A general decomposition formula for derivative prices in stochastic volatility models

Elisa Alòs
Universitat Pompeu Fabra
C/ Ramón Trias Fargas, 25-27
08005 Barcelona

Abstract

We see that the price of an european call option in a stochastic volatility framework can be decomposed in the sum of four terms, which identify the main features of the market that affect to option prices: the expected future volatility, the correlation between the volatility and the noise driving the stock prices, the market price of volatility risk and the difference of the expected future volatility at different times. We also study some applications of this decomposition.

Keywords: continuous-time option pricing model, stochastic volatility, Itô's formula, incomplete markets.

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1 Introduction

Ever since the work of Black and Scholes [BS] and Merton [M] option pricing theory has motived a great deal of attention from reseachers of various disciplines. The original work of Black and Scholes assumes that the stock prices X_t satisfy a stochastic differential equation of the form

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

where μ and σ are constants W is a standard Brownian motion. The parameter σ is called the *volatility* of the model.

It is widely recognized that this model provides a less-than-perfect description of the real world. In particular, the constant volatility assumption is clearly not true from empirical studies. If option prices in the market were conformable with this model, all the Black-Scholes implied volatilities corresponding to the same asset would coincide with the volatility parameter σ of the underlaying asset. In reality this is not the case, and he Black-Scholes implied volatility heavily depends on the calendar time, the time to maturity and the moneyness of the option.

The natural extension of the Black-Scholes model that has been pursued in the literature and in practice is to modify the specification of volatility to make it a stochastic process. In most cases, σ is repaced by $\sigma(t) = f(Y_t)$, where f is a given function and Y_t satisfies a stochastic differential equation driven by another (maybe correlated) Brownian motion . Some examples of modelling are: Hull and White ([HW]), Stein and Stein ([SS]) and Heston ([H]).

It is difficult in general to choose a specific stochastic volatility model and to develop closed forms for option pricing in this framework. Nevertheless, one feature that most models seem to like is *mean reversion*. The volatility tends to fluctuate at a high level for a while, then at a low level for a similar period, then high again, and so on. Taking into account this fact, in [FPS] the authors developed a method for approximate derivative prices that it is valid for fast mean reverting volatilities. They identify the important groupings of market parameters, which otherwise are not obvious, and they turn out that estimation of these composites from market data is extremely efficient and stable. Furthermore, the metodology is robust and it does not assume a specific volatility model. The basic idea is to work in large time intervals, where we can assume that the mean reversion is fast and then the constant-volatility Black-Scholes model (with a correction to account for random volatility) is a good approximation.

The main goal of this paper is to prove a general decomposition formula for derivative prices in the stochastic volatility framework. By means of Itô's formula we see that the option price of an european call can be decomposed as the sum of several factors depending on the basic properties of the market:

1. the expected value (at present time) of the mean-square time average of the volatility process until the expiration time,
2. the correlation between the volatility process and the Brownian motion W,
3. the market price of volatility risk, and
4. the difference between the expected value of the mean-square time average of the volatility at different times.

This formula gives us a way to understant the influence of each one of the above factors to the final option price. As a particular application we deduce an approximation option pricing formula for the case of fast mean reverting volatilities that coincides with the one presented in [FPS].

2 Preliminaries on option pricing

We will consider the following model for stock prices

$$dX_t = \mu X_t dt + \sigma_t X_t dW_t, \ t \in [0, T] \tag{1}$$

where μ is a constant, W_t is an standard Brownian motion and σ_t is a square integrable process adapted to the same filtration as W_t (that we will denote by $\mathcal{F}_t$). An *European call option* is a contract that gives its holder the right, but not the obligation, to buy one unit of an underlying asset for a predetermined *strike price* K on the *maturity time* T. The asset is assumed not to pay dividends and there are not transaction costs. It follows that the value of this contract at time T, its *payoff*, is given by the quantity $h(X_T) = (X_T - K_T)_+$. At time $t < T$ this contract has a value, known as the *derivative price* V_t, which will vary with t and the observed stock prices until time t.

In the next subsection we present the general methodology for option pricing based in the Girsanov transformation. We will denote by r the interest rate, that we will assume constant.

