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Urban Public Facility Location, Multipurpose Trips and Spatial Competition

Equilibrium and welfare analysis*

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Abstract

We study the impact of the urban location of one single public facility on spatial competition *à la Hotelling*. If transportation costs are very low compared to the value of the public service then both firms tacitly choose the facility location without moderation of price competition, in contrast to mainstream results in the literature. In this event, minimum differentiation is efficient. For intermediate values of the relative transportation rate, inefficient partially-dispersed equilibria emerge with one firm at the facility site while its competitor locates at one end of the linear city. We also analyze the welfare impacts of changes, successively, in the facility location and the transportation rate, taking into account firms relocations.

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1 Introduction

It is generally admitted in the economic literature that competition à la Hotelling leads to inefficient spatial dispersion because sellers of homogeneous goods differentiate their location in order to relax price competition. In this paper, we challenge this view from both the normative and the positive viewpoints. Using the standard spatial setting, we will indeed show that homogenous duopolists may *efficiently* agglomerate *without moderation of price competition* if one takes into account *two realistic features of urban life*.

Firstly, it is a well-documented fact that people often make multipurpose trips, that is, they decide to visit a particular site for several reasons such as the purchase of various private commodities or the consumption of a local public good[1]:

> "For example, on the same trip, a consumer buys different goods, meet friends, visits a movie theater, goes to the post office, or just wanders and looks around. (...) The fact that consumers group their purchases in order to reduce travel costs creates demand externalities which firms would exploit by locating with firms selling other goods". Fujita and Thisse (1999), p. 30.

Secondly, in many European city centres (e.g. Paris or London), a huge proportion of land is used by public facilities such as parks, museums, libraries or major transportation nodes. As a result, some urban locations characterized by a large amount of public services *around*, may be more attractive for consumers, other things being equal. This may in turn give a demand externality-like advantage to any firm that would locate at these locations, especially when consumers desire to jointly consume both type of goods (public and private) on one single trip—or, in the case of a transportation facility, when the fixed cost of travel (waiting time, parking) is directly reduced.

The most casual observations suggest indeed that public facility sites often serve as agglomeration points for firms (see also Thisse and Wildasin, 1992). This is the idea developed in the next section. We will assume that one single public facility is located somewhere within the linear city before the standard location-then-price game à la Hotelling takes place. We will especially establish that for a very low transportation cost, or a high value the public service, minimum differentiation occurs at the facility site without any moderation of price competition.

As far as we know, the effects of spatial variation in the amount of public goods on urban location decisions made by oligopolistic firms are seldom analyzed, with the exception of Thisse and Wildasin (1992). The latter paper

[1] For a survey of empirical studies, see Thill and Thomas (1987).

examines the impact of the location of one single public facility on the interdependent locational choices of firms and households. Thisse and Wildasin (1992) mainly show that if the public facility is centrally-located then one obtains symmetric equilibria ranging from the fully agglomerated outcome (both firms at the facility central site) to the fully dispersed outcome where both firms are situated at opposite ends. On the other hand, if the public facility is situated close to one end of the city then any equilibrium exhibits both firms at the center of the urban area.[2]

As compared with Thisse and Wildasin (1992), we want to relax the fixed price assumption by using the standard location-then-price game à la Hotelling. The idea behind is that price competition is a strong centrifugal force that could *destroy* the possibility of agglomeration at the facility site. Moreover, one objective of our paper is to depart from the assumption of independent trips by exploring the effect of multipurpose shopping on spatial competition—as suggested by Thisse and Wildasin (1995) themselves by the way.

The remainder of the paper is organized as follows. The following section presents the model. Section 3 solves the subgame in prices and section 4 characterizes all the equilibria of the location-then-price game. Section 5 analyzes the efficiency of the market outcomes. Section 6 looks at the effects of a change in the facility location on welfare, taking into account relocations of firms. Section 7 analyzes welfare implications of policies aiming at a decrease in transportations costs. Section 8 concludes.

2 The model

Our set-up closely follows the assumptions made by d'Aspremont et al (1979) (hereafter called AGT). Consumers are uniformly distributed along a linear city $[0, 1]$ and bear a quadratic transportation cost. The population of consumers is normalized to one. As emphasized in the preceding section, we assume that one single public facility is located somewhere in the linear city, at some location denoted l ; without loss of generality, suppose that the facility is in the second half-segment: $\frac{1}{2} \leq l \leq 1$. We denote $\alpha > 0$ the value of the service provided by that facility to any individual. In order to focus on locational aspects, we assume that α is determined outside the model and constant throughout the system.[3] Two oligopolists, indexed $i = A, B$, produce a homogenous good or service at respective prices p_i with an identical marginal cost that is normalized to 0 (standard). As suggested by Thisse and Wildasin (1992), p. 85: "as in Hotelling (1929)", one can think of "su-

[2] The idea is that there exists an equilibrium only if revenue is large enough, to such an extent that the market area effect dominates the consumption effect.

[3] This is standard in the litterature (see ReVelle, 1987) and is in the spirit of Thisse and Wildasin (1992).

permarkets selling private goods on a large scale". After having observed the location of the public facility, the two private firms play a game in two stages. In the first stage, they simultaneously choose their own location, respectively a and $1-b \in [0,1]$ with $1-b-a \geq 0$ without loss of generality. The last stage is the standard Bertrand price competition.

As emphasized in the introduction, we now modify AGT by assuming that households have a preference for multipurpose "one-stop" trips. Firstly, assume that households bear some fixed cost of transportation denoted t_f for each trip they make (e.g. waiting time, parking). The remaining part of the total transportation cost is quadratic in distance as in AGT: $t_f + tz^2$ ($t_f, t > 0$) is the travel cost incurred by a consumer who travels a distance z for the purpose of buying one unit of the private (or public) service. Secondly, the utility derived at location l depends on the value of the public service provided at that location. To make it clear, consider that firm B is located at the place of the public facility whereas A is not. Then, denoting $\overline{u}$ the value of the private good, the utility of the consumer residing at x is simply given by

$$U(x,a) = \overline{u} - p_A - t_f - t(x-a)^2 \tag{1}$$

if she shops at A, or

$$U(x,l) = \overline{u} - p_B - t_f - t(l-x)^2 + \alpha \tag{2}$$

if she shops at B and consumes a quantity α of the public service.[4] We now give examples of public services we have in mind and the corresponding interpretation of the parameter α (the *demand externality*) in the specification above.

2.1 Example 1: major transportation node

Imagine that the "public facility" at l is a major node of the urban transportation system. We can rewrite the utility derived from shopping at B when this firm is situated at l as

$$U(x,l) = \overline{u} - p_B - (t_f - \alpha) - t(l-x)^2$$

Clearly, α represents the reduction in the fixed cost of travel due to better terminal conditions (e.g. waiting time in public transit, or direct link between the highway and the parking).[5]

[4] The reservation price $\overline{u}$ is large enough to ensure that every consumer is served in any equilibrium. In fact, we checked that $\overline{u} > 3t$ is sufficient (standard).

[5] For example, in Lisbon, the hypermarket Continente lies (in the mall *Colombo*) *right* next to the North-South highway with a direct access, from the highway, to the 20-entrance parking of the mall with space for 6800 vehicles; it can also be reached by two other avenues that cross there; on this site, there are also two taxi ranks, a metro station and a bus terminal served by 24 urban and suburban bus routes. It is important to notice that the building of the highway link and the extension of the metro at that location were the results of prior decisions made by the City Council of Lisbon.

2.2 Example 2: park, museum, etc. and "one-stop" multipurpose trips

Assume now that α represents the quantity consumed of a public service provided by a facility such as a museum, a park, a post office, a library or any other government administrative buildings. Multipurpose "one-stop" trips to location l are due to nonconvexities in the transport cost as captured by t_f in equations (1)-(2) above. Observe that preference for one-stop trips is indeed very strong in our specification since those consumers visiting firm A do not decide to make a separate trip to the facility for the unique purpose of consuming the public good. This one-stop trip pattern emerges if the fixed component of the disutility of travel is large enough. In particular, it should be clear that, for $t_f > \alpha$, no consumer will make an additional trip to the facility.

In fact, some (or all) consumers may make independent trips to the facility but we assume that this occurs on "another day of the week". Indeed, the shopping behavior should be apprehended over one week or one month (see Stahl, 1987). Then, following Thill (1992), assume that households consume the private good at a constant rate and purchase it according to a fixed schedule: every Saturday, let us say, all households necessarily buy one unit of the private good (main shopping day) while in some intermediate day of the week, some consumers make one independent trip to the public facility: they consume a quantity $\beta > t_f > \alpha$ of the public good, let us say, every Wednesday.[6] The last inequality simply means that the marginal utility of the public good is decreasing: this may be due to a more binding time constraint on Saturday (a high amount of time is already spent in the shopping activity, including storage at home and holding inventory, time spent in the shop)[7] or the fact that the public service has a higher utility during the week (e.g. post office).[8] Then, our model is unchanged since, for these consumers, we simply add the same net utility $\beta - t(l-x)^2$ to both $U(x,a)$ and $U(x,b)$ in order to derive the weekly level of utility.[9] It is readily

[6] Thus without shopping, on that day, a second unit at B, should that firm be located near the facility. Clearly, if $\beta > t_f + t$ then all the population will indeed make an independent trip to the facility during the intermediate day.

[7] see Thill (1992), footnote 3.

[8] Consider also the following example: on Saturday, some consumer shops at B in the *Quartier Latin* (central Paris) and, since the transportation cost is sunk, s/he decides to wander in the park "Jardin du Luxembourg" ; or s/he might choose to visit a nearby museum. Assume also that every Sunday afternoon, our consumer systematically visits the same park with the family. Our specification expresses the idea that the net utility derived from the consumption of the public good will probably be lower on Saturday as compared to Sunday, because of different time constraints (obviously, as opposed to many retail firms in central Paris, parks and museums are open on Sunday). Due to the fixed cost of travel, those patronizing firm A on Saturday derive a higher weekly utility by visiting the park only on Sunday.

[9] Clearly, the whole population makes its decisions to visit A or B by comparing the

verified that, under the assumption $\beta > t_f > \alpha$, those consumers patronizing firm A on Saturday are better-off when they split, both spatially and over the week, their consumption of both types of good, private and public.

To sum up, our utility specification captures the main idea that people have the opportunity to consume, *on one single trip*, an (additional) amount of the public service when they patronize the firm located in the vicinity of the facility. This is the standard "demand externality" effect: shopping trips characterized by economies of scope in jointly buying independent goods on the same site induce *demand complementarities* which translates in an increase of the aggregate consumption of each good or service (see Stahl, 1987).

It is convenient to let

$$f(x_i) = \begin{cases} \alpha & \text{if } x_i = l \\ 0 & \text{if } x_i \neq l \end{cases} \tag{3}$$

where x_i is the location of firm i ($x_i = a,\ 1-b$). The utility of a consumer residing at x and shopping at firm i is then given by

$$U(x, x_i) = \overline{u} - p_i - t_f - t(x - x_i)^2 + f(x_i) \tag{4}$$

For expositional purpose let us denote the externality advantage of firm B as follows: $\Delta f(a,b) = f(1-b) - f(a)$. It is equal to α (respectively $-\alpha$) if only firm B (respectively only firm A) is at the place of the public facility ; it is equal to zero in any other case. From (4), the consumer who is indifferent between the two marketplaces resides at $\widehat{x}$ such that

$$\widehat{x} = \frac{p_B - p_A}{2t(1-b-a)} + \frac{1-b+a}{2} - \frac{\Delta f(a,b)}{2t(1-b-a)} \tag{5}$$

When the two firms are located at the same point, we assume that consumers choose randomly between them so that the demand facing any firm is a one-half share of the population (standard).[10] We now can write the demand for the product sold by firm i ($i = A, B$) as:

$$D_A(p_A, p_B; a, b) = \begin{cases} \widehat{x} & \text{if } 0 < \widehat{x} < 1,\ a \neq 1-b \\ 0 & \text{if } \widehat{x} \leq 0,\ a \neq 1-b \\ 1 & \text{if } \widehat{x} \geq 1,\ a \neq 1-b \\ \frac{1}{2} & \text{if } a = 1-b \end{cases} \tag{6}$$

$$D_B(p_A, p_B; a, b) = 1 - D_A(p_A, p_B; a, b) \tag{7}$$

Finally, respective profits are as follows:

$$\Pi_i(p_A, p_B; a, b) = p_i D_i(p_A, p_B; a, b) \qquad i = A, B \tag{8}$$

two last utility functions.

