TRE
UNIVERSITÀ DEGLI STUDI

DIPARTIMENTO DI ECONOMIA

EFFICIENCY AND PRICES IN ECONOMIES OF OVERLAPPING GENERATIONS

Gaetano Bloise

Working Paper n° 72, 2007

- I “Working Papers” del Dipartimento di Economia svolgono la funzione di divulgare tempestivamente, in forma definitiva o provvisoria, i risultati di ricerche scientifiche originali. La loro pubblicazione è soggetta all’approvazione del Comitato Scientifico.

- Per ciascuna pubblicazione vengono soddisfatti gli obblighi previsti dall’art. 1 del D.L.L. 31.8.1945, n. 660 e successive modifiche.

- Copie della presente pubblicazione possono essere richieste alla Redazione.

REDAZIONE:
Dipartimento di Economia
Università degli Studi Roma Tre
Via Silvio D'Amico, 77 - 00145 Roma
Tel. 0039-06-574114655 fax 0039-06-574114771
E-mail: dip_eco@uniroma3.it

DIPARTIMENTO DI ECONOMIA

EFFICIENCY AND PRICES IN ECONOMIES OF OVERLAPPING GENERATIONS

Gaetano Bloise*

**Dipartimento di Economia, Università degli Studi "Roma Tre"*

EFFICIENCY AND PRICES
IN ECONOMIES OF OVERLAPPING GENERATIONS

GAETANO BLOISE

Department of Economics, University of Rome III
139 Via Ostiense, I-00154 Roma (Italy)
E-mail: gaetano.bloise@uniroma3.it

Current version: September 13, 2006

Abstract. In a general economy of overlapping generations, I introduce a notion of uniform inefficiency, corresponding to the occurrence of a Pareto improvement with a small uniform destruction of resources (Debreu [11]). I provide necessary and sufficient conditions for uniform inefficiency in terms of competitive equilibrium prices. Minimal assumptions are needed for such a complete characterization; moreover, proofs reduce to simple and short direct arguments. Finally, I verify that uniform inefficiency is preserved under perturbations of the endowments, a property that has not been established for the canonical notion of inefficiency. Remarkably, an allocation is uniformly inefficient if and only if a non-vanishing canonical social security mechanism is welfare improving.
Keywords. Overlapping generations; efficiency; competitive prices; Cass Criterion; social security.
JEL Classification Numbers. D52, D61.

1. Introduction

In this paper, I introduce a notion of uniform inefficiency corresponding to the presence of a welfare improvement with any small uniform destruction of available resources (Debreu [11]). In a general economy of overlapping generations, I provide an equivalent characterization of uniform inefficiency in terms of competitive equilibrium prices. In particular, for nearly stationary competitive equilibria, uniform inefficiency occurs if and only if the implicit real rate of interest is negative in the long-run.

Several pieces of work in the literature provide conditions for efficiency in terms of equilibrium prices in economies of overlapping generations, inspired by the studies of Cass [9] and Benveniste [4, 5] on capital theory. The initial characterizations of Balasko and Shell [2] and Okuno and Zilcha [14] for canonical two-period overlapping generations economies were extended by Geanakoplos and Polemarchakis [12] to growing (or declining) economies, by Chattopadhyay and Gottardi [10] to economies with uncertainty and by Burke [6] and Molina-Abraldes and Pintos-Clapés [13] to economies with heterogeneous horizons for generations. Furthermore, in a related paper, Richard and Srivastava [15] propose a pure duality approach to economies with the double infinity of individuals and commodities, clarifying its inadequacy for economies of overlapping generations.

To the purpose of comparison, I shall briefly present the crucial elements of the characterization established in the literature omitting irrelevant details. In the simplest framework, an equilibrium allocation is Pareto

I am grateful to Herakles Polemarchakis, Pietro Reichlin and Paolo Siconolfi for their valuable suggestions and comments. I also thank participants to the 2004 PRIN Workshop held in Alghero in June 2006. Remaining errors, omissions and misunderstandings are my own responsibility.

efficient if and only if the so-called Cass Criterion holds in terms of equilibrium prices. More precisely, at a competitive equilibrium,[1]

$$\text{inefficiency of allocation if and only if } \sum_{t=0}^{\infty} \frac{1}{\|p_t\|} \text{ is finite.}$$

Such an equivalence obtains under rather technical restrictions on preferences referred to as conditions of non-vanishing curvature and bounded curvature of indifference curves. These assumptions, which are substantially innocuous over a single individual, are to be satisfied uniformly across individuals of all generations.

The notion of uniform inefficiency introduced in this note allows for a complete characterization in terms of equilibrium prices under weaker restrictions on fundamentals than those appearing in the literature. In fact, at a competitive equilibrium,

$$\text{uniform inefficiency of allocation if and only if } \liminf_{t\to\infty} \frac{\|p_t\|}{\|p_0\| + \cdots + \|p_t\|} > 0.$$

This only requires an hypothesis that rules out preferences converging to Leontief utilities, which is less restrictive than the traditional bounded curvature condition. It is easily verified that

$$\liminf_{t\to\infty} \frac{\|p_t\|}{\|p_0\| + \cdots + \|p_t\|} > 0 \text{ only if } \sum_{t=0}^{\infty} \frac{1}{\|p_t\|} \text{ is finite,}$$

which is consistent with the fact that the set of uniformly inefficient allocation is smaller than the set of simply inefficient allocations. The condition for uniform inefficiency is equivalent to the existence of some $1 > \delta > 0$ such that

$$\delta \geq \frac{\|p_0\| + \cdots + \|p_t\|}{\|p_0\| + \cdots + \|p_t\| + \|p_{t+1}\|}.$$

Thus, an equilibrium allocation is uniformly inefficient if and only if the value of the (bounded and non-vanishing) intertemporal aggregate endowment grows at a geometric rate over periods of trade.

Uniformly efficient allocations are dually characterized by supporting linear functionals defined over the relevant commodity space, exactly as in economies with finitely many individuals. However, in economies of overlapping generations, such linear functionals might not admit any sequential representation, so preventing their interpretation as competitive prices (see Richard and Srivastava [15]). Importantly, an equivalent characterization fails for simply efficient allocations. This suggests that uniform efficiency represents the natural extension of the canonical notion of efficiency to economies of overlapping generations, as uniform inefficiency, under mild restrictions, corresponds to the canonical notion of inefficiency in economies with finitely many individuals.

An economic interest for uniform inefficiency relies on its robustness to slight perturbations of endowments intertemporally, that is, the set of uniform inefficient allocations is open in the uniform topology. To the best of my knowledge, an equivalent property has not been established (and, probably, fails) for the canonical notion of inefficiency.[2] Policy intervention is motivated by a failure of efficiency in competitive markets. In this perspective, the doctrine would lack foundation if inefficiency were to depend on the precise distribution of endowments across individuals.

Finally, as a comparison with the established results in the literature, under some sort of bounded and non-vanishing curvature assumptions, I verify that a uniformly efficient allocation is not efficient only if Pareto

[1]Here, as in the following discussion, p_t represents the vector of (Arrow-Debreu) commodity prices prevailing in period t.

[2]Burke [7, 8] studies the related issue of the robustness of optimal monetary equilibrium under uniform perturbations of endowments and preferences.

improving trades vanish eventually. In addition, uniform inefficiency occurs if and only if a non-vanishing social security mechanism, consisting in a reduction of consumption in the first period for an increase in the second period, delivers a Pareto improvement.

