Optimal Second-degree Price Discrimination under Arbitrage: On the Role of Asymmetric Information among Buyers*

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Abstract

The traditional theory of second-degree price discrimination tackles individual self-selection but does not address the possibility that buyers could form a coalition to do arbitrage, that is, to coordinate their purchases and to reallocate the goods. In this paper, we design the optimal sale mechanism which takes into account both individual and coalition incentive compatibility when buyers can form a coalition under asymmetric information. We show that the monopolist can achieve the same profit regardless of whether or not buyers can form a coalition. Although, in the optimal sale mechanism, marginal rates of substitution are not equalized across buyers of different types (hence there exists potential room for arbitrage), they fail to realize the gains from arbitrage because of the transaction costs in coalition formation generated by asymmetric information.

JEL Classification: D42, D82, L12

Key Words: Second-degree Price Discrimination, Arbitrage, Coalition Incentive Compatibility, Asymmetric Information, Transaction Costs.

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1 Introduction

The theory of monopolistic screening[1] (second-degree price discrimination) studies a monopolist's optimal pricing scheme when she has incomplete information about buyers' individual preferences.[2] According to the theory, the monopolist can maximize her profit by using a menu of packages which induces each type of buyer to select the package designed for the type. While the theory tackles the self-selection issue at the individual level, it assumes away the possibility that price discrimination might induce buyers to form coalitions to do arbitrage, that is, to coordinate their purchases and to reallocate the goods they bought among themselves. Since this might reduce the seller's profit, in this paper we study the optimal sale mechanism which takes into account not only individual incentive compatibility but also coalition incentive compatibility (i.e., buyers' incentive to collectively engage in arbitrage). In particular, in addressing this fundamental and fascinating problem, we focus on the role of asymmetric information among buyers about each other's preferences.

In reality, there exists much evidence of (legal or illegal) coalitions among buyers. On the one hand, bidders' collusive behavior in auctions is well documented and auction literature has been devoting an increasing attention to the topic.[3] On the other hand, buyers often form cooperatives to jointly purchase goods.[4] One central question regarding buyer coalitions is how asymmetric information among the buyers affects coalition formation. Our major goal is to identify the transaction costs in coalition formation generated by asymmetric information and to find the sale mechanism which best exploits these transaction costs.

Consider for example the situation in which an upstream monopolist sells her goods to two downstream firms operating in separate markets. Given a menu of quantity-transfer pairs offered by the monopolist, the two downstream firms can employ two instruments to increase their joint payoffs. First, they can jointly decide which pair each buyer should choose. In our paper, this is modeled by manipulation of the reports which the buyers send into the sale mechanism. Second, they can reallocate among themselves the goods bought from the seller. We first show that under the standard optimal mechanism which neglects coalition incentive compatibility, buyers can increase their payoffs by engaging in arbitrage and this reduces the

[1] See, for instance, Maskin and Riley (1984) and Mussa and Rosen (1978) for an introduction and Rochet and Stole (2002) for a recent contribution dealing with random participation.

[2] We use 'she' to represent the monopolist and 'he' to represent a buyer or the third-party.

[3] For examples, see Caillaud and Jehiel (1998), Graham and Marshall (1987), McAfee and McMillan (1992) and Brusco and Lopomo (2002).

[4] There exist various forms of supply cooperatives to purchase some products together. For instance, Heflebower (1980) describes three types of supply cooperatives: farmers's cooperatives, consumer cooperatives and those run by urban businesses.

seller's profit. However, as the main result, we find an optimal mechanism which allows the monopolist to realize the same profit regardless of whether or not buyers can form a coalition to do arbitrage.

Consider for simplicity a two-buyer setting and suppose that the seller can produce any amount of a homogeneous product at a constant marginal cost and a buyer has either high valuation (H-type) or low valuation (L-type) for the product. Assume that types are independently and identically distributed and a buyer's type is his private information. It is well-known that in the optimal mechanism(s) without buyer coalition, the quantity allocated to H-type is equal to the first-best level while the quantity allocated to L-type is distorted downward compared to the first-best level since the payment the seller receives from H-type decreases in the quantity sold to L-type. This implies that L-type has a higher marginal surplus for the product than H-type and, if there are no transaction costs in coalition formation, buyers can increase their payoffs by reallocating some quantity from H-type to L-type (with a suitable money transfer from the latter to the former) in the state of nature in which one buyer has H-type and the other has L-type. This may alter ex ante buyers' incentives to report truthfully and reduce the seller's expected profit.

Drawing on Laffont and Martimort (1997, 2000), we model coalition formation under asymmetric information by a side-contract offered to the buyers by a third-party who maximizes the sum of buyers' payoffs. The side-contract specifies both the manipulation of the reports made into the sale mechanism and the reallocation of the goods obtained from the seller. The side-contract must satisfy budget balance, participation and incentive constraints. The incentive constraints need to hold since the third-party does not know the buyers' types; the acceptance constraints are defined with respect to the utilities the buyers obtain when playing the sale mechanism non-cooperatively.

We first consider *simple* mechanisms in which both the quantity that a buyer receives and his payment do not depend on the other buyer's report. We show that if the seller uses the simple mechanism which is optimal without buyer coalition, buyers can realize strict gains at the seller's loss by suitably arbitraging. For instance, when the both buyers have H-type (HH-coalition) they have an incentive to report HL instead of truthtelling and to reallocate quantities and transfers. To see this, note that under the optimal *simple* mechanism, H-type is indifferent between the quantity-transfer pair designed for H-type and the pair for L-type. This implies that if reallocation is impossible, HH-coalition is indifferent between reporting HL and truth-telling. However, if reallocation is feasible, under standard convexity assumptions on buyers' preferences, each buyer's payoff conditional on reporting HL strictly increases since they can share equally the total quantity and transfers. In contrast, conditional on reporting

HH, reallocation does not affect the payoffs since both buyers receive the same quantity from the seller. Therefore, HH-coalition prefers to report HL rather than HH.

After studying simple mechanisms, we consider the mechanisms in which the seller makes the payment of a buyer depend on the report of the other buyer. In particular, we focus on those transfers which keep the buyers' expected payments equal to the ones in the simple optimal mechanism, while the quantity profile is unchanged. It turns out that there exists a transfer scheme which allows the seller to deter manipulation of reports and reallocation of goods at no cost, thus letting her realize the same profit as when there is no buyer coalition. In particular, even if the marginal rates of substitution are not equalized across buyers with different types, the third party is not able to implement any efficient reallocation between H-type and L-type in HL-coalition because of the tension between incentive and participation constraints in the side-contract. The intuition for this result is as follows. Since the rent that H-type obtains by pretending to be L-type in the side mechanism increases in the quantity received by L-type, if the third-party reallocates some quantity from H-type to L-type then he is forced to concede H-type a higher rent in order to elicit a truthful report: the alternative of reducing L-type's payoff is impossible since it would induce L-type to reject the side-contract. This increase in the rent is defined as the transaction costs generated by asymmetric information. We quantify the transaction costs and show that they are larger than the gains from reallocating quantity from H-type to L-type; therefore the reallocation cannot be realized. We also show that this optimal outcome can be implemented by a menu of two-part tariffs. Finally, our main result that buyer coalition does not hurt the seller extends to more general settings: when the marginal cost is increasing, or there are n buyers, or there are three possible buyer types.

The literature about consumer coalitions mostly addresses issues different from the one we consider in this paper.[5] Alger (1999) is one exception: She studies the optimal menu of price-quantity pairs when (a continuum of) consumers are able to purchase multiple times or/and jointly in a two-type setting. She finds that with multiple purchases only, the monopolist offers strict quantity discounts while, with joint purchases only, discounts are infeasible. Her results are based on two following assumptions. First, consumer coalitions are formed under complete information among the consumers about each other's type and only consumers with the same type can form coalitions. Second, the set of mechanisms available to the seller is restricted by assuming that the quantity allocated to a consumer and his payment do not depend on the other consumers' choices. In contrast, in our model a coalition is formed under asymmetric

[5]For instance, Innes and Sexton (1993, 1994) analyze the case in which the monopolist is facing identical consumers who may form coalitions. They show that even though consumers' characteristics are homogeneous, the monopolist may price discriminate in order to deter the formation of coalitions, whereas price discrimination is unprofitable in the absence of the coalitions.

information among buyers and the seller can use complete contracts such that the quantity sold to a buyer and his payment can depend on the others' choices.

Using a third-party to model collusion under asymmetric information was first introduced in auction literature – see the first three papers mentioned in footnote 3. While that literature studies the optimal auction in a restricted set of mechanisms, usually finding the optimal reserve price for a first or second price auction, Laffont and Martimort (1997, 2000) use a more general approach in that they characterize the set of collusion-proof mechanisms and optimize in this set. In their settings reallocation is infeasible[6] and they show that if the agents' types are independently distributed, then a dominant-strategy mechanism implements the second-best outcome and eliminates any gain from joint manipulation of reports. Furthermore, this mechanism does not exploit the transaction costs created by asymmetric information. In our setting, the dominant-strategy mechanism is not collusion-proof since the coalition owns the additional instrument of quantity reallocation, but the seller can still achieve the second-best profit by fully exploiting the transaction costs in coalition formation. We also note that Laffont and Martimort limit the analysis to the two-agent-two-type setting and do not consider implementation through non-direct mechanisms.

Our paper is to some extent related to the papers studying auctions with resale. For instance, Ausubel and Cramton (1999) analyze the optimal auction when buyers can engage in resale after receiving goods from the seller and the resale is (assumed to be) always efficient. They prove that the seller maximizes his profit by allocating goods efficiently. In contrast, in our setting, buyers sign a binding side-contract before each buyer chooses how much to buy and they fail to achieve efficient reallocation because of the transaction costs.[7]

The rest of the paper is organized as follows. In Section 2, we introduce the model and in Section 3 we review as a benchmark the optimal sale mechanisms without buyer coalition. In Section 4 we prove that the simple optimal mechanism in which each buyer's allocation depends only on his own report leaves room for arbitrage such that buyer coalition reduces the seller's profit. In order to define the seller's optimization problem under collusion, still in Section 4 we introduce the (weakly) collusion-proofness principle and characterize the constraints that a collusion-proof mechanism must satisfy. In Section 5, we define and solve the seller's problem

[6] In the first paper, they consider two regulated firms producing complementary inputs. The firms have independently distributed types and collusion has bite since an exogenous restriction on the set of the principal's mechanisms is imposed. In the second paper, they consider collusion between consumers of a public good with correlated types. Consumers have incentives to collude since the principal will fully extract their rents if they behave non-cooperatively.

[7] Zheng (2002) allows resale in a one-good auction with asymmetrically distributed buyers' values and proves that an equilibrium exists which induces the same payoffs as if resale can be costlessly banned.

and prove our main result that these constraints can be satisfied without reducing the seller's profit. In Section 6, we extend the main result to more general settings. In Sections 4-6, we make some specific assumptions about buyers' off-the-equilibrium-path beliefs and behavior. In Section 7, we show that our main result is robust to relaxing these assumptions. Concluding remarks are given in Section 8. All of the proofs are left to Appendix except the proof of Proposition 5.

2 The model

2.1 Preferences, information and mechanisms

A seller (for instance, an upstream monopolist) can produce any amount $q \geq 0$ of homogeneous goods at cost $C(q)$ and sells the goods to $n \geq 2$ buyers (for instance, downstream firms operating in separate markets). Throughout the paper, we will interpret q as quantity except in Section 4, where we consider also the case in which q represents quality. Buyer i ($i = 1, ..., n$) obtains payoff $\mathcal{U}(q^i, \theta^i) - t^i$ from consuming quantity $q^i \geq 0$ of the goods and paying $t^i \in \mathbb{R}$ units of money to the seller. He privately observes his own type $\theta^i \in \Theta \equiv \{\theta_L, \theta_H\}$, where $\Delta\theta \equiv \theta_H - \theta_L > 0$. The types θ^i and θ^j are identically and independently distributed for any $i \neq j$, with $p_L \equiv \Pr\left\{\theta^i = \theta_L\right\} \in (0, 1)$ for $i = 1, ..., n$; the distribution of $(\theta^1, ..., \theta^n)$ is common knowledge. We suppose that $C(\cdot)$ and $\mathcal{U}(\cdot)$ are such that $C(0) = 0$, $C'(q) > 0$ and $C''(q) \geq 0$ for any $q \geq 0$; $\mathcal{U}(0, \theta) = 0$, $\mathcal{U}_1(q, \theta) > 0 > \mathcal{U}_{11}(q, \theta)$, $\mathcal{U}_2(q, \theta) > 0$ and $\mathcal{U}_{12}(q, \theta) > 0$ for any (q, θ), where subscripts denote partial derivatives and $\mathcal{U}_{12}(q, \theta) > 0$ is the standard Spence-Mirrlees single-crossing condition. Furthermore, we assume that $\frac{\mathcal{U}_1(0,\theta_L)}{p_L} - \frac{(1-p_L)\mathcal{U}_1(0,\theta_H)}{p_L} > C'(\bar{q})$, where $\bar{q}$ is the first best quantity for an H-type when he is the only buyer ($\bar{q}$ is defined as the unique solution to $\mathcal{U}_1(q, \theta_H) = C'(q)$). This condition guarantees that each type of buyer receives a positive quantity in the optimum without buyer coalition.[8] The reservation utility of each type of buyer i is given by $\mathcal{U}(0, \theta^i) - 0 = 0$, his payoff if he does not transact with the monopolist.

In what follows, for expositional simplicity, we focus on the case with $n = 2$ buyers, constant marginal cost $c(> 0)$ and $\mathcal{U}(q, \theta) = \theta u(q)$. However, our main result holds for any $n > 2$, any convex cost function and any $\mathcal{U}(q, \theta)$ with the properties described above and it also holds in the three-type setting with $\Theta \equiv \{\theta_L, \theta_M, \theta_H\}$. See Section 6 for all the extensions.

The seller designs a sale mechanism to maximize her expected profit. A generic sale mechanism is denoted by M and, according to the revelation principle, we can restrict our attention

[8] Our results below holds even when the seller finds it optimal to refuse to serve L-type.

to direct revelation mechanisms:

$$M = \left\{ q^i(\widehat{\theta}^1, \widehat{\theta}^2), t^i(\widehat{\theta}^1, \widehat{\theta}^2);\ i = 1, 2 \right\},$$

where $\widehat{\theta}^i \in \{\theta_L, \theta_H\}$ is buyer i's report, $q^i(\cdot)$ is the quantity he receives and $t^i(\cdot)$ is his payment to the seller. Since buyers are ex ante identical, without loss of generality we focus on symmetric mechanisms in which the quantity sold to a buyer and his payment depend only on the reports $(\widehat{\theta}^1, \widehat{\theta}^2)$ and not on his identity. Then, we can introduce the following notation to simplify the exposition: For quantities,

$$\begin{aligned} q_{HH} &= q^1(\theta_H, \theta_H) = q^2(\theta_H, \theta_H),\ q_{HL} = q^1(\theta_H, \theta_L) = q^2(\theta_L, \theta_H), \\ q_{LH} &= q^1(\theta_L, \theta_H) = q^2(\theta_H, \theta_L),\ q_{LL} = q^1(\theta_L, \theta_L) = q^2(\theta_L, \theta_L). \end{aligned}$$

$(t_{HH}, t_{HL}, t_{LH}, t_{LL}) \in \mathbb{R}^4$ are similarly defined. Let $\mathbf{q} \equiv (q_{HH}, q_{HL}, q_{LH}, q_{LL})$ denote the vector of quantities and $\mathbf{t} \equiv (t_{HH}, t_{HL}, t_{LH}, t_{LL})$ denote the vector of transfers.

The sale mechanisms we consider involve (second-degree) price discrimination. Although price discrimination can be illegal if it threatens to injure competition[9], in our context there is no such concern since the buyers operate in separate markets.

2.2 Buyer coalition

Drawing on Laffont and Martimort (1997, 2000), we model buyers' coalition formation by a side-contract, denoted by S, offered by a benevolent third-party. The third party designs S in order to maximize the sum of buyers' expected payoffs subject to incentive compatibility (since he does not observe the types) and participation constraints written with respect to the utility a buyer obtains when M is played non-cooperatively.