[10]see, for example, Thisse and Wildasin (1992), p. 89.

3 Price competition

Following the backward induction principle, let us tackle the solution of the price game. For conciseness, let us drop the arguments a and b in any price (or profit) function for the rest of this section. Suppose also for the moment that each firm expects that it will face a positive demand at dispersed locations. Then, for fixed locations, let $P_i(p_j)$ denote the best response of firm i to a price p_j set by its rival. The first-order conditions for profit maximization lead to the following system of equations which must prevail in any equilibrium *with positive demands*:

$$P_A(p_B) = \frac{p_B}{2} + \frac{t(1-b+a)(1-b-a) - \Delta f(a,b)}{2} \tag{9}$$

$$P_B(p_A) = \frac{p_A}{2} + \frac{\Delta f(a,b) + t(1+b-a)(1-b-a)}{2} \tag{10}$$

First, if both firms are "outside the facility place" then the price subgame is identical to AGT: equilibrium prices are given by the solution of (9)-(10) where $\Delta f(a,b) = 0$ [see Tirole, 1988, page 281, Eqs. (7.7)-(7.8)].

Second, assume that firm B is located at the public facility site whereas A is located somewhere on the left side ($a < l$). Whence, only those consumers patronizing firm B benefit from the provision of the public service: they get an additional level of utility $\Delta f(a,b) = \alpha$. In stark opposition to AGT, for low values of t relatively to α, the system of price functions above is simply not valid because it implies a zero-demand for the firm situated outside the facility location, even if the latter firm prices at marginal cost.[11] On the other hand, for a high value of the ratio t/α, the pair of equilibrium prices is the solution of the system of first-order equations above:

Result 1. Assume $a < 1 - b = l \leq 1$. Then, the unique pure strategy equilibrium in prices is

$$p_A^* = 0, \qquad p_B^* = \overset{\circ}{p}_B(0,a) = \alpha - t(l+a)(l-a) \tag{11}$$

if $\alpha/t \geq (l-a)(2+l+a)$, and,

$$p_A^* = t(l-a)(1 - \frac{1-l-a}{3}) - \frac{\alpha}{3} \tag{12}$$

$$p_B^* = t(l-a)(1 + \frac{1-l-a}{3}) + \frac{\alpha}{3} \tag{13}$$

if $\alpha/t < (l-a)(2+l+a)$.

[11] For example, imagine α so large that $\widehat{x} < 0$ even if firm A is close to B and prices at marginal cost.

Proof. in appendix 9.1. ■

In equilibrium (11), firm A prices at marginal cost but still, firm B monopolizes the market by setting the limit price $\overset{\circ}{p}_B$ which is the highest price such that the entry of firm A is deterred.[12] Interestingly enough, this asymmetry in price competition is very similar to the one obtained in the outside location game developed by Gabszewics and Thisse (1992), p. 290:

> "(...) one of the player is endowed with a strict externality advantage over the other one, as in the outside location game. The fact that, in this game, seller A's [in our model, seller B's] location is viewed as strictly better by all consumers than seller B's [seller A's] location, prevents the latter from using price strategies that would attract the whole market to him. This privilege is reserved for firm A [firm B]".

One obvious difference is of course that, in our model, the two firms are located within the consumers' area.

4 Equilibria

Before solving the location subgame, let us adopt the following definitions that will prove useful:

Definition 1. Should the duopolists locate together at the facility location, we shall say that the market outcome—denoted $(a, b) = FA = (l, 1 - l)$—is *fully agglomerated.*

Definition 2. If one firm locates at the facility place and its rival at a distinct location then we shall speak of *partial dispersion.*

Definition 3. If both firms locate at opposite ends of the city then the equilibrium will be said *fully dispersed* and will be denoted $FD = (0, 0)$.

We also define $\widetilde{t} = t/\alpha$ as the *relative transportation cost* and refer to $\widetilde{\alpha} = \alpha/t$ as the *relative value of the public service.*

4.1 Restoring the principle of minimum differentiation

Note that the RHS of the last inequality in result 1 achieves a maximum value of $l\,(2 + l)$ at $a = 0$. Thus:

[12] In fact, we obtain in (11) a structure of equilibrium prices very similar to Grilo and Thisse (1999), p. 598, who use the Hotelling model of product differentiation to show that collective passions for some differentiated goods (i.e. $a \neq 1 - b$ by assumption) may lead to the emergence of a dominant product which monopolizes the whole market.

Lemma 1. Let $l = 1 - b$. If the relative transportation cost is weak, i.e., if $\widetilde{\alpha} \geq l\,(2 + l)$, then there is no location $a < l$ such that firm A would capture some positive demand.

A similar condition is derived whenever firm A is alone at the place of the public facility:

Lemma 2. Let $a = l < 1 - b$. If $\widetilde{\alpha} \geq (3 - l)\,(1 - l)$, then B is necessarily inactive.

Note that for $l = \frac{1}{2}$, lemmas 1 and 2 are equivalent, and more importantly, $l\,(2 + l) > (3 - l)\,(1 - l)$ for any $l > \frac{1}{2}$. Consequently, we assume in what follows that the condition in lemma 1 holds (thus implying lemma 2). The remaining case $(3 - l)\,(1 - l) < \widetilde{\alpha} < l\,(2 + l)$ will be more conveniently solved in the next section.

In the context of lemma 1 above, any pattern of locations involving both firms outside the facility location cannot be a Nash equilibrium because each firm is incited to relocate at the site of the facility location:

Lemma 3. For $\widetilde{\alpha} \geq l\,(2 + l)$, any subgame-perfect equilibrium (SPE) exhibits at least one firm at the facility site.

Proof. In appendix 9.2. ∎

Next, notice that the facility location is a (weakly) dominant strategy for firm B since its payoff at that location is either equal to zero (if firm A locates itself at the facility location) or equal to $\overset{\circ}{p}_B(0) > 0$ if firm A locates at some $a \neq l$. The same argument holds for firm A. Hence,

Proposition 1 *(Full agglomeration) If the benefit from the public facility is high compared to transportation costs, i.e., $\widetilde{\alpha} \geq l\,(2 + l)$, then the unique SPE outcome which satisfies the dominance criterion exhibits both firms at the facility site:*

$$FA = (l, 1 - l) \quad \textit{and} \quad p_A^* = p_B^* = 0$$

Proof. In appendix 9.3. ∎

The intuition behind this full agglomeration outcome is that the facility site is the unique location where each firm can guarantee itself a positive market share—exactly half of the market—even though both firms expect a fierce (Bertrand) price competition. In particular:

Corollary 1. (*Hotelling and Bertrand). Assume that the facility is centrally-located. If the relative value of the public good is sufficiently high, i.e., if $\alpha/t > 5/4$, then both firms locate at the central site to secure half of the market (Hotelling) without any moderation of price competition (Bertrand).*

Proof. $l\,(2+l) = 5/4$. ■

In a sense, this result reconciles the Hotelling's principle of minimum differentiation and the Bertrand result in the context of demand externalities that are created by a (centrally-located) public facility. Price competition is indeed a strong dispersion force ; it has been posited in the spatial economics literature that the observed agglomeration of retailers selling similar goods should be explained by some softening of price competition—through product differentiation or any other mechanism that relaxes price competition. Our model shows that this is not necessarily true. Indeed, despite the two oligopolistic firms expect a fierce price competition in the last stage, they may agglomerate in the preliminary stage if the value of the public service provided by the (centrally-located) facility is high compared to transportation costs. The key of minimum differentiation is the desire of consumers to visit the facility (central) site in order to jointly consume both types of goods, public and private, to such an extent that a duopolist outside the main urban site would face a zero demand.

For example, in most European cities *McDonald's* is sharing the fast food restaurants market with another competitor (e.g. *Quick* in Belgium and France) situated in general next door. Products are almost homogeneous[13] and price competition is tough. This is the case in many areas of central Paris (e.g in front of the *Gare du Nord*). Near the park *Jardins de Luxembourg*, close to the museum *Pantheon*, *McDonald's* is the next-door neighbor of *Quick*. Again, we argue that the reason for this pattern of locations is not a moderation of price competition through product differentiation—which is very low in that industry. It is instead the attractive force exercised by some valuable public facility(ies) combined to the desire of consumers to engage in multipurpose trip. In the last example above, one can, on the same trip, visit a museum, eat at one of the two fast food restaurants above-mentioned and then wander in the park. In the periphery of Brussels, two competing similar supermarkets are on the same site, near a major highway link.

More importantly, the existence of such spatial demand externalities stresses the role of the city government in shaping the urban spatial structure. In particular, the centralization of public goods or any decision aiming to decrease transportation costs may foster competition to such a degree that the public facility serves as an agglomeration point (as in Thisse and Wildasin, 1992) without any moderation of price competition (this is new compared with the aforementioned authors). In particular, for a fixed transportation cost parameter, observe in proposition 1 that the more centralized the public facility, the more likely the full agglomeration outcome. We will check in section 5 below that this competitive outcome is indeed desirable from the welfare viewpoint.

[13] In a *Quick* restaurant, everything looks like *McDonald's*: the way the employees are dressed, the range of dishes/menus, the background color of the logo, and so on.

Note finally that placing one facility within the linear city amounts to incorporate a spatially-variant element of vertical differentiation in the Hotelling model. Yet, our model should not be viewed as a vertical model of differentiation—in contrast to the outside location game in Gabszewics and Thisse (1992)—since we will show below that for a low α, at equal prices, not all the consumers view the facility site as the best shopping location. In the context of proposition 1 above however, it is true that all the consumers want to shop at B. In this sense, following Lambertini (1997), we can say that the agglomeration equilibrium is "vertical".

4.2 From partial to full dispersion

Let us now look at the remaining case: $0 < \widetilde{\alpha} < l(2+l)$. In what follows, we establish that there exists a unique subgame-perfect equilibrium exhibiting some differentiation of locations. We first look at the case in which firm A is on the left side of the public facility: $a < l$. It is more convenient to analyze the case $l \leq a \leq 1-b \leq 1$ afterwards.

Suppose first that firm B is at the facility location while A is not: $a < l = 1-b$. We formally prove in the appendix that firm A is in position to find a location close to the west end where it attracts a positive demand in any equilibrium of the last stage. We then show the following result:

Lemma 4. Let $0 < \widetilde{\alpha} < l(2+l)$. If firm B is located at the facility location then firm A optimally locates itself at the west end of the line: $a^* = 0$.

Proof. In appendix 9.4. ■

The idea behind this result is very simple. First, the strategic price competition effect is identical to the one calculated in AGT.[14] Second, the market effect area, defined as the increase in market share (at fixed prices) resulting from a move towards one's rival's location, is even lower as compared with the no-facility model. As a result, the total negative effect in AGT is reinforced: for a fixed b, the more distant firm A, the higher respective payoffs.[15] Using result 1, we get respective prices as follows:

$$p_A(0, 1-l) = \frac{t}{3} l\,(2+l) - \frac{1}{3}\alpha \tag{14}$$

$$p_B(0, 1-l) = \frac{t}{3} l\,(4-l) + \frac{1}{3}\alpha \tag{15}$$

[14] see Tirole (1988).

[15] We have checked that the market area effect *in percentage* is in fact similar to the one calculated in the reference model because the firm located at the facility site lets a much lower market share to its rival (as compared with AGT).

Substituting these equations into $p_i D_i = \frac{(p_i)^2}{2t(1-b-a)}$ $(i = A, B)$, one easily obtains the corresponding payoffs as

$$\Pi_A(0, 1-l) = \frac{[tl\,(2+l) - \alpha]^2}{18tl} \tag{16}$$

$$\Pi_B(0, 1-l) = \frac{[tl\,(4-l) + \alpha]^2}{18tl} \tag{17}$$

Clearly, in such an asymmetric situation, the more valuable the public service the higher the payoff to B and the lower the payoff to A.

