_2 \in (-\infty, -z(1-z)] \\ \left(\frac{p_1 - p_2}{2z(1-z)} + \frac{1}{2} - v\right)p_1, & p_1 - p_2 \in [-z(1-z), -z(2v-q)] \\ \left(\frac{p_1 - p_2}{2(1-z)} + \frac{1}{2} - \frac{q}{2}\right)p_1, & p_1 - p_2 \in [-z(2v-q), zq] \\ \left(\frac{p_1 - p_2}{2z(1-z)} + \frac{1}{2}\right)p_1, & p_1 - p_2 \in [zq, z(1-z)] \\ 0, & p_1 - p_2 \in [z(1-z), \infty) \end{cases}$$

$$B_2 = \begin{cases} 0, & p_1 - p_2 \in (-\infty, -z(1-z)] \\ \left(\frac{p_1 - p_2}{2z(1-z)} + \frac{1}{2}\right)p_2, & p_1 - p_2 \in [-z(1-z), -z(2v-q)] \\ \left(\frac{p_1 - p_2}{2(1-z)} + \frac{1}{2} - v + \frac{q}{2}\right)p_2, & p_1 - p_2 \in [-z(2v-q), zq] \\ \left(\frac{p_1 - p_2}{2z(1-z)} + \frac{1}{2} - v\right)p_2, & p_1 - p_2 \in [zq, z(1-z)] \\ (1-v)p_2, & p_1 - p_2 \in [z(1-z), \infty) \end{cases}$$

In the next subsection, we analyze the existence of the sequential equilibrium in the circular model.

#### 4.2. Equilibrium existence

For given locations, a Nash-price equilibrium is a pair  $(p_1^*, p_2^*)$  such that each firm selects the price which maximizes profits, considering the other firm's equilibrium price as given. That is,  $(p_1^*, p_2^*)$  satisfies:

$$B_i(p_i^*, p_j^*) \geq B_i(p_i, p_j^*), \text{ for all } i \text{ and } j, i \neq j \quad (9)$$

We now characterize the equilibrium in the following proposition. The result also shows that such equilibrium does not necessarily exist for all the possible locations of the firms.<sup>8</sup>

**Proposition 2.** *Given  $v \geq 0$ , the Nash-price equilibrium in the circular model is the following:*

$$p_1^* = (1/3)(1 - z)(3 - q - v) \qquad p_2^* = (1/3)(1 - z)(3 + q - 4v) \qquad (10)$$

*Such equilibrium exists if and only if  $(v, q, z) \in A$*

In contrast with the linear model, we have now determined location regions where a Nash price equilibrium can exist. This region is clearly dependent on the length of the commercial area. In *Figure 3*, we illustrate that the equilibrium region increases with  $v$  (we analyze the particular cases where  $v = 1/4$ ,  $v = 1/3$ , and  $v = 1/2$ ). In the horizontal axis we measure  $z$ , the distance between the firms, and in the vertical axis we measure  $q$ , the double of the centrality within the commercial area. The valid region A is shown shadowed.