The paper is organized as follows. In section 2, I describe a general economy of overlapping generations, which includes the canonical cases in the literature. In section 3, I introduce the notion of uniform efficiency and provide some preliminary basic characterization. In section 4, I present the notion of competitive prices, jointly with the hypotheses corresponding to those of bounded curvature and non-vanishing curvature of indifference curves. In section 5, I provide a duality analysis by showing that uniform efficiency corresponds to supportability by means of positive linear functionals that are defined on the relevant commodity space, a closed property in the uniform topology. In section 6, which represents the major contribution of this paper, I show that uniform efficiency is equivalently characterized by a Modified Cass Criterion in terms of competitive prices, under restrictions that are substantially weaker than those in the literature. Finally, in section 7, I compare the characterization in this note with the literature, by showing that, when a uniformly efficient allocation fails efficiency, then any Pareto improvement eventually vanishes over periods of trade. All proofs are collected in the appendix.

2. Fundamentals

Infinitely many commodities are traded by infinitely many individuals. The commodity space is $L = \mathbb{R}^{\mathcal{L}}$, where $\mathcal{L}$ is a countably infinite set of (dated and, possibly, contingent) commodities.[3] There is a countably infinite set $\mathcal{G}$ of individuals. For an individual i in $\mathcal{G}$, preferences $\succeq^i$ on the *consumption space* X^i are strictly monotone, convex and continuous (in the relative product topology), where X^i is the positive cone of a *finite-dimensional* vector subspace L^i of L.

This rather general structure is complemented by assumptions on the indecomposability of the economy and on the finite overlapping of generations. First, the economy is indecomposable, that is, for every non-trivial partition $\{\mathcal{G}', \mathcal{G}''\}$ of the set of individuals $\mathcal{G}$,

$$\sum_{i \in \mathcal{G}'} L^i \cap \sum_{i \in \mathcal{G}''} L^i \neq \{0\}.$$

Second, the overlapping of individuals is finite, that is, for every finite subset $\mathcal{G}'$ of the set of individuals $\mathcal{G}$,

$$\mathcal{G}'' = \left\{ i \in \mathcal{G}/\mathcal{G}' : L^i \cap \sum_{i \in \mathcal{G}'} L^i \neq \{0\} \right\}$$

is finite.

Under the maintained assumptions, by a canonical argument initially due to Balasko, Cass and Shell [3], the economy can be represented by a sequence of generations overlapping on a single period only. Periods of trade are $\mathcal{T} = \{0, \ldots, t, \ldots\}$, so that the commodity space decomposes as $L = \oplus_{t \in \mathcal{T}} L_t$, where L_t is a

[3] Such a space is endowed with the canonical order: $z \geq x$ if and only if $z_\ell \geq x_\ell$ for every ℓ in $\mathcal{L}$. An element x of L is positive (strictly positive) if $x_\ell \geq 0$ ($x_\ell > 0$) for every ℓ in $\mathcal{L}$. For an element x of L, x^+ and x^- are, respectively, its positive and its negative part, so that $x = x^+ - x^-$. Also, $|x| = x^+ + x^-$ is the absolute value. Finally, $L_+ = \{x \in L : x \geq 0\}$ is the positive cone of L. Similar definitions apply to vector subspaces of L. For an element e of L, the principal indeal $L(e) = \{v \in L : |v| \leq \lambda |e| \text{ for some } \lambda > 0\}$ is a vector subspace of L, endowed with the e-supremum norm

$$\|v\| = \inf \{\lambda > 0 : |v| \leq \lambda |e|\}.$$

For details, I refer to Aliprantis and Border [1].

finite-dimensional vector subspace of L for every t in $\mathcal{T}$.[4] In addition, there is a non-trivial partition $\{\mathcal{G}_t\}_{t\in\mathcal{T}}$ of the set of individuals $\mathcal{G}$, where $\mathcal{G}_t$ is a finite set for every t in $\mathcal{T}$, such that L^i is a vector subspace of L_0, for every individual i in $\mathcal{G}_0$, and, for every t in $\mathcal{T}$, L^i is a vector subspace of $L_t \oplus L_{t+1}$, for every individual i in $\mathcal{G}_{t+1}$. Reinterpreting terms, finitely many commodities are traded in every period of trade t in $\mathcal{T}$, represented by the vector space L_t, and infinitely many generations, each consisting of finitely many individuals, overlap on a single period of trade, with generations $\mathcal{G}_t$ and $\mathcal{G}_{t+1}$ overlapping only in period t in $\mathcal{T}$. Clearly, such a reduction is not unique. I shall peg one of such reductions and treat it as given throughout the analysis.

An economy is *simple* if $L^i = L_0$, for every individual i in $\mathcal{G}_0$, and, for every t in $\mathcal{T}$, $L^i = L_t \oplus L_{t+1}$, for every individual i in $\mathcal{G}_{t+1}$. Thus, an economy is simple whenever individuals in the same generation desire the same set of commodities. It is worth noticing that the hypothesis of a simple economy rules out many instances of economies of overlapping generations, beginning with uncertainty if individuals are distinguished on contingencies. It is only motivated by the need of analytical tractability and could be substantially weakened at the cost of heavy notation and qualifications.[5]

3. Uniform Efficiency

An *allocation* x is an element of $X = \left\{x \in L^{\mathcal{G}} : x^i \in X^i \text{ for every } i \in \mathcal{G}\right\}$. Notice that, for an element v of $L^{\mathcal{G}}$,

$$\sum_{i\in\mathcal{G}} v^i = \sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{G}_t} v^i = \sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{G}_t\cup\mathcal{G}_{t+1}} v_t^i$$

is a well-defined element of L, as $\mathcal{G}_t$ is finite for every t in $\mathcal{T}$.

An allocation x in X is *Pareto dominated* by an allocation z in X whenever, for every individual i in $\mathcal{G}$, $z^i \succeq^i x^i$ and, for some individual i in $\mathcal{G}$, $z^i \succ^i x^i$. An allocation x in X is *efficient* if it is not Pareto dominated by an alternative allocation z in X satisfying

$$\sum_{i\in\mathcal{G}} \left(z^i - x^i\right) \leq 0.$$

This canonical notion of efficiency is weakened in order to provide a full characterization.

For a positive element e of L, an allocation x in X is *e-efficient* if it is not Pareto dominated by an alternative allocation z in X satisfying, for some $\epsilon > 0$,

$$\epsilon e + \sum_{i\in\mathcal{G}} \left(z^i - x^i\right) \leq 0.$$

Thus, an allocation is e-efficient when a Pareto improvement is ruled out by an e-uniform small destruction of available resources. To simplify presentation, whenever the positive element e of L corresponds to the aggregate endowment, $e = \sum_{i\in\mathcal{G}} x^i$, I shall refer to an e-efficient (e-inefficient) allocation simply as a *uniformly efficient* (*uniformly inefficient*) allocation. Notice that the canonical notion of efficiency corresponds to 0-efficiency.

Lemma 1 (Ordering in Efficiency)**.** *An allocation x in X is e'-efficient for a positive element e' of L only if it is e''-efficient for every positive element e'' of L satisfying $L(e') \subset L(e'')$, where, for every element e of L,*

$$L(e) = \{v \in L : |v| \leq \lambda |e| \text{ for some } \lambda > 0\}.$$

[4]An element v of L uniquely decomposes as $(v_0, \ldots, v_{t-1}, v_t, v_{t+1}, \ldots)$, where v_t is an element of L_t for every t in $\mathcal{T}$.

[5]For instance, instead of periods of trade $\mathcal{T}$, one could consider an event-tree $\mathcal{S}$ of date-events.

Uniform inefficiency is, as a matter of fact, equivalent to robust Pareto dominance by means of a redistribution of resources. This is of relevance for policy intervention, as any slight distortion in the operating of competitive markets might still be compatible with a welfare improvement upon an uniformly inefficient allocation.