Now, for $l \neq 1$, as the relative transportation cost rises, the price competition centrifugal force becomes stronger and stronger. There exists a threshold value of the relative transportation cost denoted $\widetilde{t}_0$ above which this effect dominates the externality advantage; it is then profit-maximizing for firm B to depart from the facility location, necessarily eastwards from AGT.[16] We also formally establish in the appendix that, for $\widetilde{t} > \widetilde{t}_0$, A is not incited to occupy the facility site whenever B is located at the opposite end of the city.

Finally, we prove that the remaining case we have to look at, i.e., $\frac{1}{2} < l \leq a \leq 1 - b \leq 1$, is an impossible market outcome (see appendix 9.5). Indeed, any candidate for an equilibrium exhibits firm A at the facility site $(a = l)$ and the optimal location of firm B in such a configuration—i.e., the east end—is simply dominated by the west endpoint.[17]

From all this, it follows:

Proposition 2 *(i) If the value of the public service is relatively low compared to transportation costs, the unique market outcome is a maximum differentiation of locations. (ii) For intermediate values of $\widetilde{\alpha}$, the unique pure-strategy SPE (up to a permutation of firms) exhibits asymmetric locations with one firm at the facility location and its competitor at the edge of the city:*

$$(a^*, b^*) = \begin{cases} FD = (0,0) & \textit{if } 0 < \widetilde{\alpha} \leq v(l) \\ P = (0, 1-l) & \textit{if } v(l) < \widetilde{\alpha} < l\,(2+l) \end{cases}$$

Proof. In appendix 9.6. ■

We can also summarize all the equilibria (propositions 1 and 2) as follows: whenever the relative transportation rate is very high, price competition is

[16] see lemma 4.1 in appendix 9.6 for more details.

[17] For $\widetilde{\alpha} < (3-l)(1-l)$, the explanation is the one given for lemma 4 above. Furthermore, whenever $(3-l)(1-l) < \widetilde{\alpha} < l\,(2+l)$, firm A locates at $a = l$ whereas firm B faces a zero-demand at any location $1 - b \in \,]l, 1]$ (by lemma 2). A relocation of the latter firm at the west end guarantees a positive profit. (Interchange the roles of A and B in lemma 3).

the stronger force and $FD = (0,0)$ occurs; for intermediate values of t/α, firm B balances the benefit from spatial isolation and those from being in a better environment, and chooses to locate at the facility place; finally, for a very low t/α, A cannot do any better than departing from the west end to join B at the facility location, where s/he makes no supranormal profits but captures half of the market (proposition 1).

For example, if the public facility is centrally-located then $P = (0, \frac{1}{2})$ emerges for $0.371 < \widetilde{\alpha} < \frac{5}{4}$. This result may appear surprising at first glance since even though the set-up is perfectly symmetric, the existence of a public facility at the center of the distribution of households leads to asymmetric locations. It is less surprising if one recalls results in Tabuchi and Thisse (1986) or in Combes and Linnemer (2000), for example. The demand to A exhibits a discontinuity at $\widetilde{\alpha} = \frac{5}{4}$: it tends to zero as $\widetilde{\alpha}$ increases near $\frac{5}{4}$ (still $\widetilde{\alpha} < \frac{5}{4}$) whereas it is equal to $\frac{1}{2}$ for any $\widetilde{\alpha} \geq \frac{5}{4}$ (FA).[18]

Assume now the public facility at the edge of the city $(l = 1)$. Clearly, $b = 0$ is then a dominant strategy and the fully agglomerated outcome occurs if $\widetilde{\alpha} \geq 3$ while $P = (0,0)$ prevails otherwise. In configuration P, the differentiation is maximum but the equilibrium is said to be partially-dispersed since one firm is situated at the facility site.

In addition to the importance of the relative value of the public good through the ratio α/t, the location of the facility itself and the absolute value of the transportation rate can have a significant impact on the spatial structure of the business sector.

Firstly, observe that for some values of $\widetilde{\alpha}$ the type of equilibrium that arises does not depend on the facility location at all: this is the case for $\widetilde{\alpha} > 3$ (full agglomeration) and $0.371 < \widetilde{\alpha} < 1.25$ (partial dispersion).

Secondly, for any other value of $\widetilde{\alpha}$, the location of the facility is crucial for the type of outcome that will emerge. For example, assume $\widetilde{\alpha} = 0.35$ and $l = 1/2$. Then, both firms tacitly play the fully-dispersed equilibrium and consumers don't benefit from the provision of the public good. Now, move the facility a bit to the right, at $l = 0.53$. One calculates $v(0.53) = .345 < \widetilde{\alpha}$ which means that firm B relocates at the facility site while firm A stays on the left border (configuration P). Note that firm B is better-off since $\Pi_B(0, 1 - 0.53) = .502\,t > \frac{t}{2} = \Pi_B(0,0)$.

Observe in figure 1 that, for $\widetilde{\alpha} < 3$, the range of $\widetilde{\alpha}$ in which P occurs, widens up and down as l rises. In other words, the more eastward the facility, the higher the probability that partial dispersion will emerge. The idea behind this is quite intuitive: whenever the public facility is sufficiently distant from the midpoint, it is also from firm A and price competition is

[18] $0 < D_A(0, \frac{1}{2}) = \frac{5}{12} - \frac{1}{3}\widetilde{\alpha} < 0.29$ if $0.371 < \widetilde{\alpha} < 5/4$. In any other case ($FD$ or FA), $D_A(0, \frac{1}{2}) = 1/2$. Note that the equilibrium price p_A^* is continuous in the ratio α/t: the lower the relative value of the public good, the lower the price charged by this firm (in order to secure a positive demand) as shown by (12). For $\widetilde{\alpha} \geq \frac{5}{4}$, A prices at marginal cost but is not able to attract a positive demand.

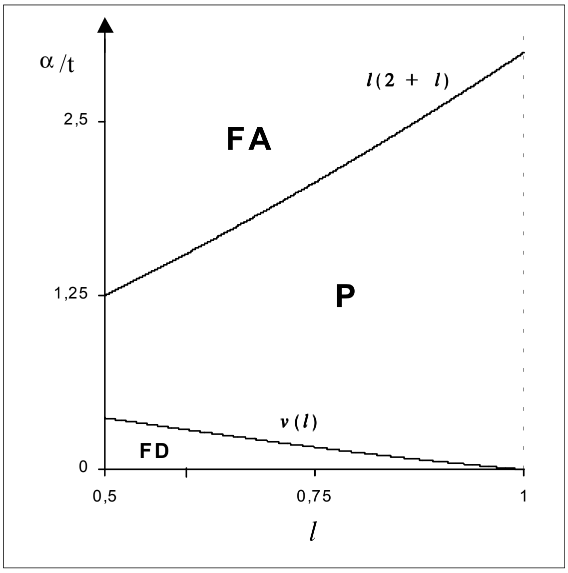


Figure 1: Equilibria

somewhat relaxed at the facility site. It follows, that firm B finds it profitable to locate at l in order to exploit the demand externality advantage.

Clearly, given α and t, firm B always prefers to observe a facility location decision which fosters the emergence of the partially-dispersed equilibrium since *its profit in such a configuration is greater than the payoff under maximum differentiation* [19]; for the opposite reason, firm A strictly prefers a symmetric maximum differentiation that yields the standard common payoff $t/2$ (provided of course that the facility is not situated exactly at the east end):

$$\Pi_A(0, 1-l) < t/2 < \Pi_B(0, 1-l) \tag{18}$$

Nevertheless, it is also easy to establish that

$$\frac{\delta}{\delta t}\Pi_i(0, 1-l) > 0 \qquad i = A, B \tag{19}$$

for any $0 < \alpha/t < l\,(2+l)$ (partial dispersion). In other words, any policy reducing the transportation rate t will decrease respective profits under partial (full) dispersion. It follows that, even though firm B prefers configuration P *for a given* t, the two firms would, in fact, be better off in a city where transportations costs are very large, to such an extent that full dispersion would

[19] Note that the payoff to firm B under partial dispersion is not necessarily maximized at $l = 1$ as shown in result 3 in appendix 9.7. Indeed, for $1 < \alpha/t < 3$, firm B prefers an interior facility location because the sum of the externality effect and the market area effect dominates the negative effect of price competition.

occur. Clearly, whichever the type of dispersion, the monopoly power generated by geographical isolation increases with transportation costs incurred by consumers, as often illustrated in the literature.[20]

Looking at the values of t and l which simultaneously maximize firm B's profit, we have proved the following:

Result 2. *Given α, the equilibrium profit of firm B is maximized whenever t approaches ∞ and $l = 1$ (partial dispersion). The corresponding equilibrium profit is $\Pi_B(0,0) \approx \frac{1}{2}t + \frac{1}{3}\alpha$; this firm captures approximately half of the market.*[21]

In other words, whenever the facility is located at the east end of the city ($l = 1$) and t is very large, firm B's profit exceeds the profit under FD (with $l < 1$) by roughly $\frac{1}{3}\alpha$. A's profit is lower than $t/2$ by the same amount.

On the contrary, A is better off when the public facility is centralized and α/t is very weak so as to make more likely the emergence of the fully-dispersed outcome.

Needless to say, the objective of the well-intentioned urban planner might be in discordance with the interests of firms. The former should also take into account the impact of changes in l (or t) on the surplus of consumers. This will be analyzed in sections 6 and 7. Yet, it is interesting to notice here that firms have strong incentives to lobby the city government, each duopolist trying to impose an opposite view on where should be situated the new facility—eastwards *versus* westwards—whereas they would agree on a *weak* transportation policy.

5 Optimality of equilibria

The objective of this section is to compare market outcomes derived earlier with the locations that a central planner would choose. In contrast to the standard welfare analysis of the Hotelling model, we cannot *a priori* assume that the socially-optimal locations will be symmetric. Indeed, the city planner might consider the possibility of locating one firm at the facility site and the other one near the left border. This would induce a large share of the population to visit firm B (and thus the facility) while it would reduce transportation costs of those residing at the outskirts of the city. For a fixed

[20] Hotelling himself gave the intuition, p. 50:

> "These particular merchants would do well, instead of organising improvement clubs and booster association to better the roads, to make transportation as difficult as possible. (...) The objective of each is merely to attain something approaching a monopoly"

[21] Respective demands are easily obtained from (34) in appendix 9.4: $D_A(0,1) = \frac{1}{2} - \frac{\alpha}{6t} \approx D_B(0,1) = \frac{1}{2} + \frac{\alpha}{6t} \approx 1/2$ as $t \to \infty$.

facility location, the central city planner must select a pattern of locations among three possible configurations:

(1) either the two firms outside the facility location (dispersion)

(2) or, both at the place of the facility (full agglomeration or configuration FA)

(3) or, one firm at the facility location (say firm B) and its rival outside the facility location (partial dispersion).

We assume that firms are forced to price at marginal cost (standard, see Tirole, 1988) and we denote $\widehat{D}_A(a,b) \equiv D_A(0,0;a,b)$ the corresponding quantity sold by firm A. One checks that the total welfare is equal to the aggregate surplus derived from the consumption of the public good (if any firm is at the facility site) minus the social transportation cost, which is given by:

$$T(a,b) = \int_0^{\widehat{D}_A(a,b)} t(x-a)^2 \, dx + \int_{\widehat{D}_A(a,b)}^1 t(1-b-x)^2 \, dx \tag{20}$$

5.1 Dispersion

In the first configuration, the social transportation cost is identical to the one calculated in AGT since no one visits the facility. We know that the socially-optimal pattern of locations is $S = (\frac{1}{4}, \frac{1}{4})$ (standard). This choice of locations simply minimizes the total transportation cost which is equal to $T(\frac{1}{4}, \frac{1}{4}) = \frac{1}{48}t$. It is convenient for the remainder of the paper to define $W^{SY}(a)$ as the total welfare in any symmetric configuration (a,a) (not only S). The expression of W^{SY} is given in the appendix (subsection 9.8.1) from which we immediately derive the value of welfare in configuration S as:

$$W^{SY}(\frac{1}{4}) = \overline{u} - \frac{1}{48}t \tag{21}$$

5.2 Full agglomeration

In the second configuration, the whole population of consumers (normalized to 1) visits the facility site where it derives a constant utility α from the provision of the public good. Using $D_i(l, 1-l) = \frac{1}{2}$, one easily derives the social transportation cost as in (45) (in appendix 9.8) and then calculates the total welfare under full agglomeration from (43) as:

$$W^{FA}(l) = \overline{u} + t\left[\widetilde{\alpha} - \frac{1}{3} + l(1-l)\right] \tag{22}$$

5.3 Partial dispersion

Note that the assumption $a < 1-b = l$ might entail a loss of generality (with the exception $l = \frac{1}{2}$ which obviously restores symmetry). Indeed, the city planner could consider to locate one firm at the facility place and its competitor on the right side in order to induce a larger share of the population to consume the public good or service. The welfare loss due to a larger amount of travel could, *a priori,* be counterbalanced by a higher surplus from the consumption of the public good. The two possibilities are analyzed in the two following subsections.