We assume that the seller is the first mover and can commit not to serve a buyer if the other buyer refuses M. This limits the strategies available to the buyer coalition: in particular, the third-party cannot employ the strategy of making only one buyer buy from the seller and share the goods bought with the other buyer.[10] Precisely, the game of seller's mechanism offer cum buyer coalition formation has the following timing.

Stage 1. Nature draws buyers' types (θ^1, θ^2); buyer i privately observes θ^i, $i = 1, 2$.

[9] This is the purpose of the Robinson-Patman Act.

[10] Alternatively, we may assume that if buyer 1 (say) does not accept M, then the seller can serve buyer 2 with a single-buyer mechanism. In this case, our results would still hold if the seller can observe whether or not a buyer uses her goods as in Rey and Tirole (1986). Since then the seller can induce buyer 2 not to resell to buyer 1 (part of) the goods he bought from the seller by specifying ex ante a high penalty for buyer 2, both buyers will buy from the seller in equilibrium.

Stage 2. The seller proposes a sale mechanism M.

Stage 3. Each buyer simultaneously accepts or rejects M. If at least one buyer refuses M, then each buyer realizes the reservation utility and the following stages do not occur.

Stage 4. If both buyers accept to play M, then the third party proposes them a direct side-contract S in order to jointly manipulate their reports into M and to reallocate between themselves the goods bought from the seller.[11]

Stage 5. Each buyer simultaneously accepts or rejects S.

Stage 6. If at least one buyer refuses S, then M is played non-cooperatively. In this case, reports are directly made in M and stages 7 and 9 below do not occur. If instead S has been accepted by both buyers, then reports are made into S.

Stage 7. As a function of the reports in S, the third party enforces the manipulation of reports into M.

Stage 8. Quantities and transfers specified in M are enforced.

Stage 9. Quantity reallocation and side-transfers specified in S (if any) take place in the buyer coalition.

Formally, a side-contract S takes the following form:

$$S = \{\phi(\widetilde{\theta}^1, \widetilde{\theta}^2), x^i(\widetilde{\theta}^1, \widetilde{\theta}^2, \widetilde{\phi}), y^i(\widetilde{\theta}^1, \widetilde{\theta}^2);\ i = 1, 2\},$$

where $\widetilde{\theta}^i \in \{\theta_L, \theta_H\}$ is buyer i's report to the third-party. $\phi(\cdot)$ is the report manipulation function which maps any pair of reports $(\widetilde{\theta}^1, \widetilde{\theta}^2)$ made by the buyers to the third-party into a pair of reports to the seller. We assume that $\phi(\cdot)$ can specify stochastic manipulations, as this convexifies the third-party's feasible set. More precisely, let $\widetilde{\phi} \in \Theta^2$ denote an outcome of $\phi(\cdot)$. Then, $\phi(\cdot)$ specifies the probability $p^\phi(\widetilde{\theta}^1, \widetilde{\theta}^2, \widetilde{\phi})$ that the third party, after receiving reports $(\widetilde{\theta}^1, \widetilde{\theta}^2)$, requires the buyers to report $\widetilde{\phi}$ to the seller. When the manipulation is deterministic, i.e., $p^\phi(\widetilde{\theta}^1, \widetilde{\theta}^2, \widetilde{\phi}) = 1$ for a $\widetilde{\phi} \in \Theta^2$, we write $\phi(\widetilde{\theta}^1, \widetilde{\theta}^2) = \widetilde{\phi}$ with some abuse of notation.

After the buyers bought goods from the seller, the third-party can reallocate them within the coalition. Let $x^i(\widetilde{\theta}^1, \widetilde{\theta}^2, \widetilde{\phi})$ represent the quantity of goods that buyer i receives from the third-party when $\widetilde{\phi}$ is reported to the seller. Finally, $y^i(\widetilde{\theta}^1, \widetilde{\theta}^2)$ denotes the monetary transfer from buyer i to the third-party; y^i does not need to depend on $\widetilde{\phi}$ because of quasi linearity of a buyer's payoff in money. Since we assume that the third party is not a source of goods or money, a side-contract should satisfy the ex post budget balance constraints for the reallocation

[11] Actually, the Revelation Principle applies to the third-party's design of S but not to the seller's design of M. Thus, the seller may wish to propose non-direct sale mechanisms. Nevertheless, as Proposition 3 in Laffont and Martimort (2000) establishes, any perfect Bayesian equilibrium outcome arising from a non-direct sale mechanism can be obtained as a perfect Bayesian equilibrium outcome induced by a direct sale mechanism.

of goods and for the side transfers:

$$\sum_{i=1}^{2} x^i(\theta^1, \theta^2, \widetilde{\phi}) = 0 \quad \text{and} \quad \sum_{i=1}^{2} y^i(\theta^1, \theta^2) = 0, \quad \text{for any } (\theta^1, \theta^2) \in \Theta^2 \text{ and any } \widetilde{\phi} \in \Theta^2.$$

After a side-contract S is proposed, a two-stage game is played by buyers: in its first stage (stage 5) each buyer accepts or rejects S; in the second stage (stage 6) the buyers report types either into M or into S depending on their decisions at the first stage. We are interested in (collusive continuation) equilibria in which both buyers accept S; thus, no learning about types occurs along the equilibrium path.[12] In Sections 4-6, we make the following assumption:[13]

Assumption WCP: Given an incentive compatible mechanism M, if buyer i vetoes S (which is an off-the-equilibrium-path event), then buyer $j \neq i$ still has prior beliefs about θ^i and the truthful equilibrium is played in M.

By definition, truthtelling is an equilibrium in M under prior beliefs if and only if M is incentive compatible. Let $U^M(\theta_j)$ $(j = L, H)$ denote the expected payoff of j-type in the truthful equilibrium in M. Then, $U^M(\theta_j)$ is the reservation utility for j-type when deciding whether to accept S or not. In Section 7, we relax this assumption WCP.

3 The optimal mechanisms without buyer coalition

In this section, we characterize the profit maximizing mechanisms when there is no buyer coalition. The seller's expected profit with mechanism $M = \{\mathbf{q}, \mathbf{t}\}$ is

$$\Pi \equiv 2p_L^2(t_{LL} - cq_{LL}) + 2p_L(1-p_L)(t_{HL} + t_{LH} - cq_{HL} - cq_{LH}) + 2(1-p_L)^2(t_{HH} - cq_{HH})$$

M should satisfy the following Bayesian incentive compatibility constraints: for H-type,

$$\begin{aligned} (BIC_H) \quad & p_L[\theta_H u(q_{HL}) - t_{HL}] + (1-p_L)[\theta_H u(q_{HH}) - t_{HH}] \\ & \geq p_L[\theta_H u(q_{LL}) - t_{LL}] + (1-p_L)[\theta_H u(q_{LH}) - t_{LH}]; \end{aligned} \tag{1}$$

for L-type,

$$\begin{aligned} (BIC_L) \quad & p_L[\theta_L u(q_{LL}) - t_{LL}] + (1-p_L)[\theta_L u(q_{LH}) - t_{LH}] \\ & \geq p_L[\theta_L u(q_{HL}) - t_{HL}] + (1-p_L)[\theta_L u(q_{HH}) - t_{HH}]. \end{aligned} \tag{2}$$

[12] Notice, however, that there also exists an equilibrium in which both buyers refuse any side mechanism: If buyer i is vetoing any side mechanism, then rejecting is a best reply for buyer j.

[13] WCP means weakly collusion-proof. The assumption makes us add the qualifier "weakly" in our definition of collusion-proof mechanisms: see Definition 2.

M should also satisfy the following individual rationality constraints: for H-type and L-type, respectively

$$(BIR_H) \quad p_L[\theta_H u(q_{HL}) - t_{HL}] + (1-p_L)[\theta_H u(q_{HH}) - t_{HH}] \geq 0; \tag{3}$$

$$(BIR_L) \quad p_L[\theta_L u(q_{LL}) - t_{LL}] + (1-p_L)[\theta_L u(q_{LH}) - t_{LH}] \geq 0. \tag{4}$$

The seller designs M to maximize Π subject to (1) to (4). We characterize the optimal mechanisms in the next proposition:

Proposition 1 *The optimal mechanisms in the absence of buyer coalition are characterized as follows.*
(a) The optimal quantity schedule $\mathbf{q}^* = (q^*_{HH}, q^*_{HL}, q^*_{LH}, q^*_{LL})$ *is given by:*
(i) $q^*_{HH} = q^*_{HL} = q^*_H$, *where* $\theta_H u'(q^*_H) = c$;
(ii) $q^*_{LH} = q^*_{LL} = q^*_L$, *where* $(\theta_L - \frac{1-p_L}{p_L}\Delta\theta)u'(q^*_L) = c$.
(b) Transfers are such that the constraints (BIC_H) *and* (BIR_L) *are binding.*

In Proposition 1, q^*_H (q^*_L) is the optimal quantity allocated to H-type (L-type), when the seller faces a single buyer. Thus, Proposition 1 states that, in the optimal mechanisms for the two-buyer case, the quantity obtained by a buyer is equal to the quantity he would receive in the one-buyer setting, independently of the report of the other buyer. In the one-buyer case, it is well known that the payment the seller obtains from H-type is decreasing in the quantity received by L-type because of (BIC_H). This induces the seller to evaluate L-type's surplus with the so-called virtual valuation $\theta^v_L \equiv \theta_L - \frac{1-p_L}{p_L}\Delta\theta < \theta_L$ instead of θ_L, and therefore to distort the quantity allocated to L-type below the first-best level since she equalizes L-type's marginal virtual surplus to marginal cost.

The facts that $q^*_{HH} = q^*_{HL} = q^*_H$, $q^*_{LH} = q^*_{LL} = q^*_L$ and (BIC_H), (BIR_L) bind imply that the expected payments of L-type and H-type, $\bar{t}_L \equiv p_L t_{LL} + (1-p_L)t_{LH}$ and $\bar{t}_H \equiv p_L t_{HL} + (1-p_L)t_{HH}$ respectively, are equal to the payments of the two types in the one-buyer setting: $\bar{t}_L = t^*_L \equiv \theta_L u(q^*_L)$ and $\bar{t}_H = t^*_H \equiv \theta_H u(q^*_H) - (\Delta\theta)u(q^*_L)$. The seller has two degrees of freedom in the choice of transfers to satisfy $\bar{t}_L = t^*_L$ and $\bar{t}_H = t^*_H$. For instance, she can set $t_{LL} = t_{LH} = t^*_L$ and $t_{HL} = t_{HH} = t^*_H$, so that each buyer's payment does not depend on the other buyer's report. In what follows, we let $M^d \equiv \{\mathbf{q}^*, \mathbf{t}^d\}$ where $t^d_{LL} = t^d_{LH} = t^*_L$ and $t^d_{HL} = t^d_{HH} = t^*_H$. In M^d, truthtelling is a dominant strategy since each buyer's payoff depends only on his own report. Basically, with M^d the seller maximizes her profit by dealing with each buyer separately.

A simple intuition sheds light on the close relation between the optimal mechanism in one-buyer case and the ones in two-buyer case.[14] If there exists a mechanism $\{\mathbf{q}', \mathbf{t}'\}$ which is

[14] We thank Raymond Deneckere for pointing this out to us.

strictly better than the mechanisms characterized by Proposition 1, then we can find a menu of two (possibly stochastic) contracts[15] which is strictly better than (q_H^*, t_H^*) and (q_L^*, t_L^*) for the single-buyer model. However, this is impossible by definition.

Last, we make an obvious (but important) observation about the optimal mechanisms in the absence of buyer coalition.

Observation: In any optimal sale mechanism without buyer coalition, HL-coalition can increase its payoff by reallocating some quantity from H-type to L-type in the absence of transaction costs.

Since $\theta_H u'(q_H^*) = (\theta_L - \frac{1-p_L}{p_L}\Delta\theta)u'(q_L^*) = c$ implies that L-type's marginal utility for goods is strictly larger than H-type's, HL-coalition has an incentive to reallocate some quantity from H-type to L-type if there exists no transaction costs in coalition formation. We emphasize that this incentive exists because the seller reduces the quantity consumed by L-type below the socially efficient level in order to extract more rent from H-type. In contrast, if she observed (θ^1, θ^2), there would be no room for arbitrage since the first-best quantity schedule (q_H^{FB}, q_L^{FB}) is such that $\theta_H u'(q_H^{FB}) = \theta_L u'(q_L^{FB}) = c$.

4 Coalition under asymmetric information

In this section we introduce formally the third party's design problem of S and then show that M^d characterized above leaves room for arbitrage, in the sense that buyers can increase their payoffs by manipulating reports and reallocating goods, at the expenses of the seller. Therefore, this section provides a motivation to look for a mechanism which performs better than M^d in the presence of buyer coalition, the issue we deal with in the next section. In particular, in this section we also show that the seller can restrict his attention to a particular set of (collusion-proof) mechanisms which we characterize..

Let $p(\theta^1, \theta^2)$ (respectively, $p(\theta^i)$ with $i = 1, 2$) denote the probability of having $(\theta^1, \theta^2) \in \Theta^2$ (respectively, the probability of having $\theta^i \in \Theta$). We recall that $p^\phi(\widetilde{\theta}^1, \widetilde{\theta}^2, \widetilde{\phi})$ denotes the probability that, after receiving reports $(\widetilde{\theta}^1, \widetilde{\theta}^2)$, the third party requires the buyers to report $\widetilde{\phi} \in \Theta^2$ to the seller. When $\widetilde{\phi}$ is reported to the seller, buyer i receives quantity $q^i(\widetilde{\phi})$ from the seller and pays $t^i(\widetilde{\phi})$ to her.

Definition 1 *A side-contract $S^* = \{\phi^*(\cdot), x^{i*}(\cdot), y^{i*}(\cdot)\}$ is coalition-interim-efficient with re-*

[15] The menu is such that conditional on the report of θ_H (θ_L), the buyer receives quantity q'_{HL} (q'_{LL}) with probability p_L and q'_{HH} (q'_{LH}) with probability $1 - p_L$ and pays $p_L t'_{HL} + (1 - p_L)t'_{HH}$ ($p_L t'_{LL} + (1 - p_L)t'_{LH}$).

spect to an incentive compatible mechanism M providing the reservation utilities $\{U^M(\theta_L), U^M(\theta_H)\}$ *if and only if it solves the following program:*

$$\max_{\phi(\cdot),x^i(\cdot),y^i(\cdot)} \sum_{(\theta^1,\theta^2)\in\Theta^2} p(\theta^1,\theta^2)[U^1(\theta^1)+U^2(\theta^2)]$$

subject to

$$U^i(\theta^i) = \sum_{\theta^j\in\Theta} p(\theta^j)\left\{\sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}(\theta^i,\theta^j,\widetilde{\phi})[\theta^i u(q^i(\widetilde{\phi})+x^i(\theta^i,\theta^j,\widetilde{\phi}))-t^i(\widetilde{\phi})]-y^i(\theta^i,\theta^j)\right\},$$

for any $\theta^i\in\Theta$ *and* $i,j=1,2$ *with* $i\neq j$;

$$(BIC^S)\ U^i(\theta^i) \geq \sum_{\theta^j\in\Theta} p(\theta^j)\left\{\sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}(\widetilde{\theta}^i,\theta^j,\widetilde{\phi})[\theta^i u(q^i(\widetilde{\phi})+x^i(\widetilde{\theta}^i,\theta^j,\widetilde{\phi}))-t^i(\widetilde{\phi})]-y^i(\widetilde{\theta}^i,\theta^j)\right\},$$

for any $(\theta^i,\widetilde{\theta}^i)\in\Theta^2$ *and* $i,j=1,2$ *with* $i\neq j$;

$$(BIR^S)\ U^i(\theta^i)\geq U^M(\theta^i),\ \textit{for any}\ \theta^i\in\Theta\ \textit{and}\ i=1,2;$$

$$(BB : x)\ x^1(\theta^1,\theta^2,\widetilde{\phi})+x^2(\theta^1,\theta^2,\widetilde{\phi})=0,\ \textit{for any}\ (\theta^1,\theta^2)\in\Theta^2\ \textit{and any}\ \widetilde{\phi}\in\Theta^2;$$
$$(BB : y)\ y^1(\theta^1,\theta^2)+y^2(\theta^1,\theta^2)=0,\ \textit{for any}\ (\theta^1,\theta^2)\in\Theta^2.$$

In words, a side-contract is coalition-interim-efficient with respect to M if it maximizes the sum of the buyers' expected utilities subject to incentive, acceptance and budget balance constraints. Let $S^0\equiv\{\phi(\cdot)=Id(\cdot),x^1(\cdot)=x^2(\cdot)=0,y^1(\cdot)=y^2(\cdot)=0\}$ denote the contract which implements no manipulation of reports, no reallocation of quantity and no side-transfer; S^0 is called the *null-side contract* and M is not affected by buyer coalition if the third-party proposes S^0. The next definition refers to this class of mechanisms.

