A MODEL FOR THE YIELD CURVE

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Abstract

The starting point is an interrogation about the non-broken character of the term structure of interest rates. Some arguments for that smooth character are presented here, all of which are based upon the assumption that market participants - arbitrageurs and speculators - always try to explore any misalignments discovered in the interest market. This led to the basic concept behind the model that the current short-term rate determines most of the value of the rate level for the subsequent period. A linear model describing that simple relationship is assumed and that constitutes the building block from where one can develop the mathematical equations necessary to work with different sets of market data.

A number of different yield curves were modelled by adjustment to real market data using this basic model, all of them showing a very high quality of the fits when measured by the non-linear ratio $\boldsymbol{R^2}$*. Nevertheless this fact still needs to be confirmed as the examples were drawn from non-independent markets and from a very short time window.*

The model can be improved by simple addition of a liquidity premium depend only upon the maturity of the rates. However, that improvement sophisticates tremendously the mathematical tractability of any real situation without any assurance that this added cost compensates for the increased quality of the fit.

The model is designed around only 3 parameters that can all be interpreted in economic terms. Two of them, in particular, bring a significant improvement over the traditional views frequently extracted from the shape of the yield curve.

Provided future tests confirm the high quality of the basic and the improved (with a liquidity premium) models, both are supportive of the expectation hypothesis (EH) and the liquidity premium hypothesis (LPH).

[1] I have to thank to Luís Catela Nunes for his valuable comments about an earlier version of this paper. Of course, any remaining errors are my own fault.

1. Introduction

The shape of the yield curve has intrigued economists for a long time and that triggered the search for a series of theories that tried to explain that particular shape. The initial attempts tried to engender a coherent view from the behaviour of the users of money - borrowers and lenders - that could justify the differences between short term and long term interest rates. When those users, by any sort of reasons, demanded more long deposits or loans than short term ones the curve should be upward sloping, and the opposite should occur when their demand for shorter operations was higher than for long ones.

That is, initially it was the general shape of the curve that was the object of the analysis, and from there, basically four schools of thought were produced - the traditional term structure theories: the Expectations Hypothesis (EH), the Liquidity Preference Hypothesis, the Market Segmentation Hypothesis and the Preferred Habitat Theory.

More recently, the search for models to price debt derivatives led to the development of a different series of approaches, all of them based on stochastic calculus. Instead of explaining the overall shape of the curve, these new theories focus on the dynamics of the curve itself, that is, how it might evolve along the time. More specifically, they produce estimates for the interest rate volatility of each maturity so that a price can be attached to that uncertainty. Here less attention is dedicated to the views of the "consumers" of money as these models tend to accept as a starting point some Brownian diffusion process for one or two interest rates without any consideration of the psychological behaviour behind such differential equations.

Beyond the explanations of the qualitative shape of the curve and the models that price the volatility of interest rates, the search here is for a *quantitative* mathematical model describing that shape in order to help practitioners extract objective financial information from the actual instruments traded in the market.

In fact, if one can define one explicit and (if possible) simple mathematical formula $\mathbf{R_T} = \mathbf{f}(\text{maturity T})$ with a certain number of parameters, it will be much simpler to fit such function to the numerical data provided by money and capital markets instruments than the common methods of segmenting the curve into a number of splines or approaching it by some form of polynomial or a sum of exponentials. One can also discount the cash-flows produced by a set of different issues of traded T-Bonds using the rates estimated by that formula and adjusting the parameters of the model in order to minimise the mean squared error between those market prices and the prices produced by the formula.

The existence of a single formula relating interest rates to maturities also simplifies the estimation of the interest rate for any maturity between two adjacent dates for which that information is available. There is no need to interpolate between those adjacent rates, and so, for a new debt instrument with cash flows between traditional dates, it becomes much simpler to find the appropriate discount factors that are compatible with the current term structure.

The model developed also brings the advantage that the parameters of the formula have an economic meaning which helps in interpreting the "consumers" expectations about future developments in the relevant interest rate market.

2. Literature

One of the most resilient ideas in this field of the yield curve is the concept that investors´ views for the future are always imbedded in the current rates for long-term loans or deposits. When Irving Fisher added the expected inflation rate to the real cost of money to obtain the current nominal interest rates, he implicitly accepted the idea that inflation expectations for the future were a determinant of today´s long-term rates. Long rates must compensate for the expected future inflation and still leave some margin to pay some real return to the investor or to inflict a real cost to the borrower.

It is true that there is a theory that negates any role to the current expectations in the market, the so called Market Segmentation Hypothesis, due to Culbertson. For him, the market is segmented along the maturity axis because the interest rate for maturity T is simply the outcome of a local equilibrium that is struck between suppliers of funds and borrowers of capital, both of whom can only invest and raise money within a very narrow range of maturities. None of them can consider the demand and the offer of capital in any other segment, even if adjacent to their natural one. This is the result of their risk aversion to the uncertainty of future interest rates and the correspondent volatility of asset prices. So, they always try to match as much as possible the maturities of their assets to the maturities of their liabilities. But that is a rather extremist vision of the world which does not seem to be confirmed by empirical research.

In fact, Lutz was more explicit than Fisher in the role played by the current expectations for the future. For him, the shape of the tem structure is a clear consequence of what is currently forecasted for the future level of short-term rates. Typically, an upward sloping term structure indicates that the overall market expects an increase in future rates, independently of whether that increase is the result of more inflation or of a more expensive real cost of money.

Even Hicks, who introduced the concept of the liquidity premium, does not reject the idea that forward rates incorporate some views for the future. He connects the shape of the curve to the preferences of the majority of the players in the market. Those preferences deter-mine that, under equal conditions, investors prefer to invest shorter rather than longer but borrowers have an opposite preference. Therefore, long-term borrowers need to bribe the suppliers of capital to move them into their preferred longer time slots. That is, they need to pay a premium for longer maturities, and that spread must be larger the longer the required funds are immobilised. This means the curve should show a permanent trend towards a positive slope without any relation to the current expectations for future short-term rates. For example, in case the market anticipates lower rates in the future, this theory only states that the yield curve will be less descendent than forecasted by EH due to the presence of that liquidity premium. Apart from that, expectations can play freely their role.

This overwhelming presence of the investors´ expectations in a number of theories has led various researchers to check them empirically. Unfortunately the different results achieved are somewhat contradictory, if not questionable.

The US money and bond markets are the most studied ones in the world and a significant number of empirical studies suggest that expectations about the future cannot be a good explanation for the shape of the yield curve, at least, in its lower end (short-term rates). For the longer end, EH may be accepted, and also for short maturities if one uses market data sampled from some years before the inception of the Federal Reserve System (FED) in 1915.

In other markets the Expectations Hypothesis seems more appropriate as is the case of Germany and even the UK.

These findings left a large question mark upon the role played by expectations in shaping the term structure, but some authors have always argued that they could never be completely scrapped as that would mean negating the proper role every Central Bank tries to play in his home money market when conducting his monetary policy. When a Central Bank changes the stance of his monetary policy, what he is doing is an exercise of influence upon the market expectations about the future level of interest rates in order to curb or to fuel the pace of "his" economy. Of course, that stance sooner or later has to be confirmed by a sudden and discrete change in his main steering rate (a discount rate, a main refinancing rate, the overnight Fed Funds Rate, etc). But that potential jump is also part of the game of influencing the expectations for the future.

Recently, Mankiw and Miron suggested an explanation for the apparent failure of the EH to work properly in the short-term segment of the US money market in this post-FED era. Since the FED keeps acting upon the market via its open market operations with the purpose of smoothing out any excess overnight (ON) Fed Funds volatility, the term spread between that ON rate and the six month rate loses any meaning in terms of anticipating future changes in money rates. Imagine, for example, that the market forecasts an increase in the ON rate in the near future and therefore prices the six months rate above the current ON level. That spread may not be later confirmed by a larger ON rate if the FED keeps acting in the market in order to maintain the initial ON rate. So, one can say that the FED active presence in the market destroys the informational contents of the above term spread between ON and 6 months as repeatedly tested in US.

Additionally and in spite of this "anti-market" attitude of the FED, the target rate of this central bank may also convene some information to the market. More specifically, the potential future modifications of that steering rate by unilateral decision of the bank - direction, amplitude and timing - has an informational contents. That is the logic behind the model developed by Balduzzi, Bertola and Foresi (BBF) where they tried to *measure* the market estimates for the size and direction of the next change in that reference interest rate.

This means that a different empirical test of US market short-term rates may confirm the presence of the expectations effect in shaping the yield curve as in fact has occurred with BBF and BBFK.

From all these reasons, this paper accepts the basic idea that human beings always look forward to anticipate the future and also that it is based on the current available information that they form rational expectations for the future of interest rates.

3. The origin of the model

a) Whatever the shape of a yield curve, the fact is that it shows always a continuum of interest rates in the sense that:

- two close by maturities show a rather similar level of interest rates; and
- a trend – a positive or negative slope – is present; even when the curve starts with a positive slope and then switches to a negative trend or vice-versa, there are no sudden jumps from one level to the next.

For example, interest rates for one month and for two months might be different, but they are never much different, and the same applies to 10 and 11 years. But one month and 10 years rates might be – and usually are – somewhat different.

However, one could follow the Market Segmentation Hypothesis and argue that those two rates (1 month and 2 months) could perfectly be rather different, should the supply and demand of funds in those two segments of the money market be so unbalanced that two rather different equilibrium rates would develop at that point in time.

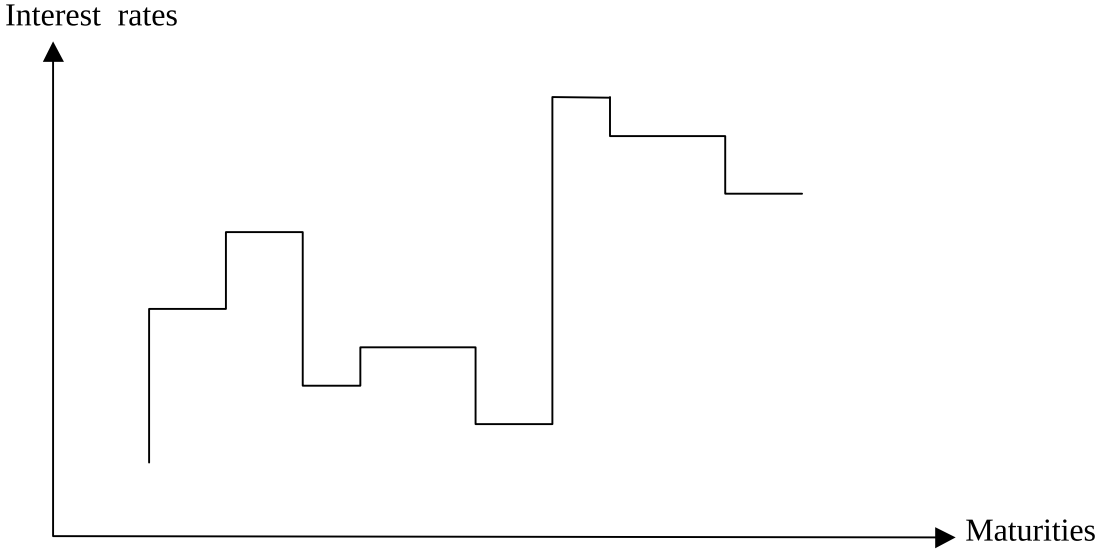


Fig. 1 Broken line type of yield curve acceptable under the Market Segmentation Hypothesis

If the financial market is somewhat segmented but yet the yield curve is a smooth and trendy line - instead of a broken one - some reasons might be behind that shape. This is the central question that led to this paper.

b) One reason might come from an inter-temporal or inter-maturity form of arbitrage explored by every investor (supplier of funds) and/or by every creditor (borrower of funds). If the rate[2] $\mathbf{R_1}$ for 1 month is *much* below $\mathbf{R_2}$ for 2 months, an investor might prefer to extend *a bit* his deposit to two months in order to benefit from that higher return. The decision to extend or not depends on the comparison between the benefits earned from such extension and the costs imposed by that reduction in his liquidity.

[2] We use capital letters to refer to current spot rates and small letters to refer to both future short-term spot rates. Of course $\mathbf{R_1} = \mathbf{r_1}$.

If that comparison favours the extension, it will be used by a number of other investors, and that shift will reduce the overall supply of funds in the 1 month bracket and will enlarge the supply of funds in the 2 months interval. Rates will be forced down for the later maturity and up for the first, and the difference between the two initial rates will be reduced.

Also borrowers, faced with $\mathbf{R_1} \ll \mathbf{R_2}$ might be able to change slightly their requirements of funds and raise for 1 month instead of for 2 months. From this demand side, the two rates will also tend to level off, provided the costs of that change of policy is more than compensated by the reduction of the rate from $\mathbf{R_2}$ to $\mathbf{R_1}$.

In case $\mathbf{R_1} \gg \mathbf{R_2}$, the suppliers may move funds to the shorter maturity and the borrowers to the longer one until the two rates approach each other, in both cases forcing the two rates to come closer one to the other. Of course, in the real life these two maturities can only be mutual alternatives if they are rather close, which means that, for example, for 1 month and 10 years this logic cannot apply.

c) A similar inter-temporal arbitrage might be explored by those working in the forward market. Once again, if the 2 months rate $\mathbf{R_2}$ is somewhat higher than the one month $\mathbf{R_1}$, the implicit rate $\mathbf{r_2}$ for the second month is much above $\mathbf{R_1}$.

This high appeal of a large return in month two will increase the demand for 1 month borrowed money and enlarge the supply of savings for 2 months deposits, forcing the first rate to go up and the second to go down, until $\mathbf{r_2}$ reaches a level low enough to stop seducing investors to save at a higher rate in the second month. Of course, this deposit shifting is more likely to occur between similar maturities due to the lower costs involved.

Even without a liquid forward market, any investor can capture this rather high forward rate by borrowing for 1 month and investing that amount for 2 months, the first month with that loaned money and the second month with his own funds.

Also from the perspective of the borrowers, one might lock in a very low implicit forward rate for the second month by borrowing immediately for two months but making no use of those funds during the first month other than depositing them at the higher rate. Market forces will once more determine an approach of the two interest rates.

Either way - $\mathbf{R_1} < \mathbf{R_2}$ or $\mathbf{R_1} > \mathbf{R_2}$ - the implicit forward rate for the second month cannot be much different from $\mathbf{R_1}$. In these two arbitrage strategies, the higher the costs incurred to shift the preferences along the maturity scale, the larger must be the difference between adjacent interest rates to induce a change of attitude by depositors and/or borrowers. So, unless those costs are huge, the forward rate $\mathbf{r_2}$ cannot be much different from the current short-term rate $\mathbf{R_1}$.

d) Besides arbitrageurs, also speculators are present in this interest rate market. Anyone that wants to deposit for two periods at a known spot rate $\mathbf{R_2}$ will consider also the alternative of starting with a one period deposit (at a known spot rate $\mathbf{R_1}$) and risk the level of the actual return $\mathbf{r_2}$ during the second period. The first alternative brings no risk, but the second has to bear the uncertainty of the value of $\mathbf{r_2}$ at inception.

If depositors forecast higher future interest rates, they may opt for the risky alternative, provided that anticipated improvement is large enough to, most likely, cover the uncertainty involved in that estimated (= average) future rate. Here the comparison is between the losses of a lower than expected rate (when the second period starts) and the potential excess return obtained if the personal estimate is correct.

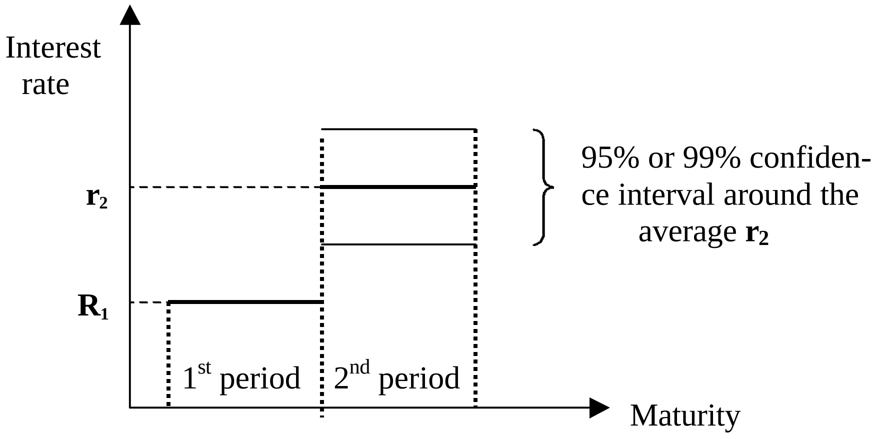


Fig. 2 A large difference between adjacent rates is not sustainable

On the other side of the fence, if borrowers forecast a large decrease of future interest rates, some of them will similarly prefer to borrow short term - one period only - and risk the future rate for the second period.

In both cases, speculators compare their estimates of future rates to the current forward rates implicit in the difference between $\mathbf{R_1}$ and $\mathbf{R_2}$ spot rates. When the deviation[3] of their estimates from the forward rate is large enough (in the scale of the "normal" volatility of future interest rates), the risky strategy may be selected by some of them. When that deviation is small, the guarantied strategy will be opted for.