2.1 Pricing with equivalent measures

Suppose that there exists a probability distribution P^* equivalent to the original one P under which the discounted stock price process $\tilde{X}_t = \mathrm{e}^{-rt} X_t$ is a martingale. It is well-known that if we price an european call by the formula

$$V_t = e^{-r(T-t)} E^* \left[(X_T - K_T)_+ \middle| \mathcal{F}_t \right], \tag{2}$$

where E^* denotes de expectation with respect to P^*, there is no arbitrage oportunity. Thus V_t is a possible price for this derivative.

Let us now construct equivalent martingale measures. As the process $\tilde{X}_t$ satisfies the equation

$$d\tilde{X}_t = (\mu - r) \tilde{X}_t dt + \sigma_t \tilde{X}_t dW_t$$

we need, in order to absorb the drift term of $\tilde{X}_t$ in its martingale term, to set

$$W_t^* = W_t + \int_0^t \frac{(\mu - r)}{\sigma_s} ds.$$

On the other hand, if we assume that $\sigma(t)$ depend on a second independent Brownian motion Z, any transformation of the form

$$Z_t^* = Z_t + \int_0^t \gamma_s ds,$$

where γ_s is a process such that the integral $\int_0^t \gamma_s ds$ is well defined, will not change the drift of $\tilde{X}_t$. By Girsanov's theorem we know that, if $\frac{(\mu - r)}{\sigma_s}$ and γ_s are adapted and bounded processes there exists a probability distribution P^* equivalent to the original one such that W_t^* and Z_t^* are independent standard Brownian motions. Notice that any allowable choice of γ leads then to an equivalent martingale measure and to a different no arbitrage derivative price. This process γ is called the *risk premium factor* or the *market price of volatility risk.*

Much research has investigated the range of possible prices in general settings. The approach that we will follow here is the same as used in [FPS], where it is assumed that the market selects a unique equivalent martingale measure under which derivative contracts are priced. Notice that the value of the market's price of the volatility risk γ can be seen only in derivative prices, since γ does not feature in the real world for the stock price.

2.2 Itô's formula

The main technical used in this paper is the following classical Itô's formula for semimartingale processes. We refer to [KS] for a more complete explanation on martingale integral calculus.

Theorem 1 *(Itô's formula) Consider*

$$A = \left\{ \left(A_t^1, ..., A_t^n \right), \ t \in [0,T] \right\}$$

a n-dimensional semimartingale. That is, $A_t^i = A_0^i + B_t^i + M_t^i$,*where for all* $i = 1, .., n$, B_t^i *is a bounded-variation process and* M_t^i *is a martingale. Then, for all* $f \in \mathcal{C}_b^{1,2,...,2}(\mathsf{R}^n)$ *and for all* $t \in [0,T]$

$$\begin{aligned} & f\left(t, A_t^1, ..., A_t^n\right) \\ = \ & f\left(t, A_0^1, ..., A_0^n\right) + \int_0^t \frac{\partial f}{\partial s}\left(s, A_s^1, ..., A_s^n\right) ds \\ & + \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}\left(s, A_s^1, ..., A_s^n\right) dB_s^i + \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}\left(s, A_s^1, ..., A_s^n\right) dM_s^i \\ & + \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_i}\left(s, A_s^1, ..., A_s^n\right) d\left\langle M^i, M^j \right\rangle_s, \end{aligned} \tag{3}$$

where the differential dM_s^i *is interpreted in the Itô sense and* $\left\langle M^i, M^j \right\rangle_s$ *denotes the covariation process of* M^i *and* M^j.

Using the characterization of option prices as conditional expectations and the above Itô's formula we will develop in the next section the main result of this paper.

3 The general decomposition formula

The main goal of this section is to decompose the price of an european option in the sum of several terms, which allows us to identify the influence of the basic properties of the market on the option price. We will use the following notation:

- $v_t^2 := \frac{1}{T-t} \int_t^T E\left(\sigma_s^2 \middle| \mathcal{F}_t\right) ds$. That is, v_t^2 will denote the expected value (at present time) of the mean-square time average of the volatility process until the expiration time.

