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BAYESIAN ANALYSIS OF THE COMPOUND COLLECTIVE MODEL; THE VARIANCE PREMIUM PRINCIPLE WITH EXPONENTIAL POISSON AND GAMMA-GAMMA DISTRIBUTIONS

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BAYESIAN ANALYSIS OF THE COMPOUND COLLECTIVE MODEL; THE VARIANCE PREMIUM PRINCIPLE WITH EXPONENTIAL POISSON AND GAMMA-GAMMA DISTRIBUTIONS

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Abstract

The distribution of the aggregate claim size is the considerable importance in insurance theory since, for example, it is needed as an input in premium calculation principles and reserve calculation which plays an important paper in ruin theory. In this paper a Bayesian study for the collective risk model by incorporating a prior distribution for both, the parameter of the claim number distribution and the parameter of the claim size distribution is made and applied to the variance premium principle. Later a sensitivity study is to carry out on both parameters using Bayesian global robustness. Despite the complicated form of the collective risk model it is shown how the robustness study can be treated in an easy way. We illustrate the results obtained with numerical examples.

JEL: C11; G22

Key Words: Bayesian Robustness, Contamination Class, Variance Principle.

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1 INTRODUCTION

In Actuarial Science, the collective risk model is described by a frequency distribution for the number of claims K and a sequence of independent and identically distributed non-negative random variables representing the size of the single claims, X_i. In order to make the model mathematically tractable the individual claim sizes are assumed to be independent from the claim counts. Then the aggregate loss X is the sum of the individual claim sizes, i.e. $X = \sum_{i=1}^{n} X_i,\ N > 0,$ and $X = 0,$ for $N = 0.$ It is easy to show (see Gerber (1979)) that the expectation and variance of X are given by

$$E(X) = E(X_i)E(N),$$
$$\text{Var}(X) = \text{Var}(X_i)E(N) + E(X_i)^2\,\text{Var}(N).$$

First Expression corresponds to the net risk premium. The second one can be used to compute the variance risk premium by using the formula $P = E(X^2)/E(X) = E(X) + \delta\text{Var}(X),\ \delta > 0.$ For a revision of the premium calculation principles the lector is remitted to the papers of Gerber (1979), Goovaerts and De Vylder (1979), Heilmann (1989), Hürlimann (1994), Rolski et al. (1999) and Young (2004); among others. In practice, the distribution of the aggregate loss X depends on a parameter or a vector of parameters which are assumed to be unknown and random; therefore the risk premium is also unknown. When Bayesian models are implemented for premium calculations principles in Actuarial Statistics a structure function (prior distribution), following a Bayesian paradigm, is assumed in a natural way for the unknown parameter in the insurer's portfolio. This portfolio is assumed to include a finite number of policies or contracts. This let us to consider that the portfolio is not homogeneous and therefore that across the policies exist a random variable whose realizations are the values of the risk parameter for policies belonging to the portfolio, and its distribution is the prior distribution. Then, assuming a prior distribution on the vector of parameters the collective premium is computed as $P^{'} = E\left(E(X)\right) = E(P)$ and $P^{'} = E(P^2)/E(P),$ for the net and variance premium principles, respectively. Here, the first expectation is taken over the parametric space of the unknown vector of parameters. If experience is available the Bayes premium can be computed in the same form as the collective premium by interchanging the prior by the posterior distribution which represents the best estimator of the unknown risk premium. Due to its simple form, the net premium is the most popular premium calculation principle used in the literature. The variance premium has the advantage with respect to the net premium that this takes account only the expected claims while the first incorporates a safety loading proportional to the variance.

In this paper a full Bayesian methodology is carried out on the collective risk model assuming a prior distribution for both, the parameter of the claim count distribution and the parameter of the claim size distribution. A study of this nature has been considered by Frangos and Vrontos (2001) for the net premium and Pai (1997) in a reinsurance context but in our knowledge never under other premium calculation principle. A similar study by considering only a prior distribution on the parameter of the claim count distribution has been treated extensively in the literature. See for example Freifelder (1974), Rolski (1999) and Gómez et al. (2002). On the other hand, Bayesian methods have been widely criticized due to their use depends strongly on the prior distribution which has a strong subjective character (see Klugman, 1987, p.318 and Rios et al., 1999). Therefore, we also focus on prior influence, measuring changes of the Bayes premium with respect to changes in the prior distribution. Most of the previous works done to deal with this problem have focused on interchanging the prior distribution by a new prior which is moved into a plausible class of distributions and the range of variation of the quantity of interest is computed. This topic is called global robustness analysis (Ríos and Ruggeri (2000) and Sivanganesan and Berger (1989); among others). In fact, robustness has been treated extensively in the actuarial literature (Eichenauer et al. (1988); Makov (1995); Ríos et al. (1999) and Gómez et al. (2002)) but never under the model proposed here, a sensitivity study on the parameters of the distribution of total claims payable by an insurer when the frequency of claims is a Poisson random variable and the claim size follows an exponential distribution. This model has been studied in deep by many authors in the literature including for example to Freifelder, (1974), Seal (1979), Gerber (1979) and Gómez et al. (2002); among others. Bayesian sensitivity of this model only has been dealt in the work of Gómez et al. (2002) but including only the robustness with respect to the parameter of the claim number distribution. Despite the complicated form of the collective risk model it is shown how the robustness study can be treated in an easy way by assuming only a single period of observation.

The rest of the paper is organized as follows: Section 2 addresses the analysis of the model to be considered. Section 3 describes the True Individual Premium, the *a priori* Premium and the *a posteriori* Premium for the Variance Premium Principle. Section 4 analyses the robustness of the *a posteriori* Premium with respect to the specification of the *a priori* distribution of λ and θ respectively. In both cases, the hypothesis of independence between the parameters is maintained. Conclusions and comment upon questions that remain open to further studies are drawn in Section 5.

2.-THE MODEL

A simple and useful model to describe the model above is to assume that the number of claims follows a Poisson distribution with parameter $\theta > 0$, i.e. $\Pr(N = n) = p(n|\theta) = e^{-\theta}\theta^n / n!,\ n = 0,1,...,$ and that the size of the single claims follows an exponential distribution with parameter $\lambda > 0$, i.e. $f\left(x_i|\lambda\right) = \lambda e^{-\lambda x_i},\ x_i > 0.$

It is well known (see Freifelder (1974), Seal (1979) and Gómez et al. (2002)) that the likelihood of the model proposed is given by

$$L(x|\theta,\lambda) = \begin{cases} \sum_{n=1}^{\infty} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} \frac{e^{-\theta}\theta^n}{n!} = \frac{1}{x} e^{-(\theta+\lambda x)} \sum_{n=1}^{\infty} \frac{(\lambda\theta x)^n}{(n-1)!n!} = \sqrt{\frac{\lambda\theta}{x}} e^{-(\theta+\lambda x)} I_1\left(2\sqrt{\lambda\theta x}\right), & x > 0, \\ e^{-\theta}, & x = 0, \end{cases} \tag{1}$$

where $I_\upsilon(x)$ is the modified Bessel function of the first kind given by

$$I_\upsilon(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+\upsilon}}{k!\Gamma(\upsilon+k+1)},\ x \in \mathbb{R},\ \upsilon \in \mathbb{R},$$

and therefore, the convergence of the series is guaranteed.

Assume that the parameters θ and λ are independent, and let us specify an *a priori* Gamma distribution for each of them (which in both cases is the conjugate *a* *priori* distribution), $\pi_{10}(\theta) = \dfrac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}$, $\pi_{01}(\lambda) = \dfrac{d^c}{\Gamma(c)}\lambda^{c-1}e^{-d\lambda}$. Therefore, the joint *a priori* distribution is $\pi_0(\theta,\lambda) = \pi_{10}(\theta)\,.\,\pi_{01}(\lambda)$ for positive θ, λ; a, b, c and d are positive, known constants.

Referring the notation to $\pi_{10}(\theta)$, the value of mathematical expectation is $\dfrac{a}{b}$, and that of the variance is $\dfrac{a}{b^2}$; the distribution is unimodal when a > 1 and in this case, the value of the mode is $\theta_0 = \dfrac{a-1}{b}$; Pearson's coefficient of asymmetry is never annulled and the central moment of order 3 is only annulled when a = 0.

By direct integration, it is straightforward to find that the marginal distribution of 'x' is expressed as follows:

$$m(x/\pi_0) = \int_\theta\int_\lambda \mathrm{L}(x/\theta,\lambda)\ \pi_0(\theta,\lambda)\ d\theta\ d\lambda =$$

$$= \frac{b^a}{\Gamma(a)}\frac{d^c}{\Gamma(c)}\frac{1}{(b+1)^a}\frac{1}{(x+d)^c}\sum_{n=1}^{\infty}\frac{x^{n-1}\Gamma(a+n)\Gamma(c+n)}{(n-1)!n!(b+1)^n(x+d)^n} =$$

$$= \frac{b^a}{\Gamma(a)}\frac{d^c}{\Gamma(c)}\frac{1}{(b+1)^a}\frac{1}{(x+d)^c}\sum_{n=1}^{\infty}T_n\,; \qquad (2)$$

denoting,

$$T_n \equiv T(n;\ x,\ a,\ b,\ c,\ d) = \frac{x^{n-1}\Gamma(a+n)\Gamma(c+n)}{(n-1)!n!(b+1)^n(x+d)^n}.$$

therefore,

$$m(x/\pi_0) = \begin{cases} \dfrac{b^a}{\Gamma(a)}\dfrac{d^c}{\Gamma(c)}\dfrac{1}{(b+1)^a}\dfrac{1}{(x+d)^c}\displaystyle\sum_{n=1}^{\infty}T_n; & x>0 \\ \left(\dfrac{b}{b+1}\right)^a; & x=0 \end{cases} \qquad (3)$$

It is straightforward to show that the series in the first row of expression (2) is a convergent series of positive terms for any positive value of a, b, c, d, x.