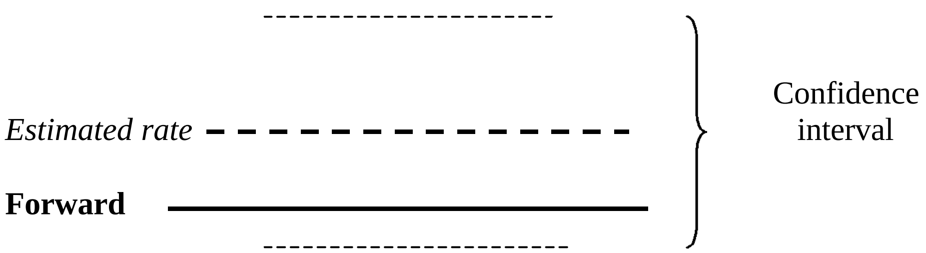


Fig. 3 The largest probability is that the future actual rate will be above the forward rate

Here risk is measured by the probability that each one´s estimate of $\mathbf{r_2}$ is sufficiently above or sufficiently below the implicit forward rate to make negligible the odds of a loss in comparison to the non-risky strategy (using directly or indirectly the forward rate).

[3] That deviation also must be in the correct direction.

Notice that, in the case of a shift in the supply of funds from 2 periods to 1 period to explore a larger forward rate, because the estimated future rate is larger then the implicit forward rate, market forces will push $\mathbf{R_2}$ upwards and $\mathbf{R_1}$ downwards, in both cases enlarging the forward rate $\mathbf{r_2}$ and, therefore, reducing the incentive for that speculative strategy. In fact, after these adjustments, the probability to earn an extra return with the risky option will be much smaller. Similarly for a shift from 2 period borrowings to 1 period loans to pay much less interests in the second period.

What the action of these speculators show is that implicit forward rates and estimated future spot rates cannot deviate much one from another.

e) Since all these three arguments indicates that $\mathbf{r_2}$ cannot be much different from $\mathbf{R_1}$, also $\mathbf{R_2}$ cannot be much different from $\mathbf{R_1}$, a fact that actual yield curves seem to fulfil. So this logic suggests that, for short term loans and deposits – say, some days, a few weeks or one or two months – the current spot rate $r_1 = \mathbf{R_1}$ must determine most of the expected value of the future spot rate $\mathbf{r_2}$ valid for an equal term starting after the end of first one.

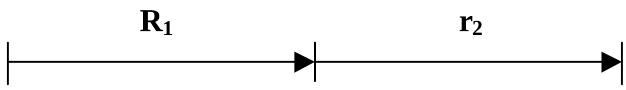


Fig. 4   Current spot rate $\boldsymbol{R_1}$ determines most of the adjacent future spot rate $\boldsymbol{r_2}$

Hence, accepting the EH and assuming a linear model for simplicity sake, one is led to write that $r_2 = a + b.r_1$. However, this gives only an average of the estimated future value $\mathbf{r_2}$ since the costs of arbitraging or the risks of speculation introduce a certain margin of uncertainty around that centre value obtained from $\mathbf{r_1}$. That is, $\mathbf{r_2}$ has some "noise" attached to it and that is expressed by an additional random term $\boldsymbol{\varepsilon}_2$

$$\boxed{r_2 = a + b.r_1 + e_2}$$

According to this logic - $\mathbf{r_1}$ determines most of the value of $\mathbf{r_2}$ - the parameter **b** must be smaller than 1, but rather close to unity, and the parameter **a** must appear in the formula to complement the result of the product $\mathbf{b.r_1}$ and make $\mathbf{r_2}$ either larger or smaller than $\mathbf{r_1}$ depending on the market views for the future:

$\mathbf{a} > (1 - \mathbf{b}).\ \mathbf{r_1}$	means a trend towards larger rates in the future
$\mathbf{a} < (1 - \mathbf{b}).\ \mathbf{r_1}$	means a trend towards smaller rates in the future

The error $\boldsymbol{\varepsilon}_2$ might follow any kind of distribution with zero mean, but it will be much simpler to assume that this error follows a Gaussian distribution with variance $\boldsymbol{\sigma^2}$:

$$e_2 = N\left[o, s^2\right]$$

The shortest time interval in current money markets comes from *overnight* deposits and loans. So the most detailed model uses rates for 24 hours loans and deposits.

This model fits well into the EH since it explains the shape of the yield curve as the result of the current set of estimates for the future short term spot rates. However, nothing has yet been said about any potential risk premium included in the current long-term spot rates. So, for the moment, forward rates are non-biased estimates of future spot rates.

4. The basic model

4.1. Extension to **n** periods

In the basic model, the overnight rate $\mathbf{r_2}$ estimated for tomorrow is determined by today's 24 hours spot rate $\mathbf{r_1}$ according to parameters **a** and **b**, but there remains some uncertainty $\boldsymbol{\varepsilon}_2$ in this forecast due to some superimposed "noise". For simplicity sake, this random variable is assumed to follow a Gaussian distribution with variance $\boldsymbol{\sigma}^2$.

For the day after tomorrow, the same arbitrage/speculation mechanisms determine also that $\boldsymbol{r}_3 = \boldsymbol{c} + \boldsymbol{d}.\boldsymbol{r}_2 + \boldsymbol{e}_3$ and the question is whether a = c, b = d and whether $\boldsymbol{\varepsilon}_3$ follows the same distribution as $\boldsymbol{\varepsilon}_2$. Probably the market agents that operate between days 2 and 3 are not exactly the same that operate between days 1 and 2, and that means that the 3 parameters might change along the yield curve. However, one can simplify the analysis by working with some average values common to all periods[4].

So, maintaining the parameters for the day after tomorrow, the current estimate for the correspondent overnight rate is

$$\boldsymbol{r}_3 = \boldsymbol{a} + \boldsymbol{b}.\boldsymbol{r}_2 + \boldsymbol{e}_3$$

which leads to

$$\boldsymbol{r}_3 = \boldsymbol{a}.(1+\boldsymbol{b}) + \boldsymbol{b}^2.\boldsymbol{r}_1 + (e_3 + \boldsymbol{b}.e_2)$$

and for day **n**

$$\boldsymbol{r_n} = \boldsymbol{a}.(1+\boldsymbol{b}+\boldsymbol{b}^2+.......+\boldsymbol{b}^{n-2}) + \boldsymbol{b}^{n-1}.\boldsymbol{r}_1 + \underbrace{(\boldsymbol{e_n} + \boldsymbol{b}.\boldsymbol{e}_{n-1} + \boldsymbol{b}^2.\boldsymbol{e}_{n-2} + ...+ \boldsymbol{b}^{n-2}.\boldsymbol{e}_2)}_{e'_n}$$

[4] Even assuming that all $\boldsymbol{\varepsilon_i}$ follow the same Gaussian distribution with the same variance $\boldsymbol{\sigma^2}$, each disturbance $\boldsymbol{\varepsilon_i}$ is a different realisation of similar random variables.

Noting that the multiplier of **a** is a geometric progression and denoting by ε'_n the added noise term

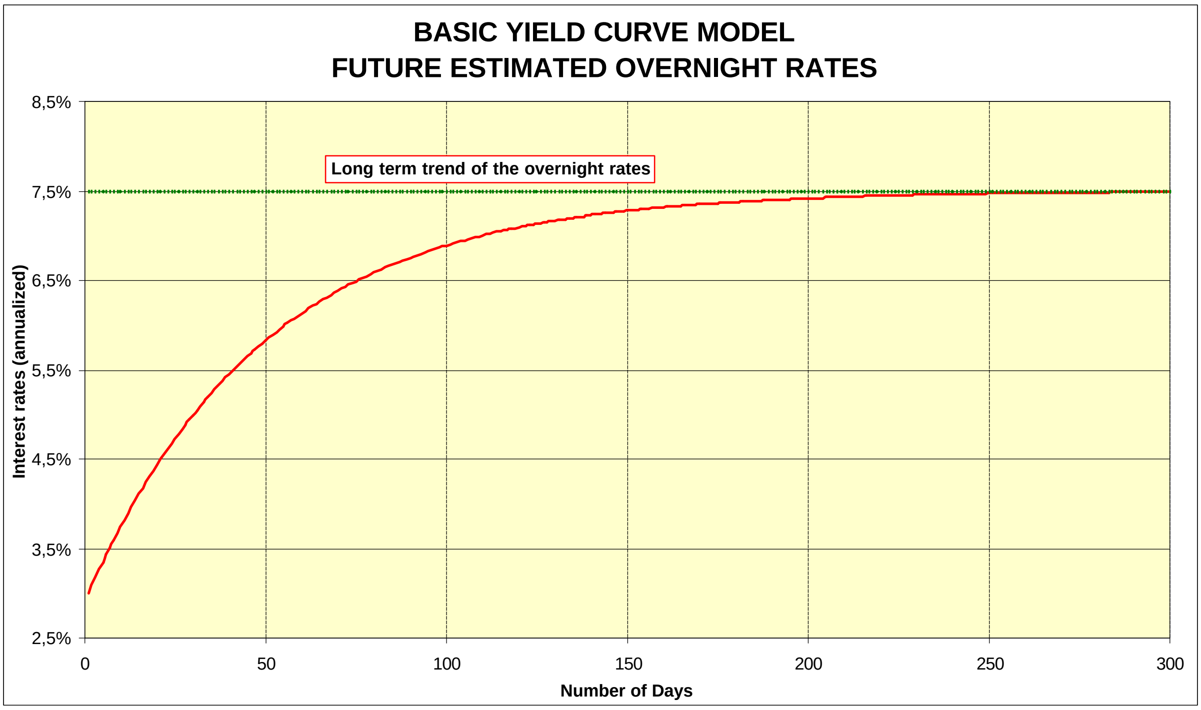
$$\boxed{r_n = a \,.\, \frac{1 - b^{\,n-1}}{1 - b} + b^{\,n-1} . r_1 + e'_n}$$

This expression gives the future overnight term rate $\mathbf{r_n}$ decomposed into 3 parts:

- since 0 < **b** < 1, the first term on right (in **a**)increases with **n** and tends to an asymptotic long-term value

$$\bar{r} = \frac{a}{1 - b}$$

- the second term (in $\mathbf{r_1}$) decreases with **n** and becomes negligible for distant future dates (large **n**): today´s short-term rate does not influence the long future



- the noise term is a linear combination of the successive disturbances embedded in each estimated value $\mathbf{r_i}$ before and up to $\mathbf{r_n}$; since it seems reasonable to accept that $\boldsymbol{\varepsilon_i}$ is i.i.d. [5]

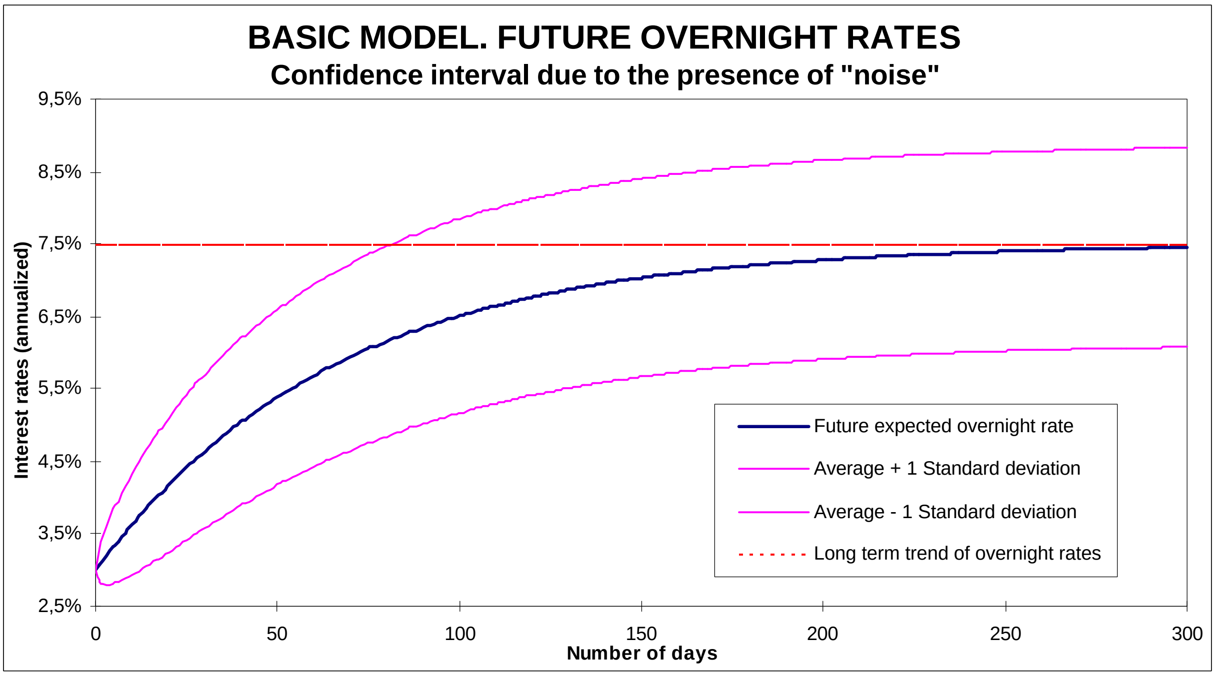
[5] There is no reason to suspect that the noise term $\boldsymbol{\varepsilon}_i$ included in the forecast of $\mathbf{r}_i$ (based on $\mathbf{r}_{i-1}$) should depend upon any previous disturbance $\boldsymbol{\varepsilon_{i-k}}$. Any of these terms simply translate the local impact of the diversity of costs among speculators and/or arbitrageurs. Therefore, one might simplify the analysis and accept that all $\boldsymbol{\varepsilon}_i$ follow the same statistical distribution (same $\boldsymbol{\sigma}$) and that each one is independent from any other.

$$E\left[e_i, e_j\right] = \begin{cases} s^2 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases}$$

that combination still follows a Gaussian distribution[6]

$$e'_n \sim N\left[0, s^2 . \frac{1-(b^2)^{n-1}}{1-b^2}\right]$$

This shows that the uncertainty of $\mathbf{r_n}$ grows the further one looks into the future, but with an asymptotic maximum given by $\sigma^2/(1\text{-}b^2)$.



1.2. Interpretation of the parameters of the model

The long term rate $\bar{r}$ is the level of overnight rates estimated for dates far away into the future. It is an average value that suffers the effect of an additive noise ε'_n that may deviate the actual rate from that estimated average.

Parameter **b** indicates how much of $\mathbf{r_1}$ is included in $\mathbf{r_2}$, but the expression for $\mathbf{r_2}$ can be rewritten in a different way

$$r_2 = r_1 + (1-b).(\bar{r} - r_1) + e_2 = r_1 + h.(\bar{r} - r_1) + e_2 \qquad \text{with} \qquad h = 1 - b$$

[6] See annex for derivation of the variance of this noise as seen from today.

which makes clear that η indicates how much of the difference between the current over-night rate $\mathbf{r_1}$ and the long term 24 hours rate $\bar{r}$ is covered during the first day.

And since η% of that difference is "jumped" in the first day, the entire transition from $\mathbf{r_1}$ to $\bar{r}$ would be covered in $\boldsymbol{\tau}$ = 1/η days should that first "velocity" be maintained unchanged in the fol-lowing days. Of course, the more **b** approaches 1 the less $\mathbf{r_2}$ can be different from $\mathbf{r_1}$ and the slower will be the transition from $\mathbf{r_1}$ to $\bar{r}$.

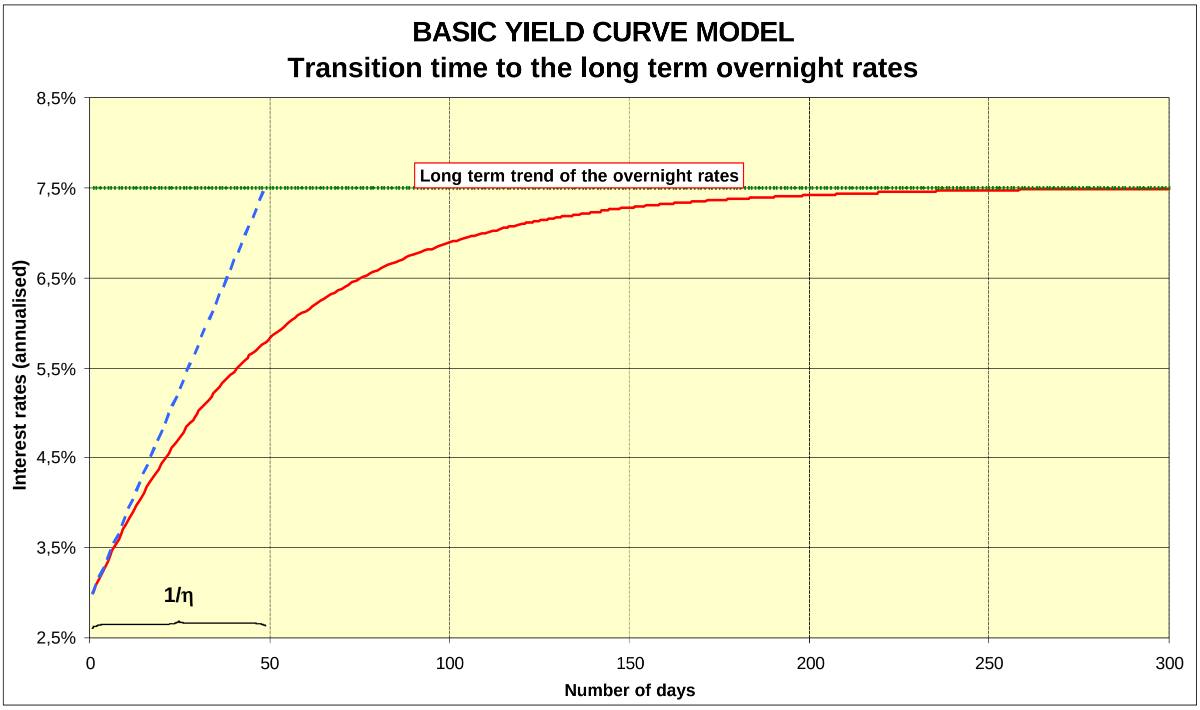
We see that this model is completely defined by the following 3 parameters (the noi-se component requires a fourth parameter[7], the volatility **σ**):

- initial or current overnight rate $\mathbf{r_1}$
- long term estimated overnight rate $\bar{r}$
- transition time $\boldsymbol{\tau}$ (in our case, days) between $\mathbf{r_1}$ and $\bar{r}$.

$$\boldsymbol{r_n} = \bar{\boldsymbol{r}} + (1-h)^{\boldsymbol{n}-1}.(\boldsymbol{r}_1 - \bar{\boldsymbol{r}}) + e'_{\boldsymbol{n}}$$

or better

$$\boxed{\boldsymbol{r_n} = \bar{\boldsymbol{r}} + (1-\frac{1}{t})^{\boldsymbol{n}-1}.(\boldsymbol{r}_1 - \bar{\boldsymbol{r}}) + e'_{\boldsymbol{n}}} \qquad \text{with} \qquad \begin{cases} \bar{\boldsymbol{r}} = \dfrac{\boldsymbol{a}}{1-\boldsymbol{b}} = \dfrac{\boldsymbol{a}}{h} \\ t = \dfrac{1}{h} = \dfrac{1}{1-\boldsymbol{b}} \end{cases}$$



[7] Below it is found that a 5th parameter (k) might be necessary in connection with this noise factor.