Definition 2 *An incentive compatible mechanism M is weakly*[16] *collusion-proof if* S^0 *is coalition-interim-efficient with respect to M.*

In the rest of this section, we consider two interpretations of q, quality or quantity,[17] and examine whether or not M^d is weakly collusion-proof in each case. The next proposition states our result:

[16] The qualifier "weakly" comes from our assumption WCP in section 2.2.

[17] For instance, in Mussa and Rosen (1978), q represents quality. Alger (1999) considers both interpretations although she focus on quantity interpretation.

Proposition 2 *Suppose the seller offers M^d. Then*
(a) when q represents quality, M^d is weakly collusion-proof;
(b) when q represents quantity, there exists a side-contract S^d which increases the payoff of each type of buyer (and reduces the seller's profit) compared to when M^d is played truthfully; in S^d, HH-coalition reports HL to the seller, HL-coalition reports LL and then quantities are reallocated within the coalitions.

Consider first the case in which q represents quality and hence reallocation of q is impossible or the case in which q represents quantity but buyers cannot reallocate it (for instance, electricity, gas, water). In these cases, the only instrument of the coalition is manipulation of reports. Then, Proposition 2(a) establishes that M^d is weakly collusion-proof. This result easily follows from the property that in M^d a buyer's payoff is independent of the other buyer's report and no agent has an individual incentive to report untruthfully since (BIC_H) and (BIC_L) are satisfied. Therefore, the sum of the buyers' payoffs is maximized by truthtelling in every state of nature and the null side-contract satisfies (BIC^S), (BIR^S) and budget balance constraints; thus, S^0 is coalition-interim-efficient. Notice that collusion has no bite even though it occurs under symmetric information among buyers. We note that Laffont and Martimort (1997, 2000) obtain similar findings (Proposition 11 and Proposition 6, respectively) when they show that there exists a dominant-strategy optimal mechanism which eliminates any gain from joint manipulation of reports if the agents' types are independently distributed.

We now turn to the case in which q represents quantity and buyers can manipulate their reports and reallocate quantity. In what follows, for simplicity of discussion, we suppose that buyers have symmetric information at the time of collusion, which is equivalent to saying that the third party does not need to satisfy (BIC^S) or that there are no transaction costs in coalition formation. This simplification is innocuous since the underlying logic holds true even when buyers form the coalition under asymmetric information. One simple way to see why the possibility of reallocation overturns the result of Proposition 2(a) is to notice that actually – when reallocation is infeasible – coalition HH (HL) is indifferent between truthtelling and reporting HL (LL) under M^d. Since reallocation makes the coalition more powerful, it is quite intuitive that now incentives to manipulate reports exist.

To be more clear, we here graphically illustrate the result of Proposition 2(b). In Figure 1, points A and B represent the two quantity-transfer pairs (q_L^*, t_L^*) and (q_H^*, t_H^*) respectively in mechanism M^d. If HH-coalition reports truthfully, each buyer will achieve B. If it reports HL and reallocates evenly the total quantity and the total payment, each buyer will obtain C, with $q^C = \frac{q_L^*+q_H^*}{2}$ and $t^C = \frac{t_L^*+t_H^*}{2}$. One can easily see from Figure 1 that each H-type strictly prefers C to B since C lies on a better indifference curve than B. Formally, C is preferred to

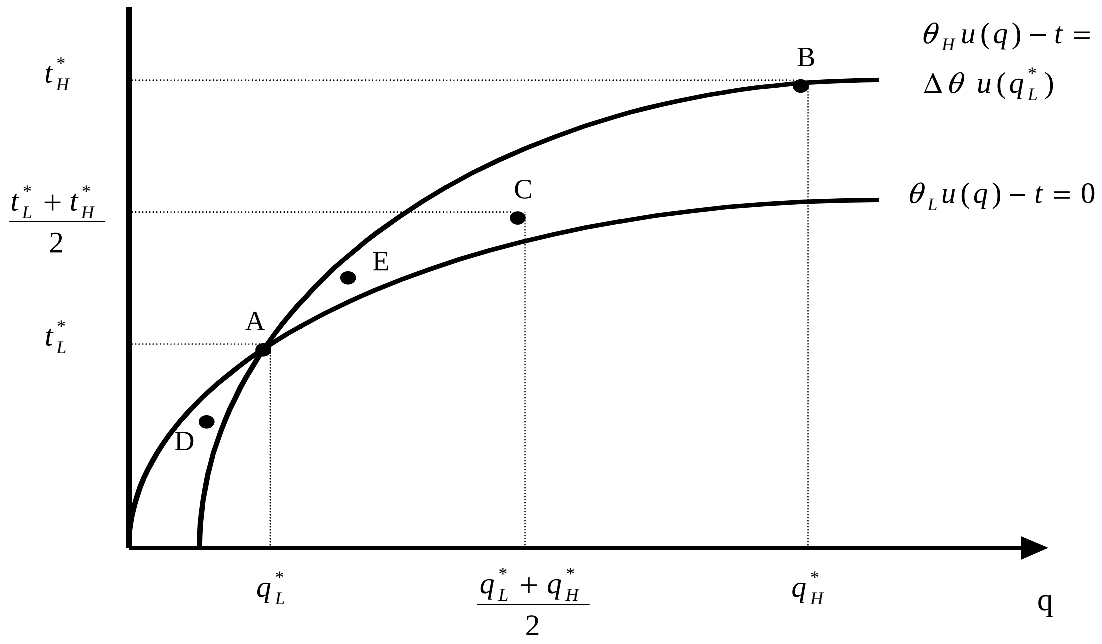

Figure 1: Gains from reallocation under the mechanism M^d

A or B since C is a convex combination of A and B, H-type is indifferent between A and B and his preferences are strictly quasi-convex.

For HL-coalition, if it reports LL and does not reallocate quantity, each buyer achieves A and obtains the same payoff as with truthtelling. However, since H-type's marginal surplus for goods is higher than L-type's one when both receive the same quantity, each buyer can achieve higher payoffs by reallocating some goods from L-type to H-type (with an appropriate money transfer from H-type to L-type): for instance, they can achieve D for L-type and E for H-type.

Since, according to Proposition 2(b), the seller earns a lower profit than under no coalition formation when she offers M^d, it is natural to inquire whether there exist better mechanisms than M^d. The following proposition simplifies our analysis, since it shows that in order to find the best mechanism for the seller we can restrict our attention to the set of weakly collusion-proof mechanisms.

Proposition 3 (weakly collusion-proofness principle) *There is no loss of generality in restricting the seller to offer weakly collusion-proof mechanisms in order to characterize the outcome of any perfect Bayesian equilibrium of the game of seller's mechanism offer cum coalition formation such that a collusive equilibrium occurs on the equilibrium path.*

The idea behind Proposition 3 is the following: since the third-party has no informational or instrumental advantage over the seller and is subject to the incentive, acceptance and budget balance constraints, any outcome that can be implemented by allowing coalitions to manipulate reports and/or to reallocate goods can be mimicked by the seller in a collusion-proof way without loss.

The next proposition characterizes the set of weakly collusion-proof mechanisms.[18] Before stating the proposition, it is useful to define the following variables θ_L^ϵ, $q_H^\epsilon(x)$ and $q_L^\epsilon(x)$, in which $\epsilon \in [0,1)$ and $x > 0$:

$$\theta_L^\epsilon \equiv \theta_L - \frac{1-p_L}{p_L}(\Delta\theta)\epsilon,$$

$$q_H^\epsilon(x) \equiv \arg\max_{z\in[0,x]} \theta_H u(z) + \theta_L^\epsilon u(x-z) \quad \text{and} \quad q_L^\epsilon(x) \equiv x - q_H^\epsilon(x) \tag{5}$$

We note that $q_H^\epsilon(x)$ is uniquely defined since $\theta_H u(z) + \theta_L^\epsilon u(x-z)$ is a strictly concave function of z. In particular, $(q_H^\epsilon(x), q_L^\epsilon(x))$ is the efficient allocation of a total quantity $x > 0$ between a buyer with valuation θ_H and a buyer with valuation θ_L^ϵ.

Proposition 4 *An incentive compatible sale mechanism $M = \{\mathbf{q}, \mathbf{t}\}$ is weakly collusion-proof if and only if there exists $\epsilon \in [0,1)$ such that*
(a) the following coalition incentive constraints are satisfied: for HH coalition,

$$(CIC_{HH,HL}) \quad 2\theta_H u(q_{HH}) - 2t_{HH} \geq 2\theta_H u(\frac{q_{HL}+q_{LH}}{2}) - t_{HL} - t_{LH}, \tag{6}$$

$$(CIC_{HH,LL}) \quad 2\theta_H u(q_{HH}) - 2t_{HH} \geq 2\theta_H u(q_{LL}) - 2t_{LL}; \tag{7}$$

for HL coalition,

$$(CIC_{HL,HH}) \quad \theta_H u(q_{HL}) + \theta_L^\epsilon u(q_{LH}) - t_{HL} - t_{LH} \geq \theta_H u(q_H^\epsilon(2q_{HH})) + \theta_L^\epsilon u(q_L^\epsilon(2q_{HH})) - 2t_{HH}, \tag{8}$$

$$(CIC_{HL,LL}) \quad \theta_H u(q_{HL}) + \theta_L^\epsilon u(q_{LH}) - t_{HL} - t_{LH} \geq \theta_H u(q_H^\epsilon(2q_{LL})) + \theta_L^\epsilon u(q_L^\epsilon(2q_{LL})) - 2t_{LL}; \tag{9}$$

for LL coalition,

$$(CIC_{LL,HH}) \quad 2\theta_L^\epsilon u(q_{LL}) - 2t_{LL} \geq 2\theta_L^\epsilon u(q_{HH}) - 2t_{HH}, \tag{10}$$

$$(CIC_{LL,HL}) \quad 2\theta_L^\epsilon u(q_{LL}) - 2t_{LL} \geq 2\theta_L^\epsilon u(\frac{q_{HL}+q_{LH}}{2}) - t_{HL} - t_{LH}; \tag{11}$$

[18] We here focus on weakly collusion-proof mechanisms where L-type's Bayesian individual incentive constraint is not binding. We prove in Section 5 that the seller is not going to offer a mechanism M such that L-type's incentive constraint binds in the side-contract which is optimal with respect to M.

(b) the following no arbitrage constraint (which relies on (5)) is satisfied

$$q_{HL} = q_H^{\epsilon}(q_{HL} + q_{LH}), \tag{12}$$

(c) if $\epsilon > 0$, then H-type's incentive constraint in the side mechanism is binding.

Notice that each coalition incentive constraint takes into account the reallocation of the goods: If both agents report the same types to the third party, each buyer receives half of the total quantity available (see (6)-(7) and (10)-(11)) while if the reports are different, the total quantity is allocated according to (5) (see (8)-(9)). When all the coalition incentive constraints are satisfied, the third-party does not manipulate the buyers' reports into M. Then, no room for reallocation exists if $\theta^1 = \theta^2$ since the seller allocates the same quantity to each buyer. If $\theta^1 \neq \theta^2$, then the third party will not reallocate the goods bought from the seller after making truthful reports if and only if the no-arbitrage constraint (12) is satisfied.

In (8)-(12), $\epsilon \in [0, 1)$ appears. Roughly speaking, ϵ is the Lagrange multiplier of (BIC_H^S), H-type's incentive constraint in the third-party's design problem of S, and it can be positive when (BIC_H^S) is binding.[19] The seller has some flexibility in choosing ϵ since S^0 is optimal for the third party if and only if it satisfies the necessary and sufficient conditions for optimality in the third party's problem for at least one ϵ in $[0, 1)$.

In the presence of complete information within the coalition, the side mechanism does not need to satisfy any individual incentive constraint. Therefore, the coalition incentive and the no-arbitrage constraints under complete information are obtained from (6)-(12) by taking ϵ equal to 0 and the third party realizes whatever gains from cooperative actions if there is any. When the coalition forms under asymmetric information, it may be costly to satisfy (BIC_H^S) because of a well-known tension between (BIC_H^S) and (BIR_L^S); ϵ measures how costly it is. The coalition incentive constraints under asymmetric information differ from the constraints under complete information since L-type's valuation θ_L is replaced by the virtual value θ_L^{ϵ}. The latter is smaller than θ_L for $\epsilon > 0$ since, as the quantity allocated to L-type (by the third party) increases, it is more difficult to satisfy (BIC_H^S). The value of θ_L^{ϵ} affects the coalition incentive constrains through two channels. First, given a quantity consumed by L-type, the third-party evaluates his surplus with θ_L^{ϵ} instead of θ_L. Second, this in turn affects the third-party's decision to reallocate the goods given a total quantity available to a coalition.

One might argue that the seller could ask the buyers for the information that they may have learned during the course of coalition formation. However, there is no loss in restricting the seller to use mechanisms such as those defined in subsection 2.1 since we show that she can nevertheless deter buyer coalition at no cost.

[19] Precisely, $\epsilon = \frac{\delta}{a+\delta}$ where δ is the Lagrange multiplier of (BIC_H^S) and $a > 0$.

5 The optimal weakly collusion-proof mechanisms

In this section, we analyze the optimal weakly collusion-proof mechanism. Observe that when the third party proposes S^0, (i) the Bayesian incentive constraints (BIC^S) in the side mechanism reduce to (BIC_H) and (BIC_L) introduced in section 3; (ii) the acceptance constraints (BIR^S) in the side mechanism are automatically satisfied with equality. Hence, the seller's maximization program under collusion - denoted by (P) - is defined as follows:

$$\max_{\{\mathbf{q},\mathbf{t},\epsilon\}} \quad \Pi \text{ subject to (1)-(4) and (6)-(12).}$$

Since (P) has more constraints than the seller's program without collusion, the seller cannot earn more profit in the presence of collusion than in its absence. However, the next proposition states that the profit level is the same in the two cases. More precisely, it provides a transfer schedule which, paired with the quantity profile $\mathbf{q}^*$ of Proposition 1, yields the seller the profit she obtains in the absence of collusion.

Proposition 5 *Let $\mathbf{t}^{**}$ be such that (BIR_L), (BIC_H), $(CIC_{HH,HL})$ and $(CIC_{HL,LL})$ bind when $\mathbf{q} = \mathbf{q}^*$ and $\varepsilon = 1$.*[20] *Then $M^{**} \equiv \{\mathbf{q}^*, \mathbf{t}^{**}\}$ is both an optimal mechanism in the absence of buyer coalition and weakly collusion-proof.*

Proof. We basically prove that the seller can satisfy all the constraints imposed by weak collusion-proofness without any loss.

We first notice that $\mathbf{q}^*$ satisfies the no-arbitrage constraint (12) with $\varepsilon = 1$. In fact, when $\varepsilon = 1$ both the seller and the third-party have the same virtual valuation of L-type, $\theta_L - \frac{1-p_L}{p_L}\Delta\theta$; hence the third-party has no incentive to modify the quantity allocation $\mathbf{q}^*$ decided by the seller. Then, we can find a (unique) transfer profile $\mathbf{t}^{**}$ such that (BIR_L), (BIC_H), $(CIC_{HH,HL})$ and $(CIC_{HL,LL})$ bind when $\mathbf{q} = \mathbf{q}^*$ and $\varepsilon = 1$ (see the appendix). We remark that this is possible because satisfying (BIR_L) and (BIC_H) with equality absorbs only two degrees of freedom from the transfer schedule $\mathbf{t}$. By Proposition 1, $M^{**} \equiv \{\mathbf{q}^*, \mathbf{t}^{**}\}$ is optimal in the absence of coalition since (BIR_L) and (BIC_H) bind.