We now provide an example in which the marginal distribution $m(x|\pi_0)$ is determined.[4]

<u>Example 1</u>.- Numerical illustration and graphical representation of the Marginal Distribution.

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[4] Programs created within the *Matemática* software package, used to calculate this marginal distribution and other future examples, are available to any person who requests them.

Assume the following *a priori* distributions are specified, $\theta \to Gamma(2,7)$ y $\lambda \to Gamma(5,3)$. Then,

$$\text{m}(\text{ x } / \pi_0) = \begin{cases} \dfrac{7.7519}{(x+3)^5} \displaystyle\sum_{n=1}^{\infty} \dfrac{x^{n-1}(n+4)(n+3)(n+2)(n+1)^2 n}{8^n (x+3)^n} & \text{x} > 0 \\ 0.7656 & \text{x} = 0 \end{cases}$$

the values of which are incorporated into the following table:

(Table 1)

(Figure 1)

The *a posteriori* distribution of (θ, λ) given the sampling observation 'x', is obtained as follows:

$$\pi_0(\theta, \lambda / x) = \frac{L(x/\theta, \lambda)\pi_0(\theta, \lambda)}{m(x/\pi_0)} =$$

$$= \begin{cases} \dfrac{\theta^{a-1}\lambda^{c-1}e^{-(b+1)\theta}e^{-(x+d)\lambda}\dfrac{1}{x}\displaystyle\sum_n \dfrac{(\lambda x\theta)^n}{(n-1)!n!}}{\dfrac{1}{(b+1)^a}\dfrac{1}{(x+d)^c}\displaystyle\sum_n T_n}; & x > 0 \\ \dfrac{(b+1)^a d^c}{\Gamma(a)\Gamma(c)}\theta^{a-1}e^{-(b+1)\theta}\lambda^{c-1}e^{-d\lambda}; & x = 0 \end{cases} . \quad (4)$$

The series of the numerator in the first row in Expression (4) is the same as that in Expression (1) and the series of the denominator is the same as that in Expression (2).

3.- THE VARIANCE PREMIUM PRINCIPLE

The following Lemmas 1 and 2 are well known (see, for example, Gómez (1996)). We reproduce them here for the sake of completeness, and merely sketch out the proof.

Lemma 1
The True Individual Premium, P, is equal to

$$\text{P} = \frac{\theta + 2}{\lambda}. \quad (5)$$

Proof
The True Individual Premium is defined as:

$$P = \frac{E\left[X^2\right]}{E[X]} = E[X] + \frac{Var[X]}{E[X]}.$$

By mathematical operations, numerator is

$$E\left[X^2\right] = \int x^2 L(x/\theta,\lambda)\,dx = \int x^2 e^{-\theta} e^{-\lambda x} \sum_{n=1}^{\infty} \frac{\lambda^n x^{n-1}\theta^n}{(n-1)!n!} dx =$$

$$= \sum_{n=1}^{\infty} \frac{e^{-\theta}\lambda^n\theta^n}{(n-1)!n!} \int x^{n+1} e^{-\lambda x} dx = \frac{e^{-\theta}}{\lambda^2} \sum_{n=1}^{\infty} \frac{\theta^n (n+1)}{(n-1)!} =$$

$$= \frac{e^{-\theta}}{\lambda^2} e^{\theta}\left(\theta^2 + 2\theta\right) = \frac{\theta^2 + 2\theta}{\lambda^2};$$

In an analogous way, denominator is,

$$E[X] = \int x L(x/\theta,\lambda)\,dx = \int x e^{-\theta} e^{-\lambda x} \sum_{n=1}^{\infty} \frac{\lambda^n x^{n-1}\theta^n}{(n-1)!n!} dx =$$

$$= \sum_{n=1}^{\infty} \frac{e^{-\theta}\lambda^n\theta^n}{(n-1)!n!} \int x^n e^{-\lambda x} dx = \frac{\theta}{\lambda};$$

By substitution in the definition of P we obtain the expression we were looking for.

■

Lemma 2

The *a priori* premium, P', is obtained with the following expression:

$$P' = \frac{d\left[a(a+1) + 4ab + 4b^2\right]}{b(c-2)(a+2b)}. \qquad (6)$$

Proof

The *a priori* Premium is defined as $P' = \dfrac{\int P^2 \pi_0(\theta,\lambda)\,d\theta d\lambda}{\int P \pi_0(\theta,\lambda)\,d\theta d\lambda}$ with $P = \dfrac{\theta+2}{\lambda}$.

The numerator of P' is obtained by integration:

$$\int P^2 \pi_0(\theta,\lambda)\,d\theta d\lambda = \int_\theta \int_\lambda \frac{(\theta+2)^2}{\lambda^2} \pi_0(\theta,\lambda)\,d\theta d\lambda =$$

$$= \frac{b^a d^c}{\Gamma(a)\Gamma(c)} \left[\int_\theta (\theta+2)^2 \theta^{a-1} e^{-b\theta} d\theta\right] \left[\int_\lambda \frac{1}{\lambda^2} \lambda^{c-1} e^{-d\lambda} d\lambda\right] =$$

$$= \frac{d^2\left[a(a+1)+4ab+4b^2\right]}{b^2(c-1)(c-2)}.$$

The denominator of P' is obtained in a similar form,

$$\int P\pi_0(\theta,\lambda)\,d\theta d\lambda = \int_\theta\int_\lambda \frac{\theta+2}{\lambda}\pi_0(\theta,\lambda)\,d\theta d\lambda = \frac{(a+2b)d}{b(c-1)}.$$

Just by substitution we obtain the expression in the definition of P'.

■

<u>Lemma 3</u>

The *a posteriori* premium, P*, is expressed by

$$P^*\left[\pi_0(\theta,\lambda/x)\right] =$$

$$= \begin{cases} \dfrac{(x+d)\displaystyle\sum_{n=1}^{\infty}\frac{(a+n+1)(a+n)+4(a+n)(b+1)+4(b+1)^2}{(c+n-2)(c+n-1)}T_n}{(b+1)\displaystyle\sum_{n=1}^{\infty}\frac{a+2b+n+2}{c+n-1}T_n}; & x>0 \\ \dfrac{\left(a^2+4b^2+4ab+5a+8b+4\right)d}{(a+2b+2)(b+1)(c-2)}; & x=0 \end{cases}. \qquad (7)$$

<u>Proof</u>

The *a posteriori* premium is defined as P* = $\dfrac{\int P^2\pi_0(\theta,\lambda/x)\,d\theta d\lambda}{\int P\pi_0(\theta,\lambda/x)\,d\theta d\lambda}$ with

P = $\dfrac{\theta+2}{\lambda}$.

For x>0, by integration we obtain the numerator of P*

$$\int P^2\pi_0(\theta,\lambda/x)\,d\theta d\lambda = \int_\theta\int_\lambda \frac{(\theta+2)^2}{\lambda^2}\pi_0(\theta,\lambda/x)\,d\theta d\lambda =$$

$$= \frac{b^a d^c}{\Gamma(a)\Gamma(c)m(x/\pi_0)}\sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!n!}\left[\int_\theta (\theta+2)^2\theta^{a+n-1}e^{-(b+1)\theta}d\theta\right]\left[\int_\lambda \lambda^{c+n-3}e^{-(x+d)\lambda}d\lambda\right] =$$

$$= \frac{(x+d)^2}{(b+1)^2}\;\frac{\displaystyle\sum_{n=1}^{\infty}\frac{(a+n+1)(a+n)+4(a+n)(b+1)+4(b+1)^2}{(c+n-2)(c+n-1)}T_n}{\displaystyle\sum_{n=1}^{\infty}T_n},$$

and the denominator of P*:

$$\int P\pi_0(\theta,\lambda/x)\,d\theta d\lambda = \int_\theta\int_\lambda \frac{\theta+2}{\lambda}\pi_0(\theta,\lambda/x)\,d\theta d\lambda =$$

$$= \frac{b^a d^c}{\Gamma(a)\Gamma(c)m(x/\pi_0)} \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!n!}\left[\int_\theta(\theta+2)\theta^{a+n-1}e^{-(b+1)\theta}d\theta\right]\left[\int_\lambda \lambda^{c+n-2}e^{-(x+d)\lambda}d\lambda\right]=$$

$$= \frac{x+d}{b+1}\;\frac{\sum_{n=1}^{\infty}\frac{a+2b+n+2}{c+n-1}T_n}{\sum_{n=1}^{\infty}T_n}.$$

In an analogous form, for x=0, we obtain the numerator

$$\int_\theta\int_\lambda \frac{(\theta+2)^2}{\lambda^2}\pi_0(\theta,\lambda/0)\,d\theta d\lambda =$$

$$= \left(\frac{b+1}{b}\right)^a \int_\theta(\theta+2)^2 e^{-\theta}\pi_{10}(\theta)\,d\theta\int_\lambda \frac{1}{\lambda^2}\pi_{01}(\lambda)\,d\lambda =$$

$$= \frac{a^2+4b^2+4ab+5a+8b+4}{(b+1)^2}\frac{d^2}{(c-1)(c-2)},$$

and the denominator of P*:

$$\int_\theta\int_\lambda \frac{\theta+2}{\lambda}\pi_0(\theta,\lambda/0)\,d\theta d\lambda =$$

$$=\left(\frac{b+1}{b}\right)^a \int_\theta(\theta+2)e^{-\theta}\pi_{10}(\theta)\,d\theta\int_\lambda \frac{1}{\lambda}\pi_{01}(\lambda)\,d\lambda=$$

$$= \frac{(a+2b+2)d}{(b+1)(c-1)}.$$

So the expression in the definition of P* is attached by substitution.