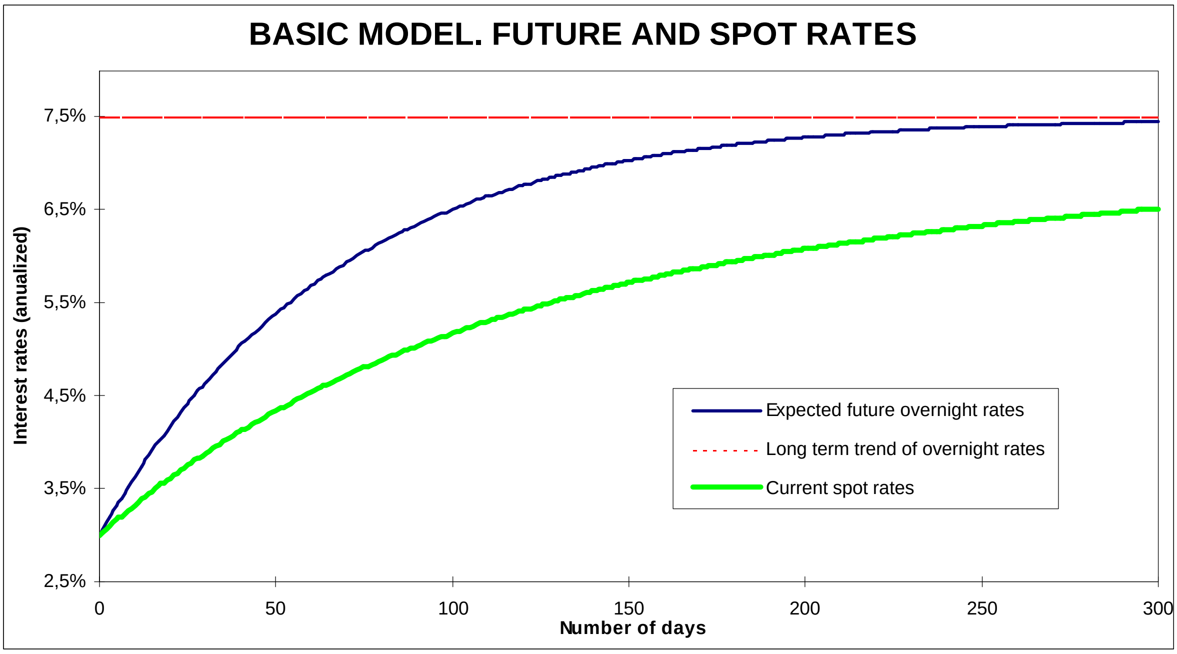
1.3. Estimated spot rates

For a low level of annual interest rates - say 3% or 5% - the correspondent *daily* rates are always very small numerical figures. This allows a simplification when compounding the estimated future overnight rates to obtain the estimated spot rates appropriate for each particular maturity. If fact, for two days, the spot rate is

$$1 + R_2 = \sqrt{(1+r_1).(1+r_2)} \quad \Rightarrow \quad R_2 \cong \frac{r_1 + r_2}{2}$$

and, for three days, is

$$1 + R_3 = \sqrt[3]{(1+r_1).(1+r_2).(1+r_3)} \quad \Rightarrow \quad R_3 \cong \frac{r_1 + r_2 + r_3}{3}$$



Adopting this linear aproximation, the estimated 4 days spot rate is given by

$$R_4 = \frac{1}{4}.\left\{ r_1 + \underbrace{(a + b.r_1 + e_2)}_{r_2} + \left[\underbrace{a.(1+b) + b^2.r_1 + (e_3 + b.e_2)}_{r_3}\right] + \left[\underbrace{a.(1+b+b^2) + b^3.r_1 + (e_4 + b.e_3 + b^2.e_2)}_{r_4}\right]\right\}$$

and rearranging the terms

$$R_4 = \frac{1}{4}\left\{a.\left[1 + (1+b) + (1+b+b^2)\right] + (1+b+b^2+b^3)r_1 + \left[e_4 + e_3.(1+b) + e_2.(1+b+b^2)\right]\right\}$$

This expression helps in writing the formula for the **n** period spot rate $\mathbf{R_n}$

$$\boldsymbol{R}_n = \frac{1}{\boldsymbol{n}}\sum_1^{\boldsymbol{n}} \boldsymbol{r}_i = \frac{1}{\boldsymbol{n}}.\left\{\begin{array}{c} \boldsymbol{a}.\left[1+(1+\boldsymbol{b})+\cdots+(1+\boldsymbol{b}+\cdots+\boldsymbol{b}^{\boldsymbol{n}-2})\right]+(1+\boldsymbol{b}+\cdots+\boldsymbol{b}^{\boldsymbol{n}-1}).\boldsymbol{r}_1 + \\ + \underbrace{\left[\boldsymbol{e}_{\boldsymbol{n}} + \boldsymbol{e}_{\boldsymbol{n}-1}.(1+\boldsymbol{b})+\cdots+\boldsymbol{e}_2.(1+\boldsymbol{b}+\cdots+\boldsymbol{b}^{\boldsymbol{n}-2})\right]}_{e''_n} \end{array}\right\}$$

Noting the presence of a number of geometric progressions, the above formula for the **n** period spot rate can be simplified[8] to

$$\boldsymbol{R}_{\boldsymbol{n}} = \frac{\boldsymbol{a}}{1-\boldsymbol{b}}.\left[\frac{\boldsymbol{n}-1}{\boldsymbol{n}} - \frac{\boldsymbol{b}}{\boldsymbol{n}}.\frac{1-\boldsymbol{b}^{\boldsymbol{n}-1}}{1-\boldsymbol{b}}\right] + \frac{1}{\boldsymbol{n}}.\frac{1-\boldsymbol{b}^{\boldsymbol{n}}}{1-\boldsymbol{b}}.\boldsymbol{r}_1 + e''_{\boldsymbol{n}}$$

or, using the parameters $\bar{r}$, $\mathbf{r_1}$ and $\boldsymbol{\tau}$

$$\boxed{\boldsymbol{R}_{\boldsymbol{n}} = \bar{\boldsymbol{r}} + \frac{t}{\boldsymbol{n}}.(\boldsymbol{r}_1 - \bar{\boldsymbol{r}}).\left[1-(1-\frac{1}{t})^{\boldsymbol{n}}\right] + e''_{\boldsymbol{n}}}$$

where the noise component still follows a Gaussian distribution

$$e''_{\boldsymbol{n}} \sim \boldsymbol{N}\left[0, \frac{s^2}{\boldsymbol{n}^2.(1-\boldsymbol{b})^2}\left((\boldsymbol{n}-1) - 2\boldsymbol{b}.\frac{1-\boldsymbol{b}^{\boldsymbol{n}-1}}{1-\boldsymbol{b}} + \boldsymbol{b}^2.\frac{1-\boldsymbol{b}^{2.(\boldsymbol{n}-1)}}{1-\boldsymbol{b}^2}\right)\right]$$

These two formulae indicate that:

- the yield curve (spot rates) evolves upwards or downwards with **n** depending on the relationship between the initial overnight rate $\mathbf{r_1}$ and the long term rate $\bar{r}$ but, in any case, it evolves monotonically to

$$\bar{r} = \frac{a}{1-b}$$

- the variance of the noise term ε''_n also depends on **n** and although it might "hover" above $\boldsymbol{\sigma}^2$ for a small number of days, in the long run, it tends to decrease[9] due to the normal averaging effect of the successive overnight rates that are added up to obtain the long term spot rate.

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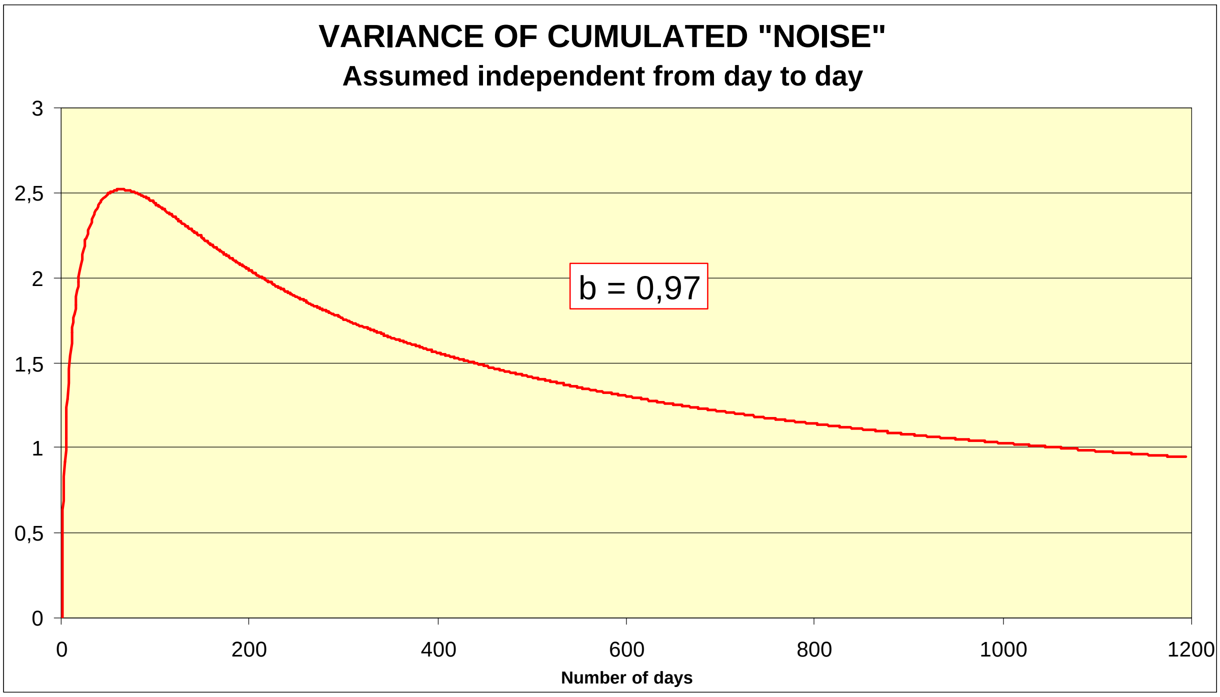
[8] See the derivation in the Annex.

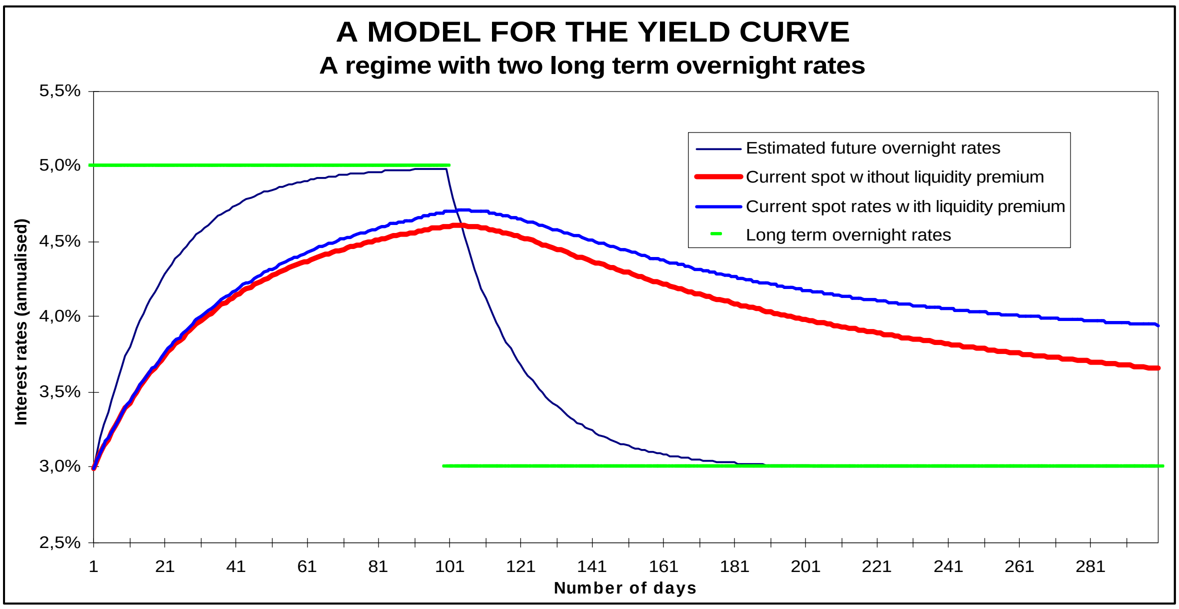
[9] In fact it goes to zero as **n** goes to infinity.



It is curious to note that although $\mathbf{r_n}$ suffers for increasing uncertainty when **n** increases, the error of $\mathbf{R_n}$ decreases with **n**. That is, the market forecasts better a long term *spot* rate than medium term spot rates.

4.4. Double regime (two different set of parameters)

Some times the market may estimate an increase in the rates in the near future that are to be followed by some reduction to more modest rates some time after. The yield curve will then start with a positive slope to inflect downwards somewhere in the future to approach smaller levels of interest rates. Other situations show the opposite, that is, an initial negative slope followed by an inversion to a positive one.