In order to prove that M^{**} satisfies all the coalition incentive constraints, let $V_m^\epsilon(x)$ denote the total virtual surplus[21] that a coalition having m number of buyers with H-type derives from consuming a total quantity $x > 0$; $m \in \{0, 1, 2\}$ is viewed as the "type" of the coalition.

[20] Although ϵ belongs to $[0, 1)$, we allow ϵ to take the value equal to one since we are interested in the Sup of the seller's profit.

[21] For instance, $V_1^\epsilon(x) \equiv \max_{z\in[0,x]} \theta_H u(z) + \theta_L^\epsilon u(x - z)$. In $V_0^\epsilon(x)$ and $V_1^\epsilon(x)$, a L-type's surplus is evaluated with θ_L^ϵ. $V_2^\epsilon(x)$ is independent of ϵ since there is no L-type in HH-coalition.

We regard each coalition as a consolidated agent and V_m^ϵ as the surplus function of type m. Then notice that (i) $2q^*_{HH} > q^*_{HL} + q^*_{LH} > 2q^*_{LL}$; (ii) the following single crossing condition holds: $\frac{\partial V_2(x)}{\partial x} > \frac{\partial V_1^\epsilon(x)}{\partial x} > \frac{\partial V_0^\epsilon(x)}{\partial x}$ for any $x > 0$ and any $\epsilon \geq 0$ (see Lemma 1 in the appendix). These two properties, together with the fact that in M^{**} the local downward coalition incentive constraints bind, allow us to use a standard result from the theory of monopolistic screening [see Section 3 in Maskin and Riley (1984)] to conclude that all the coalition incentive constraints are satisfied. ■

Proposition 5 says that the seller can implement the quantity profile $\mathbf{q}^*$ as when there is no buyer coalition and can deter collusion at no cost, thus realizing the same profit as without collusion. Hence, under asymmetric information, the ability to form a coalition does not help the buyers to increase their payoffs. In particular, even though the third party aims at maximizing the buyers' payoffs and marginal rates of substitution are not equalized across buyers in HL-coalition, no side mechanism can implement a desirable reallocation when the seller uses M^{**}. We provide an intuition in two steps for this result.

No reallocation occurs if there is no manipulation of reports To give an intuition of why the third-party fails to efficiently reallocate the goods, suppose that the buyers do not manipulate their reports. Then, we can show that no reallocation of quantity occurs under M^{**}. Obviously, no room for reallocation exists within the coalitions HH and LL since the seller allocates the same quantity to each buyer in these homogenous coalitions. However, in the case of HL-coalition, potential room for arbitrage exists since L-type's marginal utility for the goods is larger than H-type's. To understand why no reallocation occurs in this coalition, it is important to recall that under asymmetric information, a side mechanism needs to satisfy both (BIC^S) and (BIR^S). Since (BIC^S_H) binds in the side mechanism which is optimal with respect to M^{**} and the information rent H-type obtains by pretending to be L-type to the third-party increases in the quantity received by L-type, the third party evaluates L-type's surplus not with θ_L but with a virtual valuation smaller than θ_L. Furthermore, since the third party has the same prior beliefs about the buyers' types as the seller and also (BIR^S_L) binds, H-type's rent as a function of the quantity received by L-type increases with the same slope both in the third-party's problem and in the seller's problem with no coalition. Therefore, the third party evaluates L-type's surplus with the same virtual valuation θ^v_L as the seller[22] and consequently he has no incentive to modify the allocation $\mathbf{q}^*$ at which H-type's marginal surplus is equal to L-type's virtual marginal surplus.

[22] See the proof of Proposition 4 in Appendix for the formal derivation of L-type's virtual value from the third-party's point of view.

Alternatively, we can explain the no-reallocation result by directly computing the transaction costs created by asymmetric information and showing that they are larger than the gains from reallocation.[23] Consider reallocating a quantity $\Delta q \in (0, q_H^*]$ from H-type to L-type within HL-coalition. First, the gains from reallocation are given by $G \equiv \theta_L [u(q_L^* + \Delta q) - u(q_L^*)] - \theta_H [u(q_H^*) - u(q_H^* - \Delta q)]$, which is positive, at least for a small Δq, from the inequality $\theta_L u'(q_L^*) > \theta_H u'(q_H^*)$. Second, the reallocation also increases H-type's rent since it increases the quantity consumed by L-type; we define this increase in rent as the transaction costs TC created by asymmetric information. In order to compute TC, suppose that an H-type pretends to be L-type to the third-party while the other buyer reports truthfully. Then, the expected surplus of the former is equal to $(1 - p_L)\theta_H u(q_L^* + \Delta q) + p_L \theta_H u(q_L^*)$ while his expected payment is equal to $(1 - p_L)\theta_L u(q_L^* + \Delta q) + p_L \theta_L u(q_L^*)$, determined by the binding L-type's participation constraint in the side-mechanism. Therefore, H-type's expected rent is $\Delta\theta [(1 - p_L)u(q_L^* + \Delta q) + p_L u(q_L^*)]$, higher than his rent $\Delta\theta u(q_L^*)$ when $\Delta q = 0$, and $TC = \Delta\theta(1 - p_L) [u(q_L^* + \Delta q) - u(q_L^*)]$. Last, the third-party can implement the reallocation only if the expected gain from reallocation $2p_L(1 - p_L)G$ are larger than the expected transaction costs $2(1 - p_L)TC$. Since $2p_L(1 - p_L)G < 2(1 - p_L)TC$ holds for any $\Delta q \in (0, q_H^*]$, we conclude that reallocation is infeasible.

No manipulation of reports is profitable In order to understand why no manipulation is implemented given M^{**}, it is useful to define $V_m^1(x)$ (as in the proof of proposition 5) as the total surplus a coalition with m buyers with H-type derives from a total quantity $x > 0$ after optimally allocating x within the coalition with $\varepsilon = 1$. As we mentioned above, the third party evaluates the surplus of L-type with θ_L^v instead of θ_L. Therefore, we have $V_1^1(x) \equiv \max_{z\in[0,x]} \theta_H u(z) + \theta_L^v u(x - z)$ and $V_0^1(x) \equiv 2\theta_L^v u(\frac{x}{2})$, while $V_2(x) \equiv 2\theta_H u(\frac{x}{2})$ as under symmetric information.

As a first step, we below focus on the two downward manipulations which are mentioned in Proposition 2(b). When $\mathbf{q} = \mathbf{q}^*$, HH-coalition prefers truthful report to reporting HL if and only if the following inequality holds:

$$V_2(2q_H^*) - 2t_{HH} \geq V_2(q_H^* + q_L^*) - t_{HL} - t_{LH} \tag{13}$$

HL-coalition reports truthfully rather than LL if and only if

$$V_1^1(q_H^* + q_L^*) - t_{HL} - t_{LH} \geq V_1^1(2q_L^*) - 2t_{LL} \tag{14}$$

[23] Makowski and Mezzetti (1994) and Williams (1999) use an argument similar to ours to prove (non-) existence of efficient mechanisms in environments which include Myerson-Satterthwaite (1983)'s one seller-one buyer setting as a special case.

We notice that the transfers in M^d violate both (13) and (14), but the seller can find transfers which satisfy (13) and (14) and make (BIC_H) and (BIR_L) bind. On the one hand, a suitable decrease in t_{HH} and an increase in t_{HL}, both with respect to t_H^*, allow to satisfy (13) while keeping (BIC_H) binding, as it is necessary to achieve the same profit as without collusion. On the other hand, an increase in t_{LL} and a decrease in t_{LH}, both with respect to t_L^*, allow to satisfy (14) while keeping (BIR_L) still binding. Formally, the seller can use two degrees of freedom in transfers to satisfy (13) and (14) at no cost while using the remaining two degrees freedom to leave (BIR_L) and (BIC_H) binding. Indeed, the transfers $\mathbf{t}^{**}$ in Proposition 5 are defined as the (unique) profile of transfers which satisfies all (BIR_L), (BIC_H), $(CIC_{HH,HL})$ and $(CIC_{HL,LL})$ with equality: Consistently with the intuition suggested above, we find $t_{LH}^{**} < t_L^* < t_{LL}^{**}$ and $t_{HH}^{**} < t_H^* < t_{HL}^{**}$.

We graphically explain how $\mathbf{t}^{**}$ deters HH-coalition from reporting HL.[24] In figure 2, A and B are defined as in figure 1 and represent the two quantity-transfer pairs under M^d. Under M^{**}, after reporting truthfully, each buyer in HH-coalition obtains the pair B', which is better than B since $t_{HH}^{**} < t_H^*$ holds while, after reporting HL and sharing equally the quantity and the transfer, each buyer obtains the pair C' i.e., $(\frac{q_L^*+q_H^*}{2}, \frac{t_{LH}^{**}+t_{HL}^{**}}{2})$. Since by construction H-type is indifferent between B' and C', the HH-coalition will report truthfully under M^{**}.

We now argue that also the coalition incentive constraints we neglected are satisfied by M^{**}. For this purpose, we note that (i) (13) and (14) (the local downward coalition incentive constraints) bind in M^{**} (ii) a single crossing condition for coalitions holds[25]: $\frac{\partial V_2(x)}{\partial x} > \frac{\partial V_1^1(x)}{\partial x} > \frac{\partial V_0^1(x)}{\partial x}$ for any $x > 0$ and $\epsilon \geq 0$ (iii) the quantity profile for coalitions is monotone: $2q_{HH}^* > q_{HL}^* + q_{LH}^* > 2q_{LL}^*$. Therefore, we can use a standard result from the theory of monopolistic screening [see Section 3 in Maskin and Riley (1984)] to conclude that (6)-(11) are satisfied.

It is interesting to notice that there exist infinitely many transfer schemes $\widehat{\mathbf{t}}$ such that $\left\{\mathbf{q}^*, \widehat{\mathbf{t}}\right\}$ is optimal under no coalition and weakly collusion-proof (for instance, it is possible to strictly satisfy (6)-(11) without reducing the profit).[26] However, the following inequalities

$$\widehat{t}_{LH} < t_L^* < \widehat{t}_{LL} \qquad \text{and} \qquad \widehat{t}_{HH} < t_H^* < \widehat{t}_{HL} \tag{15}$$

must be satisfied by any such $\widehat{\mathbf{t}}$. The inequalities mean that upon reporting a type, each buyer faces a lottery which determines his payment as a function of the report of the other buyer. In particular, facing an L-type is bad news because then the payment is higher than when

[24] We only examine $(CIC_{HH,HL})$ because representing $(CIC_{HL,LL})$ in Figure 2 is much more difficult since, in HL-coalition, goods are not reallocated evenly as in HH-coalition.

[25] See the proof of lemma 1 in the appendix for the details.

[26] In Section 7, we exploit this multiplicity to find an optimal weakly collusion-proof mechanism which is strategically more robust than M^{**}.

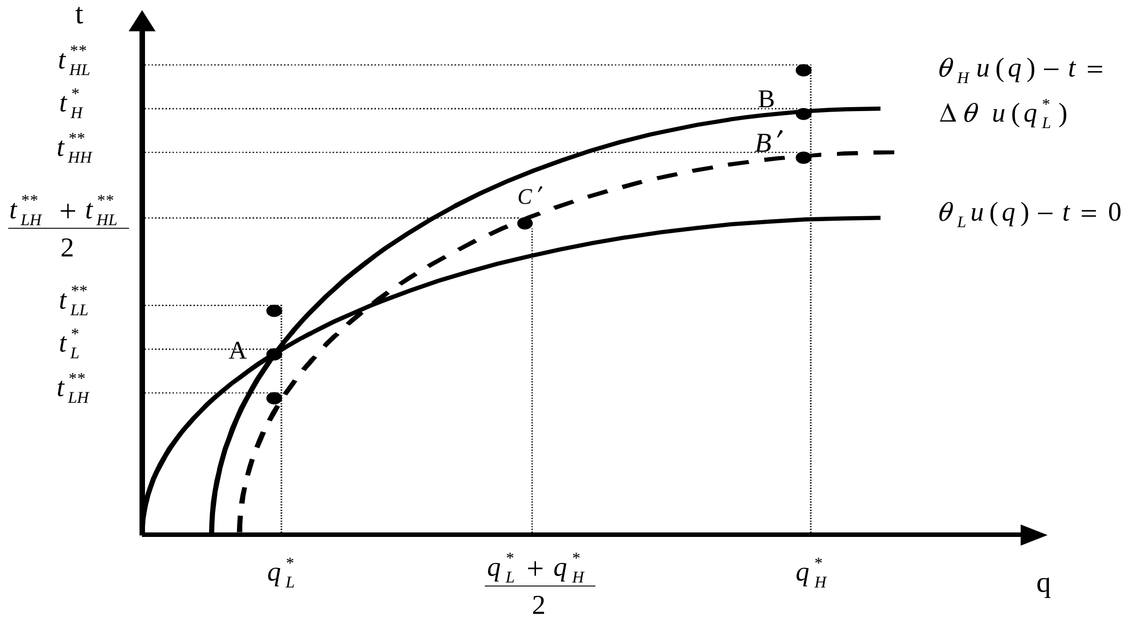

Figure 2: Transfers inducing an HH-coalition to report truthfully

facing an H-type. This feature results from the seller's desire to deter coalitions' downward manipulation of reports, as we below argue in proving (15).

To show (15), let $\widehat{t}_{HL} = t^*_H + a$, $\widehat{t}_{LL} = t^*_L + b$, $\widehat{t}_{HH} = t^*_H - \frac{p_L a}{1-p_L}$ and $\widehat{t}_{LH} = t^*_L - \frac{p_L b}{1-p_L}$: therefore $\widehat{\mathbf{t}}$ satisfies (BIC_H) and (BIR_L) with equality. Define $\alpha \equiv V_2(2q^*_H) - V_2(q^*_H + q^*_L) - (t^*_H - t^*_L)$ and $\beta \equiv V^1_1(q^*_H + q^*_L) - V^1_1(2q^*_L) - (t^*_H - t^*_L)$. Then, $(CIC_{HH,HL})$ and $(CIC_{HL,LL})$ at $\mathbf{q} = \mathbf{q}^*$ reduce respectively to

$$(1-p_L)\alpha \geq -a - p_L(a-b) \qquad \text{and} \qquad (1-p_L)\beta \geq -b + (1-p_L)(a-b)$$

Therefore, the set of (a,b) which satisfy $(CIC_{HH,HL})$ and $(CIC_{HL,LL})$ is given by $Z \equiv \left\{(a,b) \in R^2 \mid \frac{(1-p_L)\alpha+(1+p_L)a}{p_L} \geq b \geq \frac{(1-p_L)a-(1-p_L)\beta}{2-p_L}\right\}$ and the point at which $\frac{(1-p_L)\alpha+(1+p_L)a}{p_L} = b = \frac{(1-p_L)a-(1-p_L)\beta}{2-p_L}$ holds corresponds to the transfers $\mathbf{t}^{**}$ of M^{**}. Figure 3 represents Z graphically and, since $\alpha < 0$, $\beta < 0$[27] and $\frac{1+p_L}{p_L} > \frac{1-p_L}{2-p_L}$, any $(a,b) \in Z$ should be such that $a > 0$ and $b > 0$. Therefore, for any mechanism which is optimal under no coalition and satisfies (13)-(14) (necessary conditions for weakly collusion-proofness), its transfers $\widehat{\mathbf{t}}$ must satisfy (15). This implies in particular that (i) M^d is not weakly collusion-proof since $a = b = 0$ in M^d (ii) ex

[27] It is straightforward to verify that $\alpha < 0$. For the proof of $\beta < 0$ see the Appendix, immediately before the proof of Proposition 6.