5.3.1 Firm A on the left of the facility site

Assume that the city planner first looks at the case $a < l = 1-b$. Only $1 - \widehat{D}_A(a,l)$ proportion of the population visit the centrally-located firm B and receive the benefit from the provision of the public good. Therefore, the total welfare is given by:

$$W^P(a, 1-l) = \overline{u} + \left[1 - \widehat{D}_A(a, 1-l)\right] \alpha \ - T(a, 1-l) \tag{23}$$

where

$$\widehat{D}_A(a, 1-l) = \frac{1}{2}(a+l) - \frac{\widetilde{\alpha}}{2\,(l-a)} = \frac{l^2 - a^2 - \widetilde{\alpha}}{2(l-a)} \tag{24}$$

after substituting $p_A = p_B$ into (5).[22] We next assume:

Condition 1 In any partially-dispersed configuration the city planner might choose, the demand to A is positive: $\widehat{D}_A(a, 1-l) > 0 \quad \Leftrightarrow \quad \widetilde{\alpha} < l^2 - a^2 \quad \Leftrightarrow \quad a < \sqrt{(l^2 - \widetilde{\alpha})}$

In other words, the value of the public good must not be too high and/or the location chosen by the city planner should not be *too close* to the facility.[23] In particular, we assume $\widetilde{\alpha} < l^2$ otherwise the whole population would visit firm B at the facility location (whatever a may be); from the strict total welfare viewpoint, would be indifferent between the full agglomeration situation and the partially-dispersed equilibrium $(a, \frac{1}{2})$: the welfare is identical in both cases and is equal to $W^P(\frac{1}{2}, \frac{1}{2})$. Indeed, in both configurations, the whole population visits the facility site and derives the same net utility from the consumption of both goods, public and private, whatever the number of

[22] Note that $\widehat{D}_A$ is concave and reaches a global maximum at $\widetilde{a} = l - \sqrt{\widetilde{\alpha}}$ where $\widetilde{a} = \widehat{D}_A(\widetilde{a}, 1-l)$. Indeed, if firm A moves to the right then it looses some consumers in its close hinterland because the travel cost to this firm rises while the net benefit from visiting B is unchanged. In other words, the facility location becomes relatively more attractive and the demand to A is strictly decreasing on the right of $\widetilde{a}$.

[23] Clearly, since prices are equal, there always exist some consumers situated between A and B, possibly very close to the facility location, who patronize firm B. So necessarily, $D_B(a, 1-l) > 0$.

firms (one or two) at that location. The only difference is that under partial dispersion, A does not produce any output. Put differently, for $\widetilde{\alpha} > l^2$, the problem of choosing a location for firm A outside the facility location is simply meaningless because this firm would not capture any positive demand. We thus suppose that, in such circumstances, the city planner prefers the fully-centralized outcome where firm A produces a positive output.

Before determining the welfare-maximizing firm A's location, let us analyze one important component of the total welfare. After some manipulation, one obtains the expression of the social transportation cost as in (46) (see appendix 9.8.3). For $\alpha = 0$, the social transportation cost is minimized at $a = \frac{1}{3}l$. If the location of the facility (and thus firm B's location) is fixed at $\frac{3}{4}$, we get the standard result of AGT: $a = \frac{1}{4}$. If B is being imposed the central location then we get $a = \frac{1}{6}$ as transportation-cost-minimizing location. This is not surprising since in the (no-facility) model of Hotelling (with fixed prices), the socially-optimal locations of 3 firms is $\frac{1}{6}$, $\frac{1}{2}$, $\frac{5}{6}$.

Of course, one can increase the second component of the welfare function by inducing a higher aggregate consumption of the public good—i.e., by inducing more people to visit firm B. One shows that for $\widetilde{\alpha} < \frac{4}{9}l^2$ the demand to A is increasing on $[0, \frac{1}{3}l]$. For $\widetilde{\alpha}$ too high, more precisely for $\frac{8}{9}l^2 < \widetilde{\alpha}$ $(< l^2)$, one checks that $\widehat{D}_A(a, 1-l) = 0$ at any $a \geq \frac{1}{3}l$. Hence, in these two cases, the welfare-maximizing location will necessarily lie within the segment $\left[0, \frac{1}{3}l\right[$. For $\frac{4}{9}l^2 < \widetilde{\alpha} < \frac{8}{9}l^2$, it is a little bit more complex to predict, at this stage, what will be the socially-optimal location of A because the demand function has a positive value at $\frac{1}{3}l$ and exhibits a maximum on the left of that location (see figure 2).[24] We can however reasonably expect that, again, the city planner will choose a location in the interior of $[0, \frac{1}{3}l]$ because a move eastwards translates into higher transportation costs.

We prove in the appendix that the unique value of a which maximizes the welfare function under partial dispersion, both locally and globally, is given by

$$\widehat{a} = \frac{2}{3}l - \frac{1}{3}\sqrt{l^2 + 3\widetilde{\alpha}} \tag{25}$$

with $0 < \widehat{a} \leq \frac{1}{3}l$ as expected. One checks that condition 1 is satisfied near $\widehat{a}$, i.e., the demand facing A is positive (see appendix 9.9).[25]

[24] Indeed, departing from $\frac{1}{3}l$, there exists two possible types of relocation that potentially increases welfare (through a rise in the aggregate consumption of the public good, see figure 2): either near the left border, or within some segment $[a_1, a_2]$ with $a_1 > \frac{l}{3}$ and a_2 satisfying $\widehat{D}_A(a_2, 1-l) = 0$.

[25] As $\widetilde{\alpha} \to 0$, we have $\widehat{a} \to \frac{1}{3}l$ which re-establish consistently the transportation cost-minimizing location (as indicated earlier). Notice also that the demand to A is at most $\frac{2}{3}$ whenever the public facility is located at the east end of the line and $\widetilde{\alpha} \to 0$ (firm A being optimally-located at $\frac{1}{3}$).

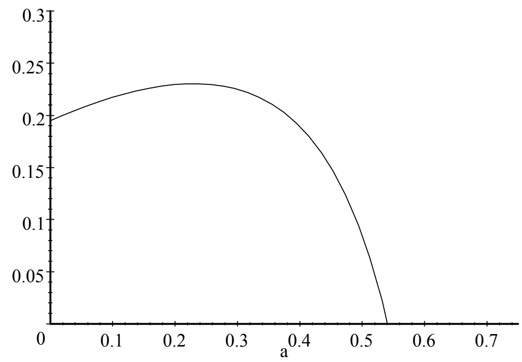


Figure 2: Demand to A ($l = 1 - b = 0.75$)

5.3.2 Firm A at the facility site

We show in the appendix that any configuration involving $a = l < 1 - b$ is suboptimal from the welfare viewpoint. We first determine the socially-optimal location $b = \widehat{b}$ subject to $a = l < 1 - b \leq 1$ and then we show that the corresponding value of the welfare is lower than $W^P(\widehat{a}, 1 - l)$ calculated in the preceding section. We conclude that in any asymmetric pattern of locations involving B at $l > \frac{1}{2}$, the omniscient central planner would locate A at $\widehat{a} < l$ as defined in (25) above. We shall speak of *optimal partial dispersion* or configuration $OP = (\widehat{a}, 1 - l)$.

5.4 Optimal pattern of locations

Proposition 3 *Assume that the public facility is located somewhere in the second half-segment.*

- *For a very low $\widetilde{\alpha}$, the optimal locations are given by the (standard) transportation-cost-minimizing solution:*

$$\widetilde{\alpha} < y(l) \quad \Rightarrow \quad (\widehat{a}, \widehat{b}) = S = (1/4, 1/4)$$

- *On the other hand, if the relative value of the public good is very high then the socially-optimal pattern of locations exhibits both firms at the facility site:*

$$\widetilde{\alpha} \geq l^2 \quad \Rightarrow \quad (\widehat{a}, \widehat{b}) = FA = (l, 1 - l)$$

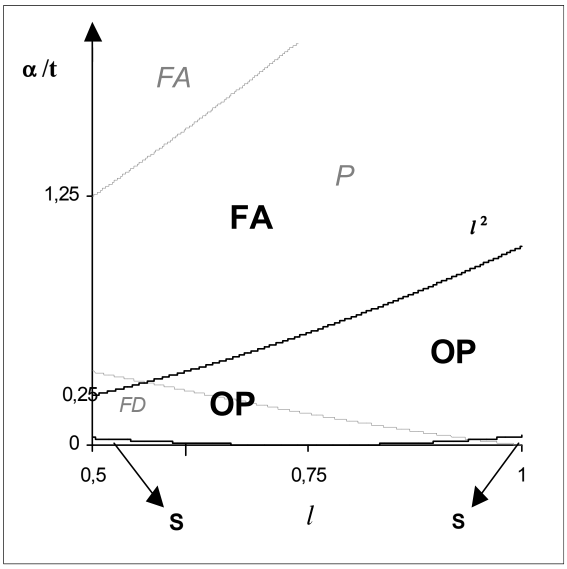
- *For intermediate values of the relative transportation cost parameter, the city planner finds a compromise between saving on transportation costs and inducing a high proportion of the population to consume the* *public good ; s/he locates one firm at the facility place and its rival at a more suburban interior site:*

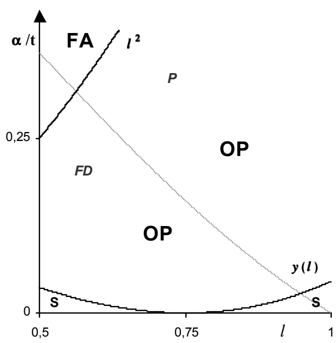
$$(\widehat{a}, \widehat{b}) = OP = (\frac{2}{3}l - \frac{1}{3}\sqrt{l^2 + 3\widetilde{\alpha}}, 1 - l)$$

- *When it emerges as a market outcome, full agglomeration is socially-optimal.*
- *Under laissez-faire, any dispersion, partial or full, is excessive: when the market outcome is $P = (0, 1 - l)$ it should be FA or OP (or even S whenever l is close to the east end and α near 0). When dispersion is maximal, the city planner would prefer it partial with one firm at the facility site, or $S = (1/4, 1/4)$ for a very weak $\widetilde{\alpha}$ (or even FA whenever l is close to the midpoint and $0.25 \leq l^2 < \widetilde{\alpha} < 0.371$).*[26]

Proof. In the appendix. ■

In some circumstances, firm B optimally locates at the facility—a move which certainly fosters the consumption of the public service—but its rival is still suboptimally located at the outskirts. For a low $\widetilde{\alpha} < y(l)$ and l approaching 0.75, it becomes more and more interesting to induce $OP \neq S$ with firm B at the facility site, in order to exploit the externality. Indeed, when l is close to 0.75, OP retains a high proportion of the social benefits of the configuration $S = (1/4, 1/4)$ while adding the aggregate surplus derived from the consumption of the public good.[27]





[26] In one particular situation, the partially-dispersed market outcome is "nearly" socially-optimal: if $\widetilde{\alpha} \approx l^2$ and P emerges, then $\widehat{a} \approx 0 = a^*$ (i.e., $OP \approx P$).