The model can approximate such shapes by assuming the existence of two “long” term levels for the overnight rates - $\bar{r}_1$ and $\bar{r}_2$ - and two transition times - $\boldsymbol{\tau_1}$ and $\boldsymbol{\tau_2}$:

- the first **n** periods are characterised by, $\mathbf{r_1}$, $\mathbf{a_1}$ and $\mathbf{b_1}$ so that the future spot overnight rates are determined by

$$\boldsymbol{r_n} = \boldsymbol{a}_1 . \frac{1 - \boldsymbol{b}_1^{\boldsymbol{n}-1}}{1 - \boldsymbol{b}_1} + \boldsymbol{b}_1^{\boldsymbol{n}-1} . \boldsymbol{r}_1 + e'_{\boldsymbol{n}} \qquad \text{where} \qquad \begin{cases} \bar{\boldsymbol{r}}_1 = \dfrac{\boldsymbol{a}_1}{1 - \boldsymbol{b}_1} \\ \tau_1 = \dfrac{1}{1 - \boldsymbol{b}_1} \end{cases}$$

and the current spot rates calculated by

$$\boldsymbol{R_n} = \frac{\boldsymbol{a}_1}{1 - \boldsymbol{b}_1} . \left[\frac{\boldsymbol{n} - 1}{\boldsymbol{n}} - \frac{\boldsymbol{b}_1}{\boldsymbol{n}} . \frac{1 - \boldsymbol{b}_1^{\boldsymbol{n}-1}}{1 - \boldsymbol{b}_1} \right] + \frac{1}{\boldsymbol{n}} . \frac{1 - \boldsymbol{b}_1^{\boldsymbol{n}}}{1 - \boldsymbol{b}_1} . \boldsymbol{r}_1 + e''_{\boldsymbol{n}}$$

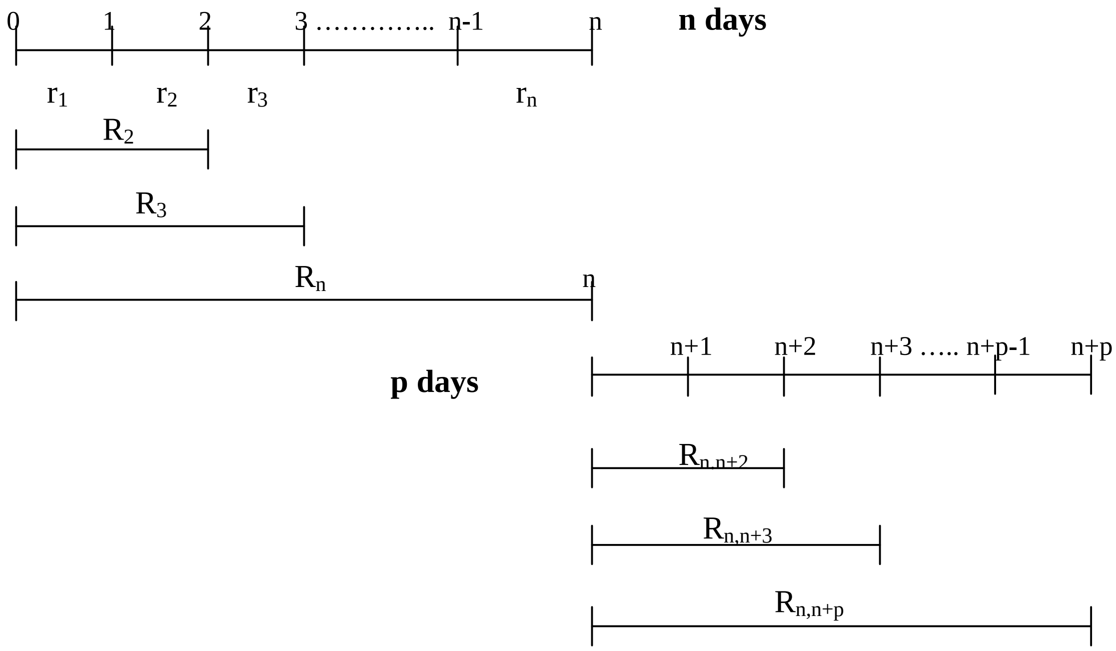


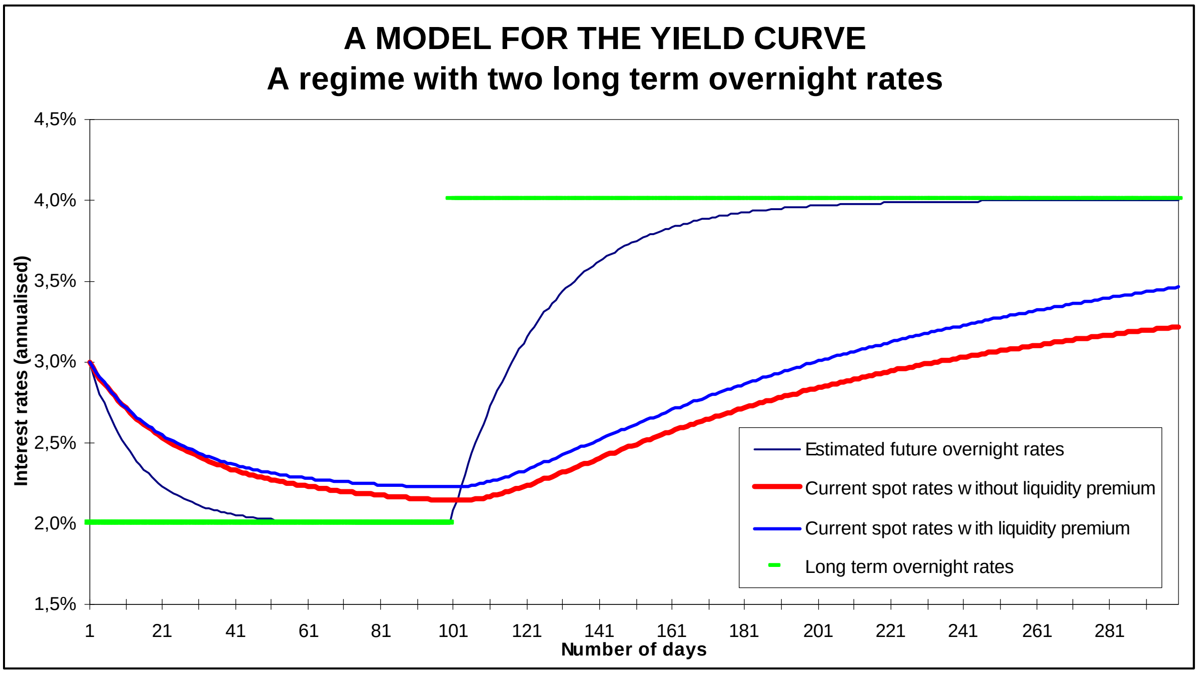
Fig 5. Double regime model

- and the second group of **p** days is characterised by parameters $\mathbf{r_{n+1}}$, $\mathbf{a_2}$ and $\mathbf{b_2}$ where $\mathbf{r_{n+1}}$ plays here the role of $\mathbf{r_1}$

$$\boldsymbol{r}_{\boldsymbol{n}+\boldsymbol{p}} = \boldsymbol{a}_2 . \frac{1 - \boldsymbol{b}_2^{\boldsymbol{p}-1}}{1 - \boldsymbol{b}_2} + \boldsymbol{b}_2^{\boldsymbol{p}-1} . \boldsymbol{r}_{\boldsymbol{n}+1} + \text{noise} \qquad \text{where} \qquad \begin{cases} \bar{\boldsymbol{r}}_2 = \dfrac{\boldsymbol{a}_2}{1 - \boldsymbol{b}_2} \\ \tau_2 = \dfrac{1}{1 - \boldsymbol{b}_2} \end{cases}$$

To calculate the spot rates for maturities falling into this second regime, the simplest approach is to use the arithmetic average of all overnight rates from day **1** until day (**n** + **p)**.

Beyond $\overline{r_1} \neq \overline{r_2}$, there is no reason to believe that the two "velocities" of transition τ_1 and τ_2 should be equal. Therefore, the parameters **a** and **b** will probability change from the first set of **n** days to the second set of **p** days.



4.5. Liquidity premium

Up to now we have assumed implicitly the Perfect Expectations Hypothesis version of the EH by calculating the current spot rate $\mathbf{R_n}$ for **n** days as the simple average of all the future overnight rates - from $\mathbf{r_1}$ until $\mathbf{r_n}$. The increased uncertainty $e_i^{;}$ of future overnight rates $\mathbf{r_i}$ did not affect the expected value of the long-term spot rate $\mathbf{R_n}$.

However longer term investments impose a loss of liquidity to the creditor which means that to bribe investors into those loans, borrowers need to pay some spread on top of the rate obtained by simply "adding" the successive short term rates forecasted for each future day of that loan.

In fact, someone who lends for two periods ($\mathbf{R_2}$) faces the uncertainty of the rate in effect in the second period ($\mathbf{r_2}$) should he be forced to liquidate that investment beforehand: he never knows the market value of his investment when forced to sell. Therefore it is the uncertainty of $\mathbf{r_2}$ that somehow will determine that liquidity spread. That uncertainty might well be found in the standard deviation σ_2 of the noise $e_2^{;} = e_2$ embedded in $\mathbf{r_2}$.

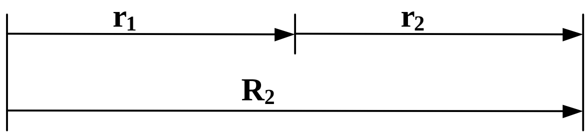


Fig. 6 Comparison between the spot rate for 2 days and the cascaded 2 one day rate

Accepting a linear relation for simplicity sake[10]

premium $p_2 = k.\,\sigma$

where **k** is a constant of proportionality dependent upon the more or less intense risk aversion of the market players. Therefore, for two days, the spot rate $\mathbf{R_2}$ is given by

$$R_2 = \frac{1}{2}.(r_1 + r_2) + p_2$$

For 3 days, there must exist an additional premium $\mathbf{p_3}$ (on top of the sum of $\mathbf{R_2}$ to $\mathbf{r_3}$) to induce investors to lend money for 3 days (R_3) instead of 2+1 days

$$R_3 = \frac{1}{3}.(2.R_2 + r_3) + p_3 = \frac{1}{3}.(r_1 + r_2 + r_3 + 2.p_2) + p_3$$

where, from the noise of e_3', one can write

$$p_3 = k.\frac{s}{\sqrt{1-b^2}}.\sqrt{1-b^4}$$

For **n** days, the spot rate estimated taking into account the liquidity premium is

$$(R_n)_{with\ premium} = \frac{\sum_1^n r_i}{n} + \frac{2.p_2 + 3.p_3 + 4.p_4 + \cdots + n.p_n}{n}$$

that is

$$\boxed{(R_n)_{with\ premium} = (R_n)_{without\ premium} + \frac{1}{n}.\sum_2^n i.p_i}$$

And since

[10] There is no risk $\mathbf{p_1}$ because $\mathbf{r_1} = \mathbf{R_1}$ is the current (known) spot rate.

$$p_i = k.\frac{s}{\sqrt{1-b^2}}.\sqrt{1-b^{2(i-1)}}$$

the liquidity premium included in $\mathbf{R_n}$ is given by

$$premium = \frac{k.s}{n.\sqrt{1-b^2}}.\left[2.\sqrt{1-b^2}+3.\sqrt{1-b^4}+4.\sqrt{1-b^6}+\cdots+n.\sqrt{1-b^{2.(n-1)}}\right]$$

4.6. Forward rates for **p** days

Usually the financial market supplies interest rate data related to the yield curve for a set of maturities, typically weeks, months and/or years. There is not much information for daily maturities. For example, every day Telerate supplies 14 spot rates from the Euro money market:

Eonia	1 W	1 M	2 M	3 M	4 M	5 M
6M	7M	8 M	9 M	10 M	11 M	12 M

In the case of the Eurodollar futures the market does not supply a set of spot rates but rather a collection of forward rates. Although all these rates are for deposits lasting 91 days, some future deposits begin while some former ones are still running. There is some overlapping of maturates and in particular the first deposit may start less than 91 days ahead of today:

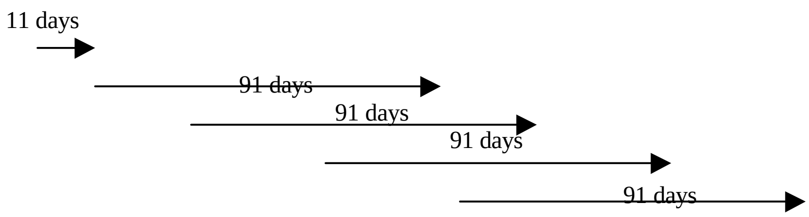


Fig 7. Eurodollar Futures and the overlapping of the nominal deposits

To work out these cases where the successive maturities do not differ by one single day there are two alternatives:

a) one uses the numerical expression of the forward rate for an operation lasting **p** days (p=91 in the case of the eurodollar futures) but starting after **n** initial days. It requires[11] the computation of both $\mathbf{r_{n+1}}$ and $\mathbf{R_{n,n+p}}$:

$$\boldsymbol{R}_{n,n+91} = \bar{\boldsymbol{r}} + (\boldsymbol{r}_{n+1} - \bar{\boldsymbol{r}}).\frac{t}{91}.\left[1 - (1 - \frac{1}{t})^{91}\right] + e''_{91}$$

b) the other works with forward rates (for 91 days deposits) obtained from the futures prices as if they were overnight rates r_i, that is, the model works with periods of 91 days instead of 24 hours; all expressions for $\mathbf{r_n}$ and $\mathbf{R_n}$ for the overnight model are used here assuming that $\mathbf{r_1}$ - the rate implicit in the price of the first contract alive - is the current spot rate for an immediate 91 days operation.

Since the first alternative can best explore the entire set of data obtained from the traded contracts, even those overlapping futures, it is the one recommended.

The above expression of $\mathbf{R_{n,n+91}}$ can be modified in order to use $\mathbf{r_1}$ instead of $\mathbf{r_{n+1}}$:

$$\begin{cases} \boldsymbol{R}_{n,n+91} - \bar{\boldsymbol{r}} = (\boldsymbol{r}_{n+1} - \bar{\boldsymbol{r}}).\dfrac{t}{91}.\left[1 - (1 - \dfrac{1}{t})^{91}\right] \\ \\ \boldsymbol{r}_{n+1} - \bar{\boldsymbol{r}} = (\boldsymbol{r}_1 - \bar{\boldsymbol{r}}).(1 - \dfrac{1}{t})^{\boldsymbol{n}} \end{cases}$$

which, by simple substitution, yields the desired expression for the forward rate as a function of $\mathbf{r_1}$:

$$\boxed{\boldsymbol{R}_{n,n+91} = \bar{\boldsymbol{r}} + (\boldsymbol{r}_1 - \bar{\boldsymbol{r}}).(1 - \frac{1}{t})^{\boldsymbol{n}}.\frac{t}{91}.\left[1 - (1 - \frac{1}{t})^{91}\right]}$$

[11] Note that in this formula, the total uncertainty of $\boldsymbol{R}_{n,n+91}$ results from the noise included in $\mathbf{r_{n+1}}$ plus the explicit noise e''_{91} added by the "new" 91 overnight rates.

5. Applications of the model

5.1. Introduction

The quality of this model can only be gauged by applying it to real market data in order to describe actual term structures by means of a single mathematical formula with 3 parameters. Of course, these parameters must be estimated from that data by the best fitting a curve to the available sampled points.

Unfortunately, the model produces a non-linear function of the maturity **i** and of $\boldsymbol{\tau}$ and that requires the adjustment to follow a numerical methodology due to the absence of closed form equations for those 3 parameters. However, simply minimising the root mean square of the deviations between the curve and the observed points is not enough as it is also necessary to force the average error to be zero[12]:

- Min $\sum (\boldsymbol{R_i} - \boldsymbol{\hat{R}_i})^2$
- $\sum (\boldsymbol{R_i} - \boldsymbol{\hat{R}_i}) = 0$

For that purpose, a number of examples were used: Lisbor, Euribor, Euro$, US T-Bills, US Strips.

5.2. Margin of error of the estimated parameters

The quality of the fit can be measured by the non-linear ratio $\mathbf{R^2}$ defined as

$$\boldsymbol{R}^2 = 1 - \frac{\sum (\boldsymbol{error_i})^2}{\sum (\boldsymbol{R_i} - \boldsymbol{\overline{R}})^2}$$

One could also proceed and compute the margin of error of the estimated values obtained for the 3 parameters. But the calculation of the standard error of each of those 3 estimates implies overcoming two barriers:

- the model is non-linear in one of the parameters[13] (τ) although it may look linear in the two other parameters

[12] This null condition for the average error is required to be consistent with the basic model where $\boldsymbol{E}\,[e_i] = 0$, and therefore also $\boldsymbol{E}\,[e_{i'}] = 0$ and $\boldsymbol{E}\,[e_{i''}] = 0$.

[13] $\boldsymbol{r}_1 = \boldsymbol{R}_1 =$ current overnight rate

$\overline{\boldsymbol{r}} = \frac{\boldsymbol{a}}{1-\boldsymbol{b}}$ long term overnight rate

$t = \frac{1}{1-\boldsymbol{b}}$ transition time from current rate to the long term one

$$\boldsymbol{R}_i = \bar{\boldsymbol{r}} + (\boldsymbol{r}_1 - \bar{\boldsymbol{r}}).\overbrace{\frac{t}{\boldsymbol{i}}.\left[1-(1-\frac{1}{t})^{\boldsymbol{i}}\right]}^{\boldsymbol{F}_i} + e_i'' \quad \Rightarrow \quad \boldsymbol{R}_i = \bar{\boldsymbol{r}} + (\boldsymbol{r}_1 - \bar{\boldsymbol{r}}).\boldsymbol{F}_i + e_i''$$

- the disturbances e_i'' are heteroskesdastic and very much auto-correlated

$$\boldsymbol{Cov}\left[e_n'';e_{n-k}''\right] = \frac{s^2}{\boldsymbol{n}.(\boldsymbol{n}-\boldsymbol{k}).(1-\boldsymbol{b})^2}.\left\{(\boldsymbol{n}-\boldsymbol{k}-1)-\boldsymbol{b}.(1+\boldsymbol{b}^{\boldsymbol{k}}).\frac{1-\boldsymbol{b}^{\boldsymbol{n}-\boldsymbol{k}-1}}{1-\boldsymbol{b}}+\boldsymbol{b}^2.\frac{1-(\boldsymbol{b}^2)^{\boldsymbol{n}-\boldsymbol{k}-1}}{1-\boldsymbol{b}^2}\right\} \neq 0$$

In addition, not always this simple model is recommended to best describe the sampled points. Sometimes, market data suggests the use of a double regime model where, besides two sets of parameters, one also has also to estimate the transition moment from one regime to the next. Also for cases with very long-term data (e.g. 30 years) the use of the version with a liquidity premium to better estimate the asymptotic rate $\bar{\boldsymbol{r}}$ turns the computation into a practically intractable task. In all these cases the ratio $\mathbf{R^2}$ can always be directly calculated, but it is much more time consuming to compute the standard error of every parameter. Therefore, in this first approach, the quality of the fit was restricted to the ratio $\mathbf{R^2}$.

In all cases the adjustment of the curve to the sampled points was obtained via a trial and error method based on the "solver" instruction of the Excel language.