Lemma 2 (Robust Pareto dominance)**.** *For a strictly positive element e of L corresponding to the aggregate endowment, in an economy with a bound on the cardinality of generations, an allocation x in X is e-inefficient* if and only if *there are $\lambda > 0$ and an alternative allocation z in X satisfying $\sum_{i \in \mathcal{G}} \left(z^i - x^i\right) \leq 0$ and, for every individual i in $\mathcal{G}$, $y^i \succeq^i x^i$ for every y^i in X^i with $\left|y^i - z^i\right| \leq \lambda e$.*

4. Supporting Prices

A *price* p is an element of P, the positive cone of L, with evaluation

$$p \cdot x = p \cdot \sum_{t \in \mathcal{T}} x_t = \limsup_{t \in \mathcal{T}} p \cdot x_0 + \cdots + p \cdot x_t.$$

Values are allowed to be infinite. However, a price p in P defines a positive linear functional on L^i for every individual i in $\mathcal{G}$.

An allocation x in X is *supported* by price p in P if, for every individual i in $\mathcal{G}$,

$$z^i \succ^i x^i \text{ implies } p \cdot \left(z^i - x^i\right) > 0,$$

for every z^i in X^i. By local non-satiation of preferences, for every individual i in $\mathcal{G}$,

$$z^i \succeq^i x^i \text{ implies } p \cdot \left(z^i - x^i\right) \geq 0,$$

for every z^i in X^i. Also, notice that, by monotonicity of preferences, a supporting price p in P is strictly positive.

Supportability is reinforced by stronger properties in part of the analysis.

For a positive element e of L, an allocation x in X, with supporting price p in P, is *e-smoothly supported* by price p in P if, for every $1 > \rho > 0$, there is $\lambda > 0$ such that, for every individual i in $\mathcal{G}$,

$$\rho p \cdot \left(z^i - x^i\right)^+ \geq p \cdot \left(z^i - x^i\right)^- \text{ implies } z^i \succeq^i x^i,$$

for every z^i in X^i with $\left|z^i - x^i\right| \leq \lambda e$. In fact, smooth supportability requires that, locally, the (translated) convex cone

$$\left\{z^i \in X^i : \rho p \cdot \left(z^i - x^i\right)^+ \geq p \cdot \left(z^i - x^i\right)^-\right\}$$

be contained in the weakly preferred set $\left\{z^i \in X^i : z^i \succeq^i x^i\right\}$. This is a mild requirement for a single individual, so that the restriction is substantial only insofar as it holds uniformly for all individuals. This drawback is common to many other characterizations of efficiency in overlapping generations through supporting prices. In fact, it is similar to the traditional assumption of uniformly bounded curvature of indifference curves.

Remark 1 (Smooth supportability)**.** *If preferences are smooth, smoothly strictly increasing and smoothly quasi-concave, the requirement of smooth supportability is satisfied for every single individual at an interior consumption plan.*

For a positive element e of L, an allocation x in X, with supporting price p in P, is *e-strictly supported* by price p in P if, for some $\lambda > 0$, there is $\delta > 0$ such that, for every individual i in $\mathcal{G}$,

$$z^i \succeq^i x^i \text{ implies } p \cdot \left(z^i - x^i\right) \geq \delta \left\|z^i - x^i\right\|^2 p \cdot e^i,$$

for every z^i in X^i with $\left|z^i - x^i\right| \leq \lambda e$, where e^i is the projection of e in L into L^i and the norm is the e-supremum norm. Strict supportability requires that, locally, the weakly preferred set $\left\{z^i \in X^i : z^i \succeq^i x^i\right\}$ be contained in the convex set

$$\left\{z^i \in X^i : p \cdot \left(z^i - x^i\right) \geq \delta \left\|z^i - x^i\right\|^2 p \cdot e^i\right\}.$$

Strict supportability corresponds to the hypothesis of non-vanishing curvature of indifference curves in the literature.

Remark 2 (Strict supportability). *If preferences are smooth, smoothly strictly increasing and smoothly strictly quasi-concave, the requirement of strict supportability is satisfied for every single individual at an interior consumption plan.*

Both smooth and strict supportability are to hold uniformly across individuals for some element e of L. For instance, e in L could be the *unit* element, $e = (1, 1, 1, \ldots, 1, \ldots)$, jointly with the assumption that for some $\lambda > 0$, $x^i \leq \lambda e$ for every individual i in $\mathcal{G}$. Alternatively, e in L could be the aggregate endowment, $e = \sum_{i \in \mathcal{G}} x^i$, jointly with the assumption of a bound on the cardinality of generations.

5. Duality

An interesting feature of the modified notion of efficiency is that it admits an equivalent characterization in terms of supporting linear functionals. Relevantly, such an equivalence fails for the canonical notion of efficiency, that is, removing the *interiority* assumption. In particular, an inefficient allocation might still admit a supporting linear functional.

For a positive element e of L, endow the vector space $L(e)$ with the e-supremum norm, where

$$L(e) = \{v \in L : |v| \leq \lambda |e| \text{ for some } \lambda > 0\}.$$

Let $L'(e)$ denote the norm dual of $L(e)$, that is, the vector space of all norm continuous linear functionals φ on $L(e)$, where the duality operation is denoted by $\varphi \odot v$ for every v in $L(e)$. An allocation x in X is e-supported by a linear functional $\varphi > 0$ in $L'(e)$ whenever, for every allocation z in X with $\sum_{i \in \mathcal{G}} \left(z^i - x^i\right)$ in $L(e)$,

$$z \succ x \text{ implies } \varphi \odot \sum_{i \in \mathcal{G}} \left(z^i - x^i\right) \geq 0,$$

where $z \succ x$ means that allocation z in X Pareto dominates allocation x in X. In fact, for every non-trivially positive element e of L, e-efficiency is equivalent to e-supportability.

Lemma 3 (Duality). *For a non-trivially positive element e of L, an allocation x in X is e-efficient* if and only if *it is e-supported by a linear functional $\varphi > 0$ in $L'(e)$. In addition, for a positive element e of L, an allocation x in X is e-supported by a linear functional $\varphi > 0$ in $L'(e)$ only if it is v-efficient for every positive element v of $L(e)$ satisfying $\varphi \odot v > 0$.*

For a positive element e of L, consider the space of allocations with aggregate endowment bounded by (some expansion of) e in L, that is,

$$X(e) = \left\{ x \in X : \sum_{i \in \mathcal{G}} x^i \in L(e) \right\}.$$

This space of allocations is endowed with the metric induced by the e-supremum norm, so that, for every (z, x) in $X(e) \times X(e)$,

$$d(z, x) = \sup_{i \in \mathcal{G}} \inf \left\{ \lambda > 0 : \left| z^i - x^i \right| \leq \lambda e \right\}.$$

For a positive element e of L, an allocation x in $X(e)$ is *e-robustly inefficient* if there is $\epsilon > 0$ such that any alternative allocation z in $X(e)$ is inefficient provided that $d(z, x) < \epsilon$. That is, if any slight perturbation of that allocation, in the e-supremum norm, leads to an inefficient allocation.

For a strictly positive element e of L corresponding to the aggregate endowment, a remarkable property of e-inefficiency is that, under an additional hypothesis of uniform continuity on preferences, the set of such allocations is open in the e-supremum norm. So, if an allocation is e-inefficient, it is e-robustly inefficient.

For a positive element e of L, preferences are said to be *e-uniformly continuous* if, given any pair of allocations (z, x) in $X \times X$, for every $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that, for every individual i in $\mathcal{G}$,

$$z^i \succeq^i x^i \text{ implies } z^i + \epsilon e^i \succeq^i x^i + \delta(\epsilon) e^i,$$

where e^i is the algebraic projection of e into L^i.