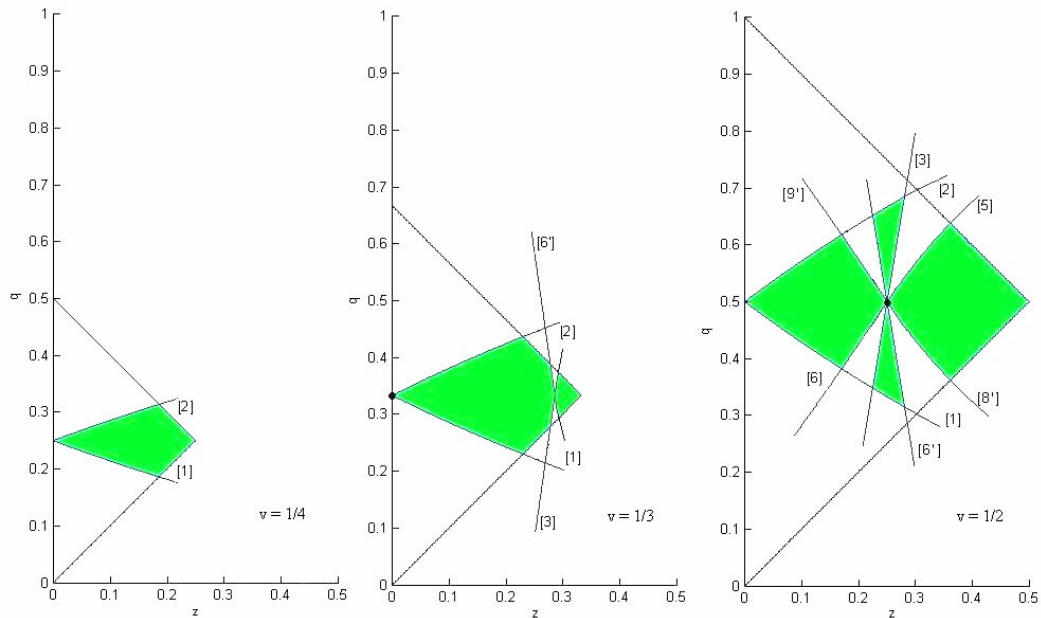


Figure 3: Equilibrium regions for  $v=1/4$ ,  $v=1/3$  and  $v=1/2$ .

<sup>8</sup> The set A in the proposition is properly defined in the Appendix.

## 5. Concluding comments

In this paper, we have shown that, under location restrictions of both firms and consumers, the sequential equilibrium may fail to exist. This result is particularly important since we have considered the case of quadratic transportation costs, an assumption which undoubtedly leads to existence and uniqueness of the equilibrium in the two traditional linear and circular models of spatial competition.

We have found that, while there is an equilibrium region in the circular model for which a Nash price equilibrium may exist, however there are no feasible firms' locations in the linear model for which we can obtain such equilibrium. In other words, while it is still possible to explain some degree of differentiation in the circular model under location restrictions, however this is not possible under linear specifications.

It is interesting to note that we have imposed two types of restrictions in firms and consumers' locations. First, we have established an area where firms must locate. Second, we have prevented consumers from locating in that area. This second restriction is the one which breaks the sequential equilibrium down in both models. In fact, it is rather simple to prove that equilibrium exists under firms' location restrictions but no consumers' location restrictions.

## 6. Appendix

**Proof of Proposition 1.** We first analyze whether there exists a Nash-price equilibrium in the region  $R_2 = \{(p_1, p_2) / p_1 - p_2 \in [-z(2 - q), z(q - 1 - v)]\}$ . If equilibrium were to exist in this region, it would be the following:

$$p_1^* = (1/3)z(2 + q - 4v) \quad p_2^* = (1/3)z(4 - q - 2v) \quad (11)$$

and the corresponding profits would be

$$B_1(p_1^*, p_2^*) = (1/18)z(2 + q - 4v)^2 \quad B_2(p_1^*, p_2^*) = (1/18)z(4 - q - 2v)^2$$

Considering (11), we have  $p_1^* - p_2^* = (2/3) z (q - 1 - v)$ . We now have to verify that  $(p_1^*, p_2^*) \in R_2$ . It is clear that  $-z(2 - q) < p_1^* - p_2^*$ , since, by our assumptions,  $q < 2(2 - v)$ . However,  $p_1^* - p_2^* \leq z(q - 1 - v)$  would require  $q \geq 1 + v$ , which, by our assumptions holds only when  $q = 1 + v$ . Therefore, this would mean that  $p_1^* = p_2^* + z(q - 1 - v)$ , and consequently, the indifferent consumer would be located at  $v + 1/2$ .