■

<u>Example 2.</u>.- Numerical illustration and graphical representation of the a Posteriori Premium.

We continue to consider the *a priori* procedure used in Example 1, such that

$$P^*\left[\pi_0(\theta, \lambda / x)\right] = \begin{cases} \dfrac{(x+5)\sum_{n=1}^{\infty} \dfrac{x^{n-1}(n^2+37n+326)(n+4)(n+3)(n+2)(n+1)^2 n}{(n^2+3n+2)8^n (x+3)^n}}{8\sum_{n=1}^{\infty} \dfrac{(n+3)(n+2)(n+1)^2 n}{8^n (x+3)^n}} & x > 0 \\ 2{,}2638 & x = 0 \end{cases}$$

The following table extracts the values of the a posteriori premium for different values of x, the value of the a priori premium in the mentioned example and its corresponding graph.

(Table 2)
(Figure 2)

4.- ANALYSIS OF ROBUSTNESS

In this section, we examine, independently, the analysis of Bayesian robustness for each of the two parameters θ and λ, and of the likelihood, with respect to the specified *a priori* distribution.

The analysis carried out is based on contamination classes (see Berger (1994); Sivaganesan (1988), (1989), (1991); and Sivaganesan and Berger (1987), (1989)), in which it is assumed that the *a priori* distribution of the parameter, denominated ϕ, belongs to a class of possible distributions of probability defined by the contamination of a singular *a priori* distribution, considering various contaminant classes. Specifically, this approach consists in assuming that a singular *a priori* distribution $\pi(\phi)$ is specified for the parameter ϕ, but that there exists a degree of uncertainty concerning this specification, this uncertainty being quantified by the amount ε; in other words, it can only be specified that the *a priori* distribution of ϕ belongs to a class of probability distributions taking the following form:

$$G_\phi(\pi,\varepsilon) = \left\{\pi^c(\phi) = (1-\varepsilon)\pi(\phi) + \varepsilon q(\phi); q \in Q\right\} \tag{8}$$

where

$\pi(\phi)$ is the singular *a priori* distribution specified for ϕ;

$\varepsilon \in [0,1]$ is the degree of contamination; and

Q is the class of contaminant distributions of probability, the definition of which incorporates non-renounceable aspects of the *a priori* distribution of ϕ.

An extreme case would be: Q_1 ={all the distributions}. Another case we will examine is that of Q_2 = {all the unimodal distributions with the same mode as $\pi(\phi)$ }.

We write $G_\phi^{(i)}(\pi,\varepsilon)$, with i = 1 and 2, to indicate that the contaminant class is Q_i.

The aim of the present study is to analyze the range of variation of the magnitude of interest, which in this case is the *a posteriori* premium $P^*\left[\pi_0(\theta,\lambda/x)\right]$:

- on the one hand, when the *a priori* distribution of λ varies within a class of contamination distributions, for different degrees of contamination, i.e. for different values of ε. The corresponding *a priori* distributions are expressed as

$\pi_0^{2c}(\theta,\lambda)=\pi_{10}(\theta)\pi^c(\lambda)$, with $\pi^c(\lambda)\in G_\lambda^{(i)}(\pi_{01},\varepsilon)$

- on the other hand, when the *a priori* distribution of θ varies within a class of contamination distributions, for different degrees of contamination, i.e. for different values of ε. The corresponding *a priori* distributions are expressed as

$\pi_0^{1c}(\theta,\lambda)=\pi^c(\theta)\pi_{01}(\lambda)$, with $\pi^c(\theta)\in G_\theta^{(i)}(\pi_{10},\varepsilon)$

Throughout the analysis, we maintain the hypothesis that λ and θ are independent. For the purposes of the present study, the following results are useful:

Lemma 4

If A > 0 and f(x) and g(x) are continuous functions with g(x) ≥ 0, then,

$$\sup_{dF(x)}\left[\inf_{dF(x)}\right]\frac{B+\int f(x)\,dF(x)}{A+\int g(x)\,dF(x)}=\sup_{x}\left[\inf_{x}\right]\frac{B+f(x)}{A+g(x)} \qquad (9)$$

where the upper (lower) is taken for all the probability distributions dF(x), and where A, B, f(x), g(x) are such that the upper (lower) of $\frac{B+f(x)}{A+g(x)}$ is obtained for any value of x.

Proof

This result is well known; see, for example, Sivaganesan and Berger (1987).

■

Lemma 5

Let $q(\phi)$ be a unimodal distribution with mode in ϕ_0 and let $h(\phi)$ be a function of ϕ; then

$$\int_\phi h(\phi)q(\phi)\,d\phi=\int_z h^*(z)\,dF(z),$$

where F(z) is a distribution function and $h^*(z)=\begin{cases}\frac{1}{z}\int_{\phi_0}^{\phi_0+z}h(\phi)\,d\phi; & z\neq 0\\ h(\phi_0); & z=0\end{cases}$.

Proof

This result, for unimodal distributions, based on the characterization by Khintchine (see Feller, 1978), is well known, see for example Sivaganesan and Berger (1989).

■

Lemma 6

For any pair of real numbers a and b such that a<b and $\forall n\in Z^+$, the following equality is found:

$$\int_a^b e^{-\lambda x}\lambda^n\,d\lambda=\sum_{k=0}^{n}\frac{n!}{(n-k)!}\frac{1}{x^{k+1}}\left[a^{n-k}e^{-ax}-b^{n-k}e^{-bx}\right].$$

Proof

The proof is obtained by induction, utilizing integration by parts.

■

Lemma 7

For $\pi_0^{2c}(\theta,\lambda)=\pi_{10}(\theta)\pi^c(\lambda)$, with $\pi^c(\lambda)\in G_\lambda^{(i)}(\pi_{01},\varepsilon)$ we have that

$$P^*\left[\pi_0^{2c}(\theta,\lambda/x)\right] = \frac{(1-\varepsilon)C_1+\varepsilon\int\limits_\lambda g_1(\lambda)q(\lambda)d\lambda}{(1-\varepsilon)C_2+\varepsilon\int\limits_\lambda g_2(\lambda)q(\lambda)d\lambda}. \tag{10}$$

where,

$$g_1(\lambda) = \frac{1}{\lambda^2}\int\limits_\theta(\theta+2)^2 L(x/\theta,\lambda)\pi_{10}(\theta)d\theta =$$

$$= \begin{cases} C_{11}\dfrac{1}{\lambda^2}e^{-\lambda x}\left[\sum\limits_{n=1}^{\infty}R_n\lambda^n n^2+C_{13}\sum\limits_{n=1}^{\infty}R_n\lambda^n n+C_{14}\sum\limits_{n=1}^{\infty}R_n\lambda^n\right]; & x>0 \\ C_{15}\dfrac{1}{\lambda^2}; & x=0 \end{cases}; \tag{11}$$

$$g_2(\lambda) = \frac{1}{\lambda}\int\limits_\theta(\theta+2)L(x/\theta,\lambda)\pi_{10}(\theta)d\theta =$$

$$= \begin{cases} C_{21}\dfrac{1}{\lambda}e^{-\lambda x}\left[\sum\limits_{n=1}^{\infty}R_n\lambda^n n+C_{23}\sum\limits_{n=1}^{\infty}R_n\lambda^n\right]; & x>0 \\ C_{24}\dfrac{1}{\lambda}; & x=0 \end{cases}; \tag{12}$$

$$C_1 \equiv C_1(a,b,c,d,x)= \int\limits_\lambda g_1(\lambda)\pi_{01}(\lambda)d\lambda =$$

$$= \begin{cases} C_{12}\left[\sum\limits_{n=1}^{\infty}\dfrac{T_n n^2}{(n+c-1)(n+c-2)}+\sum\limits_{n=1}^{\infty}\dfrac{C_{13}T_n n}{(n+c-1)(n+c-2)}+\sum\limits_{n=1}^{\infty}\dfrac{C_{14}T_n}{(n+c-1)(n+c-2)}\right]; & x>0 \\ \dfrac{b^a\left(a^2+4b^2+4ab+a+8b+4\right)d^2}{(b+1)^{a+2}\left(c^2-3c+2\right)}; & x=0 \end{cases}; \tag{13}$$

$$C_2 \equiv C_2(a,b,c,d,x)= \int\limits_\lambda g_2(\lambda)\pi_{01}(\lambda)d\lambda =$$