5.3. Some typical cases

The following examples provide a first measure of the quality of this model by estimating the parameters for different countries and for different segments of the term structure. In all of them the quality of the fit appears to be rather high, but one must not forget that the yield data used came from basically "one single picture" of the world of interest rates where all samples were collected almost at the same time. Mind also that national markets are increasingly interconnected which is relevant when the time frame is short.

a) Eonia and Euribor[14]

In Euroland, these rates are a daily measure of the short-term part of the interest rate market (up to 12 months). Due to the precautions taken against the Y2K bug, the yield curve has been showing a distortion for the two standard maturities that settle just before and just after the end of the year (1999). December 29 was the first day without the presence of that non-recurrent disturbance at any point along the curve.

[14] Source of data: Internet page "*Reference Interest Rates of the EURO area´s Money Market*" from Banco de Portugal. Eonia rate is calculated by ECB and all Euribor rates are calculated by Bridge-Telerate.

This is the single case where the market provides direct information about the value of $\mathbf{r_1}$ leaving as unknowns only $\bar{r}$ and $\boldsymbol{\tau}$. This enabled to adjust a curve either forcing r_1 to be equal to Eonia or leaving this variable free. The 14 maturities lead to the following parameters of the curve:

EURIBOR & EONIA				
29-Dec-99	$\mathbf{r_1}$	$\bar{r}$	$\boldsymbol{\tau}$	$\mathbf{R^2}$
Without Liquidity Premium				
Estimated Average	3,080%	5,145%	359 days	0,996
Est. Value with $\mathbf{r_1}$ = Eonia	3,040%	4,619%	221 days	0,991
With Liquidity Premium				
Estimated Value	3,080%	5,146%	359 days	0,996
Est. Value with $\mathbf{r_1}$ = Eonia	3,040%	4,609%	220 days	0,991

Introducing the liquidity premium does not change visibly the quality of the fit due to the short maturities involved where that premium still does not show its effects. The two different fitted curves are so similar that the graph has the two curves completely overlapping. Refer to the Annex for the complete data, the adjusted curve and the summary of the calculations.

b) The French Capital Market[15]

The French government debt market is one of the largest in Euroland and so much so that it can be assumed that any liquidity factors that might exist do not disturb heavily the level of the different interest rates. It also trades debt instruments with very large maturities (up to 30 years). Data from the available 15 maturities (no overnight rate) is published by the French Treasury and the parameters of the fitted curve are the following:

FRENCH TREASURIES				
30-Nov-99	$\mathbf{r_1}$	$\bar{r}$	$\boldsymbol{\tau}$	$\mathbf{R^2}$
Without Liquidity Premium				
Estimated Average	2,795%	5,897%	811 days	0,979
With Liquidity Premium				
Estimated Value	2,755%	5,662%	695 days	0,983

Here the introduction of the liquidity premium - $k.\sigma$ = 0,00085% p.a. - improves slightly the quality of the adjustment and reduces the spread between $\mathbf{r_1}$ and $\bar{\boldsymbol{r}}$. Refer to the Annex for the complete data, the adjusted curve and a summary of the calculations.

[15] Source of data: Internet page "*French government yield curve*" from "France Trésor".

c) The USA Capital Market (Strips)[16]

The use of the strip market allows one to work directly with spot rates (not YTM rates from bonds) and also supplies a large sample (60 points). In order to minimise the impact of any potential liquidity effects upon the market rates, only those strips produced by bond and note principals were considered[17]. This might be particularly relevant for very long maturites were tradability tends to dry out.

AMERICAN STRIPS **Notes and Bonds Principal only**			
01-Dec-99	**Without Liquidity Premium**		**R^2 = 0,922**
First regime	$\mathbf{r_1}$	$\bar{r}_1$	$\boldsymbol{\tau_1}$
Estimated Average	5,480%	6,967%	948 days
Second regime	$\mathbf{r_{n+1}}$	$\bar{r}_2$	$\boldsymbol{\tau_2}$
Estimated Average	*6,963%*[18]	5,125%	2 372 days

Here a curve generated by two sets of parameters (double regime) was adjusted to the sample with a crossover between Feb 2015 and Aug 2015. Refer to the Annex for the complete data, the adjusted curve and a summary of the calculations.

d) Eurodollar Futures[19]

The market here does not supply a set of spot interest rates but rather a collection of future rates for 91 days deposits starting at the end of the future contacts. Therefore one has to work with an adapted formula of the model:

$$\boldsymbol{R}_{n,n+91} = \bar{\boldsymbol{r}} + (\boldsymbol{r}_1 - \bar{\boldsymbol{r}}).(1-\frac{1}{t})^{n}.\frac{t}{91}.\left[1-(1-\frac{1}{t})^{91}\right]$$

where $\bar{r}$ is the long run overnight spot rate, $\mathbf{r_1}$ is the current overnight rate, $\boldsymbol{\tau}$ is the transition time and $\mathbf{R_{n,n+91}}$ is the current future rate for a deposit that will start on the day **n+1**.

The advantage of the Eurodollar futures is the large number of contracts open for trading at any time. One gains access to a large sample of interest rates for maturites up to 10 years ahead.

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16 Source of data: "*U.S. Government Strips*" published in the issue of the Wall Street Journal Europe of Saturday, 3 - 4 December 1999.

17 Mind that strips from coupons tend to be less tradable than the strips from principals due to their small outstanding nominal amounts.

18 This is an approximation by taking as $\mathbf{r_{n+1}}$ the last overnight rate $\mathbf{r_n}$ from the initial regime.

19 Source of data: Internet page "*Prices*" from Chicago Mercantile Exchange.

EURODOLLAR FUTURE CONTRACT			
8-Jul-99	r_1	$\bar{r}$	τ
Without Liquidity Premium			$R^2 = 0{,}961$
Estimated Average	5,400%	7,650%	1 464 days

Refer to the Annex for the complete data, the adjusted curve and a summary of the calculations.

6. Conclusions

The basic idea that the current short term spot rate (for one day, one week or so) must determine most of the value of the next period´s rate seems to be powerful enough to produce a rather flexible building block. Armed with it and combining it in different ways, one can construct an equation that not only describes the general shape of the yield curve but can also be calibrated to fit the actual rates sampled from the market via different financial instruments. In fact, in all the few examples tested the model led to curves that run very close to the sampled points as one can gauge by the very high $\mathbf{R^2}$ values obtained.

That basic model has three quantitative parameters to be estimated from market data, all of which have an economic interpretation:

- $\mathbf{r_1}$: the current short term rate (typically, the overnight rate)
- $\bar{r}$: the asymptotic future level of that short term rate (in the long run)
- τ: the "velocity" of transition from the current rate $\mathbf{r_1}$ to that future level $\bar{r}$ of interests and which is expressed by the number of days required to "complete" that jump.

Additionally and from a pragmatic point of view, the last two parameters may help in two areas:

- τ is a quantitative measure of the expectations of the market about the proximity of a future change in the short term level of interest rates; this is a relevant information when the monetary policy conducted by a central bank is based on a short term target rate that is continuously steered by that bank; in particular, modifications in the size of τ indicate the imminence or not of a jump in that target rate;
- as a future short-term rate, $\bar{r}$ is a more explicit information about the future level of interest rates than the traditional long-term spot rates since these last ones are a mixture of the rates expected to exist from today until that long period; additionally, the introduction of the liquidity premium allows to obtain an interest rate $\bar{r}$ completely clean from that premium and so closer to the expected overnight rate.

The model is also very flexible in the sense that it can be adapted:

- to describe regimes with only one set of parameters - positive or negative slope term structures - along with regimes with two sets of parameters - initial positive slope followed by an inversion on that derivative of the curve and vice-versa;

- to adjust curves to market data produced by different financial instruments: a collection of spot rates, a set of eurodollar futures prices, etc[20];
- to estimate, without interpolation, the interest rates for non-standard maturities.

It is likely that the model can be improved by adding a liquidity premium which is solely dependent upon the maturities of the interest rates. That requires the introduction of 2 additional parameters, but the main drawback of that improvement is the increased mathematical sophistication required in any practical use of the model. Besides, it is not yet clear that the higher quality of the adjustments with that premium compensates for the added work required to adjust the curve to any sample of points. In particular, it makes almost impractical to calculate the standard deviations of the parameters of the model.

The model draws heavily from the Expectation Hypothesis and is also improved by the addition of a concrete form of the liquidity premium. In that sense the model will be supportive of both the expectation and liquidity premium hypothesis if the high quality of the fits encountered now can be confirmed with further examples of the term structure.

In fact one cannot forget that all the examples tested in this paper came from a rather narrow set of cases where the different markets were very much interconnected - all the yield curves might be similar due to intense relationship of their markets - and also that all of the them were taken from a rather short time interval - the yield curves did not show pronounced changes in their basic shapes along that time window. Only future samples will provide significantly different curves to test more thoroughly this model.

This test of time is particularly necessary for the improved version with the liquidity premium added. First, because such a premium can only play a significant role for long maturities, suggesting further tests using long term data. Second, because all the curves used followed rather simple shapes and that has very much contributed to the high quality of the fittings encountered. With very high $\mathbf{R^2}$ values obtained without that premium, any improvements brought about by the introduction of that sophistication may not be very significant, if useful at all.

Finally, another test to the model might come from the margin of errors of the estimates of the 3 parameters: how much accurate are the estimates obtained from real data ?

7. References

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[20] For example, one can develop mathematical expressions for the discount factors used to value coupon bonds and estimate the 3 parameters that minimise the errors to the market prices of a portfolio of those bonds.

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ANNEX

1. Sum of the first n terms of a Geometric Progression

A formula for the following sum of **n** terms can be obtained

$$S_n = \underbrace{1 + b + b^2 + ... + b^{n-1}}_{n\ terms}$$

by noting that

$$\begin{cases} S_n = 1 + b + b^2 + ... + b^{n-1} \\ b.S_n = b + b^2 + b^3 + ...b^n \end{cases} \Longrightarrow \quad b.S_n - S_n = b^n - 1$$

and the final result is

$$\boxed{S_n = \frac{1 - b^n}{1 - b}}$$

Note that the power of **b** in the numerator is the number **n** of terms in the sum.

2. Uncertainty of r_n

Summing the contribution of the **n** first days (n-1 terms) for the total “noise” of $\mathbf{r_n}$:

$$e'_n = e_n + b.e_{n-1} + b^2.e_{n-2} + ... + b^{n-2}.e_2$$

Assuming that the noise of each overnight rate $\mathbf{r_n}$ is independent from all the others and that the Gaussian distribution is stable (one single standard deviation common to all days)

$$Var\left[e'_n\right] = s_n^2 + b^2.s_{n-1}^2 + b^4.s_{n-2}^2 + ... + b^{2(n-2)}.s_2^2$$

$$= s^2.\left[1 + b^2 + b^4 + ... + b^{2(n-2)}\right] = s^2.\frac{1 - (b^2)^{n-1}}{1 - b^2}$$

and finally

$$\boxed{\boldsymbol{Var}\left[e'_{\boldsymbol{n}}\right] = s^{\,2}.\frac{1-(\boldsymbol{b}^2)^{\boldsymbol{n}-1}}{1-\boldsymbol{b}^2}}$$

3. Term in “a” in the spot rate R_n

This term is the sum of **n-1** terms

$$S = a + a.(1+b) + a.(1+b+b^2) + \ldots + a.(1+b+b^2+\ldots+b^{n-2})$$

and noting that

$$1+b = \frac{1-b^2}{1-b}$$

$$1+b+b^2 = \frac{1-b^3}{1-b}$$

$$\cdots$$

$$1+b+b^2+\ldots+b^{n-2} = \frac{1-b^{n-1}}{1-b}$$

it can be written

$$\boldsymbol{S} = \boldsymbol{a}.\left[\frac{1-\boldsymbol{b}}{1-\boldsymbol{b}} + \frac{1-\boldsymbol{b}^2}{1-\boldsymbol{b}} + \frac{1-\boldsymbol{b}^3}{1-\boldsymbol{b}} + \ldots + \frac{1-\boldsymbol{b}^{\boldsymbol{n}-1}}{1-\boldsymbol{b}}\right]$$

$$= \boldsymbol{a}.\underbrace{\left[\frac{1}{1-\boldsymbol{b}} + \frac{1}{1-\boldsymbol{b}} + \frac{1}{1-\boldsymbol{b}} + \ldots + \frac{1}{1-\boldsymbol{b}}\right]}_{\boldsymbol{n}-1\ \textit{terms}} - \boldsymbol{a}.\left[\frac{\boldsymbol{b}}{1-\boldsymbol{b}} + \frac{\boldsymbol{b}^2}{1-\boldsymbol{b}} + \frac{\boldsymbol{b}^3}{1-\boldsymbol{b}} + \ldots + \frac{\boldsymbol{b}^{\boldsymbol{n}-1}}{1-\boldsymbol{b}}\right]$$

and finally

$$\boxed{= \frac{\boldsymbol{a}}{-\boldsymbol{b}.}.\left[\qquad 1) - \boldsymbol{b}\,\frac{1-\boldsymbol{b}^{\boldsymbol{n}\ 1}}{1}\right]}$$

4. Uncertainty of the spot rate R

The “noise” in the spot rate for **n**

$$
\begin{aligned}
e''_{\mathrm{n}} &= \frac{1}{\boldsymbol{n}}.\left[e_{\boldsymbol{n}} + e_{\boldsymbol{n}-1}.(1+\boldsymbol{b}) + ... + e_2.(1+\boldsymbol{b}+....+\boldsymbol{b}^{\boldsymbol{n}-2})\right] \\
&= \frac{1}{\boldsymbol{n}}.\left[e_{\boldsymbol{n}}.\frac{1-\boldsymbol{b}}{1-\boldsymbol{b}} + e_{\boldsymbol{n}-1}.\frac{1-\boldsymbol{b}^2}{1-\boldsymbol{b}} + ... + e_2.\frac{1-\boldsymbol{b}^{\boldsymbol{n}-1}}{1-\boldsymbol{b}}\right]
\end{aligned}
$$

which still is a Gaussian distribution with variance given by

$$
\boldsymbol{Var}\left[e''_{\boldsymbol{n}}\right] = \frac{s^2}{\boldsymbol{n}^2.(1-\boldsymbol{b})^2}.\left[\overbrace{(1-\boldsymbol{b})^2 + (1-\boldsymbol{b}^2)^2 + (1-\boldsymbol{b}^3)^2 + ... + (1-\boldsymbol{b}^{\boldsymbol{n}-1})^2}^{\boldsymbol{n}-1}\right]
$$

$$
= \frac{s^2}{\boldsymbol{n}^2.(1-\boldsymbol{b})^2}.\left[(\boldsymbol{n}-1) - 2\boldsymbol{b}.(1+\boldsymbol{b}+\boldsymbol{b}^2+\boldsymbol{b}^3+...+\boldsymbol{b}^{\boldsymbol{n}-2}) + (\boldsymbol{b}^2+\boldsymbol{b}^4+\boldsymbol{b}^6+....+\boldsymbol{b}^{2(\boldsymbol{n}-1)})\right]
$$

and finally

$$
\boxed{\boldsymbol{Var}\left[e''_{\boldsymbol{n}}\right] = s^2_{\boldsymbol{R}_n} = \frac{s^2}{\boldsymbol{n}^2.(1-\boldsymbol{b})^2}.\left[(\boldsymbol{n}-1) - 2\boldsymbol{b}.\frac{1-\boldsymbol{b}^{\boldsymbol{n}-1}}{1-\boldsymbol{b}} + \boldsymbol{b}^2.\frac{1-\boldsymbol{b}^{2(\boldsymbol{n}-1)}}{1-\boldsymbol{b}^2}\right]}
$$

5. Covariance between $e''_{\boldsymbol{n}}$ and $e''_{\boldsymbol{n}-\boldsymbol{k}}$

Using matrix notation, one can write that the noise for two terms in **n** and in **n-k** are:

$$
e''_{\boldsymbol{n}} = \frac{1}{\boldsymbol{n}.(1-\boldsymbol{b})}.\begin{bmatrix} e_{\boldsymbol{n}}.(1-\boldsymbol{b}) \\ e_{\boldsymbol{n}-1}.(1-\boldsymbol{b}^2) \\ \vdots \\ e_2.(1-\boldsymbol{b}^{\boldsymbol{n}-1}) \end{bmatrix}
\qquad
e''_{\boldsymbol{n}-\boldsymbol{k}} = \frac{1}{(\boldsymbol{n}-\boldsymbol{k}).(1-\boldsymbol{b})}.\begin{bmatrix} e_{\boldsymbol{n}-\boldsymbol{k}}.(1-\boldsymbol{b}) \\ e_{\boldsymbol{n}-\boldsymbol{k}-1}.(1-\boldsymbol{b}^2) \\ \vdots \\ e_2.(1-\boldsymbol{b}^{\boldsymbol{n}-\boldsymbol{k}-1}) \end{bmatrix}
$$

which leads to the following covariance between $e''_{\boldsymbol{n}}$ and $e''_{\boldsymbol{n}-\boldsymbol{k}}$

$$E\left[e''_n . e''_{n-k}\right] = \frac{1}{n.(n-k).(1-b)^2} . E\left\{ \left[e_{n-k}.(1-b^{k+1}) \quad e_{n-k-1}.(1-b^{k+2}) \quad \cdots \quad e_2.(1-b^{n-1}) \right] \begin{bmatrix} e_{n-k}.(1-b) \\ e_{n-k-1}.(1-b^2) \\ \vdots \\ e_2.(1-b^{n-k-1}) \end{bmatrix} \right\}$$

As all ε_i are supposed to have the same variance σ^2

$$E\left[e''_n . e''_{n-k}\right] = \frac{s^2}{n.(n-k).(1-b)^2} . \left\{ (1-b^{k+1}).(1-b) + (1-b^{k+2}).(1-b^2) + \cdots + (1-b^{n-1}).(1-b^{n-k-1}) \right\}$$

Noting that:

$$E\left[e''_n . e''_{n-k}\right] = \frac{s^2}{n.(n-k).(1-b)^2} . \left\{ \begin{array}{c} n-k-1 \\ -b-b^2-b^3-\cdots-b^{n-k-1} \\ -b^{k+1}-b^{k+2}-\cdots-b^{n-1} \\ +b^{k+2}+b^{k+4}+\cdots+b^{2n-k-2} \end{array} \right\}$$

this can be simplified into

$$\boxed{Cov\left[e''_n ; e''_{n-k}\right] = \frac{s^2}{n.(n-k).(1-b)^2} . \left\{ (n-k-1) - b.(1+b^k).\frac{1-b^{n-k-1}}{1-b} + b^2.\frac{1-(b^2)^{n-k-1}}{1-b^2} \right\}}$$

Mind that for **k** = 0 we have the "auto-covariance" of order **n**, that is $Var\left[e''_n\right]$

$$Var\left[e''_n\right] = \frac{s^2}{n^2.(1-b)^2} . \left\{ (n-1) - 2.b.\frac{1-b^{n-1}}{1-b} + b^2.\frac{1-(b^2)^{n-1}}{1-b^2} \right\}$$

which is consistent with the formula already obtained for the variance. Note that σ^2 has to be estimated from the **n** errors e''_i between the sample and the adjusted curve.