Using the same argument, we can also conclude that there exists no equilibrium in the region  $R_4 = \{(p_1, p_2) / p_1 - p_2 \in [z(q - 1 + v), zq]\}$

Finally, in region  $R_3 = \{(p_1, p_2) / p_1 - p_2 \in [z(q - 1 - v), z(q - 1 + v)]\}$ , there exists no equilibrium either. If it were to exist, it would be such that:

$$p_1^* = p_2 + z(q - 1 + v) \quad p_2^* = p_1 + z(q - 1 - v),$$

since firms' profits are increasing with  $p_1$  in region  $R_3$ . However, there exists a contradiction between  $p_1^*$  and  $p_2^*$ , since  $v > 0$ .

Therefore, there exists no Nash price equilibrium in this model, except when  $v = 0$ . This also implies  $z = 0$  and consequently,  $p_1^* = p_2^* = 0$ , which corresponds to Bertrand's solution.

**Proof of Proposition 2.** For given locations, if a Nash price equilibrium were to exist, it would be such that  $p_1 - p_2 \in [-z(2v - q), zq]$ . Considering (9), the expressions for the equilibrium prices and the corresponding firms' profits at the equilibrium are:

$$p_1^* = (1/3)(1 - z)(3 - q - 2v) \quad p_2^* = (1/3)(1 - z)(3 + q - 4v) \quad (12)$$

$$B_1(p_1^*, p_2^*) = (1/18)(1 - z)(3 - q - 2v)^2 \quad B_2(p_1^*, p_2^*) = (1/18)(1 - z)(3 + q - 4v)^2$$

To guarantee that (12) is a Nash equilibrium, we need to ensure that the prices differences belong to the appropriate range and that each price is a best response for each firm given the other firm's price. That is:

$$(i) \quad p_1 - p_2 \in [z(q - 1 + v), zq]$$

$$(ii) \quad B_1(p_1^*, p_2^*) \geq B_1(p_1, p_2^*), \text{ for all } p_1 \geq 0$$



$$(iii) B_2(p_1^*, p_2^*) \geq B_2(p_1, p_2), \text{ for all } p_2 \geq 0$$

Considering (12), we obtain that condition (i) is satisfied if and only if:

$$\frac{q}{2v} \geq \frac{1-z}{2+z} \quad (13)$$

$$\frac{q}{2v} \leq \frac{1+2z}{2+z} \quad (14)$$

To ensure (ii), we fix  $p_2^*$  and first analyze firm 1's best response in the region

$R_{32} = \{p_1 / p_1 - p_2^* \in [-z(I-z), -z(2v-q)]\}$ . Define:

$$p_1^{**} = \arg \max_{p_1 \in R_{32}} B_1(p_1, p_2^*) = \frac{1+z}{6} [(3+q-4v) + 3z(1-2v)] \quad (15)$$

First, we guarantee that  $p_1^{**} \in R_{32}$ , which is true if and only if:

$$\frac{q}{2v} \leq (2-3z) - \frac{3}{2v}(1+3z) \quad (16)$$

$$\frac{q}{2v} \geq \frac{4v-3+2vz+6z-3z^2+6vz^2}{2v(1+5z)} \quad (17)$$

Now, we ensure that  $B_1(p_1^*, p_2^*) \geq B_1(p_1^{**}, p_2^*)$  by means of the following condition:

$$4z(3-q-2v)^2 - [3z(I-2v) + (3+q-4v)]^2 \geq 0 \quad (18)$$

If (16) is not satisfied, then  $p_1^{**} \leq p_2^* - z(I-z)$ . Therefore,  $B_1(p_1^*, p_2^*) \geq B_1(p_2^* - z(I-z), p_2^*)$  if and only if:

$$q^2 + 2q(5v-6) + [-20v^2 + 6v(5-3z) - 9(I-2z)] \geq 0 \quad (19)$$

If (17) is not satisfied, we then have  $p_1^{**} \leq p_2^* - z(2v-q)$ . In this case, we have  $B_1(p_1^*, p_2^*) \geq B_1(p_1, p_2^*)$  for all  $p_1 \in R_{32}$ .