Proposition 1 (Robustness)**.** *For a strictly positive element e of L, in an economy with e-uniformly continuous preferences, the set of e-efficient allocations x in $X(e)$ is closed in (the metric induced by) the e-supremum norm, provided that $\sum_{i \in \mathcal{G}} e^i$ is an element of $L(e)$.*[6]

6. Equivalent Characterization

Supporting prices convey information about efficiency of the allocation of resources. In fact, I shall here provide an equivalent characterization of uniform inefficiency in terms of supporting prices. In particular, under rather mild additional restrictions, I shall show that uniform inefficient is equivalent to the following Modified Cass Criterion:

$$\liminf_{t \in \mathcal{T}} \frac{p \cdot e_t}{p \cdot e_0 + \cdots + p \cdot e_t} > 0.$$

In order to understand the full implications of the proposed equivalence, just assume that prices evolve according to a steady (gross) rate of growth $\delta > 0$, that is, for every t in $\mathcal{T}$, $p \cdot e_{t+1} = \delta p \cdot e_t$, with $p \cdot e_0 = 1$. Hence,

$$\frac{p \cdot e_t}{p \cdot e_0 + \cdots + p \cdot e_t} = \frac{\delta^t}{1 + \delta + \cdots + \delta^{t-1} + \delta^t} = \delta^t \frac{1 - \delta}{1 - \delta^{t+1}}.$$

Therefore, as it can be easily verified, an allocation is uniform inefficient if and only if the real rate of interest is negative ($\delta > 1$).

[6] This last requirement is satisfied if, for instance, there is a bound on the cardinality of generations, as

$$\sum_{i \in \mathcal{G}} e^i \leq \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{G}_t \cup \mathcal{G}_{t+1}} e^i_t \leq \sum_{t \in \mathcal{T}} \#(\mathcal{G}_t \cup \mathcal{G}_{t+1}) e_t \leq \left(\sup_{t \in \mathcal{T}} \#(\mathcal{G}_t \cup \mathcal{G}_{t+1}) \right) e.$$

The Modified Cass Criterion, that is used to establish equivalence, implies that, for some $1 > \delta > 0$,

$$\delta \geq \frac{p \cdot e_0 + \cdots + p \cdot e_t}{p \cdot e_0 + \cdots + p \cdot e_t + p \cdot e_{t+1}}$$

holds at every t in $\mathcal{T}$. Thus, the value of aggregate endowment grows at a geometric rate. It can be easily showed that, for some $1 > \beta > 0$,

$$\liminf_{t \in \mathcal{T}} \frac{p \cdot e_t}{p \cdot e_0 + \cdots + p \cdot e_t} > 0 \text{ implies } \sum_{t \in \mathcal{T}} \frac{1}{\beta^t p \cdot e_t} \text{ is finite.}$$

Thus, the necessary and sufficient condition in this note is stricter than that defined by the traditional Cass Criterion, as the set of uniformly inefficient allocations is contained in the set of inefficient allocations.

Necessity requires no additional restriction beyond the hypothesis that, in every period of trade, the value of e_t exceeds the value of consumptions of individuals in their first period of economic activity, $\sum_{i \in \mathcal{G}_{t+1}} x_t^i$. This requirement is satisfied if e corresponds to the aggregate endowment, $e = \sum_{i \in \mathcal{G}} x^i$.

Proposition 2 (Necessity)**.** *For a strictly positive element e of L satisfying*

$$e \geq \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{G}_{t+1}} x_t^i,$$

an allocation x in X, with supporting price p in P, is e-inefficient only if

$$\liminf_{t \in \mathcal{T}} \frac{p \cdot e_t}{p \cdot e_0 + \cdots + p \cdot e_t} > 0.$$

To obtain sufficiency, the hypothesis of smooth supportability is employed and, indeed, when smooth supportability fails, counterexamples to the claim in proposition 3 can be easily constructed. This substitutes for the requirement of bounded curvatures of indifference curves that emerged in the literature. In addition, as the proof is constructive, the assumption of a simple economy allows for a direct control on the Pareto improvement. Finally, in every period of trade, consumptions of individuals in their first period of economic activity, $\sum_{i \in \mathcal{G}_{t+1}} x_t^i$, exceed the corresponding values of ηe_t for some $\eta > 0$. This is again consistent with the assumption that e corresponds to the aggregate endowment, $e = \sum_{i \in \mathcal{G}} x^i$.

Proposition 3 (Sufficiency)**.** *For a strictly positive element e of L satisfying*

$$\sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{G}_{t+1}} x_t^i \geq \eta e, \text{ for some } \eta > 0,$$

in a simple economy, an allocation x in X, with e-smoothly supporting price p in P, is e-inefficient if

$$\liminf_{t \in \mathcal{T}} \frac{p \cdot e_t}{p \cdot e_0 + \cdots + p \cdot e_t} > 0.$$

The following proposition clarifies the exact relation between competitive prices and supporting linear functionals for uniformly efficient allocations. In fact, supporting linear functionals correspond to some appropriate limits of competitive prices.

Proposition 4 (Prices and supporting linear functionals)**.** *For a strictly positive element e of L satisfying*

$$e \geq \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{G}_{t+1}} x_t^i \geq \eta e, \text{ for some } \eta > 0,$$

in a simple economy, an allocation x in X, with e-smoothly supporting price p in P, is e-efficient if and only if *it is e-supported by a linear functional $\varphi > 0$ in the (weak*) closure of $\{\varphi^t\}_{t\in\mathcal{T}} \subset L'(e)$, where, for every t in $\mathcal{T}$, the linear functional φ^t in $L'(e)$ is defined by*

$$\varphi^t \odot v = \frac{p \cdot v_0 + \cdots + p \cdot v_t}{p \cdot e_0 + \cdots + p \cdot e_t}, \text{ for every } v \text{ in } L(e).$$

A final proposition shows equivalence in uniformities. In particular, if a Pareto improvement does not obtain with a small destruction of the aggregate endowment, neither does with a small destruction of the endowment of only commodities whose relative price does not vanish along periods of trade.

Proposition 5 (Equivalent uniformities)**.** *For a strictly positive element e of L satisfying*

$$e \geq \sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{G}_{t+1}} x_t^i \geq \eta e, \text{ for some } \eta > 0,$$

in a simple economy, an allocation x in X, with e-smoothly supporting price p in P, is e-efficient only if it is v-efficient for every positive element v of $L(e)$ satisfying

$$\inf_{t\in\mathcal{T}} \frac{p \cdot v_t}{p \cdot e_t} > 0.$$

7. Comparison with the Literature

I shall here verify under which conditions uniform efficiency is consistent with a failure of efficiency at equilibrium. Under a sort of curvature assumptions, it turns out that this occurs only if there is no Pareto improvement with uniformly positive trades. Thus, every Pareto improvement eventually vanishes in the long-period.

Proposition 6 (Vanishing welfare improving trades)**.** *For a strictly positive element e of L corresponding to the aggregate endowment and satisfying*

$$\sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{G}_{t+1}} x_t \geq \eta e, \text{ for some } \eta > 0,$$

in a simple economy with a bound on the cardinality of generations, an e-efficient allocation x in X, with e-smoothly and e-strictly supporting price p in P, is Pareto dominated by an alternative allocation z in X satisfying $\sum_{i\in\mathcal{G}} (z^i - x^i) \leq 0$ only if

$$\inf_{t\in\mathcal{T}} \sum_{i\in\mathcal{G}_t} \|z^i - x^i\| = 0,$$

where the norm is the e-supremum norm.

A relevant implication of vanishing trades is that the canonical social security system would not deliver a welfare improvement upon a uniformly efficient allocation which is not efficient. Oppositely, when allocation is uniformly inefficient, a welfare improvement obtains through an extremely simple policy consisting in transferring part of the endowment of every individual from the first period to the second period of economic activity, remunerating such a transfer at the (corrected) real rate of interest prevailing in the market.