- $M_t := \int_0^T E\left(\sigma_s^2 \middle| \mathcal{F}_t\right) ds$. Notice that $v_t^2 = \frac{1}{T-t}\left(M_t - \int_0^t \sigma_s^2 ds\right)$.
- $C_{BS}(t,x;\sigma)$ will denote the Black-Scholes function with constant volatility equal to σ, current stock price x, time to maturity $T-t$, strike price K adn interest rate r. That is,

$$C_{BS}(t,x,\sigma) = xN(d_1) - Ke^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\log(x/K) + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma(T-t)},$$

and

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy.$$

- $\mathcal{L}_{BS}(\sigma)$ will denote the Black-Scholes differential operator with volatility σ :

$$\mathcal{L}_{BS}(\sigma) = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right).$$

It is well-known that $\mathcal{L}_{BS}(\sigma)\, C_{BS}(\cdot,\cdot;\sigma) = 0$.

We will assume also the following hypothesis:

(H) For all $t \in [0,T]$, $M_t = M_t^* + \Lambda_t$ where, under P^*, M_t^* is a martingale and Λ_t is a bounded variation process.

Now we are in a position to prove the main result of this paper.

Theorem 2 *Assume the model (1), where $\sigma = \{\sigma_s,\ s \in [0,T]\}$ is an adapted and square integrable process such that hypothesis (H) holds. Then, for all $t \in [0,T]$*

$$\begin{aligned} V_t &= C_{BS}(t, X_t, \nu_t) + E^*\left\{ \int_t^T e^{-r(s-t)} X_s^2 \frac{\partial^2 P_0}{\partial x^2}(s, X_s, v_s)\, d\Lambda_s \right. \\ &+ \int_t^T e^{-r(s-t)} X_s \frac{\partial}{\partial x}\left(X_s^2 \frac{\partial^2 P_0}{\partial x^2}(s, X_s, v_s)\right) \sigma_s d\langle W, M\rangle_s \\ &+ \left. \int_t^T e^{-r(s-t)} \left(X_s^2 \frac{\partial^2}{\partial x^2}\left(X_s^2 \frac{\partial^2 P_0}{\partial x^2}(s, X_s, v_s)\right)\right) d\langle M, M\rangle_s \middle| \mathcal{F}_t \right\} \end{aligned} \quad (4)$$

Proof. Notice that $C_{BS}(T, X_T, \nu_T) = V_T$. As $e^{-rt}V_t$ is a P^*-martingale we can write

$$e^{-rt}V_t = E^*\left(e^{-rT}V_T \middle| \mathcal{F}_t\right) = E^*\left(e^{-rT}C_{BS}(T, X_T, \nu_T) \middle| \mathcal{F}_t\right) \quad (5)$$

Let us consider the process $e^{-rt}C_{BS}(t, X_t, \nu_t)$. Applying Itô's formula (3)and taking into account that

$$\frac{\partial C_{BS}}{\partial \sigma}(s, X_s, v_s)\frac{1}{2(T-s)v_s} = X_s^2 \frac{\partial^2 C_{BS}}{\partial x^2}(s, X_s, v_s)$$

we deduce that

$$\begin{aligned}
& e^{-rT}C_{BS}(T, X_T, \nu_T) \\
= \; & e^{-rt}C_{BS}(t, X_t, \nu_t) \\
& + \int_t^T e^{-rs}\left(\mathcal{L}_{BS}(v_s) + \frac{1}{2}\left(\sigma_s^2 - v_s^2\right) X_s^2 \frac{\partial^2}{\partial x^2}\right) C_{BS}(s, X_s, v_s)\, ds \\
& + \int_t^T e^{-rs}\left(\frac{\partial C_{BS}}{\partial x}\right)(s, X_s, v_s)\, \sigma_s X_s dW_s \\
& + \int_t^T e^{-rs} X_s^2 \frac{\partial^2 C_{BS}}{\partial x^2}(s, X_s, v_s)\left(\nu_s^2 - \sigma_s^2\right) ds \\
& + \int_t^T e^{-rs} X_s^2 \frac{\partial^2 C_{BS}}{\partial x^2}(s, X_s, v_s)\, dM_s^* \\
& + \int_t^T e^{-rs} X_s^2 \frac{\partial^2 C_{BS}}{\partial x^2}(s, X_s, v_s)\, d\Lambda_s \\
& + \int_t^T e^{-rs} \frac{\partial}{\partial x}\left(X_s^2 \frac{\partial^2 C_{BS}}{\partial x^2}(s, X_s, v_s)\right) \sigma_s X_s d\langle W, M_s\rangle \\
& + \int_t^T e^{-rs}\left(X_s^2 \frac{\partial^2}{\partial x^2}\left(X_s^2 \frac{\partial^2 C_{BS}}{\partial x^2}(s, X_s, v_s)\right)\right) d\langle M, M\rangle_s, \qquad (6)
\end{aligned}$$