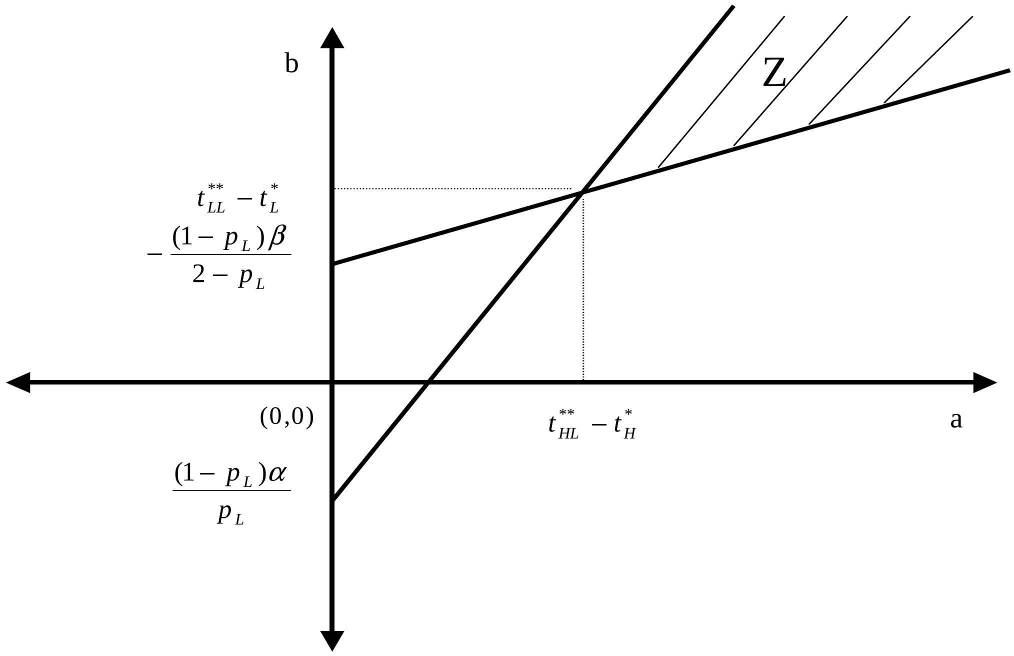

Figure 3: Transfers in the optimal collusion-proof mechanisms: a necessary condition

post individual rationality is violated for L-type in any mechanism which is optimal under no coalition and weakly collusion-proof since $t_L^* = \theta_L u(q_L^*) < \widehat{t}_{LL}$ holds.

Remark 1 (symmetric information in the coalition): Even though we focus on the role of asymmetric information among buyers, it is interesting to inquire the consequences of collusion taking place under symmetric information. For instance, suppose that the third party owns a technology that allows him to elicit credible reports from the buyers as in Baron and Besanko (1999).[28] In this case the side mechanism does not need to satisfy (BIC^S), implying that the third party evaluates L-type's surplus with the real valuation θ_L rather than with the virtual value θ_L^v. The coalition incentive constraints under symmetric information are similar to those under asymmetric information except that now, in defining $V_1^\varepsilon(x)$ and $V_0^\varepsilon(x)$, θ_L is used instead of θ_L^ε. Still, the seller can deter manipulation of reports at no cost as under asymmetric information. However, reallocation within HL-coalition takes place unless $\theta_H u'(q_{HL}) = \theta_L u(q_{LH})$, a condition which reduces the seller's profit with respect to the case without buyer coalition.

[28] They assume that the third-party who organizes an informational alliance can verify the private information of each agent forming the alliance.

Remark 2 (correlation): Proposition 5 does not hold if θ^1 are θ^2 are correlated. Indeed, in a correlated environment the seller earns the first best profit in the absence of buyer coalition (see Crémer and McLean (1985)), but that is not possible under coalition formation. We examined a specific case in which the payoff of type θ from consuming quantity $q \in [0, \theta]$ is $\theta q - \frac{1}{2}q^2$ and $\Pr\left\{\theta^1 = \theta^2 = \theta_H\right\} = \Pr\left\{\theta^1 = \theta^2 = \theta_L\right\}$, $\Pr\left\{\theta^1 = \theta_H, \theta^2 = \theta_L\right\} = \Pr\left\{\theta^1 = \theta_L, \theta^2 = \theta_H\right\}$. Now, a trade-off about the value of ε arises. On the one hand, as in the case of independent types, a large ε helps to discriminate H-type from L-type. On the other hand, the constraint $(CIC_{HL,LL})$ binds and it is tightened as ε increases. For the case of small and positive correlation, it turns out that the trade-off is optimally resolved by setting ε strictly below 1. We also obtain, as in Laffont and Martimort (2000), that the solution is continuous in the degree of correlation; furthermore, the optimal values of q_{HL} and q_{LH} are decreasing with respect to the degree of correlation, while q_{LL} is increasing.

Two-part tariffs Two-part tariffs are sometimes proposed as a simple way to implement non-linear tariffs, or as a "real-life" mechanism as opposed to abstract direct mechanisms. In the model with no buyer coalition, it is easy to see that the optimal outcome can be implemented by a menu of two-part tariffs such that each type of buyer chooses the tariff designed for his type and buys the quantity q_H^* or q_L^* according to his type. We note that the two-part tariff designed for L-type needs a kink in order to prevent H-type from choosing the tariff designed for L-type and buying more than q_L^*.[29]

The next proposition states that a more complicated menu of two-part tariffs can be used to implement the optimal outcome when coalition formation is possible. We continue to assume that the seller can commit not to serve a buyer if the other buyer does not buy anything from the seller.[30] Let the seller offer tariffs $T_H = \{(A_{HH}, p_{HH}), (A_{HL}, p_{HL})\}$ and $T_L = \{(A_{LH}, p_{LH}), (A_{LL}, p_{LL})\}$ where, for instance, A_{HL} and p_{HL} represent the fixed fee and the marginal price that a buyer choosing T_H pays if the other buyer chooses T_L. In particular, we consider the tariffs $\{T_H^{**}, T_L^{**}\}$ such that

$$\begin{cases} A_{jk}^{**} = t_{jk}^{**} - cq_j^*, \text{ for } j, k \in \{H, L\}, \\ p_{jk}^{**} = c \text{ for } q \leq q_j^* \text{ and } p_{jk}^{**} = \theta_H u'(q_L^*) \text{ for } q > q_j^* \text{ for } j, k \in \{H, L\}. \end{cases} \tag{16}$$

[29] The two-part tariff for H-type takes the form $A_H + pq$ with $A_H = t_H^* - cq_H^*$ and $p = c$. Since the tariff for L-type needs a kink at the point $q = q_L^*$, the seller has some discretion in choosing the marginal price. For instance, she can use $A_L + pq$ with $A_L = t_L^* - cq_L^*$ and $p = c$ for $q \leq q_L^*$, $p = \theta_H u'(q_L^*)$ for $q > q_L^*$.

[30] As we said in footnote 10, our results hold even if this assumption does not hold but the seller can observe whether or not a buyer uses her goods. This makes it impossible for a buyer to obtain a positive amount of the goods without paying any fixed fee to the seller as in Rey and Tirole (1986).

Proposition 6 *Suppose that the seller offers* $\{T_H^{**}, T_L^{**}\}$ *instead of* M^{**}*. Then, regardless of whether or not the buyers can form a coalition,*
(a) each buyer accepts the offer,
(b) j*-type of buyer, with* $j \in \{H, L\}$*, chooses the tariff* T_j^{**} *and buys quantity* q_j^**.*

The menu (16) is such that (i) the fixed fee a buyer pays depends on the tariff chosen by the other buyer (which is necessary since $\mathbf{t}^{**}$ requires this sort of dependence) (ii) the tariff each buyer faces has a kink.[31] The kink is necessary in order to deter downward manipulation of reports. For instance, suppose there is no kink in T_H^{**}. Then, since $A_{HH}^{**} > A_{HL}^{**} + A_{LH}^{**}$ holds, a HH-coalition has an incentive to coordinate the buyers' purchases such that only one buyer chooses T_H^{**}, he buys more than q_H^* and shares it with the other buyer who chooses T_L^{**}.[32] This deviation is prevented by the increase in the marginal price at $q = q_H^*$ - the kink - from c to $\theta_H u'(q_L^*)$.

6 Extensions

In the previous sections, for simplicity we considered the two-buyer-two-type setting with $C(q) = cq$ and $\mathcal{U}(q, \theta) = \theta u(q)$. However, Proposition 5 can be extended to an environment with n buyers and two types, or with two buyers and three types, or with general cost and utility functions which satisfy the conditions introduced in subsection 2.1.

6.1 The case of $n > 2$ buyers

When the seller faces $n > 2$ buyers, we assume that the only feasible coalition is the grand coalition, the one including all the buyers. More precisely, we suppose that if at least one buyer rejects the side mechanism, then the sale mechanism is played non-cooperatively with prior beliefs (i.e., we keep assumption WCP). This assumption is justified when any attempt to organize a coalition - after the grand coalition was rejected - is sufficiently time consuming such that it is impossible for the third party to design a new side mechanism which is tailored for the buyers who accepted the original side mechanism. Clearly, this assumption is not needed if $n = 2$ but it makes the model quite tractable when $n > 2$.

[31] Actually, no kink is needed when both buyers choose T_H^{**}: we can have $p_{HH} = c$ for all $q \geq 0$. However, in this case both the fixed fee *and* the marginal price paid by a buyer choosing T_H^{**} depends on the tariff chosen by the other buyer, while in (16) the marginal price depends only on his choice.

[32] Likewise, if there were no kink in T_L^{**}, the buyer who pretended to be L-type may buy more than q_L^* and then share with the other buyer.

Without loss of generality, we restrict our attention to symmetric sale mechanisms, which are now introduced. Let q_{Lm} ($m = 0, 1, ..., n-1$) denote the quantity allocated to each L-type by the seller when the profile of reports $\hat{\theta} \equiv (\hat{\theta}^1, ..., \hat{\theta}^n) \in \Theta^n$ includes exactly m number of H-types. The variables q_{Hm}, t_{Hm} and t_{Lm} are defined similarly. Let $\mathbf{q}_n \equiv (q_{L0}, ..., q_{Ln-1}, q_{H1}, ..., q_{Hn})$ and $\mathbf{t}_n \equiv (t_{L0}, ..., t_{Ln-1}, t_{H1}, ..., t_{Hn})$, so that a sale mechanism is given by $M_n = \{\mathbf{q}_n, \mathbf{t}_n\}$. Any optimal mechanism $\{\mathbf{q}_n^*, \mathbf{t}_n\}$ without buyer coalition is such that $q_{Lm}^* = q_L^*$ and $q_{Hm}^* = q_H^*$ for any m and the expected payment of L-type and H-type is equal to $\theta_L u(q_L^*)$ and $\theta_H u(q_H^*) - (\Delta\theta) u(q_L^*)$, respectively.

Proposition 3, the weakly collusion-proofness principle, applies to this setting. Here we generalize Proposition 4 by describing the conditions under which an incentive compatible mechanism M_n is weakly collusion-proof. In order to do that, we need to investigate how goods are reallocated by the third party in an m-coalition – a coalition with m number of H-types and $n - m$ number of L-types – when $x(> 0)$ is the total quantity available to the coalition. Since $u'' < 0$, in any $m-$coalition the third-party allocates the same quantity to each buyer of the same type. Precisely, if quantity z is allocated to each H-type, then each L-type receives $\frac{x-mz}{n-m}$; clearly, if $m = n$ (or $m = 0$) then each H-type (L-type) receives $\frac{x}{n}$. The quantity allocated to H-type is $q_{Hm}^\epsilon(x)$ defined as

$$q_{Hm}^\epsilon(x) \equiv \arg \max_{z \in [0, \frac{x}{m}]} m\theta_H u(z) + (n-m)\theta_L^\epsilon u(\frac{x-mz}{n-m}), \quad m = 1, ..., n-1$$

Hence, the no-reallocation condition for an m-coalition (if $q_{Lm} > 0$) is:

$$\theta_H u'(q_{Hm}) = \theta_L^\epsilon u'(q_{Lm}) \tag{17}$$

If (17) is satisfied by M_n, then an m-coalition which reports truthfully in M_n has no incentive to alter the allocation determined by the seller. Notice that

$$V_m^\epsilon(x) \equiv \max_{z \in [0, \frac{x}{m}]} m\theta_H u(z) + (n-m)\theta_L^\epsilon u(\frac{x-mz}{n-m}), \quad m = 1, ..., n-1$$

is the gross payoff for an m-coalition when it owns the total quantity x, given ε. For n-coalition and 0-coalition we have $V_n(x) = \theta_H u(\frac{x}{n})$ and $V_0^\epsilon(x) = \theta_L^\epsilon u(\frac{x}{n})$, respectively. As in the proof of Proposition 5, we regard each coalition as a consolidated agent and V_m^ϵ is the surplus function of type m. For an m-coalition, manipulating its reports is equivalent to reporting a number $m'(\neq m)$ of buyers with H-type. The next proposition summarizes the coalition incentive and the no-arbitrage constraints.

Proposition 7 *An incentive compatible sale mechanism M_n is weakly collusion-proof if and only if there exists $\epsilon \in [0, 1)$ such that* *(a) the following coalition incentive constraints are satisfied:*

$$\begin{aligned} & V^{\epsilon}_{m}[mq_{Hm} + (n-m)q_{Lm}] - mt_{Hm} - (n-m)t_{Lm} \\ \geq \ & V^{\epsilon}_{m}[m'q_{Hm'} + (n-m')q_{Lm'}] - m't_{Hm'} - (n-m')t_{Lm'} \text{ for any } (m,m') \in \{0,1,...,n\}^2 \end{aligned}$$

(b) the no-arbitrage condition (17) holds for $m = 1, ..., n-1$.
(c) if $\epsilon > 0$, *then* H*-type's incentive constraint in the side mechanism is binding.*

The next proposition establishes that the buyer coalition does not create any loss to the seller, as in the case of $n = 2$.

Proposition 8 *Given the quantity schedule* $\mathbf{q}^*_n$, *there exists transfers* $\mathbf{t}^{**}_n$ *such that* $M^{**}_n \equiv \{\mathbf{q}^*_n, \mathbf{t}^{**}_n\}$ *is optimal under no buyer coalition and is also weakly collusion-proof.*

Remark 3 (transaction costs): We can compare the expected gains from arbitrage with the transaction costs generated by asymmetric information for the n-buyer case. Suppose for instance that the third-party reallocates quantity such that when there are m number of H-types, each L-type receives $\Delta q \in \left(0, \frac{m}{n-m}q^*_H\right]$. Then, the expected gains from arbitrage is given by:

$$\begin{aligned} G(n,m) \ = \ & \binom{n}{m}(p_L)^{n-m}(1-p_L)^m \left\{(n-m)\theta_L \left[u(q^*_L + \Delta q) - u(q^*_L)\right]\right. \\ & \left. -m\theta_H \left[u(q^*_H) - u(q^*_H - \Delta q \frac{n-m}{m})\right]\right\}. \end{aligned}$$

The transaction costs are given by:

$$TC(n,m) \equiv n\binom{n-1}{m}(p_L)^{n-m-1}(1-p_L)^{m+1}(\Delta\theta)\left[u(q^*_L + \Delta q) - u(q^*_L)\right].$$

We have $TC(n,m) - G(n,m) > 0$ for any $\Delta q \in \left(0, \frac{m}{n-m}q^*_H\right]$. In particular, given $\Delta q > 0$, $TC(2m,m) = kG(2m,m)$ holds where k (> 1) does not depend on m.

6.2 The case of three types

Mechanism design problems under collusion often turn out to be qualitatively more complicated when there are more than two types than when there are only two types. For instance, Laffont and Martimort (1997, 2000) limit their analysis to the two-type setting since it is difficult to determine the binding coalition incentive constraints when there are more than two types. Here we briefly explain how – in our model – Proposition 5 extends to the three-type setting.

The main difficulty is related to the fact that the single-crossing condition for coalitions holds only partially.

Now the valuation θ^i of buyer i lies in $\Theta \equiv \{\theta_L, \theta_M, \theta_H\}$, with $\Delta_H \equiv \theta_H - \theta_M > 0$, $\Delta_M \equiv \theta_M - \theta_L > 0$ and $\theta_L > 0$. The types θ^1 and θ^2 are identically and independently distributed with $p_L \equiv \Pr\left\{\theta^i = \theta_L\right\} > 0$, $p_M \equiv \Pr\left\{\theta^i = \theta_M\right\} > 0$ and $p_H \equiv \Pr\left\{\theta^i = \theta_H\right\} > 0$. In the absence of buyer coalition, the virtual values of M-type and L-type are given by:

$$\theta_M^v \equiv \theta_M - \frac{p_H}{p_M}(\theta_H - \theta_M) \qquad \theta_L^v \equiv \theta_L - \frac{p_H + p_M}{p_L}(\theta_M - \theta_L)$$

Clearly, $\theta_H > \max\{\theta_M^v, \theta_L^v\}$ but the order between θ_M^v and θ_L^v depends on the parameters; if $\theta_M^v \geq \theta_L^v$, then virtual values are said to be monotonic; if $\theta_M^v < \theta_L^v$, then let $\bar{\theta}_{ML}^v \equiv \frac{p_L\theta_L^v + p_M\theta_M^v}{p_L + p_M}$. In any case, we assume that $\min\{\theta_M^v u'(0), \theta_L^v u'(0)\} > c$, so that each type receives a positive quantity in case of no coalition.