Proposition 7 (Simple welfare improving policy)**.** *For a strictly positive element e of L with*

$$e \geq \sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{G}_{t+1}} x_t^i \geq \sum_{t\in\mathcal{T}} \bigwedge_{i\in\mathcal{G}_{t+1}} x_t^i \geq \eta e, \text{ for some } \eta > 0,$$

in a simple economy with generations of constant cardinality, an e-inefficient allocation x in X, with e-smoothly supporting price p in P, is Pareto dominated by the alternative allocation z in X defined, for every t in $\mathcal{T}$, by

$$\begin{aligned} z_t^i &= x_t^i + \epsilon\left(\frac{p \cdot e_0 + \cdots + p \cdot e_t}{p \cdot e_t}\right) e_t, \text{ for every } i \text{ in } \mathcal{G}_t, \\ z_t^i &= x_t^i - \epsilon\left(\frac{p \cdot e_0 + \cdots + p \cdot e_t}{p \cdot e_t}\right) e_t, \text{ for every } i \text{ in } \mathcal{G}_{t+1}, \end{aligned}$$

where $\epsilon > 0$ is sufficiently small.

References

[1] C.D. Aliprantis and K.C. Border. *Infinite Dymensional Analysis: A Hitchhiker's Guide.* Springer-Verlag, 1999.

[2] Y. Balasko and K. Shell. The overlapping generations model, I: the case of pure exchange without money. *Journal of Economic Theory*, 23, 281-306, 1980.

[3] Y. Balasko, D. Cass and K. Shell. Existence of competitive equilibrium in a general overlapping-generations model. *Journal of Economic Theory*, 23, 307-322, 1980.

[4] L.M. Benveniste. A complete characterization of efficiency for a general capital accumulation model. *Journal of Economic Theory*, 12, 325-337, 1976.

[5] L.M. Benveniste. Pricing optimal distributions to overlapping generations: a corollary to efficiency pricing. *Review of Economic Studies*, 53, 301-206, 1986.

[6] J.L. Burke. Inactive transfers policy and efficiency in general overlapping generations economies. *Journal of Mathematical Economics*, 16, 201-222, 1987.

[7] J.L. Burke. Existence of a Pareto-optimal equilibrium in nearly-stationary overlapping-generations economies. *Economic Theory*, 5, 247-261, 1995.

[8] J.L. Burke. The robustness of optimal equilibrium among overlapping generations. *Economic Theory*, 14, 311-329, 1999.

[9] D. Cass. On capital overaccumulation in the aggregative neoclassical model of economic growth: a complete characterization. *Journal of Economic Theory*, 4, 200-223, 1972.

[10] S. Chattopadhyay and P. Gottardi. Stochastic OLG models, market structure, and optimality. *Journal of Economic Theory*, 89, 21-67, 1999.

[11] G. Debreu. The coefficient of resource utilization. *Econometrica*, 19, 273-292, 1951.

[12] J.D. Geanakoplos and H.M. Polemarchakis. Overlapping generations. In W. Hildenbrand and H. Sonnenschein (eds.), *Handbook of Mathematical Economics*, vol. IV, North-Holland, New York, 1891-1960, 1991.

[13] A. Molina-Abraldes and J. Pintos-Clapés. A complete characterization of Pareto optimality for general OLG economies. *Journal of Economic Theory*, 113, 235-252, 2003.

[14] M. Okuno and I. Zilcha. On the efficiency of a competitive equilibrium in infinite horizon monetary economies. *Review of Economic Studies*, 47, 797-807, 1980.

[15] S.F. Richard and S. Srivastava. Equilibrium in economies with infinitely many consumers and infinitely many commodities. *Journal of Mathematical Economics*, 17, 9-21, 1988.

Appendix

Proof of lemma 1. Supposing not, then allocation x in X is Pareto dominated by an allocation z in X satisfying, for some $\epsilon > 0$,

$$\frac{\epsilon}{\lambda} e' + \sum_{i \in \mathcal{G}} \left(z^i - x^i\right) \leq \epsilon e'' + \sum_{i \in \mathcal{G}} \left(z^i - x^i\right) \leq 0,$$

where $\lambda > 0$ is such that $e' \leq \lambda e''$, which is consistent as e' is an element of $L\left(e''\right)$. This shows a contradiction. □

Proof of lemma 2. Let be $\mu > 0$ be greater than twice the cardinality of each generation, $\sup_{t\in\mathcal{T}} \#\mathcal{G}_t$, and let e^i denote the algebraic projection of e in L into L^i. If allocation x in X is e-inefficient, then it is Pareto dominated by an alternative allocation v in X satisfying, for some $\epsilon > 0$,

$$\epsilon e + \sum_{i\in\mathcal{G}} \left(v^i - x^i\right) \leq 0.$$

Let z in X be defined, for every i in $\mathcal{G}$, by

$$z^i = v^i + \frac{\epsilon}{\mu} e^i$$

and notice that

$$\sum_{i\in\mathcal{G}} \left(z^i - x^i\right) \leq \epsilon \sum_{i\in\mathcal{G}} \frac{1}{\mu} e^i + \sum_{i\in\mathcal{G}} \left(v^i - x^i\right) \leq \epsilon e + \sum_{i\in\mathcal{G}} \left(v^i - x^i\right) \leq 0.$$

Letting $\lambda = \epsilon/\mu$, it follows that, for every y^i in X^i with $\left|y^i - z^i\right| \leq \lambda e$, $y^i \geq z^i - \lambda e^i \geq v^i$, so that, by monotonicity of preferences, $y^i \succeq^i x^i$.

As far as the reverse implication is concerned, assuming $1 > \lambda > 1$, consider the allocation y in X defined, for every i in $\mathcal{G}$, by $y^i = (1-\lambda) z^i$, so that $\left|y^i - z^i\right| = \lambda z^i \leq \lambda e$. Thus, allocation y in X Pareto dominates allocation x in X and, aggregating,

$$\sum_{i\in\mathcal{G}} \left(y^i - x^i\right) \leq (1-\lambda) \sum_{i\in\mathcal{G}} \left(z^i - x^i\right) - \lambda \sum_{i\in\mathcal{G}} x^i \leq -\lambda e,$$

so proving the claim. □

Proof of remark 1. Assume that the utility function $u^i : X^i \to \mathbb{R}$ is smooth, smoothly strictly increasing and smoothly quasi-concave. By supportability, at an interior consumption plan, there is $\mu^i > 0$ such that, for every h^i in L^i,

$$\partial u^i \left(x^i\right) \cdot h^i = \mu^i p \cdot h^i.$$

At no loss of generality, assume that $\mu^i = 1$. In addition, by Taylor Decomposition, there is $\lambda^* > 0$ such that, for every z^i in X^i with $\left|z^i - x^i\right| \leq \lambda^* e$,

$$u^i \left(z^i\right) - u^i \left(x^i\right) = p \cdot \left(z^i - x^i\right) + o\left(z^i - x^i\right),$$

where

$$\lim_{z^i \to x^i} \frac{o\left(z^i - x^i\right)}{\left\|z^i - x^i\right\|} = 0.$$

Pegging any $1 > \rho > 0$, I shall show that the requirement of e-smooth supportability is satisfied by some $\lambda^* > \lambda > 0$.