from where it follows that

$$\begin{aligned}
& e^{-rT}C_{BS}(T, X_T, \nu_T) \\
= \; & e^{-rt}C_{BS}(t, X_t, \nu_t) \\
& + \int_t^T e^{-rs}\left(\frac{\partial C_{BS}}{\partial x}\right)(s, X_s, v_s)\, \sigma_s X_s dW_s \\
& + \int_t^T e^{-rs} X_s^2 \frac{\partial^2 C_{BS}}{\partial x^2}(s, X_s, v_s)\, dM_s^* \\
& + \int_t^T e^{-rs} X_s^2 \frac{\partial^2 C_{BS}}{\partial x^2}(s, X_s, v_s)\, d\Lambda_s \\
& + \int_t^T e^{-rs} \frac{\partial}{\partial x}\left(X_s^2 \frac{\partial^2 C_{BS}}{\partial x^2}(s, X_s, v_s)\right) \sigma_s X_s d\langle W, M_s\rangle \\
& + \int_t^T e^{-rs}\left(X_s^2 \frac{\partial^2}{\partial x^2}\left(X_s^2 \frac{\partial^2 C_{BS}}{\partial x^2}(s, X_s, v_s)\right)\right) d\langle M, M\rangle_s. \qquad (7)
\end{aligned}$$

Then, taking conditional expectation with respect to $\mathcal{F}_t$ in both sides of (7) we deduce that

$$\begin{aligned}
& E^*\left(e^{-rT}C_{BS}(T,X_T,\nu_T)\middle|\,\mathcal{F}_t\right) \\
= \; & e^{-rt}C_{BS}(t,X_t,\nu_t) \\
& +E^*\left\{\int_t^T e^{-rs}X_s^2\frac{\partial^2 C_{BS}}{\partial x^2}(s,X_s,v_s)\,d\Lambda_s\right. \\
& +\int_t^T e^{-rs}X_s\frac{\partial}{\partial x}\left(X_s^2\frac{\partial^2 C_{BS}}{\partial x^2}(s,X_s,v_s)\right)\sigma_s d\left\langle W,M\right\rangle_s \\
& +\left.\int_t^T e^{-rs}\left(X_s^2\frac{\partial^2}{\partial x^2}\left(X_s^2\frac{\partial^2 C_{BS}}{\partial x^2}(s,X_s,v_s)\right)\right)d\left\langle M,M\right\rangle_s\middle|\,\mathcal{F}_t\right\}.
\end{aligned}$$

Now, multiplying by e^{rt} and taking into account(5)the result follows. ■

4 Applications

The pourpose of this section is to comment how the above decomposition formula can be applied to estimate option prices in different frameworks.

4.1 Approximate option pricing formulas in the fast mean-reverting case

Suppose that the volatility model can be written as $\sigma_t = f(Y_t)$, being Y_t an Ornstein-Uhlenbeck process of the form

$$dY_t = -\alpha(m - Y_t)\,dt + c\sqrt{\alpha}dB_t, \tag{8}$$

where $\alpha > 0$, m and c are constants and B_t is a Brownian motion. The coefficient α is called the *rate of mean reversion* and m is the *long-rum mean level* of Y. The drift term pulls Y toward m, so we sould expect that σ_t is pulled toward the mean value of $f(Y_t)$ with respect to the long-run distribution of Y. The solution of (8) can be explicitly written in terms of its starting value y as

$$Y_t = m + (y - m)\,e^{-\alpha t} + c\sqrt{\alpha}\int_0^t e^{-\alpha(t-s)}dB_s, \tag{9}$$

from where we deduce that Y_t is a Gaussian process such that

$$E(Y_t) = m + (y - m)\,e^{-\alpha t}, Var(Y_t) = c^2\left(1 - e^{-2\alpha t}\right)$$

and, for all $s > t$,

$$Cov(Y_t, Y_s) = c^2 e^{-\alpha(s-t)}\left(1 - e^{-2\alpha t}\right).$$

Under its invariant distribution, $Y_t \sim \mathcal{N}\left(m, c^2\right)$ and $Cov(Y_t, Y_s) = c^2 e^{-\alpha|s-t|}$. For every real function f, we will denote by $\left\langle f^2\right\rangle$ the expectation of f^2 under this invariant distribution. We notice that the decorrelation is of exponential type, that is, the process Y_t has *short-memory.* From this property we can deduce the following result