[27] Observe in the figure above that for a given $l = 0.75 = 1 - b$ and $\alpha = 0$ (OP), the area S consistently vanishes.

6 Optimal location of the facility

We have already discussed the effects of a change in l on the spatial structure of the business sector. Here, we tackle the problem of the optimal location of the public facility from the welfare viewpoint. We suppose that the choice of l is made in a stage prior to the non cooperative location-then-price game. This allows us to look at business relocation effects which are not calculated in the standard cost-benefit analysis of the building of a new facility.[28] Since α is constant, we can neglect the cost of providing of the public service, mainly the fixed cost of building the facility. Indeed, this cost only shifts the welfare functions without changing the results below. The total welfare is thus again equal to the total surplus derived from the aggregate consumption of the private good (plus the consumption of the public good, should some firm(s) be located at l) minus the social transportation cost. The latter component is given by (20) where $\widehat{D}_A$ is replaced by the demand to A under laissez-faire:

$$D_A(a^*, b^*) = \begin{cases} D_A(0,0) = \frac{1}{2} & \text{if } \widetilde{\alpha} < v(l) \\ D_A(0, 1-l) = \frac{l(2+l)-\widetilde{\alpha}}{6l} & \text{if } v(l) < \widetilde{\alpha} < l\,(2+l) \\ \frac{1}{2} & \text{if } \widetilde{\alpha} > l\,(2+l) \end{cases} \tag{26}$$

from propositions 1 and equation (34) in the appendix. When the market outcome is fully-agglomerated, firms price at marginal cost and the expressions of respective demands and welfare are the same as in the preceding section (where firms were forced to set prices to zero). One easily checks that the transportation cost function (45) is minimized when the city planner centralizes the public facility. Hence, the value of the social surplus is immediately obtained from (22):

$$W^{FA}(\frac{1}{2}) = \overline{u} + \alpha - \frac{t}{12} \tag{27}$$

Now, in the fully-dispersed subgame perfect outcome, the social transportation cost function and the total welfare function have been established again in the preceding section. Indeed, by symmetry, $D_A(a,b) = \widehat{D}_A(a,b) = \frac{1}{2}$ whenever $a = b \neq 1-l$. Moreover, the value of the social transportation cost is equal to $\frac{t}{12}$, i.e., the value calculated under full agglomeration above. Indeed, in both configurations FA and FD, the first half of the population patronizes firm A which is situated at one end of the half-segment (at $\frac{1}{2}$ or at 0). The only difference is that, under FD, no one consumes the public good. Thus, we deduce the welfare under full dispersion from (27) as

$$W^{SY}(0) = \overline{u} - \frac{t}{12} \tag{28}$$

[28] see also, in different contexts, Combes and Linnemer (2000), p. 16 and Thisse and Wildasin (1992).

If the city planner aims to induce such a pattern of firms' locations, s/he must simply locate the facility at any l satisfying $\widetilde{\alpha} < v(l)$.

Now, in the eyes of the city planner, each one of the configurations FA and FD "solely" competes with partial dispersion (see figure 1). Indeed, the latter configuration leads to a more balanced pattern of shopping trips while retaining a high social benefit for those who both shop at B and consume the public good (recall proposition 3). As opposed to the two preceding configurations, in the partially-dispersed (*laissez faire*) equilibrium, the transportation cost functions and the total welfare—denoted W_*^P, are different from the ones established in the optimality analysis.[29] Indeed, the overall urban system is now affected by non cooperative price decisions through changes in respective equilibrium demands. We now state the main proposition of this section:

Proposition 4 *Given α and t, the unique socially-optimal location of the facility is*

$$\widehat{l} = \begin{cases} 0.5916 \leq v^{-1}(\alpha/t) < 1 & \text{if } 0 < \alpha/t < 0.291 \\ \frac{1}{2} < m^{-1}(\alpha/t) < 0.5916 & \text{if } 0.291 < \alpha/t < 0.42 \\ \frac{1}{2} & \alpha/t > 0.42 \end{cases}$$

The market outcome is partially-dispersed, except for $\widetilde{\alpha} > 1.25$ where firms are induced to agglomerate at the facility central site.

Proof. In the appendix. ■

In the following figure, the optimal location of the facility is depicted by the bold line and the arrows indicate increases in the total welfare under partial dispersion or full agglomeration.[30] Observe that the city planner necessarily decides to induce some positive consumption of the public service (FD is dismissed). The curve $m(l)$ plots the first order condition for an interior maximum (the second-order condition being met). In fact, for $0 < \widetilde{\alpha} < 0.291$, in order to avoid the emergence of FD, the city planner should place the facility (obviously in the P area) *almost* on $v(l)$, that is, at some location $v^{-1}(\widetilde{\alpha}) + \epsilon$, where $\epsilon > 0$ is arbitrarily small.[31] In the competitive fully-agglomerated outcome, firms do not earn any (supranormal) profits at the central site. This pattern of locations maximizes the consumer's surplus which is equal to: $W^{FA}(\frac{1}{2}) = \overline{u} + \alpha - \frac{t}{12}$. Clearly, the higher α (and the lower t), the higher consumer welfare in the fully agglomerated outcome.

[29]see equations (53) and (54) in the appendix.

[30]Recall here that P stands for *partial dispersion* under laissez-faire and should not be confused with the optimal dispersion OP analyzed before.

[31]Example: for $\alpha = .160\,576\,211\,t$, let $\epsilon = 0.0001$ and locate the facility at $v^{-1}(\alpha/t)+\epsilon = 0.750001$ so that the profit under P is equal to $0.500\,000\,029t > t/2$, that is just enough to dismiss FD.

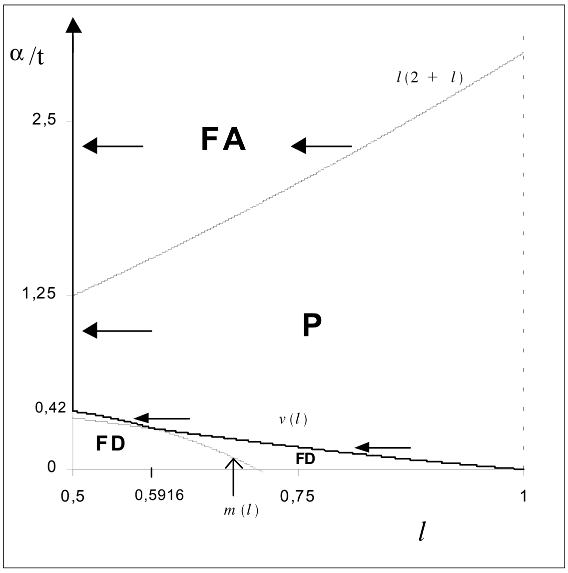


Figure 3: Optimal facility location

7 Policies aiming at a decrease in t

Once the public facility has been built at some location l, the city planner can still affect significantly the urban structure by modifying the quality of the transportation system (which determines the rate t). Many improvements of the urban transportation networks can be achieved in short run: comfort, safety, frequency of service (for public mass transit). The city planner might also add new lines or increase the number of buses, for example. As also suggested by Thisse and Wildasin (1992), p. 102:

> "Some tax policies, such as gasoline taxes, can also affect travel costs. (...) as do highway and bridge tolls and the pricing of public transportation".

For fixed firms' locations, a reduction in t clearly improves the total welfare. Nevertheless, this is misleading since firms may relocate. In particular, as argued before, the city planner could prefer to foster some dispersion in order to get a more balanced system of households trips. Yet, we proved the following intuitive non-trivial result:

Proposition 5 *Whatever the value of the public good and the location of the facility, the total welfare is globally-maximized at* $t = 0$ *(full agglomeration).*

Finally, as argued before, it is easier to change the transportation cost parameter once the public facility has been built than to do the reverse.

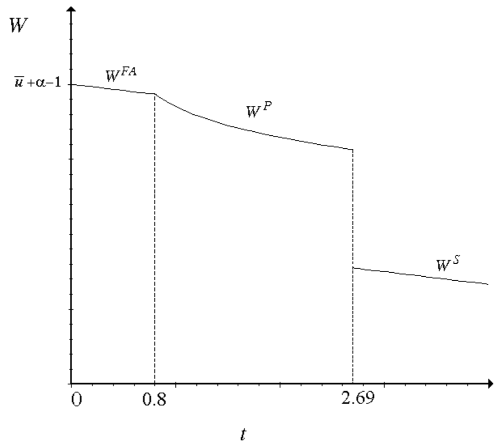


Figure 4: Impact of t on welfare

Therefore, it is natural to suppose that the choice of l is prior to any transportation policy decision and, from propositions 4 and 5, we conclude:

Proposition 6 *Assume that* $t \in]0, \infty[$. *From the welfare viewpoint, it is optimal to centralize the provision of the public service* ($l = 1/2$) *and then to reduce the transportation cost near zero.*[32]

8 Conclusion

The clustering of firms selling a similar product or service is often observed in real urban life. Our paper has explored one possible explanation to this phenomenon, namely the fact that a public facility may serve as agglomeration point, as already suggested by Thisse and Wildasin (1992). We have added this centripetal force to the standard spatial setting, assuming that people have a preference for "one-stop" multipurpose trips to the facility site. Minimum differentiation at the facility place—what we have called full agglomeration—may indeed emerge as an *efficient* market outcome, *without relaxation of price competition.* The second part of this result may surprise at first glance since it is opposed to the argument generally put forward in the literature—which precisely stresses on the need of relaxing price competition somehow or other. Yet, the idea is very simple: for low transportation costs compared to the value of the public service, the facility location is a

[32]**Proof.** Indeed, W^A reaches a maximum values of $\overline{u}+\alpha$ at $t = 0$, wherever the facility may be. Thus, the choice of l is relevant only if $t > 0$. For $t \simeq 0$, the backwards-induction (optimal) facility location is $l = 1/2$. ■

dominant strategy, and *de facto* the only place that guarantees a positive market share.

Another striking result is the fact that for a wide range of the ratio *transportation cost/value of the public service*, asymmetric equilibria emerge even if the public facility, when built at the midpoint, balances perfectly the demand between the two half-segments. In these equilibria, only one oligopolistic firm locates at the facility place in order to exploit the externality advantage. Yet, any dispersion, partial or full, is excessive from the city planner viewpoint—even though, in some circumstances, one firm optimally locates at the facility.

Finally, we have stressed the role of the city planner in shaping the business spatial structure. In particular, we have established the non-trivial result that the lower the transportation cost, the higher the total welfare. If the city planner is able to lower transportation costs "near zero", then it is optimal to locate the facility at the midpoint. On the other hand, if there exists some impediments to the reduction of urban transportation costs, then the city planner would be well-advised to locate the facility at a more eastern site. Indeed, such a configuration results in a more balanced pattern of households trips while retaining some welfare benefits from the provision of the public service to a large share of the population.

Of course, the results of our paper must not be viewed as a guide to real policy. However, our analysis has confirmed that any cost-benefit analysis of the building of a new public facility, or any evaluation of transportation policy (which is sensitive to the facility location), should take into account the impact on strategic locational choices made by oligopolists (see also Thisse and Wildasin, 1992, 1995).

One limitation of our model is the fact that multipurpose shopping is not reasonable for some public facilities such as hospitals or schools; as emphasized by Thisse and Wildasin (1995), p. 408: *"you do not buy shoes on the way to the hospital"*. We hope to modify the present framework in order to incorporate independent trips for this type of facility. We expect the establishment of the existence of equilibrium in prices to be non-trivial.

Future research is also needed in order to determine the socially-optimal level of provision of public service (i.e., optimal facility size or quality). One should distinguish between pure and congested local public goods. Congestion at the site level could make the emergence of a partially-dispersed outcome more desirable than agglomeration. Also, it would be interesting to incorporate two or three major transportation nodes, as well as two or three competing juridictions for some types of facility such libraries.