6. Linear regression to estimate the 3 parameters

Using the operational parameters, $\mathbf{r_1}$, τ and $\bar{r}$ the current spot rate $\mathbf{R_i}$ for **i** days

$$R_i = \bar{r} + \frac{t}{i}.(r_1 - \bar{r}).\left[1 - (1 - \frac{1}{t})^i\right] + e''_i$$

can be rewritten as a linear function of a new variable $\mathbf{F_i}$

$$\boldsymbol{R_i} = \bar{\boldsymbol{r}} + (\boldsymbol{r}_1 - \bar{\boldsymbol{r}}).\boldsymbol{F_i} + e_i'' \quad \text{where} \qquad \boldsymbol{F_i} = \frac{t}{\boldsymbol{i}}.\left[1-(1-\frac{1}{t})^{i}\right]$$

or, using matrix language,

$$\boldsymbol{R} = \begin{bmatrix}1 & \boldsymbol{F}\end{bmatrix}.b + e = \boldsymbol{X}.b + e \qquad \text{where} \qquad b = \begin{bmatrix}\bar{\boldsymbol{r}} \\ \boldsymbol{r}_1 - \bar{\boldsymbol{r}}\end{bmatrix}$$

This suggests that one can estimate the two parameters $\bar{r}$ and (r_1- $\bar{r}$) via a linear regression of $\mathbf{R_i}$ on $\mathbf{F_i}$. For each value of **τ**, the above "regression" can produce the best values of $\mathbf{r_1}$ and $\bar{r}$ and the solution is the one which leads to the smallest mean square error. So this a mixture of trial and error to estimate **τ** and a closed solution for the 2 other parameters.

Due to the existence of auto-correlation an heteroskesdasticity between the disturbances e_i'', one has to use GLS instead of OLS. Writing

$$\boldsymbol{E}\left[e^{\boldsymbol{T}}, e\right] = s^{2}.\Omega$$

GLS leads to

$$\hat{b} = (\boldsymbol{X}^{\boldsymbol{T}}.\Omega^{-1}.\boldsymbol{X})^{-1}.(\boldsymbol{X}^{\boldsymbol{T}}.\Omega^{-1}).\boldsymbol{R}$$

Note that the non-linearity in **τ** may allow for multiple stability points (zero first derivatives) and therefore it is necessary to assure that the obtained point is an absolute minimum and not a local minimum.

The matrix of variances and co-variances between the different noise terms of $\mathbf{R_i}$ is

$$\begin{bmatrix} s_1^2 & s_{12} & s_{1\boldsymbol{n}} \\ s_{12} & s_2^2 & s_{2\boldsymbol{n}} \\ & & \\ s_{12} & s_{2\boldsymbol{n}} & s_{\boldsymbol{n}}^2 \end{bmatrix} = \begin{bmatrix} \mathrm{Var}(e_1'') & \mathrm{Covar}(e_1'';e_2'') & \mathrm{Covar}(e_1'';e_{\boldsymbol{n}}'') \\ \mathrm{Covar}(e_1'';e_2'') & \mathrm{Var}(e_2'') & \mathrm{Covar}(e_2'';e_{\boldsymbol{n}}'') \\ & & \\ \mathrm{Covar}(e_1'';e_{\boldsymbol{n}}'') & \mathrm{Covar}(e_2'';e_{\boldsymbol{n}}'') & \mathrm{Var}(e_{\boldsymbol{n}}'') \end{bmatrix}$$

Sometimes it is used the average variance from the **n** points sample

$$\bar{s}^{2} = \frac{\sum_{\boldsymbol{i}=1}^{\boldsymbol{n}} s_{\boldsymbol{i}}^{2}}{\boldsymbol{n}}$$

to transform the covariance matrix into

$$\begin{bmatrix} s_1^2 & s_{12} & s_{1n} \\ s_{12} & s_2^2 & s_{2n} \\ & & \\ s_{12} & s_{2n} & s_n^2 \end{bmatrix} = \overline{s}^2.[\Omega]$$

7. Standard error of the 3 parameters

a) Standard error of $\boldsymbol{r_1}$ *and* $\overline{r}$

Under GLS, the standard deviation of the 2 parameters included in the vector β is

$$\boldsymbol{Var}[b] = \overline{s}^2.(\boldsymbol{X}^T.\Omega^{-1}.\boldsymbol{X})^{-1}.\Omega.(\boldsymbol{X}^T.\Omega^{-1}.\boldsymbol{X})$$

where it is still necessary to estimate the variance σ^2 from market data.

b) Standard error of $\boldsymbol{t}$

Due to the non-linearity of $\mathbf{R_i} = \mathbf{f}(\tau)$, the standard deviation of this parameter can only be estimated using an approximate equation from the real rates $\mathbf{R_i}$ of a sample. Denominating

$$\boldsymbol{S} = \sum_{\boldsymbol{i}} (\boldsymbol{R_i} - \hat{\boldsymbol{R}}_{\boldsymbol{i}})^2$$

the variance of τ can estimated from

$$\boldsymbol{Var}[t] \cong 2.s_e^2.\left(\frac{\ddot{a}^2\boldsymbol{S}}{dt^2}\right)^{-1}$$

The second derivative of **S** is

$$\frac{d\boldsymbol{S}}{dt} = -2.\sum_{\boldsymbol{i}=1}^{\boldsymbol{n}}(\boldsymbol{R}_i - \hat{\boldsymbol{R}}_i)\frac{d\hat{\boldsymbol{R}}_i}{dt} \quad \Rightarrow \quad \frac{d^2\boldsymbol{S}}{dt^2} = -2.\sum_{\boldsymbol{i}=1}^{\boldsymbol{n}}(\boldsymbol{R}_i - \hat{\boldsymbol{R}}_i).\frac{d^2\hat{\boldsymbol{R}}_i}{dt^2} + 2\sum_{\boldsymbol{i}=1}^{\boldsymbol{n}}(\frac{d\hat{\boldsymbol{R}}_i}{dt})^2$$

which leads to

$$\boxed{\begin{aligned}\frac{d^2\boldsymbol{S}}{dt^2} &= 2.\left(\boldsymbol{r}_1-\bar{\boldsymbol{r}}\right)^2.\sum_{\boldsymbol{i}=1}^{\boldsymbol{n}}\left[\frac{1-(1-1/t)^{\boldsymbol{i}}}{\boldsymbol{i}}-\frac{(1-1/t)^{\boldsymbol{i}-1}}{t}\right]^2+\\ &+2.\left(\boldsymbol{r}_1-\bar{\boldsymbol{r}}\right).\sum_{\boldsymbol{i}=1}^{\boldsymbol{n}}(\boldsymbol{R}_{\boldsymbol{i}}-\hat{\boldsymbol{R}}_{\boldsymbol{i}}).\frac{(\boldsymbol{i}-1).(1-1/t)^{\boldsymbol{i}-2}}{t^3}\end{aligned}}$$

c) Standard deviation of t *for the case of Eurodollar Futures*

The forward rate for the period between day **n** and **n+91** as a function of $\mathbf{r_1}$ is

$$\boldsymbol{R}_{\boldsymbol{n},\boldsymbol{n}+91}=\bar{\boldsymbol{r}}+(\boldsymbol{r}_1-\bar{\boldsymbol{r}}).(1-\frac{1}{t})^{\boldsymbol{n}}.\frac{t}{91}.\left[1-(1-\frac{1}{t})^{91}\right]$$

which is not much different from the correspondent equation for $\mathbf{R_n}$. Fitting this curve to a sample o actual rates requires minimising **S**

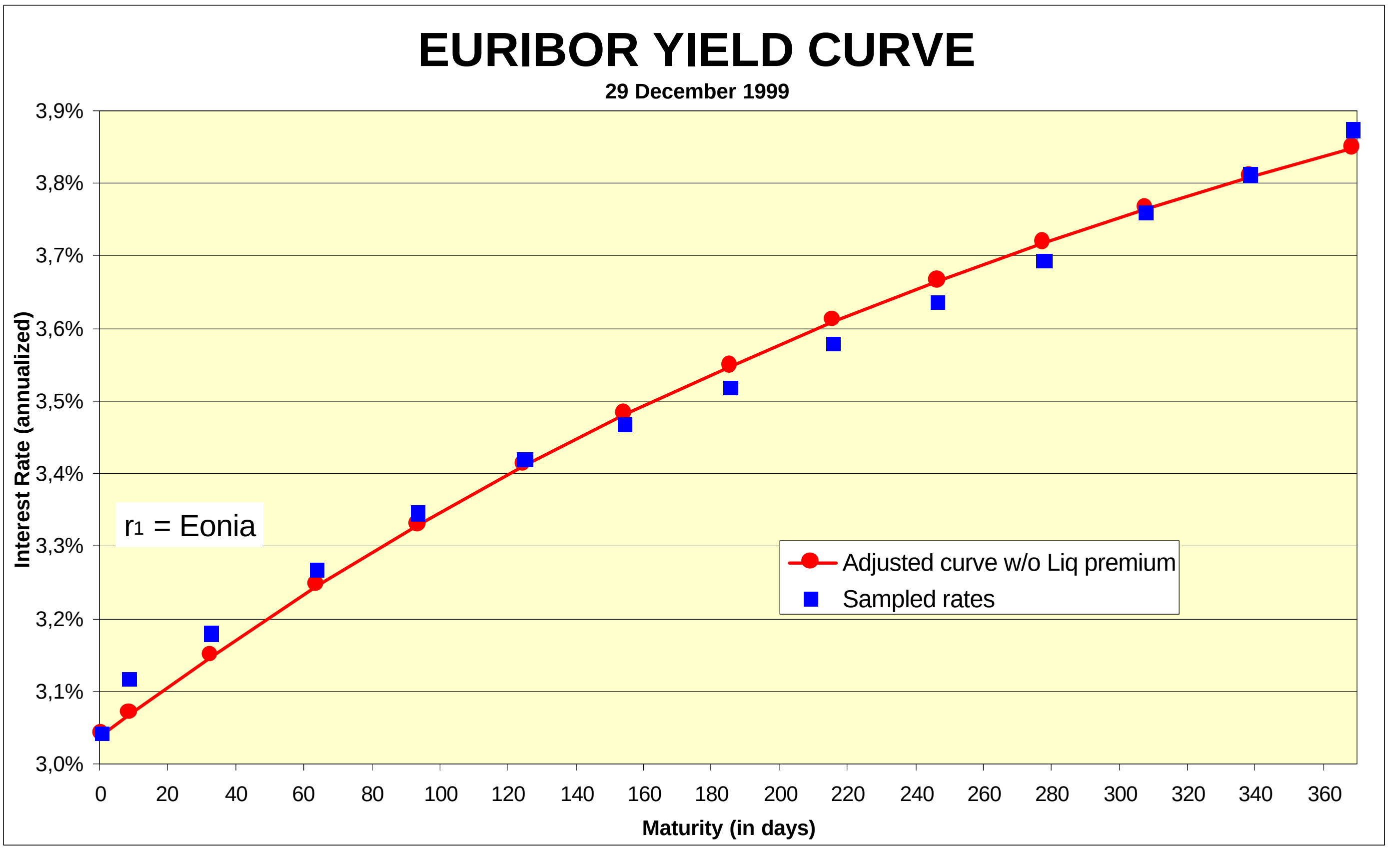
$$\boldsymbol{S}=\sum_{\boldsymbol{i}}(\boldsymbol{R}_i-\hat{\boldsymbol{R}}_i)^2$$

The standard deviation of the estimate of τ requires the computation of the second derivative of **S** in order of τ:

$$\frac{d^2\boldsymbol{S}}{dt^2}=-2.\sum_{\boldsymbol{i}=1}^{\boldsymbol{n}}(\boldsymbol{R}_i-\hat{\boldsymbol{R}}_i).\frac{d^2\hat{\boldsymbol{R}}_i}{dt^2}+2\sum_{\boldsymbol{i}=1}^{\boldsymbol{n}}(\frac{d\hat{\boldsymbol{R}}_i}{dt})^2$$

for which one requires both the first and the second derivative of $\mathbf{r_i}$ in order to τ:

$$\left\{\begin{aligned}\frac{d\hat{\boldsymbol{R}}_{\boldsymbol{i}}}{dt}&=\frac{\boldsymbol{r}_1-\bar{\boldsymbol{r}}}{91}.\left\{\frac{\boldsymbol{i}}{t}.\left(1-\frac{1}{t}\right)^{\boldsymbol{i}-1}.\left[1-(1-\frac{1}{t})^{91}\right]+(1-\frac{1}{t})^{\boldsymbol{i}}.\left[1-(1-\frac{1}{t})^{91}\right]-\frac{91}{t}.(1-\frac{1}{t})^{90+\boldsymbol{i}}\right\}\\ \frac{d^2\hat{\boldsymbol{R}}_{\boldsymbol{i}}}{dt^2}&=\frac{\boldsymbol{r}_1-\bar{\boldsymbol{r}}}{91}.\left\{\frac{\boldsymbol{i}.(\boldsymbol{i}-1)}{t^3}.(1-\frac{1}{t})^{\boldsymbol{i}-2}.\left[1-(1-\frac{1}{t})^{91}\right]-\frac{91.90}{t^3}.(1-\frac{1}{t})^{89+\boldsymbol{i}}\right\}\end{aligned}\right.$$


EURIBOR YIELD CURVE
29 December 1999
Interest Rate (annualized)
3,9%
3,8%
3,7%
3,6%
3,5%
3,4%
3,3%
3,2%
3,1%
3,0%
r1 = Eonia
Adjusted curve w/o Liq premium
Sampled rates
0
20
40
60
80
100
120
140
160
180
200
220
240
260
280
300
320
340
360
Maturity (in days)

EURIBOR YIELD CURVE

29 Dec 1999

Forcing r_1 to be = EONIA
Without Liquidity Premium

	r_1 =	3,0400%	p.a.
R^2 = 0,99066	τ =	221	days
	r_m =	4,6185%	p.a.