Lemma 3 *Consider Y_t the Ornstein-Uhlenbeck process defined by (8). Assume that the volatility process is given by $\sigma_t = f(Y_t)$, where $f \in \mathcal{C}_b^2(\mathsf{R})$. Then $\left|\nu_t^2 - \langle f^2 \rangle\right| \leq C_t (\alpha (T-t))^{-1}$, for some positive and $\mathcal{F}_t$-measurable random variable C_t.*

Proof. Notice that

$$Y_s = m + (Y_t - m) e^{-\alpha(s-t)} + c\sqrt{\alpha} \int_t^s e^{-\alpha(s-r)} dB_r. \tag{10}$$

Then can write

$$\begin{aligned}
& E\left(\sigma_s^2 \middle| \mathcal{F}_t\right) - \langle f^2 \rangle \\
= \; & E\left\{ f^2\left(m + (Y_t - m) e^{-\alpha(s-t)} + c\sqrt{\alpha} \int_t^s e^{-\alpha(s-r)} dB_r\right)\right. \\
& \left. - f^2\left(m + c\sqrt{\alpha}\int_t^s e^{-\alpha(s-r)} dB_r\right) \middle| \mathcal{F}_t \right\} \\
& + E\left(f^2\left(m + c\sqrt{\alpha}\int_t^s e^{-\alpha(s-r)} dB_r\right)\right) - \langle f^2 \rangle
\end{aligned} \tag{11}$$

As $f \in \mathcal{C}_b^2(\mathsf{R})$ we can write

$$\begin{aligned}
& \left| f^2\left(m + (Y_t - m) e^{-\alpha(s-t)} + c\sqrt{\alpha} \int_t^s e^{-\alpha(s-r)} dB_r\right)\right. \\
& \left. - f^2\left(m + c\sqrt{\alpha}\int_t^s e^{-\alpha(s-r)} dB_r\right) \right| \\
\leq \; & c \left| (Y_t - m) \right| e^{-\alpha(s-t)}.
\end{aligned} \tag{12}$$

On the other hand,

$$\begin{aligned}
& E\left(f^2\left(m + c\sqrt{\alpha}\int_t^s e^{-\alpha(s-r)} dB_r\right)\right) - \langle f^2 \rangle \\
= \; & E\left[f^2\left(m + c\sqrt{\alpha}\int_t^s e^{-\alpha(s-r)} dB_r\right)\right. \\
& \left. - f^2\left(m + c\sqrt{\alpha}\int_{-\infty}^s e^{-\alpha(s-r)} dB_r\right)\right] \\
\leq \; & C\sqrt{\alpha} E\left(\left|\int_{-\infty}^t e^{-\alpha(s-r)} dB_r\right|^2\right)^{\frac{1}{2}} \\
\leq \; & \frac{c e^{-\alpha(s-t)}}{\sqrt{2\pi}}.
\end{aligned} \tag{13}$$

Now, taking into account (11), (12) and (13) it follows that

$$\begin{aligned} \left|\nu_t^2 - \langle f^2\rangle\right| &= \left|\frac{1}{T-t}\int_t^T E\left(\sigma_s^2\middle|\mathcal{F}_t\right) - \langle f^2\rangle\right| ds \\ &\le C_t\frac{1}{T-t}\int_t^T e^{-\alpha(s-t)}ds \\ &\le \frac{C_t}{\alpha\left(T-t\right)}, \end{aligned}$$

and now the proof is complete. ■

Remark 4 *The above lemma proves that the term $C_{BS}(t,X_{t,}\nu_t)$ can be approximated by $C_{BS}(t,X_{t,}\sqrt{\langle f^2\rangle})$, being this approximation of order $\left(\alpha\left(T-t\right)\right)^{-1}$. This means that if the rate of mean reversion and the time interval are big enought, this approximation is good. The advantatge is that the contant $\sqrt{\langle f^2\rangle}$ can be easily estimated from historical volatilities, without assuming an specific volatility model.*

Once we have found an easy-to-evaluate approximation to $C_{BS}(t,X_{t,}\nu_t)$ the natural questions are if the remaining terms have an important influence in the option price, and if yes, if it is possible to find a good approximation for them. In the next lemma we find an answer to the first question.