9 Appendix

9.1 Proof of result 1

One readily checks from equation (5) that if

$$\frac{t}{\alpha} > \frac{1}{(l-a)\,(2+l+a)} \tag{29}$$

then A is in position to attract some consumers by setting a price $p_A = P_A(p_B)$ (possibly very small) as expressed in (9). On the other hand, if the last inequality is not satisfied then firm B optimally monopolizes the whole market by setting a price

$$\overset{\circ}{p}_B(p_A) = p_A + \alpha - t(l+a)(l-a) \tag{30}$$

such that $D_A = 0$.[33] In such an event, one verifies that the mirror game in prices converges to $p_A = 0$, $\overset{\circ}{p}_B(0)$. If (29) does not hold that the equilibrium prices is simply the solution of (9)-(10) as expressed in (12)-(13). Q.E.D.

9.2 Proof of lemma 3

By contradiction, assume that $\widetilde{a}$ and $1-\widetilde{b} \neq l$ is an equilibrium of the first stage. From AGT, $\widetilde{a} = \widetilde{b} = 0$ necessarily and respective maximum profits are equal to $\frac{t}{2} = \frac{p_i}{2}$ ($i = A, B$, see AGT). On the other hand, by virtue of lemma 1, the payoff to B after relocation at l is:

$$\overset{\circ}{p}_B(0) = \alpha - tl^2 \tag{31}$$

We have:

$$\overset{\circ}{p}_B(0) > \tfrac{t}{2} \quad \Leftrightarrow \quad \alpha - tl^2 = t(\widetilde{\alpha} - l^2) > \tfrac{t}{2} \quad \Leftrightarrow \quad \widetilde{\alpha} > \tfrac{1}{2} + l^2$$

which is clearly satisfied for any $\widetilde{\alpha} > l\,(2+l)$ and establishes a contradiction. Q.E.D.

9.3 Proof of proposition 1

Assume first $a < l$. Let us show that $\Pi_B(a, 1-l) > t/2 \geq \Pi_B(a,b)$. Since $\Pi_B(a, 1-l) = \overset{\circ}{p}_B(0) = \alpha - t(l+a)(l-a)$ is increasing with a it suffices to establish that $\alpha/t - l^2 > 1/2$. This has been done in the proof of lemma 3 (Q.E.D).

[33] Assume that (29) does not hold. Then, for fixed locations and a fixed price p_A, the one-variable profit function $\Pi_B(p_A,.)$ is continuous on $[0,\infty[$: it is the 45^oline on $\left[0, \overset{\circ}{p}_B(p_A)\right]$ and it is strictly concave, decreasing on $\left[\overset{\circ}{p}_B(p_A), \Pi_B^{-1}(0)\right[$. We have: $\Pi_B(p_A,.) = \overset{\circ}{p}_B(p_A) > P_B(p_A)$, that is, $\overset{\circ}{p}_B(p_A)$ is a global maximum of $\Pi_B(p_A,.)$.

Second, assume $a > l = 1-b$ and $\widetilde{\alpha} \geq (3-l)(1-l)$. Firm B monopolizes the market (exchange A with B, and a with $1-b$ in lemma 2) and optimally sets the following price: $\widetilde{p}_B(0,a) \underset{def}{\equiv} \alpha - t(a-l)(2-l-a)$. Let us prove again that it is greater than the maximum profit under full dispersion: $\widetilde{p}_B(0,a) > t/2$? One checks that (i) $\widetilde{p}_B$ is decreasing with a and (ii) $\widetilde{p}_B(0,l) = \alpha > t/2$ so that, necessarily, $\widetilde{p}_B(0,a) > t/2 \geq \Pi_B(a,b)$ for all $a > l$.

Finally, if $a = l$ then clearly $\Pi_B(l, 1-l) = 0 = \Pi_B(l,b) \quad$ where $b \neq 1-l$.

To sum up: $\Pi_B(a, 1-l) \geq \Pi_B(a,b) \quad$ for all $b \in [0,1]$, that is, $b = 1-l$ is a weakly dominant strategy for firm B. By symmetry, $a = l$ is a weakly dominating strategy for firm A. Q.E.D.

9.4 Proof of lemma 4

Let us show that firm A can always find a location close to the west end—i.e., in some interval $[0, s(\widetilde{\alpha}, l)[$, such that it attracts a positive demand in any equilibrium of the last stage. Such a segment of potential locations for A must satisfy (29), that is: $(l-a)(2+l+a) > \widetilde{\alpha}$. The relevant root of the second-degree equation in the last inequality is:

$$a = -1 + \sqrt{(1+l)^2 - \widetilde{\alpha}} \underset{def}{\equiv} s(\widetilde{\alpha}, l) \tag{32}$$

One easily checks that: (i) $(1+l)^2 - \widetilde{\alpha} > 0$, (ii) $s(\widetilde{\alpha}, l)$ is decreasing with $\widetilde{\alpha}$ and (iii) $0 \leq s(\widetilde{\alpha}, l) < l$.[34] Thus, A is always in position to find a location near the left border where it captures a positive demand after solving for price competition. (Q.E.D).

Now let us find the optimal location $a^* \in [0, s(\widetilde{\alpha})[$. Firstly, for fixed locations, the equilibrium price set by firm A is as follows:

$$p_A(a, 1-l) = \frac{\alpha}{3}\left[\widetilde{t}\,(l-a)\,(2+a+l) - 1\right] \tag{33}$$

after using result 1; the demand to A is given by:

$$D_A(a, 1-l) = \frac{p_A(a,l)}{2t(1-l-a)} = \frac{\left[\widetilde{t}\,(l-a)\,(2+a+l) - 1\right]}{6\widetilde{t}(l-a)} \tag{34}$$

after substitution of $b = 1-l$. From the two last expressions, one easily computes the reduced-form profit function of firm A. However, it is easier to analyze the sign of the partial derivative with respect to a as follows:

$$\frac{\delta}{\delta a}(p_A D_A) = \frac{p_A}{2t(l-a)}\left[2\frac{\delta p_A}{\delta a} + \frac{p_A}{(l-a)}\right] \tag{35}$$

Indeed, the demand to A, and consequently p_A, is necessarily positive in any outcome of the last stage and it suffices to analyze the sign of the

[34] $s\,(l(2+l), l) = 0$, and $s(\widetilde{\alpha}, l) \to l$ as $\widetilde{\alpha} \to 0$.

bracketed term, that we denote $G(a)$. One calculates $\frac{\delta p_A}{\delta a} = -\frac{2}{3}t\,(1+a) < 0$, and substituting this and Eq.(33) into Eq.(35), one obtains after some manipulations:

$$G(a) = \tfrac{1}{3}\left[t\,(l-2-3a) - \tfrac{\alpha}{l-a}\right] < 0$$

(since $a < l < 1$). This achieves the proof of lemma 4. Q.E.D.

9.5 Exclusion of the case $l \leq a \leq 1-b \leq 1$

Assume $l \leq a < 1-b \leq 1$. From AGT, the optimal location of firm A is $a^* = l$.

- Case #1: $(3-l)\,(1-l) < \widetilde{\alpha} < l\,(2+l)$

From lemma 2, the demand to firm B is equal to zero for all $b \in \left]0, 1-l\right]$. However, firm B can increase its profit by relocating at the left end of the city, playing the role of firm A in proposition 2. (Q.E.D).

- Case #2: $\widetilde{\alpha} < (3-l)\,(1-l)$

In any outcome, the demand to firm B must be positive since otherwise it would be incited to relocate at the left end of the city. We know that firm B can indeed always find a location close to the right end such that $D_B(l,b) > 0$ is satisfied.[35] Firm A benefits from the best environment: $\Delta f(a,b) = -\alpha$. Substituting this into equations, we derive the unique Nash equilibrium in prices as:

$$p_A^*(l,b) = t(1-b-l)(1+\tfrac{l-b}{3}) + \tfrac{\alpha}{3}$$
$$p_B^*(l,b) = t(1-b-l)(1+\tfrac{b-l}{3}) + \tfrac{-\alpha}{3}$$

After substitution into (5), we get the demand facing firm B as a unique function of locations:

$$D_B(l,b) = \frac{1}{6}\frac{t\,(1-b-l)\,(b+3-l)-\alpha}{t\,(1-b-l)} \tag{36}$$

The value of the profit to firm B is as follows:

$$\Pi_B(l,b) = \tfrac{1}{18}\tfrac{(t(1-b-l)(b+3-l)-\alpha)^2}{t(1-b-l)}$$

and we also get

$$\tfrac{d\Pi_B}{db}(l,b) = \tfrac{1}{18}\left(t\,(1-b-l)\,(b+3-l)-\alpha\right)\tfrac{-t(1-b-l)(3b+l+1)-\alpha}{t(-1+b+l)^2}$$

[35] In particular, $D_B(l,0) = \frac{1}{6}\frac{(1-l)(3-l)-\widetilde{\alpha}}{(1-l)} > 0$ since $\widetilde{\alpha} < (1-l)\,(3-l)$.

Since $D_B(l,b) > 0$ by assumption, observe that

$$t\,(1-b-l)\,(b+3-l)-\alpha>0$$

necessarily. Hence, $\frac{d\Pi_B}{db}(l,b) < 0 \Rightarrow b^* = 0$. The value of the profit is immediately obtained as being:

$$\Pi_B(l,0)=\frac{\left[t\,(1-l)\,(3-l)-\alpha\right]^2}{18t\,(1-l)} \tag{37}$$

The payoff to firm A is higher than $\Pi_B(l,0)$ and is given by:

$$\Pi_A(l,0)=\frac{\left[t\,(1-l)\,(3+l)+\alpha\right]^2}{18t(1-l)} \tag{38}$$

Next, consider a relocation of B at the west end of the city ($b = 1$) and solve for price competition. The reduced-form payoff function of a firm situated at the left end while its competitor is at the facility location is given by (16):

$$\Pi_B(l,1)=\frac{\left[tl\,(2+l)-\alpha\right]^2}{18tl} \tag{39}$$

Hence, we derive the following ratio of payoffs to B:

$$\frac{\Pi_B(l,1)}{\Pi_B(l,0)}=\frac{(l(2+l)-\widetilde{\alpha})^2}{((1-l)(3-l)-\widetilde{\alpha})^2}\frac{(1-l)}{l}$$

Let us prove that it is greater than unity. Observe that $\frac{(1-l)}{l} < 1$ for $l > \frac{1}{2}$. We can however rewrite the above ratio as follows:

$$\left[\frac{l(2+l)-\widetilde{\alpha}}{(1-l)(3-l)-\widetilde{\alpha}}\frac{(1-l)}{l}\right]^2\frac{l}{1-l}$$

Clearly, it would suffice to show that the bracketed term is greater than 1:

$$\frac{l(2+l)-\widetilde{\alpha}}{(1-l)(3-l)-\widetilde{\alpha}}\frac{(1-l)}{l}>1 \quad\Leftrightarrow\quad (2+l)-\frac{\widetilde{\alpha}}{l}>(3-l)-\frac{\widetilde{\alpha}}{1-l}$$
$$\Leftrightarrow\quad (2+l)+\frac{\widetilde{\alpha}}{1-l}>(3-l)+\frac{\widetilde{\alpha}}{l} \quad\Leftrightarrow\quad (2+l)>(3-l)+\widetilde{\alpha}\left(\frac{1}{l}-\frac{1}{1-l}\right)$$
$$\Leftrightarrow\quad (2+l)>(3-l)+\widetilde{\alpha}\frac{1-2l}{l(1-l)}$$

which is necessarily checked for $l > \frac{1}{2}$. Thus $\Pi_B(l,1) > \Pi_B(l,0)$: B is incited to relocate at the west end (Q.E.D). For $l = \frac{1}{2}$, B is indifferent between either ends of the line. This achieves the proof that $\frac{1}{2} < l \leq a \leq 1-b \leq 1$ is an impossible market outcome. Q.E.D.

9.6 Proof of proposition 2

9.6.1 Lemma 4.1.

Assume $a = 0$ and $l \neq 1$. There exists a threshold value $\widetilde{t}_0$ such that $\widetilde{t} > \widetilde{t}_0 \quad \Rightarrow \quad \Pi_B(0, 1-l) < \frac{t}{2}$: B locates at the east end ($b = 0$).