$$= \begin{cases} C_{22}\left[\sum\limits_{n=1}^{\infty}\dfrac{T_n n}{n+c-1}+C_{23}\sum\limits_{n=1}^{\infty}\dfrac{T_n}{n+c-1}\right]; & x>0 \\ \dfrac{b^a(a+2b+2)d}{(b+1)^{a+1}(c-1)}; & x=0 \end{cases}; \tag{14}$$

denoting,

$$R_n \equiv R_n(a,b,x) = \frac{x^{n-1}\Gamma(n+a)}{(n-1)!n!(b+1)^n};$$

$$C_{11} \equiv C_{11}(a,b) = \frac{b^a}{(b+1)^{a+2}\Gamma(a)};$$

$$C_{12} \equiv C_{12}(a,b,c,d,x) = \frac{b^a d^c}{\Gamma(a)\Gamma(c)(b+1)^{a+2}(x+d)^{c-2}};$$

$$C_{13} \equiv C_{13}(a,b) = 2a+4b+5;$$

$$C_{14} \equiv C_{14}(a,b) = a^2+4b^2+4ab+5a+8b+4;$$

$$C_{15} \equiv C_{15}(a,b) = \frac{(a^2+4b^2+4ab+5a+8b+4)b^a}{(b+1)^{a+2}}$$

$$C_{21} \equiv C_{21}(a,b) = \frac{b^a}{(b+1)^{a+1}\Gamma(a)};$$

$$C_{22} \equiv C_{22}(a,b,c,d,x) = \frac{b^a d^c}{\Gamma(a)\Gamma(c)(b+1)^{a+1}(x+d)^{c-1}};$$

$$C_{23} \equiv C_{23}(a,b) = a+2b+2.$$

$$C_{24} \equiv C_{24}(a,b) = \frac{b^a(a+2b+2)}{(b+1)^{a+1}}$$

Proof

$$P^*\left[\pi_0^{2c}(\theta,\lambda/x)\right] = \frac{\int P^2\pi_0^{2c}(\theta,\lambda/x)\,d\theta d\lambda}{\int P\pi_0^{2c}(\theta,\lambda/x)\,d\theta d\lambda} = \frac{\int\limits_\theta\int\limits_\lambda \frac{(\theta+2)^2}{\lambda^2}\pi_0^{2c}(\theta,\lambda/x)\,d\theta d\lambda}{\int\limits_\theta\int\limits_\lambda \frac{\theta+2}{\lambda}\pi_0^{2c}(\theta,\lambda/x)\,d\theta d\lambda} =$$

$$= \frac{\int\limits_\theta\int\limits_\lambda \frac{(\theta+2)^2}{\lambda^2}\frac{L(x/\theta,\lambda)\pi_0^{2c}(\theta,\lambda)}{m(x/\pi_0^{2c})}\,d\theta d\lambda}{\int\limits_\theta\int\limits_\lambda \frac{\theta+2}{\lambda}\frac{L(x/\theta,\lambda)\pi_0^{2c}(\theta,\lambda)}{m(x/\pi_0^{2c})}\,d\theta d\lambda} =$$

$$= \frac{\int\limits_{\theta}\int\limits_{\lambda} \frac{(\theta+2)^2}{\lambda^2} L(x/\theta,\lambda)\pi_0^{2c}(\theta,\lambda)\,d\theta d\lambda}{\int\limits_{\theta}\int\limits_{\lambda} \frac{\theta+2}{\lambda} L(x/\theta,\lambda)\pi_0^{2c}(\theta,\lambda)\,d\theta d\lambda} =$$

$$= \frac{\int\limits_{\lambda} \frac{1}{\lambda^2}\pi^c(\lambda)\left[\int\limits_{\theta}(\theta+2)^2 L(x/\theta,\lambda)\pi_{10}(\theta)\,d\theta\right]d\lambda}{\int\limits_{\lambda} \frac{1}{\lambda}\pi^c(\lambda)\left[\int\limits_{\theta}(\theta+2) L(x/\theta,\lambda)\pi_{10}(\theta)\,d\theta\right]d\lambda} =$$

$$= \frac{\int\limits_{\lambda} g_1(\lambda)\pi^c(\lambda)d\lambda}{\int\limits_{\lambda} g_2(\lambda)\pi^c(\lambda)d\lambda} = \frac{\int\limits_{\lambda} g_1(\lambda)\left[(1-\varepsilon)\pi_{01}(\lambda)+\varepsilon q(\lambda)\right]d\lambda}{\int\limits_{\lambda} g_2(\lambda)\left[(1-\varepsilon)\pi_{01}(\lambda)+\varepsilon q(\lambda)\right]d\lambda} =$$

$$= \frac{(1-\varepsilon)\int\limits_{\lambda} g_1(\lambda)\pi_{01}(\lambda)\,d\lambda+\varepsilon\int\limits_{\lambda} g_1(\lambda)q(\lambda)\,d\lambda}{(1-\varepsilon)\int\limits_{\lambda} g_2(\lambda)\pi_{01}(\lambda)\,d\lambda+\varepsilon\int\limits_{\lambda} g_2(\lambda)q(\lambda)\,d\lambda} =$$

$$= \frac{(1-\varepsilon)C_1+\varepsilon\int\limits_{\lambda} g_1(\lambda)q(\lambda)\,d\lambda}{(1-\varepsilon)C_2+\varepsilon\int\limits_{\lambda} g_2(\lambda)q(\lambda)\,d\lambda} .$$

Operating for $g_1(\lambda)$, we get Expression (11).

$$g_1(\lambda) = \frac{1}{\lambda^2}\int\limits_{\theta}(\theta+2)^2 L(x/\theta,\lambda)\pi_{10}(\theta)\,d\theta =$$

$$= \frac{b^a}{\Gamma(a)}e^{-\lambda x}\ \frac{1}{\lambda^2}\ \sum_{n=1}^{\infty}\frac{\lambda^n x^{n-1}}{(n-1)!n!}\int\limits_{\theta}(\theta+2)^2\,\theta^{n+a-1}e^{-(b+1)\theta}d\theta =$$

$$= \frac{b^a}{\Gamma(a)}e^{-\lambda x}\ \frac{1}{\lambda^2}\ \sum_{n=1}^{\infty}\frac{\lambda^n x^{n-1}}{(n-1)!n!}\left[\frac{\Gamma(n+a+2)}{(b+1)^{n+a+2}}+4\frac{\Gamma(n+a+1)}{(b+1)^{n+a+1}}+4\frac{\Gamma(n+a)}{(b+1)^{n+a}}\right];$$

For $g_2(\lambda)$, C_1 and C_2 expressions (12), (13) y (14) are obtained in a similar form to $g_1(\lambda)$.

■

It is straightforward to show that the series in the expressions (11), (12), (13) and (14) are convergent series of positive terms for any positive value of a, b, c, d, x.

Lemma 8

For $\pi_0^{1c}(\theta,\lambda)=\pi^c(\theta)\pi_{01}(\lambda)$, with $\pi^c(\theta)\in G_\theta^{(i)}(\pi_{10},\varepsilon)$ we find that,

$$P*\left[\pi_0^{1c}(\theta,\lambda/x)\right] = \frac{(1-\varepsilon)D_1+\varepsilon\int_\theta h_1(\theta)q(\theta)d\theta}{(1-\varepsilon)D_2+\varepsilon\int_\theta h_2(\theta)q(\theta)d\theta}, \tag{15}$$

where

$$h_1(\theta) = (\theta+2)^2\int_\lambda \frac{1}{\lambda^2}L(x/\theta,\lambda)\pi_{01}(\lambda)d\lambda =$$

$$= \begin{cases} D_{11}e^{-\theta}(\theta+2)^2\left[\sum_{n=1}^{\infty}\theta^n S_n\right]; & x>0 \\ \dfrac{d^2}{c^2-3c+2}(\theta+2)^2 e^{-\theta}; & x=0 \end{cases} \tag{16}$$

$$h_2(\theta) = (\theta+2)\int_\lambda \frac{1}{\lambda}L(x/\theta,\lambda)\pi_{01}(\lambda)d\lambda =$$

$$= \begin{cases} D_{21}(\theta+2)e^{-\theta}\left[\sum_{n=1}^{\infty}(n+c-2)\theta^n S_n\right]; & x>0 \\ \dfrac{d}{c-1}(\theta+2)e^{-\theta}; & x=0 \end{cases}; \tag{17}$$

$$D_1 = \int_\theta h_1(\theta)\pi_{10}(\theta)d\theta =$$

$$= \begin{cases} D_{12}\left[\sum_{n=1}^{\infty}\frac{(n+a+1)(n+a)T_n}{(n+c-1)(n+c-2)}+\sum_{n=1}^{\infty}\frac{D_{13}(n+a)T_n}{(n+c-1)(n+c-2)}+\sum_{n=1}^{\infty}\frac{D_{14}T_n}{(n+c-1)(n+c-2)}\right]; & x>0 \\ \dfrac{b^a d^2\left(a^2+4b^2+4ab+5a+8b+4\right)}{(b+1)^{a+2}\left(c^2-3c+2\right)}; & x=0 \end{cases}; \tag{18}$$