Sample Date 29 DEC 99		Number of Days	R_i	Errors Simple	Squared
EONIA	3,0400%	1	3,040%	0,0000%	2,330E-32
1 W	3,1150%	9	3,068%	0,0468%	2,187E-07
1 M	3,1800%	33	3,149%	0,0310%	9,617E-08
2 M	3,2660%	64	3,245%	0,0209%	4,368E-08
3 M	3,3450%	94	3,330%	0,0149%	2,215E-08
4 M	3,4170%	125	3,411%	0,0065%	4,212E-09
5 M	3,4660%	155	3,482%	-0,0158%	2,490E-08
6 M	3,5170%	186	3,549%	-0,0323%	1,045E-07
7 M	3,5770%	216	3,609%	-0,0323%	1,046E-07
8 M	3,6350%	247	3,666%	-0,0314%	9,838E-08
9 M	3,6920%	278	3,719%	-0,0268%	7,167E-08
10 M	3,7590%	308	3,766%	-0,0065%	4,232E-09
11 M	3,8120%	339	3,810%	0,0019%	3,764E-10
12 M	3,8730%	369	3,850%	0,0231%	5,342E-08
				Sum of Errors	
				0,0000%	8,470E-07

R. C. 00.Mar.06

EURIBOR YIELD CURVE

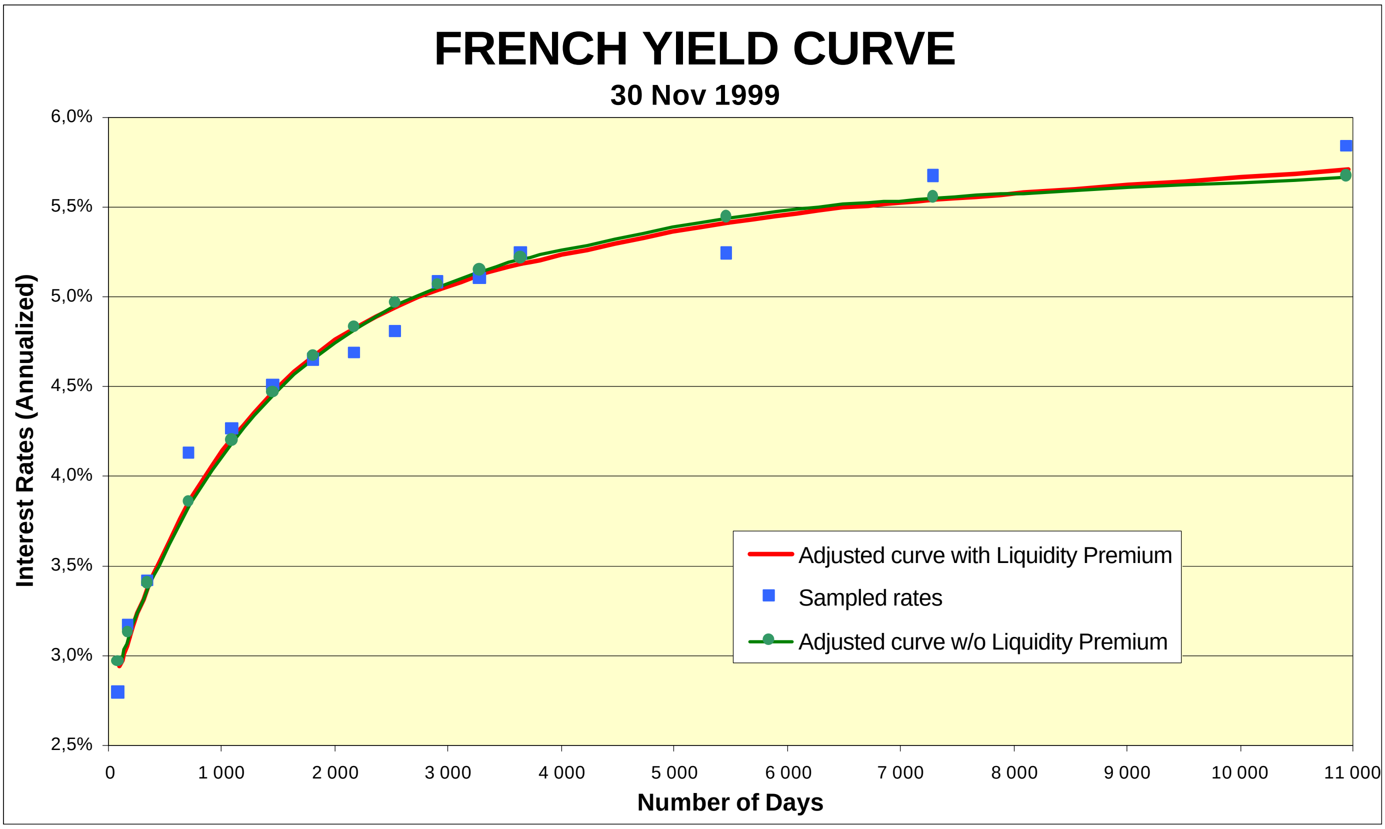
29 Dec 1999

Forcing r_1 to be = EONIA
With Liquidity Premium

	r_1 =	3,0400%	p.a.	a =	0,021%	p.d.
	r_m =	4,6086%	p.a.	b =	0,9955	
R^2 = 0,99066	τ =	220	days	$\eta = 1 - b$	0,454%	
	$\kappa.\sigma$ =	0,00050%	p.a.	$\kappa.\sigma$ =	0,000001%	p.d.

Sample Date 29 DEC 99		Number of Days	Fitted Rates W/O Premium	Liquidity Premium	Fitted Curve	Error Simple	Squared
EONIA	3,0400%	1	3,040%	0,0000%	3,0400%	0,0000%	3,510E-32
1 W	3,1150%	9	3,068%	0,0000%	3,0682%	0,0468%	2,187E-07
1 M	3,1800%	33	3,149%	0,0001%	3,1490%	0,0310%	9,621E-08
2 M	3,2660%	64	3,245%	0,0003%	3,2451%	0,0209%	4,365E-08
3 M	3,3450%	94	3,330%	0,0004%	3,3301%	0,0149%	2,210E-08
4 M	3,4170%	125	3,410%	0,0006%	3,4105%	0,0065%	4,182E-09
5 M	3,4660%	155	3,481%	0,0008%	3,4818%	-0,0158%	2,498E-08
6 M	3,5170%	186	3,548%	0,0011%	3,5493%	-0,0323%	1,046E-07
7 M	3,5770%	216	3,608%	0,0013%	3,6094%	-0,0324%	1,047E-07
8 M	3,6350%	247	3,665%	0,0015%	3,6664%	-0,0314%	9,838E-08
9 M	3,6920%	278	3,717%	0,0018%	3,7188%	-0,0268%	7,162E-08
10 M	3,7590%	308	3,764%	0,0020%	3,7655%	-0,0065%	4,207E-09
11 M	3,8120%	339	3,808%	0,0022%	3,8100%	0,0020%	3,866E-10
12 M	3,8730%	369	3,847%	0,0024%	3,8499%	0,0231%	5,355E-08
						Sum of Errors	
						0,0000%	*8,47E-07*

R. C. 00.Mar.06


FRENCH YIELD CURVE
30 Nov 1999
6,0%
5,5%
5,0%
4,5%
4,0%
3,5%
3,0%
2,5%
Interest Rates (Annualized)
0
1 000
2 000
3 000
4 000
5 000
6 000
7 000
8 000
9 000
10 000
11 000
Number of Days
Adjusted curve with Liquidity Premium
Sampled rates
Adjusted curve w/o Liquidity Premium

FRENCH YIELD CURVE

30-Nov-99

Without the Liquidity Premium

r_1 =	2,7950%	p.a.
r_m =	5,8967%	p.a.
τ =	811	days

a =	0,007%	p.d.
b =	0,9988	
$\eta = 1 - b$	0,123%	

R^2 = 0,97865

Actual		Number of Days	R_i	Error Simple	Error Squared
30-11-1999					
3 months	2,79%	91	2,9608%	0,1708%	2,9189E-06
6 months	3,16%	182	3,1167%	-0,0433%	1,8779E-07
1 year	3,41%	366	3,3989%	-0,0111%	1,2214E-08
2 years	4,12%	731	3,8515%	-0,2685%	7,2100E-06
3 years	4,25%	1 096	4,1947%	-0,0553%	3,0590E-07
4 years	4,50%	1 461	4,4583%	-0,0417%	1,7402E-07
5 years	4,64%	1 827	4,6639%	0,0239%	5,6927E-08
6 years	4,68%	2 192	4,8254%	0,1454%	2,1144E-06
7 years	4,80%	2 557	4,9544%	0,1544%	2,3848E-06
8 years	5,08%	2 922	5,0588%	-0,0212%	4,4973E-08
9 years	5,09%	3 288	5,1445%	0,0545%	2,9691E-07
10 years	5,24%	3 653	5,2153%	-0,0247%	6,0925E-08
15 years	5,24%	5 475	5,4375%	0,1975%	3,9015E-06
20 years	5,67%	7 306	5,5522%	-0,1178%	1,3869E-06
30 years	5,83%	10 959	5,6670%	-0,1630%	2,6561E-06
				Sum of Errors	
				0,00000%	***2,37E-05***

With the Liquidity Premium

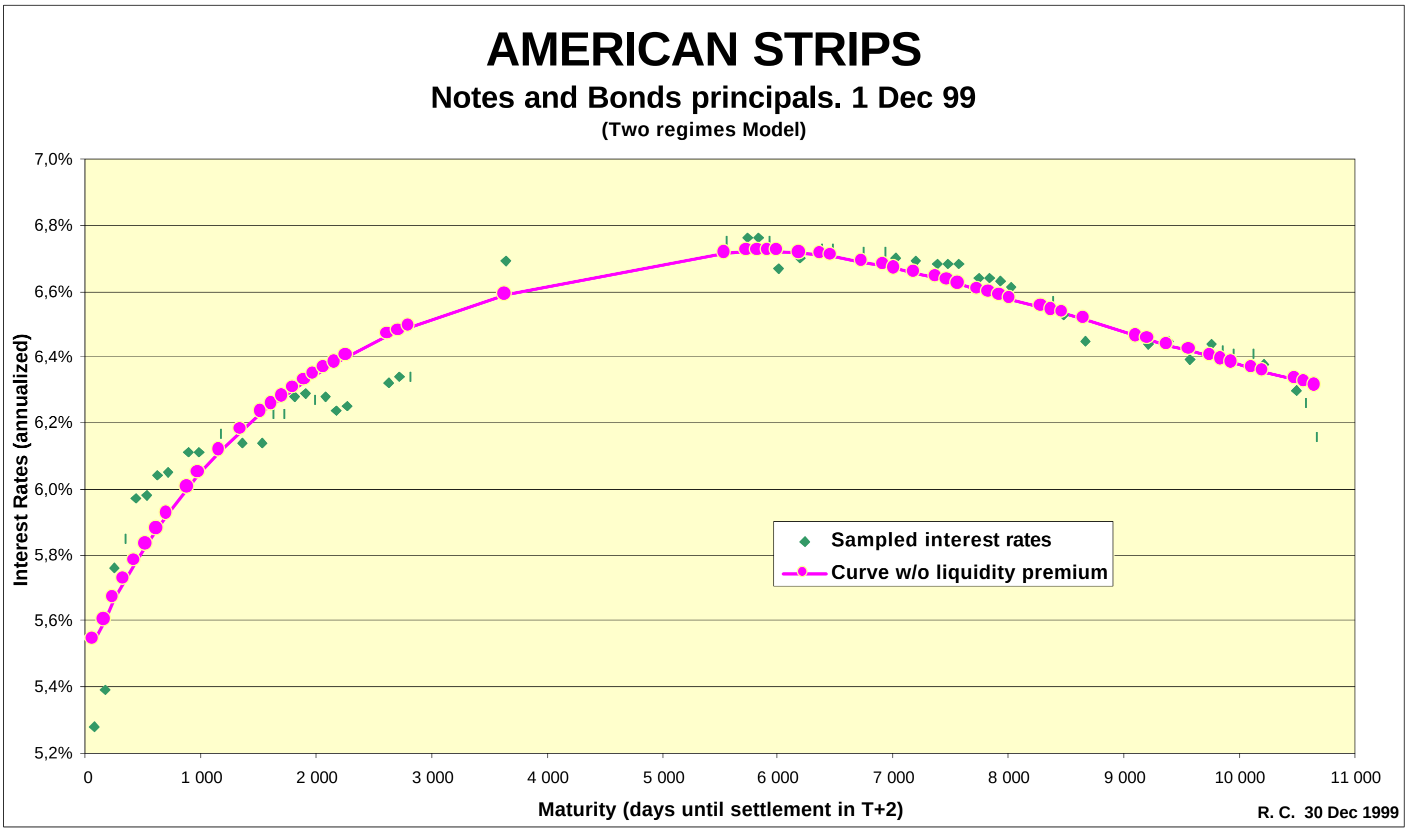
r_1 =	2,7552%	p.a.
r_m =	5,6622%	p.a.
τ =	695	days
$\kappa.\sigma$ =	0,00085%	p.a.

a =	0,008%	p.d.
b =	0,9986	
$\eta = 1 - b$	0,144%	
$\kappa.\sigma$ =	0,000002%	p.d.

R^2 = 0,98266

Actual		Number of Days	Fitted Rates W/O Premium	Liquidity Premium	R_i	Error Simple	Error Squared
30-11-1999							
3 months	2,79%	91	2,9358%	0,0008%	2,9366%	-0,1466%	2,148E-06
6 months	3,16%	182	3,1033%	0,0021%	3,1054%	0,0546%	2,983E-07
1 year	3,41%	366	3,4014%	0,0055%	3,4069%	0,0031%	9,922E-10
2 years	4,12%	731	3,8636%	0,0133%	3,8769%	0,2431%	5,911E-06
3 years	4,25%	1 096	4,1998%	0,0215%	4,2213%	0,0287%	8,236E-08
4 years	4,50%	1 461	4,4486%	0,0298%	4,4784%	0,0216%	4,670E-08
5 years	4,64%	1 827	4,6365%	0,0380%	4,6745%	-0,0345%	1,192E-07
6 years	4,68%	2 192	4,7802%	0,0461%	4,8263%	-0,1463%	2,142E-06
7 years	4,80%	2 557	4,8924%	0,0542%	4,9466%	-0,1466%	2,148E-06
8 years	5,08%	2 922	4,9815%	0,0622%	5,0437%	0,0363%	1,321E-07
9 years	5,09%	3 288	5,0535%	0,0702%	5,1237%	-0,0337%	1,137E-07
10 years	5,24%	3 653	5,1123%	0,0782%	5,1905%	0,0495%	2,447E-07
15 years	5,24%	5 475	5,2936%	0,1178%	5,4114%	-0,1714%	2,937E-06
20 years	5,67%	7 306	5,3858%	0,1575%	5,5434%	0,1266%	1,604E-06
30 years	5,83%	10 959	5,4780%	0,2366%	5,7146%	0,1154%	1,333E-06
						Sum of Errors	
						0,0000%	***1,926E-05***

R. C. 99.12.29


AMERICAN STRIPS
Notes and Bonds principals. 1 Dec 99
(Two regimes Model)
7,0%
6,8%
6,6%
6,4%
6,2%
6,0%
5,8%
5,6%
5,4%
5,2%
Interest Rates (annualized)
Sampled interest rates
Curve w/o liquidity premium
0
1 000
2 000
3 000
4 000
5 000
6 000
7 000
8 000
9 000
10 000
11 000
Maturity (days until settlement in T+2)
R. C. 30 Dec 1999

AMERICAM YIELD CURVE

From Treasury Strips (Notes & Bonds principal only)

W.S.J. Europe, 3 - 4 December, 1999

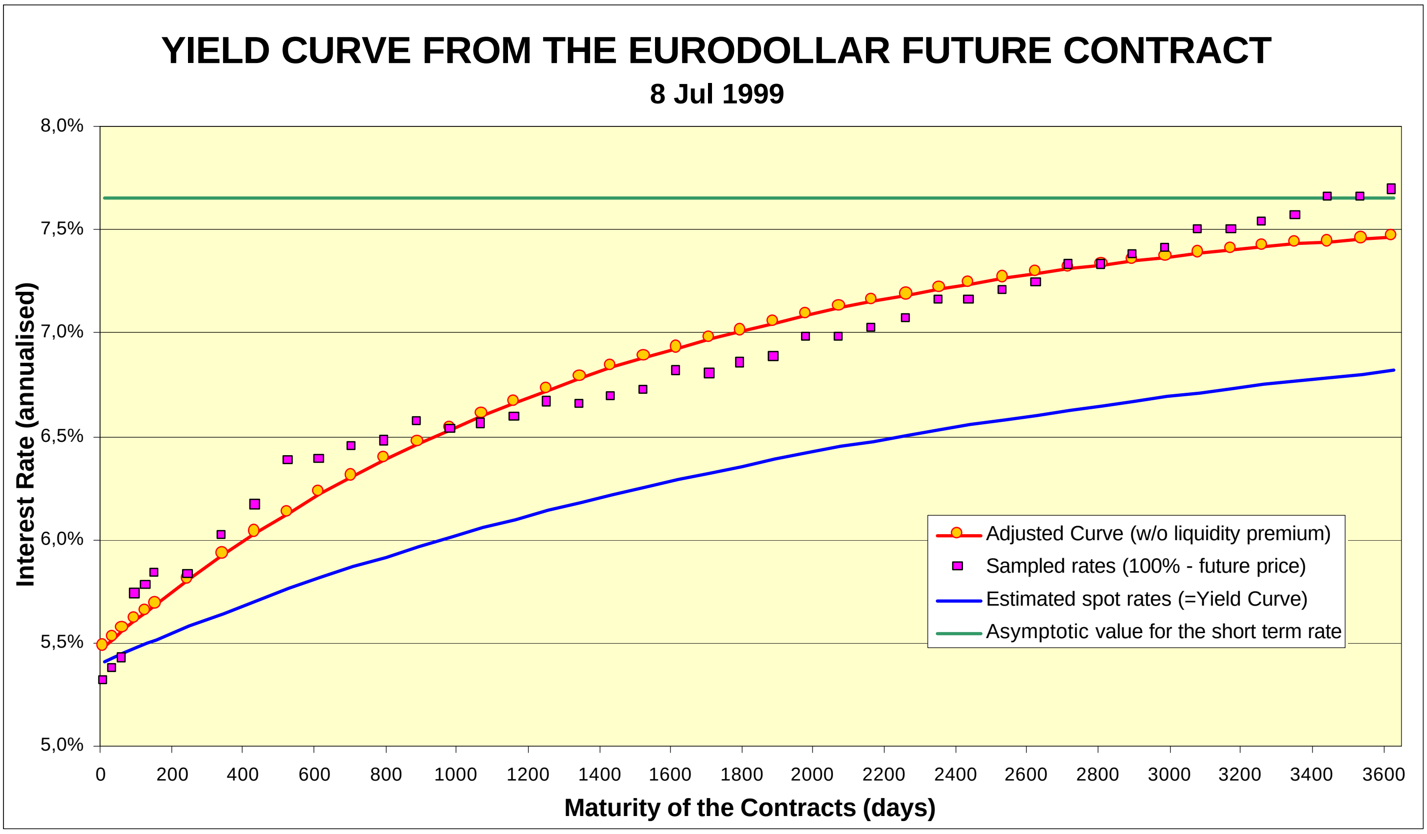
MODEL WITHOUT LIQUIDITY PREMIUM

PARAMETERS		
r_1 =	5,4803%	p.a.
r_{m1} =	6,9660%	p.a.
r_{m2} =	5,1246%	p.a.
τ_1 =	948	days
τ_2 =	2 372	days