Proof. Assume $l \neq 1$. Since, the endpoint dominates any location outside the midpoint, it suffices to compare equation (17) with the profit under maximum differentiation (AGT), that is, $\Pi_B(0,0) = \frac{t}{2}$:

$\frac{\left(\frac{t}{3}l(4-l)+\frac{1}{3}\alpha\right)^2}{2tl} < \frac{t}{2} \quad \Leftrightarrow \quad \left(\frac{\widetilde{t}}{3}l\left(4-l\right)+\frac{1}{3}\right)^2 - \widetilde{t}^2 l < 0$

The relevant root of the polynomial in the last inequality is

$$\widetilde{t}_0 = \frac{l^2 - 4l - 3\sqrt{l}}{l\left(l-1\right)\left(l^2 - 7l + 9\right)} \tag{40}$$

(the second root is negative). Define also $v(l) = 1/\widetilde{t}_0$ the corresponding threshold value of $\widetilde{\alpha}$. Since the above polynomial is concave in $\widetilde{t}$, we deduce that for any $\widetilde{t} > \widetilde{t}_0$ ($\Leftrightarrow \widetilde{\alpha} < v(l)$), $\Pi_B(0, 1-l) < \frac{t}{2}$. If $l = 1$, then $b = 0$ is clearly a dominant strategy for firm B. ■

9.6.2 proof of (ii)

Assume $v(l) < \widetilde{\alpha} < l\left(2+l\right)$ and $l \neq 1$. First, we have proved that $\Pi_B(0, 1-l) > \frac{t}{2} \geq \Pi_B(0,b)$, $\forall b \neq 1-l$. Thus $b^* = 1-l$ maximizes $\Pi_B(0,.)$ on $[0,1]$. Second, assume that $b^* = 1-l$ is fixed. By lemma 4, $a^* = 0$ maximizes $\Pi_A(., 1-l)$ on $[0,l[$. Moreover, A is worse off on the right side of (or at) the facility location, as established in subsection 9.5 (reverse the role of A and B). Thus, $(0, 1-l)$ is a pure strategy Nash equilibrium in locations. It is unique since $a = l < 1-b$ cannot be an equilibrium. (Q.E.D).

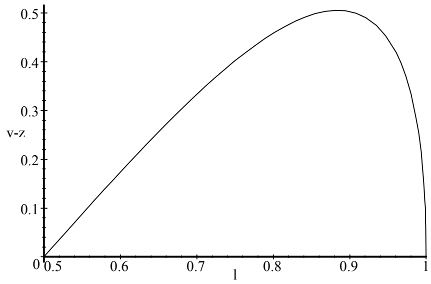
9.6.3 proof of (i)

Let $0 < \widetilde{\alpha} < v(l)$ and assume $b^* = 0$. From AGT, $\frac{\delta}{\delta a}\Pi_A(a,0) < 0$: the optimal location of A on $[0,l[$ is $a = 0$. For the same reason, the optimal location of A on $[l,1]$ is $a = l$. One easily checks that $v(l) < (3-l)(1-l)$,[36] which implies that the demand to B is positive after solving for price competition (see subsection 9.5, case #2 above). The payoff to A is $\Pi_A(l,0)$ as given by Eq. (38). Let us show that $\widetilde{\alpha} < v(l) \Rightarrow \Pi_A(l,0) < \frac{t}{2}$. One first obtains the following implication:

$$\frac{t}{2} - \Pi_A(l,0) > 0 \quad \Leftrightarrow \quad \widetilde{\alpha} < 2l + l^2 - 3 + 3\sqrt{(-l+1)} \underset{def}{=} z(l)$$

[36] Indeed, $(3-l)(1-l) - v(l) = 3\left(1-l\right)\frac{l+\sqrt{l}(3-l)}{l(4-l)+3\sqrt{l}} > 0$.

Let us show that $v(l) < z(l)$. The following picture plots the difference $v(l) - z(l)$:



$v(l) - z(l)$

Hence, we infer that $\widetilde{\alpha} < v(l) \leq z(l)$ which implies $\frac{t}{2} - \Pi_A(l, 0) > 0$ (Q.E.D). Thus, $a^* = 0$ maximizes $\Pi_A(a, 0)$ on $[0, 1]$. It is worth noticing that for $\widetilde{\alpha} = v(1/2) = z(1/2)$, we get $\frac{t}{2} = \Pi_A(l, 0)$ that consistently establishes the equilibrium $(1/2, 0)$. We already knew that this equilibrium exists as the symmetric of $(0, 1/2)$ (interchange A and B in proposition 2 where $l = 1/2$). The other limit result, i.e., $v(1) = z(1) = 0$ is also consistent since then, necessarily, $\widetilde{\alpha} > v(1)$: this limit case simply coincides with case (ii).

Finally, assume $a^* = 0$. It has already been established that $t/2 = \Pi_B(0, 0) > \Pi_B(0, 1 - l) \geq \Pi_B(0, b)$, for all $b \neq b^* = 0$ and any $\widetilde{\alpha} < v(l)$.

We conclude that the unique pure strategy Nash equilibrium in locations is $(a^*, b^*) = (0, 0)$. (recall again that $(l, 0)$ cannot be an equilibrium). Q.E.D.

9.7 Equilibrium profits (fixed t)

First of all, observe that the payoff to A under partial dispersion is affected negatively by α. For this firm, the maximum profit is reached whenever $\alpha \to 0$ since then we tend to AGT: $\Pi_A(0, 1 - l) \to t/2$. Secondly, for firm B, we have established the following:

Result 3. (i) For a fixed value $t = \overline{t}$, the location l that maximizes the payoff to B under partial dispersion is given by:

$$l_B = \begin{cases} 1 & \text{if } \alpha/\overline{t} \in \,]0, 1] \\ 2/3 < \frac{2}{3} + \frac{1}{3}\sqrt{4 - 3\alpha/\overline{t}} < 1 & \text{if } \alpha/\overline{t} \in [1, 1.31] \\ 0.527 < -1 + \sqrt{1 + \alpha/\overline{t}} \leq 1 & \text{if } \alpha/\overline{t} \in [1.31, 3[\end{cases}$$

(ii) For a fixed value of l, the payoff to B in equilibrium is strictly increasing with α.

(iii) Firm B's profit reaches a maximum value of $\Pi_B(0, 1 - l) = 2t$ whenever $l = 1$ and $\alpha \to 3t$.

Note that $-1+\sqrt{1+\widetilde{\alpha}}$ is simply the reverse function of the separating curve $l(2+l)$.

9.8 Welfare functions

9.8.1 Dispersion

In the first configuration, the social transportation cost is identical to the one calculated in AGT since no one visits the facility:

$$T(a,b)=\frac{1}{3}t\left[a^3+\frac{1}{4}\left(1-b-a\right)^3+b^3\right] \tag{41}$$

after using $p_A=p_B \;\Rightarrow\; \widehat{D}_A=\frac{1}{2}$. The total welfare is given by:

$$W^{SY}(a)=\overline{u}-T(a,a)=\frac{1}{12}t\left[1-6a\left(1-2a\right)\right] \tag{42}$$

9.8.2 Full agglomeration

In the second configuration, the total welfare is given by:

$$\begin{aligned} W^{FA}(l) &= \overline{u}+\int_0^{\widehat{D}_A}\alpha\;dx+\int_{\widehat{D}_A}^{1}\alpha\;dx\;-\;T(l,1-l) && (43)\\ &= \overline{u}+\alpha-\;T(l,1-l) && (44)\end{aligned}$$

Using $D_i(l,1-l)=\frac{1}{2}$, one easily calculates the social transportation cost as

$$T(l,1-l)=t\left[\frac{1}{3}-l(1-l)\right] \tag{45}$$

9.8.3 Partial dispersion

Assume $a<l=1-b$. The social transportation cost is given by:

$$T(a,1-l)=t\left(\frac{1}{3}-\frac{1}{4}\left(l-a\right)\left(l+a\right)^2-l\left(1-l\right)\right)+\frac{1}{4}\frac{\alpha^2}{t(l-a)} \tag{46}$$

One obtains the expression of the total welfare as

$$W^{P}(a,1-l)=\overline{u}+\left[1-\frac{l-a}{2}+\frac{\alpha/t}{2(l-a)}\right]\alpha\;-T(a,1-l) \tag{47}$$

where $T(a,1-l)$ is given by (46).

9.9 Socially-optimal $\widehat{a}$ under partial dispersion

Step 1 Assume $\widetilde{\alpha} < l^2$. The unique candidate for a maximum of the function (47) above is easily obtained from the FOC: $\widehat{a} = \frac{2}{3}l - \frac{1}{3}\sqrt{l^2 + 3\widetilde{\alpha}}$as in (25)

Step 2 (condition 1) Let us prove first that condition 1 is satisfied near $\widehat{a}$. We must prove that

$$\widehat{a}^2 < l^2 - \widetilde{\alpha} \quad \Leftrightarrow \quad \left(\tfrac{2}{3}l - \tfrac{1}{3}\sqrt{(l^2 + 3\widetilde{\alpha})}\right)^2 < l^2 - \widetilde{\alpha}$$

$$\Leftrightarrow \quad \tfrac{5}{9}l^2 - \tfrac{4}{9}\sqrt{(l^2 + 3\widetilde{\alpha})}l + \tfrac{1}{3}\widetilde{\alpha} < l^2 - \widetilde{\alpha} \quad \Leftrightarrow \quad \tfrac{4}{3}\widetilde{\alpha} < \tfrac{4}{9}l^2 + \tfrac{4}{9}\sqrt{(l^2 + 3\widetilde{\alpha})}l$$

$$\Leftrightarrow \quad \sqrt{(l^2 + 3\widetilde{\alpha})}l > 3\widetilde{\alpha} - l^2 \qquad ?$$

Since $\widetilde{\alpha} - l^2 < 0$, it suffices to show that

$$\sqrt{(l^2 + 3\widetilde{\alpha})}l > 2\widetilde{\alpha} \quad \Leftrightarrow \quad \left(\sqrt{(l^2 + 3\widetilde{\alpha})}l\right)^2 > 4\widetilde{\alpha}^2 \quad \Leftrightarrow \quad l^4 + 3l^2\widetilde{\alpha} - 4\widetilde{\alpha}^2 > 0$$

The LHS of the last expression is equal to $\left(4\widetilde{\alpha} + l^2\right)\left(l^2 - \widetilde{\alpha}\right)$ which is positive for any $\widetilde{\alpha} < l^2$ (Q.E.D).

Step 3. (second-order condition) Let us show that $\widehat{a}$ satisfies the second-order condition for a maximum.

Denote $H = \left[0, \sqrt{(l^2 - \widetilde{\alpha})}\right[$ the region where $\widehat{D}_A(a, 1 - l)$ is positive. One first calculates the value of the demand to A at $\widehat{a} \in S$:

$$\widehat{D}_A(\widehat{a}, 1 - l) = \frac{2}{3}l - \frac{2\widetilde{\alpha}}{l + \sqrt{(l^2 + 3\widetilde{\alpha})}} \tag{48}$$

Then, one derives the second derivative of W^P as follows:

$$\frac{\delta^2 W^P}{\delta a^2}(a, l) = -t\frac{g(a, l)}{2\,(l - a)^3}$$

where $g(a) \equiv -6l^2a^2 + 8a^3l - 3a^4 + l^4 - \widetilde{\alpha}^2$. One calculates:

$$\begin{aligned} g(\widehat{a}) &= \tfrac{8}{27}l^4 + \tfrac{8}{27}l^3\sqrt{(l^2 + 3\widetilde{\alpha})} + \tfrac{4}{9}l^2\widetilde{\alpha} - \tfrac{4}{3}\widetilde{\alpha}^2 \\ &= \tfrac{4}{9}\left[\tfrac{2}{3}l^3\left(l + \sqrt{(l^2 + 3\widetilde{\alpha})}\right) + \widetilde{\alpha}\left(l^2 - 3\widetilde{\alpha}\right)\right]. \end{aligned}$$

Divide the bracketed term successively by l^2 and $h = l + \sqrt{(l^2 + 3\widetilde{\alpha})} > 0$ to get:

$$\left[\tfrac{2}{3}l + \tfrac{\widetilde{\alpha}}{h}\left(1 - 3\tfrac{\widetilde{\alpha}}{l^2}\right)\right] = \tfrac{2}{3}l + \tfrac{\widetilde{\alpha}}{h} - \tfrac{3}{h}\tfrac{\widetilde{\alpha}^2}{l^2}$$
$$= \tfrac{2}{3}l - \tfrac{2\widetilde{\alpha}}{h} + \tfrac{3\widetilde{\alpha}}{h} - \tfrac{3}{h}\tfrac{\widetilde{\alpha}^2}{l^2}$$
$$= \widehat{D}_A(\widehat{a}, 1-l) + \tfrac{3\widetilde{\alpha}}{h}\left(1 - \tfrac{\widetilde{\alpha}}{l^2}\right)$$

By assumption, the last factor in parentheses is necessarily positive. Thus $g(\widehat{a}) > 0 \Rightarrow \frac{\delta^2 W^P}{\delta a^2}(\widehat{a}, l) < 0$. Since, $\widehat{a}$ is the unique candidate satisfying the FOC on H, it is a global maximum of W^P.