Lemma 5 *Assume that for all $t\in[0,T]$ the volatility process is given by $\sigma_t = f\left(Y_t\right)$, where Y_t is an Ornstein-Uhlenbeck process defined by (8) and f is a real function in $\mathcal{C}_b^2\left(\mathsf{R}\right)$ such that the function $\frac{1}{f}$ is bounded. Assume also that, under P^*, $B_t = B_t^* + \int_0^t \gamma_s ds$ for some adapted and a.s. bounded process $\gamma = \{\gamma_s, s\in[0,T]\}$. Then Hypothesis (H) holds and, for all $t\in[0,T]$*

$$E^*\left(\int_t^T e^{-r(s-t)}X_s^2\frac{\partial^2 C_{BS}}{\partial x^2}\left(s,X_s,v_s\right)d\Lambda_s\middle|\mathcal{F}_t\right)\le c\alpha^{-\frac{1}{2}}, \tag{14}$$

$$E^*\left(\int_t^T e^{-r(s-t)}X_s\frac{\partial}{\partial x}\left(X_s^2\frac{\partial^2 C_{BS}}{\partial x^2}\left(s,X_s,v_s\right)\right)\sigma_s d\left\langle W,M\right\rangle_s\middle|\mathcal{F}_t\right)\le c\alpha^{-\frac{1}{2}} \tag{15}$$

and

$$E^*\left(\int_t^T e^{-r(s-t)}\left(X_s^2\frac{\partial^2}{\partial x^2}\left(X_s^2\frac{\partial^2 C_{BS}}{\partial x^2}\left(s,X_s,v_s\right)\right)\right)d\left\langle M,M\right\rangle_s\middle|\mathcal{F}_t\right)\le c\alpha^{-1}. \tag{16}$$

Proof. In order to simplify the proof we will assume that $m=y=0$. Now the proof will be decomposed into several steps.

Step 1. Let us see first that Hypothesis (H) holds. For all $s > t$ we can write

$$\begin{aligned} Y_s &= c\sqrt{\alpha}\int_0^t e^{-\alpha(s-r)}dB_r + c\sqrt{\alpha}\int_t^s e^{-\alpha(s-r)}dB_r \\ &= : Z(s,t) + c\sqrt{\alpha}\int_t^s e^{-\alpha(s-r)}dB_r \end{aligned}$$

which implies that

$$\begin{aligned} &\int_0^T E\left(f^2(Y_s)\middle|\mathcal{F}_t\right)ds \\ = &\int_t^T \left(\int_{\mathsf{R}} f^2(x)\, g_{Z(s,t),\frac{c}{2}\sqrt{1-e^{-2\alpha(s-t)}}}(x)\,dx\right)ds + \int_0^t f^2(Y_s)\,ds, \end{aligned}$$

where

$$g_{m,\sigma}(x) := \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$$

By classical Itô's formula (3) it is easy to deduce then that

$$\begin{aligned} &\int_0^T E\left(f^2(Y_s)\middle|\mathcal{F}_t\right)ds \\ = &\int_0^T E\left(f^2(Y_s)\right)ds + 2c\sqrt{\alpha}\int_0^t\left(\int_r^T\left(\int_{\mathsf{R}} f^2(x)\,(x - Z(s,r))\right.\right. \\ &\left.\left.\times g_{Z(s,r),\sqrt{1-e^{-\alpha(s-r)}}}(x)dx\right)e^{-\alpha(s-r)}ds\right)dB_r \\ = &\int_0^T E\left(f^2(Y_s)\right)ds + 4c\sqrt{\alpha}\int_0^t\left(\int_r^T e^{-\alpha(s-r)}\left(\int_{\mathsf{R}}(ff')(x)\right.\right. \\ &\left.\left.\times g_{Z(s,r),\sqrt{1-e^{-\alpha(s-r)}}}(x)dx\right)ds\right)dB_r \\ = &\int_0^T E\left(f^2(Y_s)\right)ds + 4c\sqrt{\alpha}\int_0^t\left(\int_r^T e^{-\alpha(s-r)}E\left(((ff')(Y_s)\ \mathcal{F}_r\right)ds\right)B_r, \end{aligned}$$