We now study firm 1's best response within  $R_{34} = \{p_1 / p_1 - p_2^* \in [zq, z(I-z)]\}$ . We define:

$$p_1^{***} = \arg \max_{p_1 \in R_{34}} B_1(p_1, p_2^*) = \frac{1-z}{6} [(3+q-4v) + 3z] \quad (20)$$

To guarantee that  $p_1^{***} \in R_{34}$ , it is necessary and sufficient to have

$$\frac{q}{2v} \leq (1-z)(4v - 3(1-z)) \quad (21)$$

Ensuring  $B_1(p_1^*, p_2^*) \geq B_1(p_1^{***}, p_2^*)$  is equivalent to having:

$$4z(3-q-2v)^2 - [3z + (3+q-4v)]^2 \geq 0 \quad (22)$$

If (21) is not satisfied, then  $B_1(p_1^*, p_2^*) \geq B_1(p_1, p_2^*)$  for all  $p_1 \in R_{34}$ .

Summarizing, condition (ii) is satisfied if and only if the following restrictions apply:

- If (16) and (17) hold, then (18) must hold.
- If (16) does not hold, then (19) must hold.
- If (21) holds, then (22) must hold.

We now study the conditions under which (iii) is satisfied. We fix  $p_1^*$  given by (12) and we analyze firm 2's best response in the region  $R'_{32} = \{ p_2 / p_1^* - p_2 \in [-z(1-z), -z(2v-q)] \}$

We define

$$p_2^{***} = \arg \max_{p_2 \in R'_{32}} B_2(p_1^*, p_2) = \frac{1-z}{6} (3z + (3-q-2v))$$

Ensuring  $p_2^{***} \in R'_{32}$  is equivalent to have

$$\frac{q}{2v} \geq \frac{3z^2 - 2v + 14vz + 3 - 6z}{2v(1+5z)} \quad (23)$$

Now,  $B_2(p_1^*, p_2^*) \geq B_2(p_1^*, p_2^{***})$  is satisfied if and only if

$$4z(4v - q - 3)^2 - [3z + (3 - q - 4v)]^2 \geq 0 \quad (24)$$

If (23) is not satisfied, we then have  $p_1^* - p_2^{***} \geq -z(2v - q)$ . In this case, we have  $B_2(p_1^*, p_2^*) \geq B_1(p_1^*, p_2)$  for all  $p_2 \in R'_{32}$ .

Now, we study firm 2's best response in  $R'_{34} = \{p_2 / p_1^* - p_2 \in [zq, z(1 - z)]\}$ . We define:

$$p_2^{**} = \arg \max_{p_2 \in R'_{34}} B_2(p_1^*, p_2) = \frac{1-z}{6} [3z(1 - 2v) + (3 - 2v - q)]$$

Now,  $p_2^{**} \in R'_{34}$  if and only if

$$\frac{q}{2v} \geq (-9z + 6vz + 3 - 2v) \quad (25)$$

$$\frac{q}{2v} \leq \frac{3z^2 - 2v + 8vz - 6vz^2 - 6z}{2v(5z + 1)} \quad (26)$$

And  $B_2(p_1^*, p_2^*) \geq B_1(p_1^*, p_2^{**})$  if and only if

$$4z(4v - q - 3) - [3z(1 - 2v) + (3 - 2v - q)]^2 \geq 0 \quad (27)$$

If (25) is not satisfied, it's easy to see that  $B_2(p_1^*, p_2^*) \geq B_1(p_1^*, p_2)$  for all  $p_2 \in R'_{34}$ .

If (26) is not satisfied, then we have to ensure  $B_2(p_1^*, p_2^*) \geq B_1(p_1^*, p_1^* - z(1 - z))$ , this expression is equivalent to

$$(3 + q - 4v)^2 - 6(1 - v) [(3 - 2v - q) - 3z]^2 \geq 0 \quad (28)$$

Summarizing, condition (iii) holds if and only if the following restrictions apply:

- If (23) holds, then (24) must hold.
- If (25) and (26) hold, then (27) must hold.
- If (26) does not hold, then (28) must hold.

The set  $A$  is composed by all the requirements needed for conditions (i), (ii) and (iii) to subsist. Thus, a price equilibrium exists if and only if  $(v, q, z) \in A$ .

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