9.10 Firm A at the facility site: welfare analysis

Assume $a = l < 1 - b \leq 1$. Firm A benefits from the best environment and substituting $\Delta f(a,b) = -\alpha$ into (5) yields the demand to firm A:

$$\widehat{D}_A(l, 1-b) = \frac{1-b+l}{2} + \frac{\widetilde{\alpha}}{2(1-b-l)} \tag{49}$$

A proportion $\widehat{D}_A(l, 1-b)$ receives the benefit from the public facility whereas the rest of the population gets the standard utility as in AGT. Consequently, the total welfare is simply given by:

$$\widetilde{W}^P(a,b) = u + \alpha\ \widehat{D}_A(l, 1-b) - T(1-l, b) \tag{50}$$

The following condition parallels condition 1:

Condition 2 In any asymmetric configuration $(a, 1-l)$ we must have: $\widehat{D}_B(a, 1-l) > 0 \quad \Leftrightarrow \quad b^2 < (1-l)^2 - \widetilde{\alpha}$

The unique candidate for a maximum is interior and given by:

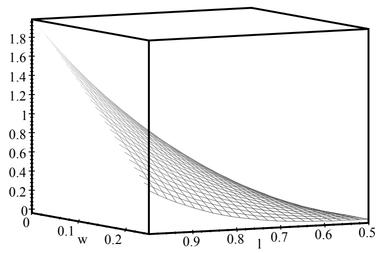
$$\widehat{b} = -\frac{2}{3}l + \frac{2}{3} - \frac{1}{3}\sqrt{(l^2 - 2l + 1 + 3\widetilde{\alpha})} \tag{51}$$

This expression describes the unique value of $\widehat{b}$ which maximizes (50), both globally and locally, subject to condition 2 and $1 - b > l$. The proof is straightforward: let $L = 1 - l$. One first observes that the expression $\widehat{b}$ is equivalent to the one of $\widehat{a}$ in (25) provided that we replace l by L:

$$\widehat{b} = \tfrac{2}{3}L - \tfrac{1}{3}\sqrt{(L^2 + 3\widetilde{\alpha})}$$

Secondly, we also easily establish that $\widehat{D}_B(l, 1-\widehat{b}) \equiv \widehat{D}_A(\widehat{a}, 1-L)$ where, again, the function $\widehat{D}_A$ has been defined above. It follows that the proof given for $\widehat{a}$ applies identically when l is replaced by L. For example, establishing $\widehat{b}^2 < (1-l)^2 - \widetilde{\alpha}$ amounts to establish that $\frac{2}{3}L - \frac{1}{3}\sqrt{(L^2 + 3\widetilde{\alpha})} < L^2 - \widetilde{\alpha}$ which is exactly step 2 in subsection 9.9 above where l is replaced by L. The same argument is valid for the second-order condition (see step 3 above).

Substituting $\widehat{b}$ into (50), one obtains the value of the total welfare $\widetilde{W}^P(l,\widehat{b})$ and the value of the difference $W^P(\widehat{a},1-l)-\widetilde{W}^P(l,\widehat{b})$ is plotted below. Clearly, for any $\frac{1}{2}\leq l\leq 1$ and $0<\widetilde{\alpha}<(1-l)^2\leq\frac{1}{4}$, welfare is greater in the configuration $(\widehat{a},1-l)$.



$W^P(\widehat{a},1-l)-\widetilde{W}^P(l,\widehat{b})$

9.11 Proof of proposition 3

We have argued that partial dispersion is impossible for $\widetilde{\alpha}>l^2$. From (21) and (22), we immediately derive another intermediate result:

$$W^{FA}(l)>W^{SY}(\frac{1}{4})\quad\Leftrightarrow\quad\widetilde{\alpha}>\frac{5}{16}-l+l^2 \tag{52}$$

For instance, if the public facility is centrally-located then $W^{FA}(\frac{1}{2})>W^{SY}(\frac{1}{4})$ if and only if $\widetilde{\alpha}>\frac{1}{16}$. Moreover, from (52) and $\frac{5}{16}-l<0$, we have immediately:

Lemma 5. If $\widetilde{\alpha}>l^2$ then the city planner chooses the fully-agglomerated pattern of locations.

Now, there remains to determine the socially-optimal pattern(s) of locations whenever $\widetilde{\alpha}<l^2$.

Lemma 6. Assume $0<\widetilde{\alpha}<l^2$. Then,

(i) $W^P(\widehat{a},1-l)>W^{FA}(l)$

(ii) $W^P(\widehat{a},1-l)<W^{SY}(\frac{1}{4})\qquad\Leftrightarrow\qquad\widetilde{\alpha}<y(l)<l^2$

where $y(l)$ is defined in footnote.[37]

[37] $y(l)=\frac{1}{3}\left(\frac{1}{8}h(l)+\frac{1}{2}\frac{(4l-3)^2}{q(l)}+l-\frac{3}{4}\right)^2-\frac{1}{3}l^2$
with $h(l)=\sqrt[3]{(v+12\sqrt{w})}$
$v=324-512l^3+1152l^2-864l$
$w=-768l^3+1728l^2-1296l+405$

Proof. (i) After some calculation and manipulation, we obtain:

$$W^P(\widehat{a}, 1-l) - W^{FA}(l) = \frac{4}{27}\frac{2q\left(l^2 - 3\widetilde{\alpha}\right) + 9\widetilde{\alpha}(l^2 - \widetilde{\alpha})}{l + \sqrt{(l^2 + 3\widetilde{\alpha})}}$$

where

$$q = l^2 + 3\widetilde{\alpha} - 2\sqrt{(l^2 + 3\widetilde{\alpha})}l = \frac{3}{2}\widehat{D}_A(\widehat{a}, 1-l)\left(l + \sqrt{(l^2 + 3\widetilde{\alpha})}\right) > 0$$

and $\widehat{D}_A(\widehat{a}, 1-l)$ is given by (49) above. Let us show that the numerator of the above fraction is positive. We get first:

$$\widetilde{\alpha} - l^2 < 0 \quad \Leftrightarrow \quad 3\widetilde{\alpha} - l^2 < 2\widetilde{\alpha} \quad \Rightarrow \quad 2q\left(l^2 - 3\widetilde{\alpha}\right) > -4q\widetilde{\alpha} \qquad (q > 0)$$

Hence, the numerator above verifies the following inequality:

$$2q\left(l^2 - 3\widetilde{\alpha}\right) + 9\widetilde{\alpha}(l^2 - \widetilde{\alpha}) > -4q\widetilde{\alpha} + 9\widetilde{\alpha}(l^2 - \widetilde{\alpha}) = \widetilde{\alpha}(9\widetilde{\alpha} - 4q + 9l^2)$$

Let us finally show that the last term in parentheses is necessarily positive:

$$9\widetilde{\alpha} - 4q + 9l^2 = 5l^2 - 4\sqrt{(l^2 + 3\widetilde{\alpha})}l + 3\widetilde{\alpha}$$

after substitution for q. For a fixed l, one easily checks that the last expression, subject to $\widetilde{\alpha} \leq l^2$, achieves a minimum value of 0 at $\widetilde{\alpha} = l^2$. Thus, $W^P(\widehat{a}, 1-l) - W^{FA}(l) > 0$ for any $\widetilde{\alpha} < l^2$ (Q.E.D). ■

For $\widetilde{\alpha} < l^2$, OP dominates the full agglomeration equilibrium in the eyes of the planner. However, for a very weak value $\widetilde{\alpha} < y(l)$, the standard symmetric pattern of locations is better from the welfare viewpoint.

9.12 Proof of proposition 4

We first replace $\widehat{D}_A$ by D_A in (20) to get the expression of the social transportation cost:

$$T^*(0, 1-l) = t\left[\frac{1}{3} - lD_B(0, 1-l)\left[1 - l + D_A(0, 1-l)\right]\right] \tag{53}$$

Second, since a proportion $D_B(0, 1-l) = 1 - D_A(0, 1-l)$ of consumers patronizes seller B, the social surplus is given by:

$$W_*^P(0, 1-l) = \overline{u} + \left[\alpha D_B(0, 1-l) \ - T(0, 1-l)\right] \tag{54}$$

For given α and t, it suffices to maximize the bracketed term with respect to l. In figure 4, $m(l)$ plots the first order condition (One checks that the second-order condition is met). We have checked that:

- $\frac{d}{dl}W_*^P(0, 1-l) < 0$ for any $\widetilde{\alpha} > m(l)$, and

- $W_*^P(0, 1-l) - W^D(0) > 0$ for any given $\widetilde{\alpha}$ and any l.

The last result indicates that partial dispersion dominates the fully dispersed equilibrium from the welfare viewpoint while the sign of the first derivative shows that the facility should be as centralized as possible. In the axis system $(l, \widetilde{\alpha})$, $m(l)$ intersects $v(l)$ at $l \simeq 0.5916$ (where $v(l) = 0.291$). The intersection of $m(l)$ with the α-axis is $m(0.5) = 0.42$.

For $\widetilde{\alpha} > 1.25$, we have proceeded as follows. First, we have established that

$$W^{FA}(\tfrac{1}{2}) > W_*^P(0, 1-l) \quad \Leftrightarrow \quad \widetilde{\alpha} < \tfrac{4}{5}l + l^2 + \tfrac{3}{5}\sqrt{(-16l^2 + 20l^3 + 5l)} \equiv k(l)$$

Second, we have checked that the curve $k(l)$ is above the "separating curve" $\widetilde{\alpha} = l(2+l)$ (except at $\frac{1}{2}$ where $k(\frac{1}{2}) = l(2+l) = 1.25$). Thus, $W^{FA}(\frac{1}{2}) > W_*^P(0, 1-l)$ when both types of equilibrium competes in the eyes of the planner. (Q.E.D).

9.13 Proof of proposition 5

Let us analyze the welfare impact of changes in t, when l and α are exogenously fixed. For each value of t/α, the market outcome is given by propositions 1 and 2. The value of the total welfare in each one of the configurations FA, P or FD, is given by (22), (28) and (54) respectively. If $\widetilde{t} \gtrapprox \frac{1}{l(2+l)}$ (partial dispersion) then almost all the residents visit the central site and derives a total utility $\overline{u} + \alpha$ from the consumption of both types of goods. It follows that $W_*^P(0, l) \to W^{FA}(l)$ as $\widetilde{t} \to \frac{1}{l(2+l)}$. One also shows that $W_*^P(0, l) > W^{SY}(0)$ for $\widetilde{t} \simeq \widetilde{t}_0$ and any $\alpha > 0$ (see also figure 4).[38] Since $W^{FA}(l)$ and W_*^P are shown to be decreasing in t:[39]

[38] For example, for $l = \frac{1}{2}$, $W_*^P(0, \frac{1}{2}) - W^S(0) = \alpha \frac{184\widetilde{t} + 80 + 5\widetilde{t}^2}{288\widetilde{t}} > 0$.

[39] $\frac{d}{dt} W_*^P = -\frac{1}{36t^2}\left(\frac{5\alpha^2}{l} - t^2\left(32l + 5l^3 - 28l^2 - 12\right)\right) < 0$.

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