$$D_2 = \int_\theta h_2(\theta)\pi_{10}(\theta)d\theta =$$

$$= \begin{cases} D_{22}\left[\sum_{n=1}^{\infty}\frac{(n+a)T_n}{n+c-1}+D_{23}\sum_{n=1}^{\infty}\frac{T_n}{n+c-1}\right]; & x>0 \\ \dfrac{b^a d(a+2b+2)}{(b+1)^{a+1}(c-1)}; & x=0 \end{cases}; \tag{19}$$

denoting,

$$S_n \equiv S_n(c,d,x) = \frac{x^{n-1}\Gamma(n+c-2)}{(n-1)!n!(x+d)^n}.$$

$$D_{11} \equiv D_{11}(c,d,x) = \frac{d^c}{(x+d)^{c-2}\Gamma(c)};$$

$$D_{12} \equiv D_{12}(a,b,c,d,x) = \frac{b^a d^c}{\Gamma(a)\Gamma(c)(b+1)^{a+2}(x+d)^{c-2}};$$

$$D_{13} \equiv D_{13}(b) = 4(b+1);$$

$$D_{14} \equiv D_{14}(b) = 4(b+1)^2;$$

$$D_{21} \equiv D_{21}(c,d,x) = \frac{d^c}{(x+d)^{c-1}\Gamma(c)};$$

$$D_{22} \equiv D_{22}(a,b,c,d,x) = \frac{b^a d^c}{\Gamma(a)\Gamma(c)(b+1)^{a+1}(x+d)^{c-1}};$$

$$D_{23} \equiv D_{23}(b) = 2(b+1).$$

Proof

The Proof is analogous to Lemma 7, so it is omitted.

■

It is straightforward to show that the series that appear in Expressions (16), (17) and (18) and (19) are convergent series of positive terms for any positive value of a, b, c, d, x.

4.1.- Analysis of robustness for the *a priori* distribution of the parameter 'individual cost of each claim'

The following results show that the range of variation of the *a posteriori* premium P*, when the *a priori* distribution of λ varies within a contamination class as $G_\lambda^{(1)}(\pi_{01},\varepsilon)$, can be obtained by calculating the upper and the lower of a real function of a real variable.

Theorem 1

The range of variation of the *a posteriori* premium when the *a priori* distribution of λ belongs to the class $G_\lambda^{(1)}(\pi_{01},\varepsilon)$ can be calculated by determining the range of variation of a function of λ. Specifically, the following equality is confirmed, and the equality is also valid when the upper is replaced by the lower.

$$\sup_{\pi^c(\lambda)\in G_\lambda^{(1)}(\pi_{01},\varepsilon)}\left[\inf_{\pi^c(\lambda)\in G_\lambda^{(1)}(\pi_{01},\varepsilon)}\right] P^*\left[\pi_0^{2c}(\theta,\lambda/x)\right] = \sup_\lambda\left[\inf_\lambda\right] \frac{(1-\varepsilon)C_1+\varepsilon g_1(\lambda)}{(1-\varepsilon)C_2+\varepsilon g_2(\lambda)}, \quad (20)$$

where, $g_1(\lambda)$, $g_2(\lambda)$, C_1 and C_2 are as in Lemma 7.

Proof
The proof is obtained by using, successively, Lemma 7 and Lemma 4.

■

Example 3..- Numerical illustration of Theorem 1
Here it is the calculation of the range of variation of the *a posteriori* premium when the *a priori* distribution of the parameter 'distribution of the severity of the accident' belongs to a class of contamination in which the contaminant class is that of all the probability distributions. In this example, we use the *a priori* data derived in Examples 1 and 2 to illustrate the result of Theorem 1.

(Table 3)

We now address the analysis of the robustness for the contamination class $G_\lambda^{(2)}(\pi_{01},\varepsilon)$, and obviously in this case it must be assumed that 'c' is greater than 1 and that the mode is $\lambda_0 = \frac{c-1}{d}$. The following result shows that the problem of searching for the upper and lower of the *a posteriori* premium when the *a priori* distribution of λ belongs to the class $G_\lambda^{(2)}(\pi_{01},\varepsilon)$ can be transformed into the search for the upper and lower, respectively, of a real function of a real variable.

Theorem 2
The range of variation of the *a posteriori* premium when the *a priori* distribution of λ belongs to the class $G_\lambda^{(2)}(\pi_{01},\varepsilon)$ can be calculated by determining the range of variation of a function of a real variable. Specifically, the following equality is confirmed, and the equality is also valid when the upper is replaced by the lower.

$$\sup_{\pi^c(\lambda)\in G_\lambda^{(2)}(\pi_{01},\varepsilon)} P^*\left[\pi^{2c}(\theta,\lambda/x)\right] = \sup_z \begin{cases} \dfrac{\frac{1-\varepsilon}{\varepsilon}C_1+g_1^*(z)}{\frac{1-\varepsilon}{\varepsilon}C_2+g_2^*(z)}; & x>0; \\ g_3^*(z); & x=0; \end{cases}, \quad (21)$$

where C_1, C_2, and C_{11}, C_{13}, C_{14}, C_{15}, C_{21}, C_{23}, C_{24}, $g_1(\lambda)$, $g_2(\lambda)$ y R_n are as in Lemma 7, and $g_1^*(z)$, $g_2^*(z)$ and $g_3^*(z)$ are given by,

$g_1^*(z) =$

$$
= \begin{cases} \dfrac{C_{11}\left(1+C_{13}+C_{14}\right)R_1 \int\limits_{\lambda_0}^{\lambda_0+z} \dfrac{e^{-\lambda x}}{\lambda} d\lambda}{z} + \left(4+2C_{13}+C_{14}\right)R_2 + \\ + \dfrac{C_{11}\sum\limits_{n=1}^{\infty}\left[\left(n+2\right)^2 + C_{13}\left(n+2\right)+C_{14}\right] n! R_{n+2}\left[U_n\left(x,\lambda_0\right)-V_n\left(x,\lambda_0,z\right)\right]}{z}; & z \neq 0; \\ g_1\left(\lambda_0\right); & z = 0; \end{cases} \quad (22)
$$

$$
g_2^*\left(z\right) =
$$

$$
\begin{cases} \dfrac{C_{21}\left(1+C_{23}\right)R_1}{x}\dfrac{e^{-\lambda_0 x}-e^{-\left(\lambda_0+z\right)x}}{z} + \dfrac{1}{z}C_{21}\sum\limits_{n=1}^{\infty}\left(n+1+C_{23}\right)n!R_{n+1}\left[U_n\left(x,\lambda_0\right)-V_n\left(x,\lambda_0,z\right)\right]; & z \neq 0; \\ g_2\left(\lambda_0\right); & z = 0; \end{cases}
$$

(23)

$$
g_3^*\left(z\right) = \begin{cases} \dfrac{\dfrac{1-\varepsilon}{\varepsilon}C_1 + C_{15}\dfrac{1}{\lambda_0\left(\lambda_0+z\right)}}{\dfrac{1-\varepsilon}{\varepsilon}C_2 + C_{24}\dfrac{1}{z}\ln\left(\dfrac{\lambda_0+z}{\lambda_0}\right)}; & z \neq 0; \\ \dfrac{\dfrac{1-\varepsilon}{\varepsilon}C_1 + \dfrac{C_{15}}{\lambda_0^2}}{\dfrac{1-\varepsilon}{\varepsilon}C_2 + \dfrac{C_{24}}{\lambda_0}}; & z = 0; \end{cases} ; \quad (24)
$$

using the notation,

$$
U_n\left(x,\lambda_0\right) = \frac{e^{-\lambda_0 x}}{x}\sum_{k=0}^{n}\frac{\lambda_0^{n-k}}{\left(n-k\right)!}\frac{1}{x^k} ;
$$

$$
V_n\left(x,\lambda_0,z\right) = \frac{e^{-\left(\lambda_0+z\right)x}}{x}\sum_{k=0}^{n}\frac{\left(\lambda_0+z\right)^{n-k}}{\left(n-k\right)!}\frac{1}{x^k} .
$$

<u>Proof</u>

From Lemma 7 we obtain

$$
\sup_{\pi^c(\lambda)\in G_\lambda^{(2)}(\pi_{01},\varepsilon)} P^*\left[\pi_0^{2c}\left(\theta,\lambda/x\right)\right] =
$$

$$= \sup_{q(\lambda)\in Q_2} \begin{cases} \dfrac{\frac{1-\varepsilon}{\varepsilon}C_1 + \int_\lambda g_1(\lambda)q(\lambda)d\lambda}{\frac{1-\varepsilon}{\varepsilon}C_2 + \int_\lambda g_2(\lambda)q(\lambda)d\lambda}; & x > 0; \\ \dfrac{\frac{1-\varepsilon}{\varepsilon}C_1 + C_{15}\int_\lambda \frac{1}{\lambda^2}q(\lambda)d\lambda}{\frac{1-\varepsilon}{\varepsilon}C_2 + C_{24}\int_\lambda \frac{1}{\lambda}q(\lambda)d\lambda}; & x = 0; \end{cases}$$

<u>Let us first consider the case in which x=0:</u>

We shall use Lemma 5 in the numerator and in the denominator from the second row of the above expression. Therefore, for $q(\lambda)\in Q_2$,

$$\int_\lambda \frac{1}{\lambda^2}q(\lambda)d\lambda = \begin{cases} \int_z \frac{1}{z}\left[\int_{\lambda_0}^{\lambda_0+z}\frac{1}{\lambda^2}d\lambda\right]dF(z); & z \neq 0; \\ \frac{1}{\lambda_0^2} \equiv \frac{d^2}{(c-1)^2}; & z = 0; \end{cases}$$

$$\int_\lambda \frac{1}{\lambda}q(\lambda)d\lambda = \begin{cases} \int_z \frac{1}{z}\left[\int_{\lambda_0}^{\lambda_0+z}\frac{1}{\lambda}d\lambda\right]dF(z); & z \neq 0; \\ \frac{1}{\lambda_0} \equiv \frac{d}{(c-1)}; & z = 0; \end{cases};$$

By applying Lemma 4, the result is obtained.