Suppose that, for every $\lambda^* > \lambda > 0$, the set

$$K^i_\lambda = \left\{z^i \in X^i : \rho p \cdot \left(z^i - x^i\right)^+ \geq p \cdot \left(z^i - x^i\right)^- ; u^i\left(x^i\right) > u^i\left(z^i\right) ; \left|z^i - x^i\right| \leq \lambda e\right\}$$

is non-empty. It follows that there is a sequence $\left\{z^i_\lambda\right\}_{\lambda^* > \lambda > 0}$, with z^i_λ lying in K^i_λ for every $\lambda^* > \lambda > 0$, satisfying

$$0 > u^i\left(z^i_\lambda\right) - u^i\left(x^i\right) = p \cdot \left(z^i_\lambda - x^i\right) + o\left(z^i_\lambda - x^i\right).$$

Therefore,

$$p \cdot \left(z^i_\lambda - x^i\right)^+ + o\left(z^i_\lambda - x^i\right) < p \cdot \left(z^i_\lambda - x^i\right)^- \leq \rho p \cdot \left(z^i_\lambda - x^i\right)^+.$$

Possibly extracting a subsequence, one might assume that the sequence

$$\left\{\frac{z^i_\lambda - x^i}{\left\|z^i_\lambda - x^i\right\|}\right\}_{\lambda^* > \lambda > 0}$$

converges to some h^i in L^i with $\left\|h^i\right\| = 1$. This, observing that

$$\lim_{\lambda \to 0} \frac{o\left(z_\lambda^i - x^i\right)}{\left\|z_\lambda^i - x^i\right\|} = 0,$$

delivers

$$p \cdot \left(h^i\right)^+ \leq p \cdot \left(h^i\right)^- \leq \rho p \cdot \left(h^i\right)^+ .$$

As p is strictly positive on L^i, $p \cdot |h| > 0$, which shows a contradiction. □

Proof of remark 2. To verify this claim, assume that the utility function $u^i : X^i \to \mathbb{R}$ is smooth, smoothly strictly increasing and smoothly strictly quasi-concave utility function.[7] By supportability, at an interior consumption plan, there is $\mu^i > 0$ such that, for every h^i in L^i,

$$\partial u^i\left(x^i\right) \cdot h^i = \mu^i p \cdot h^i.$$

At no loss of generality, assume that $\mu^i = 1$. In addition, by Taylor Decomposition, there is $\lambda^* > 0$ such that, for every z^i in X^i with $\left|z^i - x^i\right| \leq \lambda^* e$,

$$u^i\left(z^i\right) - u^i\left(x^i\right) = p \cdot \left(z^i - x^i\right) + \frac{1}{2}\left(z^i - x^i\right) \cdot \partial^2 u^i\left(x^i\right) \cdot \left(z^i - x^i\right) + o^2\left(z^i - x^i\right),$$

where

$$\lim_{z^i \to x^i} \frac{o^2\left(z^i - x^i\right)}{\left\|z^i - x^i\right\|^2} = 0.$$

For a given $\lambda^* > \lambda > 0$, consider the set

$$K_\delta^i = \left\{z^i \in X^i : p \cdot \left(z^i - x^i\right) < \delta \left\|z^i - x^i\right\|^2 p \cdot e^i ; u^i\left(z^i\right) \geq u^i\left(x^i\right); \left|z^i - x^i\right| \leq \lambda e\right\}.$$

Suppose that, for every $\delta > 0$, the set K_δ^i is non-empty, so that there is a sequence $\left\{z_\delta^i\right\}_{\delta > 0}$, with each z_δ^i lying in K_δ^i. Possibly extracting a subsequence, one might assume that such a sequence converges in X^i. If $z^i = \lim_{\delta \to 0} z_\delta^i \neq x^i$, then a contradiction emerges from $u^i\left(z^i\right) \geq u^i\left(x^i\right)$ and

$$p \cdot \left(z^i - x^i\right) \leq \lim_{\delta \to 0} \delta \left\|z_\delta^i - x^i\right\|^2 p \cdot e^i \leq \lim_{\delta \to 0} \delta \lambda^2 p \cdot e^i = 0,$$

as preferences are strictly convex. Hence, $z^i = \lim_{\delta \to 0} z_\delta^i = x^i$. Without loss of generality, assume that the sequence

$$\left\{\frac{z_\delta^i - x^i}{\left\|z_\delta^i - x^i\right\|}\right\}_{\delta > 0}$$

converges to h^i in L^i with $\left\|h^i\right\| = 1$.

Using the definition of K_δ^i,

$$0 \leq p \cdot h^i = \lim_{\delta \to 0} p \cdot \left(\frac{z_\delta^i - x^i}{\left\|z_\delta^i - x^i\right\|}\right) \leq \lim_{\delta \to 0} \delta \left\|z_\delta^i - x^i\right\| p \cdot e^i \leq \lim_{\delta \to 0} \delta \lambda p \cdot e^i \leq 0.$$

[7]Smoothly strictly quasi-concave means that, for every $\left(h^i, x^i\right)$ in $L^i \times X^i$, with $h^i \neq 0$, if $\partial u^i\left(x^i\right) \cdot h^i = 0$, then $h^i \cdot \partial^2 u^i\left(x^i\right) \cdot h^i < 0$.

Invoking Taylor Decomposition and using the definition of K_δ^i,

$$\begin{aligned}
0 &\leq \lim_{\delta\to 0} p \cdot \left(\frac{z_\delta^i - x^i}{\left\|z_\delta^i - x^i\right\|^2}\right) + \lim_{\delta\to 0} \frac{1}{2}\left(\frac{z_\delta^i - x^i}{\left\|z_\delta^i - x^i\right\|}\right) \cdot \partial^2 u^i\left(x^i\right) \cdot \left(\frac{z_\delta^i - x^i}{\left\|z_\delta^i - x^i\right\|}\right) \\
&\quad + \lim_{\delta\to 0} \frac{o^2\left(z_\delta^i - x^i\right)}{\left\|z_\delta^i - x^i\right\|^2} \\
&\leq \lim_{\delta\to 0} \delta p \cdot e^i + \frac{1}{2} h^i \cdot \partial^2 u^i\left(x^i\right) \cdot h^i \\
&\leq \frac{1}{2} h^i \cdot \partial^2 u^i\left(x^i\right) \cdot h^i.
\end{aligned}$$

This contradicts the hypothesis of smoothly strict quasi-concavity, so proving the claim. □

Proof of lemma 3. Assume that the allocation is e-efficient. Consider the set

$$C = \left\{\sum_{i\in\mathcal{G}} \left(z^i - x^i\right) : z \in Z\right\},$$

where

$$Z = \{z \in X : z \succ x\} \cap \left\{z \in X : \sum_{i\in\mathcal{G}} \left(z^i - x^i\right) \in L(e)\right\}.$$

Clearly, C is a non-empty convex set and, by hypothesis, $C \cap \operatorname{int}\left(-L_+(e)\right) = \emptyset$. By the Separation Hyperplane Theorem (Aliprantis and Border [1, Theorem 5.50]), there is a non-trivial linear functional φ in $L'(e)$ such that, for every c in C, $\varphi \odot c \geq 0$. It is also clear, by monotonicity of preferences, that $\varphi > 0$, which proves the claim as far as necessity is concerned.

As far as sufficiency is concerned, suppose not. So, there is an allocation z in X such that $z \succ x$ and, for some $\epsilon > 0$,

$$\epsilon e + \sum_{i\in\mathcal{G}} \left(z^i - x^i\right) \leq 0.$$

By monotonicity of preferences, at no loss of generality, it can be assumed that

$$\sum_{i\in\mathcal{G}} \left(z^i - x^i\right) = -\epsilon e \in \operatorname{int}\left(-L_+(e)\right).$$

Thus,

$$0 > -\epsilon\varphi \odot e \geq \varphi \odot \sum_{i\in\mathcal{G}} \left(z^i - x^i\right) \geq 0,$$

a contradiction.