As in the previous sections, we can restrict attention to symmetric direct revelation mechanisms, hence a sale mechanism is $M = \{\mathbf{q}, \mathbf{t}\}$, with $\mathbf{q} \equiv \{q_{jk}\}_{j,k=L,M,H}$, $\mathbf{t} \equiv \{t_{jk}\}_{j,k=L,M,H}$ and q_{jk} (t_{jk}) is the quantity received by a buyer (his payment) if he reports j and the other buyer reports k. Let $\bar{t}_j \equiv p_L t_{jL} + p_M t_{jM} + p_H t_{jH}$ and $\bar{u}_j \equiv p_L u(q_{jL}) + p_M u(q_{jM}) + p_H u(q_{jH})$, $j = L, M, H$. Then, the expected profit is written as

$$\begin{aligned}\Pi &= 2(p_L\bar{t}_L + p_M\bar{t}_M + p_H\bar{t}_H) - 2c[p_L^2 q_{LL} + p_L p_M(q_{LM} + q_{ML}) + p_L p_H(q_{HL} + q_{LH})] \\ &\quad -2c[p_M^2 q_{MM} + p_M p_H(q_{MH} + q_{HM}) + p_H^2 q_{HH}]\end{aligned}$$

The Bayesian incentive compatibility and participation constraints are

$$\begin{aligned}(BIC) \qquad \theta_j\bar{u}_j - \bar{t}_j &\geq \theta_j\bar{u}_{j'} - \bar{t}_{j'}, \quad j, j' = L, M, H \\ (BIR) \qquad \theta_j\bar{u}_j - \bar{t}_j &\geq 0, \quad j = L, M, H\end{aligned}$$

The seller maximizes Π subject to (BIC) and (BIR). The next proposition characterizes the optimal mechanisms in the absence of buyer coalition.

Proposition 9 *The optimal mechanisms in the absence of buyer coalition are characterized as follows*

(a) The optimal quantity schedule $\mathbf{q}^*$ *is such that:*

i) $q_{Hj}^* = q_H^*$ *for* $j = L, M, H$*, where* $\theta_H u'(q_H^*) = c$*;*

ii) $q_{Mj}^* = q_M^*$, $q_{Lj}^* = q_L^*$ *for* $j = L, M, H$ *with* $\theta_M^v u'(q_M^*) = \theta_L^v u'(q_L^*) = c$ *if* $\theta_M^v \geq \theta_L^v$ *but* $q_M^* = q_L^*$ *with* $\bar{\theta}_{ML}^v u'(q_L^*) = c$ *if instead* $\theta_M^v < \theta_L^v$*.*

(b) Transfers are such that constraints (BIC_{HM}), (BIC_{ML}) *and* (BIR_L) *bind.*

As in the two-type case, the weakly collusion-proofness principle holds. In order to characterize weakly collusion-proof mechanisms, it is useful to define i) the variables θ_H^ϵ, θ_M^ϵ and θ_L^ϵ; ii) the functions $q_j^\epsilon(x;jk)$ and $q_k^\epsilon(x;jk)$, $jk = HM, HL, ML$; iii) the functions $V_{jk}^\epsilon(x)$, $j,k = L,M,H$ as follows:

$$\theta_H^\epsilon \equiv \theta_H, \quad \theta_M^\epsilon \equiv \theta_M - \frac{p_H}{p_M}\Delta_H \epsilon_{HM}, \quad \theta_L^\epsilon \equiv \theta_L - \frac{p_H}{p_L}\Delta_M \epsilon_{ML},$$

$$q_j^\epsilon(x;jk) \equiv \arg\max_{z\in[0,x]} \theta_j^\epsilon u(z) + \theta_k^\epsilon u(x-z) \quad \text{and} \quad q_k^\epsilon(x;jk) \equiv x - q_j^\epsilon(x;jk)$$

$$V_{jk}^\epsilon(x) \equiv \max_{z\in[0,x]} \theta_j^\epsilon u(z) + \theta_k^\epsilon u(x-z), \quad j,k = L,M,H$$

where $\epsilon \equiv (\epsilon_{HM}, \epsilon_{ML}) \in [0,1) \times [0,+\infty)$ and $x > 0$.

The next proposition characterizes weakly collusion-proof mechanisms.

Proposition 10 *An incentive compatible sale mechanism M is weakly collusion-proof if and only if there exists $\epsilon \in [0,1) \times [0,+\infty)$ such that*
(a) the coalition incentive constraints are satisfied

$$V_{jk}^\epsilon(q_{jk}+q_{kj}) - t_{jk} - t_{kj} \geq V_{jk}^\epsilon(q_{j'k'}+q_{k'j'}) - t_{j'k'} - t_{k'j'}, \qquad \text{for any } j,k,j',k' = L,M,H. \tag{18}$$

(b) the no arbitrage constraints hold

$$q_{jk} = q_j^\epsilon(q_{jk}+q_{kj};jk), \qquad \text{for } jk = HM, HL, ML. \tag{19}$$

(c) if $\epsilon_{HM} > 0$ (resp. $\epsilon_{ML} > 0$), then (BIC_{HM}^S) [resp. (BIC_{ML}^S)] binds.

By exploiting Proposition 10 we can prove that the buyer coalition does not create any loss to the seller.

Proposition 11 *There exists a transfer scheme $\mathbf{t}^{**}$ such that $M^{**} \equiv \{\mathbf{q}^*, \mathbf{t}^{**}\}$ is both an optimal mechanism in the absence of collusion and weakly collusion-proof.*

We below provide an intuition of the result: the intuition is similar to the one for the two-type case although some technical details of the proof are more complicated. Given M^{**}, the virtual values of M-type and L-type from the third party's viewpoint are equal to θ_M^v and θ_L^v, the virtual valuations from the seller's viewpoint; hence the third-party will not reallocate goods conditional on that there is no manipulation of reports. Furthermore, the seller can use the six degrees of freedom in transfers in the optimal mechanisms under no coalition to satisfy (18), although the single crossing condition for coalitions holds only partially (it does not provide an order between coalitions HL and MM) and this makes more difficult to find the right transfers than in the two-type setting. We conjecture that our result will hold even when there are more than three types.

6.3 General cost function C and utility function $\mathcal{U}$

Here we show that Proposition 5 holds in the two-buyer-two-type setting if (i) the cost function satisfies $C(0) = 0$, $C'(q) > 0$ and $C''(q) \geq 0$ for any $q \geq 0$; (ii) the utility function satisfies $\mathcal{U}_1(q,\theta) > 0 > \mathcal{U}_{11}(q,\theta)$, $\mathcal{U}(0,\theta) = 0$, $\mathcal{U}_2(q,\theta) > 0$, $\mathcal{U}_{12}(q,\theta) > 0$ for any (q,θ).

Proposition 12 *In the setting of this subsection*
(i) the optimal mechanisms in the absence of buyer coalition are such that (BIR_L) *and* (BIC_H) *bind and the optimal quantity profile* $\mathbf{q}^*$ *satisfies* $\mathcal{U}_1(q^*_{HH},\theta_H) = C'(2q^*_{HH})$, $\mathcal{U}_1(q^*_{HL},\theta_H) = \frac{\mathcal{U}_1(q^*_{LH},\theta_L)}{p_L} - \frac{(1-p_L)\mathcal{U}_1(q^*_{LH},\theta_H)}{p_L} = C'(q^*_{HL} + q^*_{LH})$, $\frac{\mathcal{U}_1(q^*_{LL},\theta_L)}{p_L} - \frac{(1-p_L)\mathcal{U}_1(q^*_{LL},\theta_H)}{p_L} = C'(2q^*_{LL})$.
(ii) if $q^*_{HL} + q^*_{LH} \geq 2q^*_{LL}$, *then the optimal mechanism* M^{**} *in which* $(CIC_{HH,HL})$ *and* $(CIC_{HL,LL})$ *bind is weakly collusion-proof; if* $q^*_{HL} + q^*_{LH} < 2q^*_{LL}$, *then the optimal mechanism* M^{***} *in which* $(CIC_{HH,LL})$ *and* $(CIC_{LL,HL})$ *bind is weakly collusion-proof*

Notice that, in particular, Proposition 5 holds in an auction setting in which a single object is up for sale: $q \in [0,1]$ is the probability to win the object and $\mathcal{U}(q,\theta) = \theta q$.[33]

7 Robustness to cheap-talk and multiplicity

In this section we eliminate the assumption WCP and examine two issues which arise after the third-party's proposal of S^0 in response to M^{**}: the first is about whether or not both buyers will accept S^0 and the second is about whether they will play the truthtelling equilibrium after accepting S^0. It turns out that under a mild condition on the function u (see Proposition 13 below), both buyers accept S^0 but in M^{**} truthtelling is iteratively weakly dominated for H-type although it is strictly dominant for L-type. This motivates us to find a robust mechanism M^R in the set of optimal weakly collusion-proof mechanisms such that if M^R is proposed by the seller, then both buyers accept S^0 and truthtelling is strictly dominant for L-type and iteratively weakly dominant for H-type. In what follows, we first explain the two issues in more detail, present the results for M^{**} and then characterize M^R.

Let $\widetilde{M}$ be an optimal weakly collusion-proof mechanism offered by the seller. The first issue arises because, as we explained in Subsection 2.2, a two-stage game starts after the third party's proposal of S^0. First each buyer simultaneously announces whether he accepts or refuses S^0 and then buyers report either in S^0 if it was unanimously accepted, or in $\widetilde{M}$ otherwise. In any

[33] In this environment the seller does not need to exploit the information asymmetry between the buyers. Indeed, under no buyer coalition, there exists no potential room for arbitrage in HL-coalition because the marginal surplus of each type is constant and a corner solution achieves the first-best allocation and is optimal for the seller.

case, however, in the second stage $\widetilde{M}$ is actually played since S^0 is null. Therefore, buyer i's choice (veto or accept) in the first stage can be viewed as a preplay announcement which may signal some information about θ^i. In other words, the first stage is just a sort of cheap-talk stage in which a buyer may signal his type. We focussed above on the case in which each type of buyer accepts S^0, hence no learning occurs along the equilibrium path. Assume for a moment that it is common knowledge that buyers are going to play truthfully if S^0 is accepted (we deal with this issue below). Then, no type wishes to reject S^0 under the assumption WCP: in fact, buyers are indifferent between accepting and rejecting S^0. However, without the assumption, many off-the-equilibrium-path behavior and beliefs are possible. For instance, buyer 1 might expect that a non-truthful equilibrium of $\widetilde{M}$ (if any exists) will be played (possibly under non-prior beliefs of 2 about θ^1) in case he vetoes S^0. In other words, some type of buyer 1 might have the incentive to veto S^0 – which is a sort of out-of-equilibrium "message" – in order to manipulate buyer 2's beliefs about θ^1 and/or behavior such that he can reach a higher payoff for himself when playing $\widetilde{M}$ at the next stage.

The second issue arises when buyers have to report in S^0 after both of them accepted S^0. Reporting in S^0 is equivalent to playing non-cooperatively $\widetilde{M}$ with prior beliefs, since each buyer i has prior beliefs about θ^j $(j \neq i)$ after S^0 has been unanimously accepted. Although truthtelling is an equilibrium in $\widetilde{M}$, there may exist other equilibria which buyers may coordinate on.

The next proposition describes our results about the two issues when the seller offers M^{**}.

Proposition 13 *If $\frac{u''(x)}{u'(x)}$ is strictly increasing in x,*[34] *then in M^{**}*
(a) reporting L is strictly dominant for each L-type, while each H-type strictly prefers reporting H (L) if his opponent plays H (L);
(b) there is no belief of buyer i (after a rejection of S^0 by buyer $j(\neq i)$) which supports an equilibrium of M^{**} in which at least one type of buyer j is better off than in the truthtelling equilibrium;
(c) in the only non-truthful equilibrium, each type of buyer reports L. For buyers (and seller), the non-truthful equilibrium is strictly Pareto-dominated by the truthful one.

Although Proposition 13(b)-(c) deals with the two issues we introduced above for M^{**}, Proposition 13(a) reveals that truthtelling is iteratively weakly dominated for H-type.

We avoid this problem by designing a mechanism M^R in which truthtelling is iteratively weakly dominant for H-type. For this purpose, it is useful to examine the payoff bimatrix of

[34] When u is a Bernoulli utility function over money, this assumption on u is called "decreasing absolute risk-aversion".

the symmetric 2×2 game played by two buyers with H-type – let 1_H and 2_H denote them - when the seller offers an optimal weakly collusion-proof mechanism and each L-type plays L:[35]

$1_H \backslash 2_H$	L		H	
L	$\theta_H u(q_L^*) - t_{LL}$	$\theta_H u(q_L^*) - t_{LL}$	$(\Delta\theta) u(q_L^*)$	$\theta_H u(q_H^*) - t_{HL}$
H	$\theta_H u(q_H^*) - t_{HL}$	$(\Delta\theta) u(q_L^*)$	$(\Delta\theta) u(q_L^*)$	$(\Delta\theta) u(q_L^*)$

We see that when his opponent H-type reports L, any H-type prefers reporting H rather than L if $\theta_H u(q_H^*) - t_{HL} > \theta_H u(q_L^*) - t_{LL}$. Therefore, we look for a robust mechanism M^R in the set of optimal weakly collusion-proof mechanisms which satisfies the following condition:

$$\theta_H u(q_H^*) - t_{HL} = \theta_H u(q_L^*) - t_{LL} + \alpha, \tag{20}$$

where α is strictly positive and small. Recall that there exists a continuum of optimal weakly collusion-proof mechanisms; hence, it might be the case that at least one of them satisfies (20) for some $\alpha > 0$. The next proposition characterizes M^R and describes some of its properties.

Proposition 14 *Consider the mechanism* $M^R \equiv \{\mathbf{q}^R, \mathbf{t}^R\}$ *where* $\mathbf{q}^R = \mathbf{q}^*$ *and* $\mathbf{t}^R$ *solves the following linear system, in which* $\alpha > 0$ *and* $\beta > 0$ *are small numbers*[36]

(BIR_L), (BIC_H), $(CIC_{HH,HL})$ and (20), all written with equality	*if* $V_2(2q_H^*) - V_2(q_H^* + q_L^*) < V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)$
(BIR_L), (BIC_H), $(CIC_{HL,LL}^{\beta})$ and (20), all written with equality	*if* $V_2(2q_H^*) - V_2(q_H^* + q_L^*) \geq V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)$

Then
(a) M^R is optimal under no coalition formation and weakly collusion-proof.
(b) there is no belief of buyer i (after a rejection of S^0 by buyer $j \neq i$) which supports an equilibrium of M^R in which at least one type of buyer j is better off than in the truthtelling equilibrium;
(c) in M^R, reporting L is strictly dominant for each L-type, while each H-type strictly prefers reporting H (L) if his opponent plays L (H).

According to Proposition 14,[37] when the seller offers M^R, both buyers accept S^0 and

[35] That is the case when M^R is offered, as Proposition 14 below states.

[36] $(CIC_{HL,LL}^{\beta})$ below is obtained by adding β to the right hand side of constraint $(CIC_{HL,LL})$.

[37] We note that the result (b) in Proposition 14 [and (b) in Proposition 13] is stronger than Proposition 9 in Laffont and Martimort (2000). Indeed, their result refers to the notion of ratifiability [see Cramton and Palfrey (1995)], which allows buyer i to have only "reasonable" or "consistent" beliefs about θ^j. In contrast, we do not need any "sophisticated" argument in order to make our point: simply no beliefs of i support buyer j's rejection of S^0.

truthtelling is strictly dominant for each L-type and serially weakly dominant for each H-type. Actually, M^R admits two (asymmetric) non-truthful equilibria: One in which buyer 1 reports truthfully and each type of buyer 2 reports L and the other in which buyer 2 reports truthfully and each type of buyer 1 reports L. However, it seems reasonable to discard them because they both involve the use of iteratively weakly dominated strategies and are Pareto dominated for buyers by the truthful equilibrium. We also note that there exist a continuum of robust mechanisms since we can find a robust one for each positive small α.