which gives us that Hypothesis (H) holds with

$$M_t^* = \int_0^T E\left(f^2(Y_s)\right)ds + 4c\sqrt{\alpha}\int_0^t\left(\int_r^T e^{-\alpha(s-r)}E\left(((ff')(Y_s)\ \mathcal{F}_r\right)ds\right)B_r^*$$

and

$$\Lambda_t = 4c\sqrt{\alpha}\int_0^t\left(\int_r^T e^{-\alpha(s-r)}E\left(((ff')(Y_s)\ \mathcal{F}_r\right)ds\right)\gamma(r)\,dr.$$

Step 2. Let us prove inequality (14). The proof of (15) and (16) would be similar. It is easy to check that $x^2 \frac{\partial^2 C_{BS}}{\partial x^2}(s,x,y)$ is a bounded function. Then

$$\begin{aligned}
&\left| E^* \left(\int_t^T e^{-r(s-t)} X_s^2 \frac{\partial^2 C_{BS}}{\partial x^2}(s, X_s, v_s)\, d\Lambda_s \right| \mathcal{F}_t \right) \Bigg| \\
\leq \;& cE^* \left(\int_t^T |d\Lambda_s| \right| \mathcal{F}_t \Bigg) \\
= \;& C\sqrt{\alpha} E^* \left(\int_t^T \left(\int_r^T e^{-\alpha(s-r)} \left| E\left(((ff')(Y_s)\, \mathcal{F}_r \right) \right| ds \right) |\gamma(r)|\, dr \right| \mathcal{F}_t \Bigg) \\
\leq \;& C\sqrt{\alpha} \int_t^T \left(\int_r^T e^{-\alpha(s-r)} ds \right) dr \\
\leq \;& C\alpha^{-\frac{1}{2}},
\end{aligned}$$

and now the proof is complete. ■

The above lemma shows that $C_{BS}(t, X_t, \nu_t)$ (and then $C_{BS}(t, X_t, \sqrt{\langle f^2 \rangle})$) is an approximation of order $\alpha^{-1/2}$ for the option price. The next propostion will show us how to find an approximation of order α^{-1}.

Proposition 6 *Assume that for all $t \in [0,T]$ the volatility process is given by $\sigma_t = f(Y_t)$, where Y_t is an Ornstein-Uhlenbeck process defined by (8). Assume also that, under P^*, $B_t = B_t^* + \int_0^t h(Y_s)\, ds$, for some bounded function h. Then there exists two constants V_2 and V_3 such that*

$$\begin{aligned}
&C_{BS}(t, X_t, \sqrt{\langle f^2 \rangle}) \\
&+(T-t) \left\{ V_2 \left(X_t^2 \frac{\partial^2 C_{BS}}{\partial x^2}(t, X_t, \sqrt{\langle f^2 \rangle}) \right) \right. \\
&\left. + V_3 \left(X_t^2 \frac{\partial^2 C_{BS}}{\partial x^2}(t, X_t, \langle f^2 \rangle) + X_t^3 \frac{\partial^3 C_{BS}}{\partial x^3}\left(t, X_t, \sqrt{\langle f^2 \rangle}\right) \right) \right\}
\end{aligned} \tag{17}$$

is an approximation of order α^{-1} for the option price V_t.