As far as the second claim is concerned, simply notice that the restriction $\varphi : L(v) \to \mathbb{R}$ is a well-defined non-trivially positive linear functional in $L'(v)$, as $L(v) \subset L(e)$ and $\varphi \odot v > 0$. □

Proof of proposition 1. Let $X(e) = \left\{x \in X : \sum_{i\in\mathcal{G}} x^i \in L(e)\right\}$. Consider a sequence of e-efficient allocations x^ν in $X(e)$, converging to an allocation x in $X(e)$, in the e-supremum norm. To simplify, at no loss of generality, assume that

$$d\left(x^\nu, x\right) = \sup_{i\in\mathcal{G}} \inf\left\{\lambda > 0 : \left|x^{i\nu} - x^i\right| \leq \lambda e\right\} = \nu.$$

By lemma 3, there is a sequence of linear functionals $\varphi^\nu > 0$ in $L'(e)$ (weak*) converging, without loss of generality, to a linear functional $\varphi > 0$ in $L'(e)$ (see Alaoglu's Theorem in Aliprantis and Border [1, Theorem 6.25]). Consider any alternative allocation z in X with $\sum_{i\in\mathcal{G}}\left(z^i - x^i\right)$ in $L(e)$ such that $z \succ x$. Given any $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that, for every individual i in $\mathcal{G}$, $z^i + \epsilon e^i \succeq^i x^i + \delta(\epsilon) e^i$, so that, by monotonicity of preferences, for every $\nu > 0$ small enough, $z^i + \epsilon e^i \succeq^i x^{i\nu}$, as $x^i + \delta(\epsilon) e^i \geq x^i + \nu e^i \geq x^{i\nu}$. As $\sum_{i \in \mathcal{G}} \left(z^i + \epsilon e^i - x^{i\nu}\right)$ lies in $L(e)$, it follows that

$$\varphi^\nu \odot \sum_{i \in \mathcal{G}} \left(z^i + (\epsilon + \nu) e^i - x^i\right) \geq \varphi^\nu \odot \sum_{i \in \mathcal{G}} \left(z^i + \epsilon e^i - x^{i\nu}\right) \geq 0,$$

where the left hand-side inequality follows from the fact that $x^{i\nu} - x^i \geq -\nu e^i$ for every individual i in $\mathcal{G}$. In the limit, for any $\epsilon > 0$,

$$\varphi \odot \sum_{i \in \mathcal{G}} \left(z^i - x^i\right) \geq -\epsilon \varphi \odot \sum_{i \in \mathcal{G}} e^i.$$

As $\epsilon > 0$ vanishes, one obtains a proof of the proposition. □

Proof of proposition 2. For every t in $\mathcal{T}$, let $v^t = (v_0, \ldots, v_t)$ be the truncation of v in L at t in $\mathcal{T}$. By hypothesis, allocation x in X is Pareto dominated by an alternative allocation z in X satisfying, for some $\epsilon > 0$,

$$\epsilon e + \sum_{i \in \mathcal{G}} \left(z^i - x^i\right) \leq 0.$$

By supportability, for every t in $\mathcal{T}$,

$$p \cdot \left(\sum_{i \in \mathcal{G}} \left(z^i - x^i\right)\right)^t - p \cdot \sum_{i \in \mathcal{G}_{t+1}} \left(z^i - x^i\right)_t = p \cdot \sum_{i \in \mathcal{G}_0} \left(z^i - x^i\right) + \cdots + p \cdot \sum_{i \in \mathcal{G}_t} \left(z^i - x^i\right) \geq 0.$$

Thus,

$$\begin{aligned}
-\epsilon p \cdot e^t &\geq p \cdot \left(\sum_{i \in \mathcal{G}} \left(z^i - x^i\right)\right)^t \\
&\geq p \cdot \sum_{i \in \mathcal{G}_{t+1}} \left(z^i - x^i\right)_t \\
&\geq -p \cdot \sum_{i \in \mathcal{G}_{t+1}} \left(z^i - x^i\right)_t^- \\
&\geq -p \cdot \sum_{i \in \mathcal{G}_{t+1}} x_t^i \\
&\geq -p \cdot e_t.
\end{aligned}$$

This proves the claim, as it implies that $p \cdot e_t \geq \epsilon p \cdot e^t$ for every t in $\mathcal{T}$. □

Proof of proposition 3. For every t in $\mathcal{T}$, let $v^t = (v_0, \ldots, v_t)$ be the truncation of v in L at t in $\mathcal{T}$. By hypothesis, there is $1 > \beta > 0$ such that, for every t in $\mathcal{T}$,

$$p \cdot e_t \geq (1 - \beta) p \cdot e^t$$

and, so,

$$1 \geq (1 - \beta) \frac{p \cdot e^t}{p \cdot e_t} \geq 1 - \beta.$$

Using the first inequality and rearranging terms, it follows that, for every t in $\mathcal{T}$,

$$\beta p \cdot e_{t+1} \geq (1 - \beta) p \cdot e^t,$$

which implies

$$\beta p \cdot e^{t+1} \geq \beta p \cdot e_{t+1} + \beta p \cdot e^t \geq (1 - \beta) p \cdot e^t + \beta p \cdot e^t \geq p \cdot e^t,$$

that is,

$$\beta \frac{p \cdot e^{t+1}}{p \cdot e^{t}} \geq 1.$$

Choose $\epsilon > 0$ and $1 > \rho > 0$ so that $\rho(1-\epsilon) = \beta$ and let $\lambda > 0$ be given by the hypothesis of e-smooth supportability at $1 > \rho > 0$. At no loss of generality, assume that $\eta > \lambda > 0$.

The alternative allocation z in X is constructed as follows. (Notice that this is consistent as the economy is simple, that is, for every i in $\mathcal{G}_0$, $L^i = L_0$ and, for every i in $\mathcal{G}_{t+1}$, $L^i = L_t \oplus L_{t+1}$; in addition, for every t in $\mathcal{T}$, $e_t \wedge e_{t+1} = 0$.) Given any t in $\mathcal{T}$, for every i in $\mathcal{G}_{t+1}$

$$\sum_{i \in \mathcal{G}_{t+1}} \left(z^i - x^i\right)^- = \lambda(1-\beta) \frac{p \cdot e^t}{p \cdot e_t} e_t \leq \lambda e_t \leq \eta e_t,$$

which can be done by Riesz Decomposition Theorem (Aliprantis and Border [1, **Theorem ??**]) because $\eta e_t \leq \sum_{i \in \mathcal{G}_{t+1}} x^i_t$; in addition, for every i in $\mathcal{G}_{t+1}$,

$$\left(z^i - x^i\right)^+ = (1-\epsilon) \frac{p \cdot \left(z^i - x^i\right)^-}{p \cdot e^t} \frac{p \cdot e^{t+1}}{p \cdot e_{t+1}} e_{t+1}.$$

Finally, for every individual i in generation $\mathcal{G}_0$, let

$$z^i = x^i + \frac{1}{\#\mathcal{G}_0} \lambda(1-\epsilon)(1-\beta) \frac{p \cdot e^0}{p \cdot e_0} e_0.$$

To verify feasibility, observe that

$$\begin{aligned}
\sum_{i \in \mathcal{G}} \left(z^i - x^i\right) &= \sum_{i \in \mathcal{G}_0} \left(z^i - x^i\right)^- + \sum_{t \in \mathcal{T}} \left(\sum_{i \in \mathcal{G}_t} \left(z^i - x^i\right)^+ - \sum_{i \in \mathcal{G}_{t+1}} \left(z^i - x^i\right)^- \right) \\
&= \sum_{t \in \mathcal{T}} \left(\lambda(1-\epsilon)(1-\beta) \frac{p \cdot e^t}{p \cdot e_t} e_t - \lambda(1-\beta) \frac{p \cdot e^t}{p \cdot e_t} e_t \right) \\
&= -\epsilon \sum_{t \in \mathcal{T}} \lambda(1-\beta) \frac{p \cdot e^t}{p \cdot e_t} e_t \\
&\leq -\epsilon\lambda(1-\beta) \sum_{t \in \mathcal{T}} e_t \\
&= -\epsilon\lambda(1-\beta) e.
\end{aligned}$$

To verify welfare improvement, given any t in $\mathcal{T}$, observe that, for every i in $\mathcal{G}_{t+1}$,

$$\begin{aligned}
\rho p \cdot \left(z^i - x^i\right)^+ &= \rho(1-\epsilon) \frac{p \cdot \left(z^i - x^i\right)^-}{p \cdot e^t} \frac{p \cdot e^{t+1}}{p \cdot e_{t+1}} p \cdot e_{t+1} \\
&= \beta \frac{p \cdot e^{t+1}}{p \cdot e^t} p \cdot \left(z^i - x^i\right)^- \\
&\geq p \cdot \left(z^i - x^i\right)^-.
\end{aligned}$$