8 Concluding remarks

We found that if the seller uses simple sale mechanisms in which the quantity sold to a buyer and his payment depend solely on his own report, buyers can realize strict gains at the seller's loss by coordinating their purchases and reallocating the goods. However, we showed that when the seller judiciously designs her mechanism by exploiting the transaction costs in coalition formation, buyer coalition does not hurt her and, in particular, the buyers are unable to implement efficient arbitrage. We also showed that the optimal outcome can be implemented through a menu of two-part tariffs.

Some might find unnatural the feature of the optimal collusion-proof mechanisms that a buyer's payment depends on the other buyer's report while the quantity he receives is independent of such a report. However, this is due to the fact that we focused on the case of constant marginal cost. In a more general environment with variable marginal cost, (i) our main result still holds and (ii) even without buyer coalition, both the quantity received by a buyer and his payment will depend on the other's report under dominant strategy implementation. Furthermore, the feature that a buyer's payment depends on the other buyer's report exists in Vickrey auctions, where the price that a winner pays depends on other bidders' bids.

Our results suggest that buyer coalitions are likely to emerge either when they have better information about each other's preferences than the seller has, or when the seller is constrained to use a restricted set of contracts such that a buyer's payment cannot depend on other buyers' actions. For instance, when there are a large number of buyers (possibly a mass of buyers), the seller may have incomplete information about their number and identities. This would impose restrictions on the set of contracts available to the seller, as in Alger (1999). It would be interesting to study the case in which the seller can use only individual contracts: i.e., the quantity sold to a buyer and his payment do not depend on what other buyers do. In this setting, the collusion-proofness principle might not hold and the optimal mechanism might involve letting collusion occur.[38]

APPENDIX

Proof of Proposition 1

The arguments of the proof for the single-buyer model show that (BIC_H) and (BIR_L) bind in the optimum. After replacing in Π the transfers as obtained from (BIC_H) and (BIR_L) written with equality, (i)-(ii) emerge as necessary and sufficient conditions for the optimum and (BIC_L) and (BIR_H) are automatically satisfied. ■

Proof of Proposition 2(b)

The side mechanism $S^d = \left\{\phi^d(\theta^1,\theta^2), x^{id}(\theta^1,\theta^2,\widetilde{\phi}), y^{id}(\theta^1,\theta^2)\right\}$ mentioned in the statement of Proposition 2(b) is formally defined as follows. For simplicity, let $\phi^d_{jk} = \phi^d(\theta_j,\theta_k)$, $x^{id}_{jk,\widetilde{\phi}} = x^{id}(\theta_j,\theta_k,\widetilde{\phi})$ and $y^{id}_{jk} = y^{id}(\theta_j,\theta_k)$ with $j,k \in \{H,L\}$.

Reports manipulations: $\phi^d_{HH} = (\theta_H,\theta_L)$, $\phi^d_{HL} = \phi^d_{LH} = \phi^d_{LL} = (\theta_L,\theta_L)$.[39]

Reallocation of goods[40]: $x^{1d}_{HH} = -\frac{q^*_H - q^*_L}{2}$, $x^{2d}_{HH} = \frac{q^*_H - q^*_L}{2}$; $x^{1d}_{HL} = \hat{x} > 0$, with $\hat{x}$ close to 0, $x^{2d}_{HL} = -\hat{x}$; $x^{2d}_{LH} = -x^{1d}_{HL} = \hat{x}$; $x^{1d}_{LL} = x^{2d}_{LL} = 0$.

Side transfers: $y^{1d}_{HH} = -\frac{t^*_H - t^*_L}{2}$, $y^{2d}_{HH} = \frac{t^*_H - t^*_L}{2}$; $y^{1d}_{HL} = y^{2d}_{LH} = \hat{y}$, $y^{2d}_{HL} = y^{1d}_{LH} = -\hat{y}$; $y^{1d}_{LL} = y^{2d}_{LL} = 0$, where $\hat{y} > 0$ is still to be defined.

In words, an HH-coalition reports HL; then goods and payments are equally shared between the buyers. A coalition HL or LH reports LL; then goods are slightly reallocated from L-type to H-type and H-type pays $\hat{y}$ to L-type.

We prove that for a small $\hat{x} > 0$ there exists a $\hat{y} > 0$ such that (BIC^S) are satisfied and (BIR^S) are slack – $(BB : x)$ and $(BB : y)$ are satisfied by definition. Therefore, S^d is feasible and strictly increases the payoff of each buyer type with respect to playing M^d non-cooperatively.

[38] Another direction for extension is to consider different timing for buyer coalitions as Laffont and Martimort (1997) discuss. To focus on coordination of purchases and reallocation, we adopted the timing chosen by Laffont and Martimort (1997, 2000) but the analysis can be extended to a timing in which buyers can form a coalition after receiving the seller's offer and before deciding whether to accept or reject the offer. Independently, deQuiedt (2002) recently studied collusion with this timing in auctions.

[39] We recall that when the manipulation is deterministic, i.e., $p^{\phi}(\widetilde{\theta}^1,\widetilde{\theta}^2,\widetilde{\phi}) = 1$ for some $\widetilde{\phi} \in \Theta^2$, we write $\phi(\widetilde{\theta}^1,\widetilde{\theta}^2) = \widetilde{\phi}$ (see Section 2.2).

[40] Since the report manipulation is deterministic, we do not write $\widetilde{\phi}$ in $x^{id}_{jk,\widetilde{\phi}}$.

Let $\widehat{q}_H \equiv q_L^* + \hat{x}$, $\widehat{q}_L \equiv q_L^* - \hat{x}$ and consider constraint (BIC_H^S):

$$\begin{aligned} & p_L[\theta_H u(\widehat{q}_H) - \theta_L u(q_L^*) - \hat{y}] + (1-p_L)\{\theta_H u(\frac{q_L^* + q_H^*}{2}) - \theta_L u(q_L^*) - \frac{\theta_H}{2}[u(q_H^*) - u(q_L^*)]\} \\ \geq\ & p_L(\Delta\theta)u(q_L^*) + (1-p_L)[\theta_H u(\widehat{q}_L) - \theta_L u(q_L^*) + \hat{y}] \qquad (21) \end{aligned}$$

Let $\hat{y} = \tilde{y} \equiv \theta_H[u(q_L^*) - u(\widehat{q}_L)] > 0$, so that (i) the right hand side of (21) is equal to $U^{M^d}(\theta_H) = (\Delta\theta)u(q_L^*)$; (ii) if $\hat{x} = 0$, then (21) is strictly satisfied and therefore, when $\hat{x} > 0$ is close to 0, (21) is still strictly satisfied and (BIR_H^S) is strictly satisfied as well; (iii) (BIR_L^S) holds strictly. Given a small $\hat{x} > 0$, consider increasing $\hat{y}$ above $\tilde{y}$ until the point at which (21) binds. Then, (BIR_H^S) still holds strictly since the right hand side of (21) increased above $U^{M^d}(\theta_H)$; clearly, (BIR_L^S) holds strictly as well since now $\hat{y} > \tilde{y}$. In order to prove that (BIC_L^S) is satisfied, add up (BIC_L^S) and (BIC_H^S) (which binds) to obtain an inequality which holds strictly because $\widehat{q}_H > q_L^*$ and $\frac{q_L^* + q_H^*}{2} > \widehat{q}_L$. Therefore, S^d satisfies (BIC^S) and (BIR^S) and the payoff of each type of buyer is strictly larger than from playing M^d non-cooperatively.

Thus, with M^d, the buyer coalition strictly reduces the seller's profit because (i) in the states of nature in which reports are manipulated, the quantity sold to buyers is smaller than under truthtelling, which reduces the surplus generated by the trade and (ii) each type of buyer obtains a higher payoff than with truthtelling.[41]

Proof of Proposition 3

The proof is omitted since it is a straightforward adaptation of the proof of Proposition 3 in Laffont and Martimort (2000). ■

Proof of Proposition 4

We are interested in sale mechanisms such that L-type's incentive constraint is not binding. Since we are finding conditions under which S^0 is optimal for the third party, the incentive constraint of L-type will be slack in the side mechanism as well. In what follows, for the sake of brevity, let $x^i_{jk,\widetilde{\phi}}$ denote $x^i(\theta_j, \theta_k, \widetilde{\phi})$ with $j, k \in \{H, L\}$. Likewise, $p^{\phi}_{jk,\widetilde{\phi}}$ denotes $p^{\phi}(\theta_j, \theta_k, \widetilde{\phi})$.

[41] Actually, S^d may not be the optimal side mechanism against M^d. In particular, goods are not efficiently reallocated within HL-coalition since otherwise we are not sure of whether (BIR^S) and (BIC^S) can all be satisfied. However, if the third party chooses the optimal side mechanism against M^d, then still the profit is smaller than if M^d is played non-cooperatively.

The third-party maximizes the following objective,

$$
\begin{aligned}
&(1-p_L)^2 \textstyle\sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{HH,\widetilde{\phi}}[\theta_H u(q^1(\widetilde{\phi})+x^1_{HH,\widetilde{\phi}})-t^1(\widetilde{\phi})+\theta_H u(q^2(\widetilde{\phi})+x^2_{HH,\widetilde{\phi}})-t^2(\widetilde{\phi})]\\
&+p_L(1-p_L) \textstyle\sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{LH,\widetilde{\phi}}[\theta_L u(q^1(\widetilde{\phi})+x^1_{LH,\widetilde{\phi}})-t^1(\widetilde{\phi})+\theta_H u(q^2(\widetilde{\phi})+x^2_{LH,\widetilde{\phi}})-t^2(\widetilde{\phi})]\\
&+p_L(1-p_L) \textstyle\sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{HL,\widetilde{\phi}}[\theta_H u(q^1(\widetilde{\phi})+x^1_{HL,\widetilde{\phi}})-t^1(\widetilde{\phi})+\theta_L u(q^2(\widetilde{\phi})+x^2_{HL,\widetilde{\phi}})-t^2(\widetilde{\phi})]\\
&+p_L^2 \textstyle\sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{LL,\widetilde{\phi}}[\theta_L u(q^1(\widetilde{\phi})+x^1_{LL,\widetilde{\phi}})-t^1(\widetilde{\phi})+\theta_L u(q^2(\widetilde{\phi})+x^2_{LL,\widetilde{\phi}})-t^2(\widetilde{\phi})]
\end{aligned}
$$

subject to the following constraints.

- Budget balance constraints: for the quantity reallocation

$$\sum_{i=1}^{2} x^i(\theta^1,\theta^2,\widetilde{\phi})=0, \text{ for any } (\theta^1,\theta^2)\in\Theta^2 \text{ and any } \widetilde{\phi}\in\Theta^2;$$

for the side transfers

$$\sum_{i=1}^{2} y^i(\theta^1,\theta^2)=0, \text{ for any } (\theta^1,\theta^2)\in\Theta^2,$$

- H-type's Bayesian incentive constraint for buyer 1: $(BIC_1^S(\theta_H))$

$$
\begin{aligned}
& p_L \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{HL,\widetilde{\phi}}[\theta_H u(q^1(\widetilde{\phi})+x^1_{HL,\widetilde{\phi}})-t^1(\widetilde{\phi})-y^1_{HL}]\\
& +(1-p_L) \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{HH,\widetilde{\phi}}[\theta_H u(q^1(\widetilde{\phi})+x^1_{HH,\widetilde{\phi}})-t^1(\widetilde{\phi})-y^1_{HH}]\\
\geq\ & p_L \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{LL,\widetilde{\phi}}[\theta_H u(q^1(\widetilde{\phi})+x^1_{LL,\widetilde{\phi}})-t^1(\widetilde{\phi})-y^1_{LL}]\\
& +(1-p_L) \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{LH,\widetilde{\phi}}[\theta_H u(q^1(\widetilde{\phi})+x^1_{LH,\widetilde{\phi}})-t^1(\widetilde{\phi})-y^1_{LH}],
\end{aligned}
$$

- H-type's acceptance constraint for buyer 1: $(BIR_1^S(\theta_H))$

$$
\begin{aligned}
& p_L \textstyle\sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{HL,\widetilde{\phi}}[\theta_H u(q^1(\widetilde{\phi})+x^1_{HL,\widetilde{\phi}})-t^1(\widetilde{\phi})-y^1_{HL}]\\
& +(1-p_L) \textstyle\sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{HH,\widetilde{\phi}}[\theta_H u(q^1(\widetilde{\phi})+x^1_{HH,\widetilde{\phi}})-t^1(\widetilde{\phi})-y^1_{HH}] \geq U^M(\theta_H),
\end{aligned}
$$

- L-type's acceptance constraint for buyer 1: $(BIR_1^S(\theta_L))$

$$
\begin{aligned}
& p_L \textstyle\sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{LL,\widetilde{\phi}}[\theta_L u(q^1(\widetilde{\phi})+x^1_{LL,\widetilde{\phi}})-t^1(\widetilde{\phi})-y^1_{LL}]\\
& +(1-p_L) \textstyle\sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{LH,\widetilde{\phi}}[\theta_L u(q^1(\widetilde{\phi})+x^1_{LH,\widetilde{\phi}})-t^1(\widetilde{\phi})-y^1_{LH}] \geq U^M(\theta_L),
\end{aligned}
$$

- H-type's Bayesian incentive constraint for buyer 2 : $(BIC_2^S(\theta_H))$

• H-type's acceptance constraint for buyer 2: $(BIR_2^S(\theta_H))$

• L-type's acceptance constraint for buyer 2: $(BIR_2^S(\theta_L))$,

where $(BIC_2^S(\theta_H))$, $(BIR_2^S(\theta_H))$, $(BIR_2^S(\theta_L))$ are in the same way as $(BIC_1^S(\theta_H))$, $(BIR_1^S(\theta_H))$, $(BIR_1^S(\theta_L))$ are defined.

We introduce the following multipliers:

• $\rho^x(\theta^1, \theta^2, \widetilde{\phi})$ for the budget-balance constraint for the quantity reallocation in state $(\theta^1, \theta^2, \widetilde{\phi})$,

• $\rho^y(\theta^1, \theta^2)$ for the budget-balance constraint for the side-transfers in state (θ^1, θ^2),

• δ^i for the H-type's Bayesian incentive constraint concerning buyer i,

• v_H^i for the H-type's acceptance constraint concerning buyer i,

• v_L^i for the L-type's acceptance constraint concerning buyer i.

We define the Lagrangian function as follows:

$$\mathcal{L} = E(U_1 + U_2) + \sum_{i=1,2} \delta^i (BIC_i^S)(\theta_H) + \sum_{i=1,2} v_H^i (BIR_i^S)(\theta_H) + \sum_{i=1,2} v_L^i (BIR_i^S)(\theta_L)$$

$$+ \sum_{(\theta^1,\theta^2)\in\Theta^2} \sum_{\widetilde{\phi}\in\Theta^2} \rho^x(\theta^1, \theta^2, \widetilde{\phi})(BB:x)(\theta^1, \theta^2, \widetilde{\phi}) + \sum_{(\theta^1,\theta^2)\in\Theta^2} \rho^y(\theta^1, \theta^2)(BB:y)(\theta^1, \theta^2)$$

Step 1: Optimizing with respect to $y^i(\theta^1, \theta^2)$

After optimizing with respect to y_{HH}^i, we have:

$$\rho_{HH}^y - \delta^i(1 - p_L) - v_H^i(1 - p_L) = 0, \text{ for } i = 1, 2.$$

After optimizing with respect to y_{HL}^1 and y_{HL}^2 respectively, we have:

$$\begin{aligned} \rho_{HL}^y - \delta^1 p_L - v_H^1 p_L &= 0; \\ \rho_{HL}^y + \delta^2(1 - p_L) - v_L^2(1 - p_L) &= 0 \end{aligned}$$

After optimizing with respect to y_{LH}^1 and y_{LH}^2 respectively, we have:

$$\begin{aligned} \rho_{LH}^y + \delta^1(1 - p_L) - v_L^1(1 - p_L) &= 0; \\ \rho_{LH}^y - \delta^2 p_L - v_H^2 p_L &= 0 \end{aligned}$$

After optimizing with respect to y_{LL}^i, we have:

$$\rho_{LL}^y + \delta^i p_L - v_L^i p_L = 0, \text{ for } i = 1, 2.$$

In what follows, without loss of generality, we restrict our attention to symmetric multipliers:

$$\delta \equiv \delta^1 = \delta^2, \quad v_H \equiv v_H^1 = v_H^2, \quad v_L \equiv v_L^1 = v_L^2$$

From the above equations, we have:

$$p_L(\delta + v_H) = (1 - p_L)(v_L - \delta)$$

Step 2: Optimizing with respect to $x^i(\theta^1, \theta^2, \widetilde{\phi})$ given $p^\phi(\theta^1, \theta^2, \widetilde{\phi})$

For simplicity, let $\rho^x_{jk,\widetilde{\phi}} = \rho^x(\theta_j, \theta_k, \widetilde{\phi})$.