<u>Let us now consider the case in which x ≠ 0</u>

Again using Lemma 5 in the numerator and in the denominator from the first row of the above expression. Therefore, we obtain for $q(\lambda)\in Q_2$ and $z \neq 0$

$$\int_\lambda g_1(\lambda)q(\lambda)d\lambda = \int_z g_1^*(z)dF(z);\ \int_\lambda g_2(\lambda)q(\lambda)d\lambda = \int_z g_1^*(z)dF(z);$$

where,

$$g_1^*(z) = \frac{1}{z}\int_{\lambda_0}^{\lambda_0+z} g_1(\lambda)d\lambda =$$

$$= C_{11}\frac{1}{z}\int_{\lambda_0}^{\lambda_0+z} e^{-\lambda x}\frac{1}{\lambda^2}\sum_{n=1}^{\infty}\left(R_n n^2 + C_{13}R_n n + C_{14}R_n\right)\lambda^n d\lambda =$$

$$= \left\{ \begin{array}{l} C_{11} \frac{1}{z} \int_{\lambda_0}^{\lambda_0+z} e^{-\lambda x} \left\{ R_1 (1 + C_{13} + C_{14}) \frac{1}{\lambda} + R_2 (4 + 2C_{13} + C_{14}) + \right. \\ \left. \qquad + \sum_{n=1}^{\infty} R_{n+2} \left[(n+2)^2 + C_{13}(n+2) + C_{14} \right] \lambda^n \right\} d\lambda \end{array} \right\} =$$

$$= C_{11} R_1 (1 + C_{13} + C_{14}) \frac{1}{z} \int_{\lambda_0}^{\lambda_0+z} \frac{e^{-\lambda x}}{\lambda} d\lambda \ + \ C_{11} R_2 (4 + 2C_{13} + C_{14}) \frac{e^{-\lambda_0 x} - e^{-(\lambda_0+z)x}}{zx} \ +$$

$$+ \ C_{11} \frac{1}{z} \sum_{n=1}^{\infty} R_{n+2} \left[(n+2)^2 + C_{13}(n+2) + C_{14} \right] \int_{\lambda_0}^{\lambda_0+z} e^{-\lambda x} \lambda^n d\lambda \ =$$

$$= C_{11} R_1 (1 + C_{13} + C_{14}) \frac{1}{z} \int_{\lambda_0}^{\lambda_0+z} \frac{e^{-\lambda x}}{\lambda} d\lambda \ + \ C_{11} R_2 (4 + 2C_{13} + C_{14}) \frac{e^{-\lambda_0 x} - e^{-(\lambda_0+z)x}}{zx} \ +$$

$$+ \ C_{11} \frac{1}{z} \sum_{n=1}^{\infty} R_{n+2} \left[(n+2)^2 + C_{13}(n+2) + C_{14} \right] n! \left[U_n(x, \lambda_0) - V_n(x, \lambda_0, z) \right]$$

using Lemma 6 for the last equality.

$$g_2^*(z) = \frac{1}{z} \int_{\lambda_0}^{\lambda_0+z} g_2(\lambda) d\lambda = \frac{1}{z} \int_{\lambda_0}^{\lambda_0+z} C_{21} \frac{e^{-\lambda x}}{\lambda} \sum_{n=1}^{\infty} R_n (n + C_{23}) \lambda^n d\lambda =$$

$$= C_{21} \frac{1}{z} \int_{\lambda_0}^{\lambda_0+z} e^{-\lambda x} \left\{ R_1 (1 + C_{23}) + \sum_{n=1}^{\infty} R_{n+1} (n + 1 + C_{23}) \lambda^n \right\} d\lambda =$$

$$= \frac{C_{21} R_1 (1 + C_{23})}{x} \frac{\left[e^{-\lambda_0 x} - e^{-(\lambda_0+z)x} \right]}{z} +$$

$$+ \ C_{21} \frac{1}{z} \sum_{n=1}^{\infty} R_{n+1} (n + 1 + C_{23}) \int_{\lambda_0}^{\lambda_0+z} e^{-\lambda x} \lambda^n d\lambda \ =$$

$$= \frac{C_{21} R_1 (1 + C_{23})}{x} \frac{\left[e^{-\lambda_0 x} - e^{-(\lambda_0+z)x} \right]}{z} +$$

$$+ \ C_{21} \frac{1}{z} \sum_{n=1}^{\infty} n! R_{n+1} (n + 1 + C_{23}) \left[U_n(x, \lambda_0) - V_n(x, \lambda_0, z) \right]$$

■

It is straightforward to show, using Theorem of Mertens that the series that appear in Expressions (16), (17) and (18) and (19) are convergent series of positive terms for any positive value of a, b, c, d, x, λ_0 and z

Example 4.- Numerical illustration of Theorem 2.
Calculation of the range of variation of the *a posteriori* premium when the *a priori* distribution of the parameter 'distribution of the severity of the accident' belongs to a contamination class in which the contaminant class is that of all the unimodal probability distributions with the same mode. The data elicited are the same as in the previous examples.

In order to measure the Bayesian sensitivity, or the robustness of the intervals calculated, we use a normalized measure of relative sensitivity, defined in Sivaganesan (1991), the R.S. sensitivity factor, which is expressed as:

$$RS = \frac{\left(Sup\left(P^*\right) - Inf\left(P^*\right)\right)}{2P^*} \times 100.$$

(Table 4)

4.2.- Robustness analysis for the *a priori* distribution of the parameter 'number of claims'

In this section, we analyze the Bayesian robustness for the parameter θ of likelihood, with respect to the specified *a priori* distribution. As in the previous section, the analysis carried out is based on contamination classes.

In the following result, parallel to Theorem 1, it is apparent that the problem of searching for the upper and the lower of the *a posteriori* premium when the *a priori* distribution of θ belongs to the class $G_\theta^{(1)}\left(\pi_{10}, \varepsilon\right)$ can be transformed into the search for the upper and the lower, respectively, of a real function of the real variable θ.

Theorem 3
The range of variation of the *a posteriori* premium when the *a priori* distribution of θ belongs to the class $G_\theta^{(1)}\left(\pi_{10}, \varepsilon\right)$ can be calculated by determining the range of variation of a function of θ. Specifically, the following equality is found, which is also valid when the upper is replaced by the lower.

$$\sup_{\pi(\theta)\in G_\theta^{(1)}(\varepsilon)} \left[\inf_{\pi(\theta)\in G_\theta^{(1)}(\varepsilon)} \right] P^*\left[\pi_0^{1c}\left(\theta, \lambda / x\right)\right] = \sup_\theta \left[\inf_\theta \right] \frac{\left(1-\varepsilon\right) D_1 + \varepsilon h_1\left(\theta\right)}{\left(1-\varepsilon\right) D_2 + \varepsilon h_2\left(\theta\right)}, \qquad (25)$$

where, $h_1\left(\theta\right)$, $h_2\left(\theta\right)$, D_1 and D_2 are as in Lemma 8.

Proof
The proof is obtained by applying, successively, Lemma 8 and Lemma 4.

■

Example 5.- Numerical illustration of Theorem 3.

Here we present the calculation of the range of variation of the *a posteriori* premium when the *a priori* distribution of the parameter 'number of accidents' belongs to a class of contamination in which the contaminant class is that of all the probability distributions. Calculation of minima, maxima and R.S. sensitivity factors, taking the same *a priori* assumptions as in the previous examples, i.e. that the *a priori* parameters are *a=2*, *b=7*, *c=5*, *d=3* , and applying Theorem 3.