In addition,

$$\begin{aligned}
\left|z^i - x^i\right| &\leq \left(z^i - x^i\right)^+ + \left(z^i - x^i\right)^- \\
&\leq \lambda(1-\beta) \left(\frac{p \cdot e^t}{p \cdot e_t} e_t + (1-\epsilon) \frac{p \cdot e^{t+1}}{p \cdot e_{t+1}} e_{t+1} \right) \\
&\leq \lambda e_t + \lambda e_{t+1} \\
&\leq \lambda e.
\end{aligned}$$

Hence, by e-smooth supportability, $z^i \succeq^i x^i$. In addition, by monotonicity of preferences, for every i in the initial generation $\mathcal{G}_0$, $z^i \succ^i x^i$. This proves the claim. $\square$

Proof of proposition 4. One implication is a direct consequence of the equivalence in lemma 3. So, assume that allocation x in X is e-efficient. Because of proposition 3, for some subset $\mathcal{T}^*$ of $\mathcal{T}$,

$$\lim_{t\in\mathcal{T}^*} \frac{p\cdot e_t}{p\cdot e_0+\cdots+p\cdot e_t}=0.$$

Notice that, for every allocation z in X that Pareto dominates allocation x in X, at every t in $\mathcal{T}^*$, the following inequality holds true (see the proof of proposition 2):

$$\varphi^t \odot \sum_{i\in\mathcal{G}} \left(z^i - x^i\right) \geq -\frac{p\cdot e_t}{p\cdot e_0+\cdots+p\cdot e_t}.$$

As the value in the right hand-side of the above inequality vanishes over $\mathcal{T}^*$ and $\{\varphi^t\}_{t\in\mathcal{T}^*} \subset L'(e)$ admits an accumulation point $\varphi > 0$ in $L'(e)$, the claim is proved. $\square$

Proof of proposition 5. Notice that, for every t in $\mathcal{T}$,

$$\varphi^t \odot v = \frac{p\cdot v_0+\cdots+p\cdot v_t}{p\cdot e_0+\cdots+p\cdot e_t} \geq \inf_{t\in\mathcal{T}} \frac{p\cdot v_t}{p\cdot e_t} > 0.$$

Because of proposition 4, allocation x in X is e-supported by a linear functional $\varphi > 0$ in the closure of $\{\varphi^t\}_{t\in\mathcal{T}} \subset L'(e)$ and, because of the above inequality, $\varphi \odot v > 0$. Hence, v-efficiency follows from lemma 3. $\square$

Proof of proposition 6. As in the previous proofs, $v^t = (v_0,\ldots,v_t)$ denotes the truncation of v in L at t in $\mathcal{T}$. Let $1 > \lambda > 0$ be given by the assumption of e-strict supportability and, without loss of generality, assume that $\sup_{i\in\mathcal{G}} \left\|z^i - x^i\right\| \leq \lambda$, where the norm is the e-supremum norm. Indeed, if not, replace the allocation z in X by the alternative allocation y in X defined, for every individual i in $\mathcal{G}$, by $y^i = x^i + \lambda\left(z^i - x^i\right)$, so that

$$\sup_{i\in\mathcal{G}} \left\|y^i - x^i\right\| = \sup_{i\in\mathcal{G}} \lambda \left\|z^i - x^i\right\| \leq \lambda \left\|e\right\| = \lambda.$$

By convexity of preferences, allocation y in X Pareto dominates allocation x in X and satisfies

$$\sum_{i\in\mathcal{G}} \left(y^i - x^i\right) \leq 0.$$

In addition,

$$\inf_{t\in\mathcal{T}} \sum_{i\in\mathcal{G}_t} \left\|y^i - x^i\right\| = \lambda \inf_{t\in\mathcal{T}} \sum_{i\in\mathcal{G}_t} \left\|z^i - x^i\right\|.$$

For every t in $\mathcal{T}$, exploiting feasibility and e-strict supportability (jointly with the fact that the economy is simple, so that, for every t in $\mathcal{T}$, $e^i \geq e_t$ for every i in $\mathcal{G}_t$),

$$\begin{aligned}
0 &\geq p\cdot\left(\sum_{i\in\mathcal{G}} \left(z^i - x^i\right)\right)^t \\
&\geq p\cdot \sum_{i\in\mathcal{G}_0} \left(z^i - x^i\right) + \cdots + p\cdot \sum_{i\in\mathcal{G}_t} \left(z^i - x^i\right) + p\cdot \sum_{i\in\mathcal{G}_{t+1}} \left(z^i - x^i\right)_t \\
&\geq \delta p\cdot \sum_{i\in\mathcal{G}_0} \left\|z^i - x^i\right\|^2 p\cdot e_0 + \cdots + \delta p\cdot \sum_{i\in\mathcal{G}_t} \left\|z^i - x^i\right\|^2 p\cdot e_t + p\cdot \sum_{i\in\mathcal{G}_{t+1}} \left(z^i - x^i\right)_t \\
&\geq \delta p\cdot \sum_{i\in\mathcal{G}_0} \left\|z^i - x^i\right\|^2 p\cdot e_0 + \cdots + \delta p\cdot \sum_{i\in\mathcal{G}_t} \left\|z^i - x^i\right\|^2 p\cdot e_t - \eta p\cdot e_t.
\end{aligned}$$

Hence, for every t in $\mathcal{T}$, one obtains

$$\eta \frac{p \cdot e_t}{p \cdot e_0 + \cdots + p \cdot e_t} \geq \delta \inf_{t \in \mathcal{T}} \sum_{i \in \mathcal{G}_t} \left\| z^i - x^i \right\|^2 .$$

Because of proposition 3,

$$0 \geq \eta \liminf_{t \in \mathcal{T}} \frac{p \cdot e_t}{p \cdot e_0 + \cdots + p \cdot e_t} \geq \delta \inf_{t \in \mathcal{T}} \sum_{i \in \mathcal{G}_t} \left\| z^i - x^i \right\|^2 \geq 0,$$

which, as there is a bound of the cardinality of generations, implies

$$\inf_{t \in \mathcal{T}} \sum_{i \in \mathcal{G}_t} \left\| z^i - x^i \right\| = 0,$$

so proving the claim. □

Proof of proposition 7. Notice that, by assumptions, proposition 2 holds true. Using the notation in the proof of proposition 3 (of which the current proof is just a simplification), there is $1 > \rho > 0$ such that, for every t in $\mathcal{T}$,

$$1 \geq (1 - \rho) \frac{p \cdot e^t}{p \cdot e_t}$$

and

$$\rho p \cdot e^{t+1} \geq p \cdot e^t .$$

Let $\eta > \lambda > 0$ be given by the hypothesis of e-smooth supportability at $1 > \rho > 0$ and let $\epsilon = \lambda (1 - \rho)$. It follows that, for every t in $\mathcal{T}$,

$$\epsilon \frac{p \cdot e^t}{p \cdot e_t} \leq \lambda (1 - \rho) \frac{p \cdot e^t}{p \cdot e_t} \leq \lambda < \eta.$$

In addition, given any t in $\mathcal{T}$, for every individual i in $\mathcal{G}_{t+1}$,

$$\rho p \cdot \left(z^i - x^i \right)^+ = \epsilon \rho p \cdot e^{t+1} \geq \epsilon p \cdot e^t = p \cdot \left(z^i - x^i \right)^- ,$$

so that $z^i \succeq^i x^i$. Feasibility straightly obtains by the hypothesis of a constant cardinality of generations, because of the simple structure of the reallocation. This proves the claim. □