Proof. By going on the developement of Formula (4) we can write

$$\begin{aligned}
V_t = \;& C_{BS}(t, X_t, \nu_t) \\
& + X_t^2 \frac{\partial^2 C_{BS}}{\partial x^2}(t, X_t, v_t)\, E^* \left(\int_t^T d\Lambda_s \right| \mathcal{F}_t \Bigg) \\
& + X_t \frac{\partial}{\partial x} \left(X_t^2 \frac{\partial^2 C_{BS}}{\partial x^2}(t, X_t, v_t) \right) E^* \left(\int_t^T \sigma_s d \langle W, M \rangle_s \right| \mathcal{F}_t \Bigg) \\
& + R_t,
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
R_t \quad : \quad &= E^* \left\{ \int_t^T \left(e^{-r(s-t)} X_s^2 \frac{\partial^2 C_{BS}}{\partial x^2}(s, X_s, v_s) - X_t^2 \frac{\partial^2 C_{BS}}{\partial x^2}(t, X_t, v_t) \right) d\Lambda_s \right. \\
&+ \int_t^T \left(e^{-r(s-t)} X_s \frac{\partial}{\partial x} \left(X_s^2 \frac{\partial^2 C_{BS}}{\partial x^2}(s, X_s, v_s) \right) \right. \\
&\left. - X_t \frac{\partial}{\partial x} \left(X_t^2 \frac{\partial^2 C_{BS}}{\partial x^2}(t, X_t, v_t) \right) \right) \sigma_s d \langle W, M \rangle_s \\
&+ \left. \int_t^T e^{-r(s-t)} \left(X_s^2 \frac{\partial^2}{\partial x^2} \left(X_s^2 \frac{\partial^2 C_{BS}}{\partial x^2}(s, X_s, v_s) \right) \right) d \langle M, M \rangle_s \right| \mathcal{F}_t \Bigg\} . (19)
\end{aligned}$$

Applying again Itô's formula and following the same steps as in the proof of Lemma 5 we can see that $R_t \leq C_t \alpha^{-1}$. On the other hand, Expression (18) can be written as

$$\begin{aligned}
& C_{BS}(t, X_t, \nu_t) \\
& + (T-t) \left\{ X_t^2 \frac{\partial^2 C_{BS}}{\partial x^2}(t, X_t, v_t) E^* \left(\frac{1}{T-t} \int_t^T d\Lambda_s \middle| \mathcal{F}_t \right) \right. \\
& + X_t^2 \frac{\partial^2 C_{BS}}{\partial x^2}(t, X_t, v_t) E^* \left(\frac{1}{T-t} \int_t^T \sigma_s d \langle W, M \rangle_s \middle| \mathcal{F}_t \right) \\
& \left. + X_t^3 \frac{\partial^3 C_{BS}}{\partial x^3}(t, X_t, v_t) E^* \left(\frac{1}{T-t} \int_t^T \sigma_s d \langle W, M \rangle_s \middle| \mathcal{F}_t \right) \right\} . \qquad (20)
\end{aligned}$$

As $\gamma(r) = h(Y_r)$, using the same kind of arguments as in the proof of Lemma 3 we can see that

$$\left| E^* \left(\frac{1}{T-t} \int_t^T d\Lambda_s \middle| \mathcal{F}_t \right) - V_2 \right| \leq C_t \left(\alpha (T-t) \right)^{-1}$$

and

$$\left| E^* \left(\frac{1}{T-t} \int_t^T \sigma_s d \langle W, M \rangle_s \middle| \mathcal{F}_t \right) - V_3 \right| \leq C_t \left(\alpha (T-t) \right)^{-1} .$$

This allows us to complete the proof. ■

Remark 7 *This approximation formula coincides with the one obtained by Fouque, Papanicolau and Sircar in [FPS]. The constants V_2 and V_3 can be calibrated from implied volatility, as it is studied in [FPS].*

Remark 8 *A natural question is: Why not proceed with the above asymptotic expansion, obtaining more accurate approximations? The answer is the following: $C_{BS}(t, X_t, \bar{\sigma})$ approximates $C_{BS}(t, X_t, \nu_t)$ up to the order α^{-1}, and it is necessary to fix a specific volatility model to find better estimates for this term. That is. it is not possible to obtain higher-order approximation formulas without choosing a concrete volatility model.*

Remark 9 *In order to capture the long-memory properties of the volatility observed in some financial markets, some authors have proposed to extend the Black-Scholes model to the case where the volatility is modelled as a fractional Ornstein-Uhlenbeck process driven by a fBm (see for example [CR] and [Hu]). It is intuitive that in this case the approximation formula obtained in the last section is not going to be as accurated as in the short-memory case, because here the mean reversion is not so fast and the pricing formula will depend not only on the present stock and volatility levels but also in all the past. It becomes interesting to study how Formula (4) can be used in this case. This is the subject of the work [A].*

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