R^2 = 0,92231

01-12-1999		Strips from	MATURITY DATE			Maturity + 2	Regime		Days to	Estimated rates			Errors	
	Ask Yld	np & bp	Day	Month	Year	Days	1	2	Transition	r_1	r_2	R_1	Simple	Squared
						1			5557	5,4803%	6,963%			
feb-00	5,28%	np	15	2	2000	78	1	0		5,5964%		5,5391%	-0,2591%	6,715E-06
may-00	5,39%	np	15	5	2000	168	1	0	0	5,7205%	0	5,6039%	-0,2139%	4,577E-06
aug-00	5,76%	np	15	8	2000	260	1	0	0	5,8368%	0	5,6661%	0,0939%	8,808E-07
Nov-00	5,85%	np	15	11	2000	352	1	0	0	5,9405%	0	5,7245%	0,1255%	1,574E-06
feb-01	5,97%	np	15	2	2001	444	1	0	0	6,0354%	0	5,7794%	0,1906%	3,633E-06
may-01	5,98%	np	15	5	2001	533	1	0	0	6,1189%	0	5,8293%	0,1507%	2,271E-06
aug-01	6,04%	np	15	8	2001	625	1	0	0	6,1974%	0	5,8779%	0,1621%	2,629E-06
Nov-01	6,05%	np	15	11	2001	717	1	0	0	6,2686%	0	5,9236%	0,1264%	1,599E-06
may-02	6,11%	np	15	6	2002	898	1	0	0	6,3900%	0	6,0058%	0,1042%	1,086E-06
aug-02	6,11%	np	15	8	2002	990	1	0	0	6,4433%	0	6,0440%	0,0660%	4,352E-07
feb-03	6,17%	np	15	2	2003	1 174	1	0	0	6,5357%	0	6,1141%	0,0559%	3,122E-07
aug-03	6,14%	np	15	8	2003	1 355	1	0	0	6,6107%	0	6,1756%	-0,0356%	1,270E-07
feb-04	6,14%	np	15	2	2004	1 539	1	0	0	6,6735%	0	6,2315%	-0,0915%	8,380E-07
may-04	6,23%	np	15	6	2004	1 629	1	0	0	6,7001%	0	6,2567%	-0,0267%	7,136E-08
aug-04	6,23%	np	15	8	2004	1 721	1	0	0	6,7248%	0	6,2811%	-0,0511%	2,610E-07
Nov-04	6,28%	np	15	11	2004	1 813	1	0	0	6,7471%	0	6,3042%	-0,0242%	5,851E-08
feb-05	6,29%	np	15	2	2005	1 905	1	0	0	6,7675%	0	6,3261%	-0,0361%	1,302E-07
may-05	6,27%	np	15	6	2005	1 994	1	0	0	6,7853%	0	6,3462%	-0,0762%	5,806E-07
aug-05	6,28%	np	15	8	2005	2 086	1	0	0	6,8021%	0	6,3659%	-0,0859%	7,386E-07
Nov-05	6,24%	np	15	11	2005	2 178	1	0	0	6,8173%	0	6,3847%	-0,1447%	2,094E-06
feb-06	6,25%	np	15	2	2006	2 270	1	0	0	6,8312%	0	6,4025%	-0,1525%	2,326E-06
feb-07	6,32%	np	15	2	2007	2 635	1	0	0	6,8745%	0	6,4651%	-0,1451%	2,105E-06
may-07	6,34%	np	15	5	2007	2 724	1	0	0	6,8828%	0	6,4786%	-0,1386%	1,921E-06
aug-07	6,34%	np	15	8	2007	2 816	1	0	0	6,8906%	0	6,4919%	-0,1519%	2,309E-06
Nov-09	6,69%	bp	15	11	2009	3 639	1	0	0	6,9346%	0	6,5878%	0,1022%	1,044E-06
feb-15	6,75%	bp	15	2	2015	5 557	1	0	0	6,9625%	0	6,7139%	0,0361%	1,305E-07
aug-15	6,76%	bp	15	8	2015	5 738	0	1	5557	6,8275%	6,9625%	6,7196%	0,0404%	1,633E-07
Nov-15	6,76%	bp	15	11	2015	5 830	0	1	0	6,7627%	0	6,7208%	0,0392%	1,538E-07
feb-16	6,75%	bp	15	2	2016	5 922	0	1	0	6,7003%	0	6,7210%	0,0290%	8,435E-08
may-16	6,67%	bp	15	5	2016	6 012	0	1	0	6,6417%	0	6,7202%	-0,0502%	2,522E-07
Nov-16	6,70%	bp	15	11	2016	6 196	0	1	0	6,5284%	0	6,7162%	-0,0162%	2,625E-08
may-17	6,73%	bp	15	5	2017	6 377	0	1	0	6,4252%	0	6,7094%	0,0206%	4,243E-08
aug-17	6,73%	bp	15	8	2017	6 469	0	1	0	6,3757%	0	6,7050%	0,0250%	6,244E-08
may-18	6,72%	bp	15	5	2018	6 742	0	1	0	6,2397%	0	6,6889%	0,0311%	9,685E-08
Nov-18	6,72%	bp	15	11	2018	6 926	0	1	0	6,1564%	0	6,6758%	0,0442%	1,952E-07
feb-19	6,70%	bp	15	2	2019	7 018	0	1	0	6,1172%	0	6,6688%	0,0312%	9,765E-08
aug-19	6,69%	bp	15	8	2019	7 199	0	1	0	6,0442%	0	6,6539%	0,0361%	1,300E-07
feb-20	6,68%	bp	15	2	2020	7 383	0	1	0	5,9756%	0	6,6379%	0,0421%	1,776E-07
may-20	6,68%	bp	15	5	2020	7 473	0	1	0	5,9439%	0	6,6297%	0,0503%	2,532E-07
aug-20	6,68%	bp	15	8	2020	7 565	0	1	0	5,9127%	0	6,6211%	0,0589%	3,465E-07
feb-21	6,64%	bp	15	2	2021	7 749	0	1	0	5,8539%	0	6,6036%	0,0364%	1,326E-07
may-21	6,64%	bp	15	5	2021	7 838	0	1	0	5,8270%	0	6,5949%	0,0451%	2,034E-07
aug-21	6,63%	bp	15	8	2021	7 930	0	1	0	5,8003%	0	6,5858%	0,0442%	1,952E-07
Nov-21	6,61%	bp	15	11	2021	8 022	0	1	0	5,7746%	0	6,5766%	0,0334%	1,113E-07
aug-22	6,56%	bp	15	8	2022	8 295	0	1	0	5,7039%	0	6,5490%	0,0110%	1,211E-08
Nov-22	6,57%	bp	15	11	2022	8 387	0	1	0	5,6818%	0	6,5396%	0,0304%	9,252E-08
feb-23	6,53%	bp	15	2	2023	8 479	0	1	0	5,6606%	0	6,5301%	-0,0001%	1,904E-12
aug-23	6,45%	bp	15	8	2023	8 660	0	1	0	5,6212%	0	6,5115%	-0,0615%	3,783E-07
Nov-24	6,47%	bp	15	11	2024	9 118	0	1	0	5,5340%	0	6,4644%	0,0056%	3,135E-09
feb-25	6,44%	bp	15	2	2025	9 210	0	1	0	5,5184%	0	6,4550%	-0,0150%	2,251E-08
aug-25	6,45%	bp	15	8	2025	9 391	0	1	0	5,4895%	0	6,4366%	0,0134%	1,791E-08
feb-26	6,39%	bp	15	2	2026	9 575	0	1	0	5,4623%	0	6,4181%	-0,0281%	7,895E-08
aug-26	6,44%	bp	15	8	2026	9 756	0	1	0	5,4374%	0	6,4001%	0,0399%	1,594E-07
Nov-26	6,42%	bp	15	11	2026	9 848	0	1	0	5,4255%	0	6,3910%	0,0290%	8,408E-08
feb-27	6,41%	bp	15	2	2027	9 940	0	1	0	5,4141%	0	6,3820%	0,0280%	7,847E-08
aug-27	6,41%	bp	15	8	2027	10 121	0	1	0	5,3928%	0	6,3644%	0,0456%	2,077E-07
Nov-27	6,38%	bp	15	11	2027	10 213	0	1	0	5,3826%	0	6,3556%	0,0244%	5,951E-08
aug-28	6,30%	bp	15	8	2028	10 487	0	1	0	5,3544%	0	6,3297%	-0,0297%	8,838E-08
Nov-28	6,26%	bp	15	11	2028	10 579	0	1	0	5,3457%	0	6,3212%	-0,0612%	3,743E-07
feb-29	6,16%	bp	15	2	2029	10 671	0	1	0	5,3373%	0	6,3127%	-0,1527%	2,332E-06
													Sum of errors	
													0,00000%	4,716E-05

R. C. 30 December 1999


YIELD CURVE FROM THE EURODOLLAR FUTURE CONTRACT
8 Jul 1999
Interest Rate (annualised)
8,0%
7,5%
7,0%
6,5%
6,0%
5,5%
5,0%
0
200
400
600
800
1000
1200
1400
1600
1800
2000
2200
2400
2600
2800
3000
3200
3400
3600
Maturity of the Contracts (days)
Adjusted Curve (w/o liquidity premium)
Sampled rates (100% - future price)
Estimated spot rates (=Yield Curve)
Asymptotic value for the short term rate

EURODOLLAR FUTURE CONTRACT

CME, 8 Jul 1999, at 19h00

Without Liquidity Premium

r_1 =	5,4000%	p.a.
r_m =	7,6500%	p.a.
τ =	1 464	days

R^2 = **0,9611**

LAST TRADING DAY			STRIKE MONTH	SETTLEMENT PRICE	INTEREST RATE	DAY COUNT	Spot r_{i+1}	Forward 91 $r_{i+1,i+91}$	Errors Simple	Errors Squared	R_i	r_m
8	**7**	**1999**										
19	7	1999	Jul-99	94,6900	5,3100%	11	5,4168%	5,4841%	-0,1741%	3,031E-06	5,408%	7,65%
16	8	1999	Ago-99	94,6300	5,3700%	39	5,4591%	5,5251%	-0,1551%	2,406E-06	5,429%	7,65%
13	9	1999	Set-99	94,5750	5,4250%	67	5,5007%	5,5654%	-0,1404%	1,971E-06	5,450%	7,65%
18	10	1999	Out-99	94,2700	5,7300%	102	5,5514%	5,6146%	0,1154%	1,331E-06	5,476%	7,65%
15	11	1999	Nov-99	94,2300	5,7700%	130	5,5912%	5,6532%	0,1168%	1,365E-06	5,496%	7,65%
13	12	1999	Dez-99	94,1700	5,8300%	158	5,6302%	5,6910%	0,1390%	1,932E-06	5,516%	7,65%
13	3	2000	Mar-00	94,1750	5,8250%	249	5,7519%	5,8091%	0,0159%	2,536E-08	5,580%	7,65%
19	6	2000	Jun-00	93,9850	6,0150%	347	5,8748%	5,9283%	0,0867%	7,521E-07	5,646%	7,65%
18	9	2000	Set-00	93,8350	6,1650%	438	5,9818%	6,0320%	0,1330%	1,768E-06	5,705%	7,65%
18	12	2000	Dez-00	93,6250	6,3750%	529	6,0823%	6,1295%	0,2455%	6,025E-06	5,761%	7,65%
19	3	2001	Mar-01	93,6150	6,3850%	620	6,1768%	6,2212%	0,1638%	2,684E-06	5,815%	7,65%
18	6	2001	Jun-01	93,5600	6,4400%	711	6,2656%	6,3073%	0,1327%	1,761E-06	5,867%	7,65%
17	9	2001	Set-01	93,5250	6,4750%	802	6,3490%	6,3882%	0,0868%	7,532E-07	5,917%	7,65%
17	12	2001	Dez-01	93,4350	6,5650%	893	6,4274%	6,4643%	0,1007%	1,015E-06	5,965%	7,65%
18	3	2002	Mar-02	93,4700	6,5300%	984	6,5011%	6,5357%	-0,0057%	3,268E-09	6,011%	7,65%
17	6	2002	Jun-02	93,4400	6,5600%	1 075	6,5704%	6,6029%	-0,0429%	1,838E-07	6,056%	7,65%
16	9	2002	Set-02	93,4150	6,5850%	1 166	6,6354%	6,6660%	-0,0810%	6,558E-07	6,098%	7,65%
16	12	2002	Dez-02	93,3350	6,6650%	1 257	6,6966%	6,7253%	-0,0603%	3,634E-07	6,139%	7,65%
17	3	2003	Mar-03	93,3500	6,6500%	1 348	6,7540%	6,7810%	-0,1310%	1,716E-06	6,179%	7,65%
16	6	2003	Jun-03	93,3150	6,6850%	1 439	6,8080%	6,8334%	-0,1484%	2,202E-06	6,217%	7,65%
15	9	2003	Set-03	93,2850	6,7150%	1 530	6,8588%	6,8826%	-0,1676%	2,809E-06	6,254%	7,65%
15	12	2003	Dez-03	93,1900	6,8100%	1 621	6,9065%	6,9288%	-0,1188%	1,412E-06	6,289%	7,65%
15	3	2004	Mar-04	93,2000	6,8000%	1 712	6,9513%	6,9723%	-0,1723%	2,969E-06	6,323%	7,65%
14	6	2004	Jun-04	93,1500	6,8500%	1 803	6,9934%	7,0132%	-0,1632%	2,662E-06	6,356%	7,65%
13	9	2004	Set-04	93,1150	6,8850%	1 894	7,0330%	7,0515%	-0,1665%	2,773E-06	6,387%	7,65%
13	12	2004	Dez-04	93,0250	6,9750%	1 985	7,0701%	7,0876%	-0,1126%	1,268E-06	6,418%	7,65%
14	3	2005	Mar-05	93,0250	6,9750%	2 076	7,1051%	7,1215%	-0,1465%	2,146E-06	6,447%	7,65%
13	6	2005	Jun-05	92,9800	7,0200%	2 167	7,1379%	7,1533%	-0,1333%	1,778E-06	6,475%	7,65%
19	9	2005	Set-05	92,9350	7,0650%	2 265	7,1711%	7,1855%	-0,1205%	1,452E-06	6,505%	7,65%
19	12	2005	Dez-05	92,8450	7,1550%	2 356	7,1999%	7,2135%	-0,0585%	3,422E-07	6,531%	7,65%
13	3	2006	Mar-06	92,8450	7,1550%	2 440	7,2250%	7,2378%	-0,0828%	6,862E-07	6,555%	7,65%
19	6	2006	Jun-06	92,8000	7,2000%	2 538	7,2526%	7,2645%	-0,0645%	4,164E-07	6,581%	7,65%
18	9	2006	Set-06	92,7600	7,2400%	2 629	7,2765%	7,2878%	-0,0478%	2,281E-07	6,605%	7,65%
18	12	2006	Dez-06	92,6700	7,3300%	2 720	7,2990%	7,3096%	0,0204%	4,166E-08	6,628%	7,65%
19	3	2007	Mar-07	92,6700	7,3300%	2 811	7,3202%	7,3301%	-0,0001%	1,084E-12	6,650%	7,65%
18	6	2007	Jun-07	92,6300	7,3700%	2 902	7,3400%	7,3494%	0,0206%	4,250E-08	6,671%	7,65%
17	9	2007	Set-07	92,5950	7,4050%	2 993	7,3587%	7,3675%	0,0375%	1,406E-07	6,692%	7,65%
17	12	2007	Dez-07	92,5050	7,4950%	3 084	7,3763%	7,3845%	0,1105%	1,220E-06	6,712%	7,65%
17	3	2008	Mar-08	92,5100	7,4900%	3 175	7,3928%	7,4005%	0,0895%	8,006E-07	6,731%	7,65%
16	6	2008	Jun-08	92,4700	7,5300%	3 266	7,4083%	7,4156%	0,1144%	1,310E-06	6,749%	7,65%
15	9	2008	Set-08	92,4350	7,5650%	3 357	7,4228%	7,4297%	0,1353%	1,831E-06	6,768%	7,65%
15	12	2008	Dez-08	92,3450	7,6550%	3 448	7,4365%	7,4430%	0,2120%	4,496E-06	6,785%	7,65%
16	3	2009	Mar-09	92,3500	7,6500%	3 539	7,4494%	7,4554%	0,1946%	3,785E-06	6,802%	7,65%
15	6	2009	Jun-09	92,3100	7,6900%	3 630	7,4615%	7,4672%	0,2228%	4,965E-06	6,818%	7,65%
									Sum of errors			
									0,0000%	7,152E-05		

R. C. 00.01.03