(Table 5)

Theorem 4

The range of variation of the *a posteriori* premium when the *a priori* distribution of θ belongs to the class $G_\theta^{(2)}(\pi_{10},\varepsilon)$ can be calculated by determining the range of variation of a real function of a real variable. Specifically, the following equality is confirmed, and this is equally valid when the upper is replaced by the lower.

$$\sup_{\pi^c(\theta)\in G_\theta^{(2)}(\pi_{10},\varepsilon)} P*\left[\pi^{1c}\left(\theta,\lambda/x\right)\right] = \sup_z \begin{cases} \dfrac{\frac{1-\varepsilon}{\varepsilon}D_1+h_1^*(z)}{\frac{1-\varepsilon}{\varepsilon}D_2+h_2^*(z)}; & x>0; \\ h_3^*(z); & x=0; \end{cases}, \tag{26}$$

where D_1, D_2, D_{11}, D_{21}, $h_1(\theta)$, $h_2(\theta)$ and S_n are as in Lemma 8. U_n and V_n are as in Theorem 2 and $h_1^*(z)$, $h_2^*(z)$ and $h_3^*(z)$ are given by,

$$h_1^*(z) = \begin{cases} \dfrac{D_{11}}{z}\displaystyle\sum_{n=1}^{\infty} S_n\Big\{(n+2)!\left[U_{n+2}(1,\theta_0)-V_{n+2}(1,\theta_0,z)\right]+ \\ 4(n+1)!\left[U_{n+1}(1,\theta_0)-V_{n+1}(1,\theta_0,z)\right]+4n!\left[U_n(1,\theta_0)-V_n(1,\theta_0,z)\right]\Big\}; & z\neq 0; \\ h_1(\theta_0); & z=0; \end{cases} \tag{27}$$

$$h_2^*(z) = \begin{cases} \dfrac{D_{21}}{z}\displaystyle\sum_{n=1}^{\infty}(n+c-2)S_n\Big\{(n+1)!\left[U_{n+1}(1,\theta_0)-V_{n+1}(1,\theta_0,z)\right]+ \\ +2n!\left[U_n(1,\theta_0)-V_n(1,\theta_0,z)\right]\Big\}; & z\neq 0; \\ h_2(\theta_0); & z=0; \end{cases} \tag{28}$$

$$h_3^*(z) = \begin{cases} \dfrac{\dfrac{1-\varepsilon}{\varepsilon} D_1 + h_4(z)}{\dfrac{1-\varepsilon}{\varepsilon} D_2 + h_5(z)}; & z \neq 0; \\ \dfrac{\dfrac{1-\varepsilon}{\varepsilon} D_1 + \dfrac{d^2 (\theta_0 + 2)^2 e^{-\theta_0}}{c^2 - 3c + 2}}{\dfrac{1-\varepsilon}{\varepsilon} D_2 + \dfrac{d(\theta_0 + 2) e^{-\theta_0}}{c-1}}; & z = 0; \end{cases};$$

using the notation,

$$h_4(z) = \frac{d^2 \left[e^{-\theta_0} \left(\theta_0^2 + 6\theta_0 + 10 \right) - e^{-(\theta_0 + z)} \left((\theta_0 + z)^2 + 6\theta_0 + 6z + 10 \right) \right]}{\left(c^2 - 3c + 2 \right) z};$$

$$h_5(z) = \frac{d \left[(\theta_0 + 3) e^{-\theta_0} - (\theta_0 + z + 3) e^{-(\theta_0 + z)} \right]}{(c-1) z}.$$

Proof

Let us first consider the case in which x = 0.
We shall use Lemma 5 in the numerator, and the denominator from the second row of the previous expression. Thus, for $q(\lambda) \in Q_2$,

$$\int_\theta \frac{d^2}{c^2 - 3c + 2} (\theta + 2)^2 e^{-\theta} q(\theta) d\theta =$$

$$= \begin{cases} \displaystyle\int_z \frac{1}{z} \left[\int_{\theta_0}^{\theta_0 + z} \frac{d^2}{c^2 - 3c + 2} (\theta + 2)^2 e^{-\theta} d\theta \right] dF(z); & z \neq 0; \\ h_1(\theta_0) = \dfrac{d^2}{c^2 - 3c + 2} (\theta_0 + 2)^2 e^{-\theta_0}; & z = 0; \end{cases}$$

$$\int_\theta \frac{d}{c-1} (\theta + 2) e^{-\theta} q(\theta) d\theta = \begin{cases} \displaystyle\int_z \frac{1}{z} \left[\int_{\theta_0}^{\theta_0 + z} \frac{d}{c-1} (\theta + 2) e^{-\theta} d\theta \right] dF(z); & z \neq 0; \\ \dfrac{d}{c-1} (\theta_0 + 2) e^{-\theta_0}; & z = 0; \end{cases};$$

By using Lemma 4, it is straightforward to obtain the result.

Let us now consider the case in which x ≠ 0:

By once again using Lemma 5, and taking into account that, for $z \neq 0$ and $q(\theta) \in Q_2$ we have

$$\int_{\theta} h_1(\theta) q(\theta) d\theta = \int_{z} h_1^*(z) dF(z); \quad \int_{\theta} h_2(\theta) q(\theta) d\theta = \int_{z} h_2^*(z) dF(z);$$

where,

$$h_1^*(z) = \frac{1}{z} \int_{\theta_0}^{\theta_0+z} h_1(\theta) d\theta =$$

$$= D_{11} \frac{1}{z} \int_{\theta_0}^{\theta_0+z} e^{-\theta} (\theta+2)^2 \left[\sum_{n=1}^{\infty} S_n \theta^n \right] d\theta =$$

$$= D_{11} \frac{1}{z} \sum_{n=1}^{\infty} S_n \left[\int_{\theta_0}^{\theta_0+z} e^{-\theta} \theta^{n+2} d\theta + 4 \int_{\theta_0}^{\theta_0+z} e^{-\theta} \theta^{n+1} d\theta + 4 \int_{\theta_0}^{\theta_0+z} e^{-\theta} \theta^{n} d\theta \right]$$

$$= D_{11} \frac{1}{z} \sum_{n=1}^{\infty} S_n \left\{ (n+2)! \left[U_{n+2}(1,\theta_0) - V_{n+2}(1,\theta_0,z) \right] + \right.$$

$$\left. + \, 4(n+1)! \left[U_{n+1}(1,\theta_0) - V_{n+1}(1,\theta_0,z) \right] + 4n! \left[U_n(1,\theta_0) - V_n(1,\theta_0,z) \right] \right\}$$

using Lemma 6 for the last equality.

$$h_2^*(z) = \frac{1}{z} \int_{\theta_0}^{\theta_0+z} h_2(\theta) d\theta =$$

$$= D_{21} \frac{1}{z} \int_{\theta_0}^{\theta_0+z} (\theta+2) e^{-\theta} \left[\sum_{n=1}^{\infty} S_n (n+c-2) \theta^n \right] d\theta =$$

$$= D_{21} \frac{1}{z} \sum_{n=1}^{\infty} (n+c-2) S_n \left[\int_{\theta_0}^{\theta_0+z} e^{-\theta} \theta^{n+1} d\theta + 2 \int_{\theta_0}^{\theta_0+z} e^{-\theta} \theta^{n} d\theta \right] =$$

$$= D_{21} \frac{1}{z} \sum_{n=1}^{\infty} (n+c-2) S_n \left\{ (n+1)! \left[U_{n+1}(1,\theta_0) - V_{n+1}(1,\theta_0,z) \right] + \right.$$

$$\left. + 2n! \left[U_n(1,\theta_0) - V_n(1,\theta_0,z) \right] \right\}$$

using Lemma 6 for the last equality.

■

It is straightforward to show, using Theorem of Mertens that the series that appear in Expressions (16), (17) and (18) and (19) are convergent series of positive terms for any positive value of a, b, c, d, x, θ_0 and z.

Example 6.- Numerical illustration of Theorem 4.
Calculation of the range of variation of the *a posteriori* premium when the *a priori* distribution of the parameter 'number of accidents' belongs to a class of contamination in which the contaminant class is that of all the unimodal distributions with the same mode. Calculation of the minima, maxima and R.S. sensitivity factors, taking into account the same *a priori* assumptions as in the above examples, i.e. that the *a priori* parameters are *a=2*, *b=7*, *c=5*, *d=3* , and then applying Theorem 4.

(Table 6)

Now we represent the ranges of variation of the premium and R.S sensitivity factor in the case when x gets the value 0.5.

(Figure 3)
(Figure 4)

5.- FINALS REMARKS AND FURTHER LINES OF RESEARCH

Computing the complete Bayes premium in the collective risk model requires two prior distributions for both, the parameter of the distribution of the number of claims and the parameter of the distribution of the single claim size. Since the incorporation of these prior distributions can be criticized by the fact that the practitioner perhaps does not know them totally, a robustness study has been carried out in this paper. Due to this procedure, new results were obtained that let us to study in deep the influence on the Bayes premium by assuming single prior distributions for those parameters.

This paper leaves some other aspects open to question, which could be the subject of future study. First, since actual experience shows that the distribution of claim counts tend to have greater variance than the mean, i.e. tends to be overdispersed, the negative binomial distribution, $NB(r,\theta)$, has been proposed as a model preferable to the Poisson in Actuarial Science. Then it would be convenient to model it from a Bayesian point of view assuming a Beta prior distribution for the parameter θ, which results conjugate with respect to the negative binomial. In this case (see Rolski et al., 1999) the aggregate claim amount X is expressed in terms of the generalized Laguerre polynomial.

Second, an extension of the study proposed here would be made by considering more than one period of observation. This, perhaps, would let us to obtain Bayes credibility premiums which plays an important role known in Actuarial Science as credibility theory. This study, of course, seems to be very complicated, involving in the likelihood the product of a sequence of Bessel functions although some ideas to undertake the problem can be view in Linz (1972) and Linz and Kropp (1973).

Finally, the hypothesis of independence between θ and λ can be very restrictive. Therefore, it would be convenient to choose a bivariate distribution for both parameters, by assuming some dependence between them. A bivariate distribution with given marginals would be incorporated by taking the Sarmanov family of bivariate distribution (see Lee, 1996) or the Farlie-Gumbel Morgenstern family (see Johnson and Kotz, 1975).

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TABLES

Table 1: Marginal distribution for different values of x

x	0.1	0,5	1	2	3	4
$m(x / \pi_0)$	0,27173	0,1486	0,07532	0,0234	0,00878	0,00377

Table 2: the a priori premium and the a posteriori premium for different values of x.

x	0.1	0,5	1	2	3	4
P'	2,3	2,3	2,3	2,3	2,3	2,3
$P^*[\pi_0(\theta, \lambda / x)]$	1,8467	2,0499	2,3047	2,8161	3,3288	3,8424

Table 3: Minima, maxima and sensitivity factors. Theorema 1

	x	**0**	**0.5**	**1**	**1.5**	**2**
ε	**P***	2.2638	2.0499	2.3047	2.5602	2.8161
0.05	**Min.**	2.1718	2.012	2.2677	2.5189	2.7698
	Max.	∞	∞	∞	∞	∞
0.1	**Min.**	2.1338	1.9734	2.2299	2.4771	2.719
	Max.	∞	∞	∞	∞	∞
0.15	**Min.**	2.0944	1.9334	2.1901	2.4347	2.6705
	Max.	∞	∞	∞	∞	∞
0.2	**Min.**	2.5316	1.8919	2.1509	2.3915	2.6218
	Max.	∞	∞	∞	∞	∞
0.25	**Min.**	2.010	1.8489	2.196	2.3473	2.5729
	Max.	∞	∞	∞	∞	∞

Table 4. Minima, maxima and sensitivity factors. Theorem 2.

	x	**0**	**0.5**	**1**	**1.5**	**2**
ε	**P***	2.2638	2.0499	2.3047	2.5602	2.8161
0.05	**Min.**	2.1737	0.2659	0.1553	0.0905	0.0527
	Max.	2.2638	10.276	16.7137	26.436	40.6108
	RS.	1.99%	244%	359.23%	475.46%	720.11%
0.1	**Min.**	2.1737	0.2659	0.1553	0.0905	0.0527
	Max.	2.2638	19.4159	32.7233	47.4108	82.6045
	RS.	1.99%	467.09%	706.56%	924.15%	1465.71%
0.15	**Min.**	2.1737	0.2659	0.1553	0.0905	0.0527
	Max.	2.2638	29.6307	50.6162	82.6134	129.538
	RS.	1.99%	716.29%	1094.74%	1611.65%	2299.02%
0.2	**Min.**	2.1737	0.2659	0.1553	0.0905	0.0527
	Max.	2.2638	41.1219	70.7452	115.968	182.338
	RS.	1.99%	996.54%	1531.43%	2236.06%	3236.48%
0.25	**Min.**	2.1737	0.2659	0.1553	0.0905	0.0527
	Max.	2.2638	54.1447	93.5577	153.77	242.178
	RS.	1.99%	1314.18%	2026.35%	3000.32%	4298.95%

Table 5. Minima, maxima and sensitivity factors. Theorem 3.

	x	**0**	**0.5**	**1**	**1.5**	**2**
ε	**P***	2.2638	2.0499	2.3047	2.5602	2.8161
0.05	**Min.**	2.2487	2.0490	2.2491	2.5546	2.8106
	Max.	2.2929	2.3456	2.8258	3.3996	4.0561
	RS.	0.9762%	7.2345%	12.5139%	16.5026%	22.1139%
0.1	**Min.**	2.2337	2.0389	2.2933	2.5489	2.8047
	Max.	2.3239	2.5921	3.1971	3.8914	4.6375
	RS.	1.9922%	13.4933%	12.6077%	26.2185%	32.4725%
0.15	**Min.**	2.219	2.0323	2.2874	2.5429	2.7987
	Max.	2.3569	2.8051	3.4798	4.2331	5.0042
	RS.	3.0457%	18.8497%	25.8687%	33.0091%	39.1588%
0.2	**Min.**	2.2045	2.0261	2.2813	2.5368	2.7935
	Max.	2.3923	2.9944	3.7214	4.4949	5.2747

	RS.	4.1479%	23.6182%	31.2427%	38.2411%	44.0538%
0.25	**Min.**	2.1903	2.0196	2.2749	2.5304	2.7861
	Max.	2.4304	3.1665	3.9239	4.7085	5.4806
	RS.	5.3030%	27.9745%	35.7747%	42.5377%	47.841%

Table 6. Minima, maxima and sensitivity factors. Theorem 4.

	x	**0**	**0.5**	**1**	**1.5**	**2**
ε	**P***	2.2638	2.0499	2.3047	2.5602	2.8161
0.05	**Min.**	2.2574	0.2397	0.1449	0.0926	0.0619
	Max.	2.2858	2.2536	2.6649	3.1466	3.6985
	RS.	1.28%	49.12%	54.67%	59.64%	64.57%
0.1	**Min.**	2.2509	0.2397	0.1449	0.0926	0.0619
	Max.	2.3088	2.4261	2.9312	3.5114	4.1455
	RS.	1.89%	53.33%	60.45%	66.77%	72.5%
0.15	**Min.**	2.2445	0.2397	0.1449	0.0926	0.0619
	Max.	2.3303	2.5751	3.3064	3.9535	4.4229
	RS.	2.3586%	56.96%	68.59%	75.4%	77.43%
0.2	**Min.**	2.2382	0.2397	0.1449	0.0926	0.0619
	Max.	2.3586	2.7060	3.7214	4.4949	4.6150
	RS.	2.66%	60.16%	77.59%	85.97%	80.84%
0.25	**Min.**	2.2319	0.2397	0.1449	0.0926	0.0619
	Max.	2.3858	2.8227	3.4462	4.1012	4.7576
	RS.	3.399%	63%	71.62%	78.29%	83.37%

FIGURES

Figure 1. Marginal Distribution

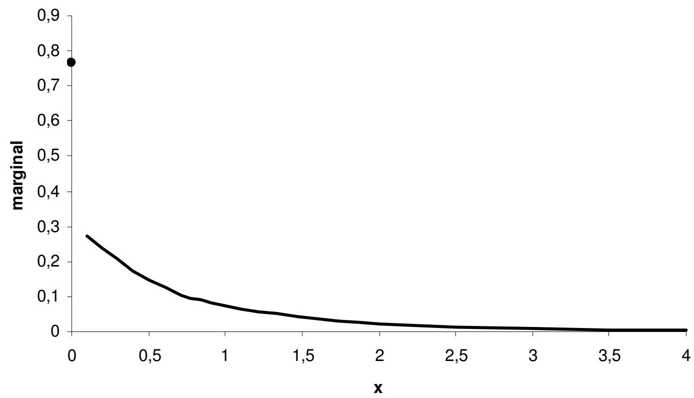


Figure 2.The a Posteriori and a Priori Premium.

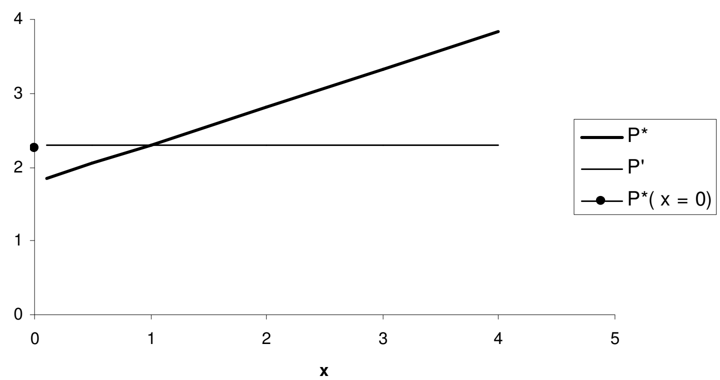
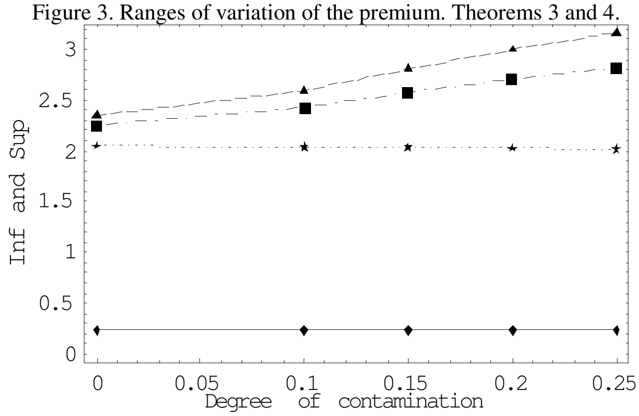
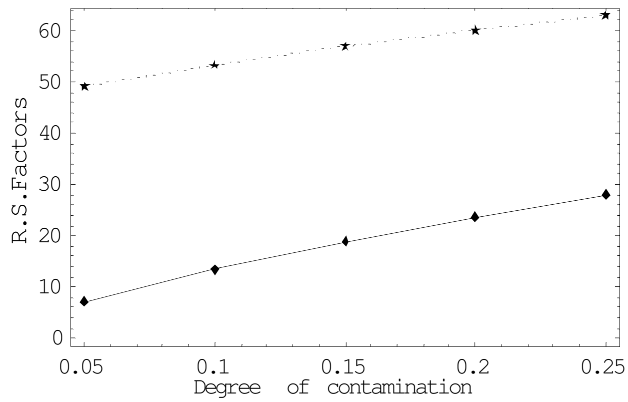


Figure 3. Ranges of variation of the premium. Theorems 3 and 4.



Stars and Triangles: all distributions. Rhombus and squares: unimodals distributions

Figure 4. Ranges of variation of R.S. sensitivity factor. Theorems 3 and 4.



Stars: unimodals distributions. Rhombus: all distributions.