After optimizing with respect to $x^i_{HH,\widetilde{\phi}}$, we have:[42]

$$\rho^x_{HH,\widetilde{\phi}} + p^\phi_{HH,\widetilde{\phi}}(1 - p_L + \delta + v_H)(1 - p_L)\theta_H u'(q^i(\widetilde{\phi}) + x^i_{HH,\widetilde{\phi}}) = 0, \text{ for } i = 1, 2, \text{ and any } \widetilde{\phi} \in \Theta^2.$$

The above equations imply that $q^1(\widetilde{\phi}) + x^1_{HH,\widetilde{\phi}} = q^2(\widetilde{\phi}) + x^2_{HH,\widetilde{\phi}}$ for any $\widetilde{\phi} \in \Theta^2$. Since $x^1_{HH,\widetilde{\phi}} + x^2_{HH,\widetilde{\phi}} = 0$ from the budget balance constraint, we have $q^i(\widetilde{\phi}) + x^i_{HH,\widetilde{\phi}} = \frac{q^1(\widetilde{\phi}) + q^2(\widetilde{\phi})}{2}$ for each $\widetilde{\phi}$. Hence, any total quantity which is available to $HH-$coalition is always split equally between the two buyers. We will see that the same result holds for $LL-$coalition.

After optimizing with respect to $x^1_{HL,\widetilde{\phi}}$ and $x^2_{HL,\widetilde{\phi}}$ respectively, we have:

$$\begin{aligned} \rho^x_{HL,\widetilde{\phi}} + p^\phi_{HL,\widetilde{\phi}}(1 - p_L + \delta + v_H)p_L\theta_H u'(q^1(\widetilde{\phi}) + x^1_{HL,\widetilde{\phi}}) &= 0, \text{ for any } \widetilde{\phi} \in \Theta^2, \\ \rho^x_{HL,\widetilde{\phi}} + p^\phi_{HL,\widetilde{\phi}}(p_L\theta_L - \delta\theta_H + v_L\theta_L)(1 - p_L)u'(q^2(\widetilde{\phi}) + x^2_{HL,\widetilde{\phi}}) &= 0, \text{ for any } \widetilde{\phi} \in \Theta^2. \end{aligned}$$

By using $p_L(\delta + v_H) = (1 - p_L)(v_L - \delta)$, we obtain from the two above equations:

$$\theta_H u'(q^1(\widetilde{\phi}) + x^1_{HL,\widetilde{\phi}}) = \left(\theta_L - \frac{1 - p_L}{p_L}(\Delta\theta)\epsilon\right) u'(q^2(\widetilde{\phi}) + x^2_{HL,\widetilde{\phi}}), \text{ for any } \widetilde{\phi} \in \Theta^2,$$

where $\epsilon \equiv \frac{\delta}{1 - p_L + \delta + v_H}$. Since $\theta^\epsilon_L = \theta_L - \frac{1 - p_L}{p_L}(\Delta\theta)\epsilon$, any total quantity available to $HL-$coalition is split according to the following condition:

$$\theta_H u'(q^1(\widetilde{\phi}) + x^1_{HL,\widetilde{\phi}}) = \theta^\epsilon_L u'(q^2(\widetilde{\phi}) + x^2_{HL,\widetilde{\phi}}), \text{ for any } \widetilde{\phi} \in \Theta^2.$$

After optimizing with respect to $x^1_{LH,\widetilde{\phi}}$ and $x^2_{LH,\widetilde{\phi}}$ respectively, we have:

$$\begin{aligned} \rho^x_{LH,\widetilde{\phi}} + p^\phi_{LH,\widetilde{\phi}}(p_L\theta_L - \delta\theta_H + v_L\theta_L)(1 - p_L)u'(q^1(\widetilde{\phi}) + x^1_{LH,\widetilde{\phi}}) &= 0, \text{ for any } \widetilde{\phi} \in \Theta^2, \\ \rho^x_{LH,\widetilde{\phi}} + p^\phi_{LH,\widetilde{\phi}}(1 - p_L + \delta + v_H)p_L\theta_H u'(q^2(\widetilde{\phi}) + x^2_{LH,\widetilde{\phi}}) &= 0, \text{ for any } \widetilde{\phi} \in \Theta^2. \end{aligned}$$

[42] In homogeneous coalitions, HH and LL, the reallocation cannot lead to corner solutions. In $HL-$coalition, instead, this is conceivable but it is not going to occur when the seller designs the sale mechanism optimally. Hence, we only consider interior solutions for the reallocation problem.

From the two above equations, we obtain:

$$\theta_H u'(q^2(\widetilde{\phi}) + x^2_{LH,\widetilde{\phi}}) = \theta^\epsilon_L u'(q^1(\widetilde{\phi}) + x^1_{LH,\widetilde{\phi}}), \text{ for any } \widetilde{\phi} \in \Theta^2.$$

After optimizing with respect to $x^i_{LL,\widetilde{\phi}}$, we have:

$$\rho^x_{LL,\widetilde{\phi}} + p^\phi_{LL,\widetilde{\phi}}(p_L\theta_L - \delta\theta_H + v_L\theta_L)p_L u'(q^i(\widetilde{\phi}) + x^i_{LL,\widetilde{\phi}}) = 0, \text{ for } i = 1,2 \text{ and any } \widetilde{\phi} \in \Theta^2.$$

The above equations imply that $q^1(\widetilde{\phi}) + x^1_{LL,\widetilde{\phi}} = q^2(\widetilde{\phi}) + x^2_{LL,\widetilde{\phi}}$. Since $x^1_{LL,\widetilde{\phi}} + x^2_{LL,\widetilde{\phi}} = 0$ from the budget balance constraint, we have $q^i(\widetilde{\phi}) + x^i_{LL,\widetilde{\phi}} = \frac{q^1(\widetilde{\phi})+q^2(\widetilde{\phi})}{2}$.

Step 3: Optimizing with respect to $\phi(\theta^1, \theta^2)$

Recall that we want to find conditions under which the third party optimally requires any coalition with $(\theta^1, \theta^2) = (\theta_j, \theta_k)$ to report (θ_j, θ_k), i.e., $\phi(\theta_j, \theta_k) = (\theta_j, \theta_k)$.

• HH coalition:

$$(\theta_H, \theta_H) \in \arg\max_{\tilde{\phi}\in\Theta^2} \left\{ 2\theta_H u(\frac{q^1(\tilde{\phi}) + q^2(\tilde{\phi})}{2}) - t^1(\tilde{\phi}) - t^2(\tilde{\phi}) \right\}.$$

• HL coalition:

$$(\theta_H, \theta_L) \in \arg\max_{\tilde{\phi}\in\Theta^2} \left\{ \begin{array}{c} \theta_H u \left[q^\epsilon_H(q^1(\tilde{\phi}) + q^2(\tilde{\phi})) \right] + \theta^\epsilon_L u \left[q^\epsilon_L(q^1(\tilde{\phi}) + q^2(\tilde{\phi})) \right] \\ -t^1(\tilde{\phi}) - t^2(\tilde{\phi}) \end{array} \right\}.$$

• LH coalition:

$$(\theta_L, \theta_H) \in \arg\max_{\tilde{\phi}\in\Theta^2} \left\{ \begin{array}{c} \theta^\epsilon_L u \left[q^\epsilon_L(q^1(\tilde{\phi}) + q^2(\tilde{\phi})) \right] + \theta_H u \left[q^\epsilon_H(q^1(\tilde{\phi}) + q^2(\tilde{\phi})) \right] \\ -t^1(\tilde{\phi}) - t^2(\tilde{\phi}) \end{array} \right\}.$$

• LL coalition:

$$(\theta_L, \theta_L) \in \arg\max_{\tilde{\phi}\in\Theta^2} \left\{ 2\theta^\epsilon_L u(\frac{q^1(\tilde{\phi}) + q^2(\tilde{\phi})}{2}) - t^1(\tilde{\phi}) - t^2(\tilde{\phi}) \right\}.$$

Finally, notice that the above conditions are equivalent to (6)-(11).

Some missing elements in the proof to Proposition 5

The transfers $\mathbf{t}^{**}$ in M^{**} are given as follows**:**

$$\begin{aligned} t^{**}_{HL} &= \frac{(1+p_L)\theta_L - (3-p^2_L)\theta_H}{2} u(q^*_L) + \theta_H \frac{p_L(3-p_L)}{2} u(q^*_H) \\ &\quad + (1-p_L)(2-p_L)\theta_H u(\frac{q^*_H + q^*_L}{2}) + \frac{p_L(1-p_L)}{2} V^1_1(2q^*_L), \end{aligned}$$

$$t^{**}_{LH} = \frac{(p_L+3)\theta_L + (2p_L+p_L^2-1)\theta_H}{2}u(q_L^*) + \theta_H\frac{p_L(1-p_L)}{2}u(q_H^*)$$
$$-p_L(1-p_L)\theta_H u(\frac{q_H^*+q_L^*}{2}) - \frac{p_L(1+p_L)}{2}V_1^1(2q_L^*),$$

$$t^{**}_{HH} = \frac{(p_L+2)\theta_L - (1-p_L)(2+p_L)\theta_H}{2}u(q_L^*) + \theta_H\frac{2+2p_L-p_L^2}{2}u(q_H^*)$$
$$-p_L(2-p_L)\theta_H u(\frac{q_H^*+q_L^*}{2}) - \frac{p_L^2}{2}V_1^1(2q_L^*),$$

$$t^{**}_{LL} = \frac{(p_L^2+2p_L-1)\theta_L^1}{2}u(q_L^*) - \theta_H\frac{(1-p_L)^2}{2}u(q_H^*) + (1-p_L)^2\theta_H u(\frac{q_H^*+q_L^*}{2})$$
$$+\frac{1-p_L^2}{2}V_1^1(2q_L^*).$$

Lemma for the proof of Proposition 5

Lemma 1 *A single crossing condition for coalitions holds:*

$$\frac{\partial V_2(x)}{\partial x} > \frac{\partial V_1^\epsilon(x)}{\partial x} > \frac{\partial V_0^\epsilon(x)}{\partial x} \text{ for any } x>0 \text{ and } \epsilon \geq 0.$$

Proof. We have $V_2(x) = 2\theta_H u(\frac{x}{2})$ and $V_0^\epsilon(x) = 2\theta_L^\epsilon u(\frac{x}{2})$; hence $\frac{\partial V_2(x)}{\partial x} = \theta_H u'(\frac{x}{2})$ and $\frac{\partial V_0^\epsilon(x)}{\partial x} = \theta_L^\epsilon u'(\frac{x}{2})$. For an HL-coalition, let us consider for simplicity interior allocations (but the proof is easily adapted to the non-interior case). Then $q_H^\epsilon(x)$ and $q_L^\epsilon(x)$ are such that $\theta_H u'[q_H^\epsilon(x)] = \theta_L^\epsilon u'[q_L^\epsilon(x)]$ and the envelope theorem implies $\frac{\partial V_1^\epsilon(x)}{\partial x} = \theta_H u'[q_H^\epsilon(x)] = \theta_L^\epsilon u'[q_L^\epsilon(x)]$. Since u' is strictly decreasing and $\theta_H > \theta_L^\epsilon$, we have $q_H^\epsilon(x) > \frac{x}{2} > q_L^\epsilon(x)$; hence $\frac{\partial V_2(x)}{\partial x} = \theta_H u'(\frac{x}{2}) > \theta_H u'[q_H^\epsilon(x)] = \theta_L^\epsilon u'[q_L^\epsilon(x)] > \theta_L^\epsilon u(\frac{x}{2}) = \frac{\partial V_0^\epsilon(x)}{\partial x}$. ■

Proof that β defined at page 20 is negative

Let $g(z) \equiv V_1^1(2q_L^*+z) - V_1^1(2q_L^*) - \theta_H\left[u(q_L^*+z) - u(q_L^*)\right]$; we want to show that $g(q_H^*-q_L^*) < 0$ because $\beta = g(q_H^*-q_L^*)$. Since $g(0)=0$ and we can prove that $g'(z) < 0$ $\forall z \in [0, q_H^*-q_L^*)$, we obtain $g(q_H^*-q_L^*) < 0$. We find $g'(z) \equiv \theta_H u'\left[q_H^1(2q_L^*+z)\right] - \theta_H u'(q_L^*+z)$. $g'(z) < 0$ is equivalent to $q_H^1(2q_L^*+z) > q_L^*+z$, which holds for $\forall z \in [0, q_H^*-q_L^*)$ since otherwise we have $\theta_H u'\left[q_H^1(2q_L^*+z)\right] \geq \theta_H u'(q_L^*+z) > \theta_H u'(q_H^*) = \theta_L^v u'(q_L^*) > \theta_L^v u'\left[q_L^1(2q_L^*+z)\right]$, a violation of $\theta_H u'[q_H^1(x)] = \theta_L^v u'[q_L^1(x)]$ for $x = 2q_L^*+z$.

Proof of Proposition 6

We assume here that the seller proposes the menu of two-part tariffs $\{T_H^{**}, T_L^{**}\}$. We consider first the case without buyer coalition and then examine the case with buyer coalition.

Proof that with $\{T_H^{**}, T_L^{**}\}$, under buyers' noncooperative behavior the same outcome arises as with M^{**}. For the sake of brevity, let $\bar{p} \equiv \theta_H u'(q_L^*)$ and recall that if buyers choose tariffs (T_j^{**}, T_k^{**}), then the buyer who chose T_j^{**} (resp. T_k^{**}) faces the marginal price $\bar{p}$ for quantities above q_j^* (resp. q_k^*) and the marginal price c for quantities below q_j^* (resp. q_k^*).

Suppose that buyer 2 chooses the tariff designed for his type and consider the decision problem of H-type of buyer 1 (buyers are ex ante symmetric). If he selects T_H^{**}, then he buys the quantity which maximizes $\theta_H u(q) - cq1_{[q \le q_H^*]} - [cq_H^* + \bar{p}(q - q_H^*)]\, 1_{[q > q_H^*]}$ and the maximum is achieved at $q = q_H^*$. Given that his expected fixed fee is $p_L A_{HL}^{**} + (1 - p_L) A_{HH}^{**}$, his expected payoff is $(\Delta\theta) u(q_L^*)$. Suppose now that he selects T_L^{**}. Then, he would buy the quantity which maximizes $\theta_H u(q) - cq1_{[q \le q_L^*]} - [cq_L^* + \bar{p}(q - q_L^*)]\, 1_{[q > q_L^*]}$. The maximum is attained at $q = q_L^*$. Given that the expected fixed fee is $p_L A_{LL}^{**} + (1 - p_L) A_{LH}^{**}$, his expected payoff is again $(\Delta\theta) u(q_L^*)$.

Consider now L-type of buyer 1. If he chooses T_L^{**}, he buys the quantity which maximizes $\theta_L u(q) - cq1_{[q \le q_L^*]} - [cq_L^* + \bar{p}(q - q_L^*)]\, 1_{[q > q_L^*]}$. The solution is $q = q_L^*$ and his expected payoff (taking into account fixed fees) is 0. If he chooses T_H^{**}, then he will consume quantity q_L^{FB} such that $\theta_L u'(q_L^{FB}) = c$ and his expected payoff would be,

$$\theta_L u(q_L^{FB}) - cq_L^{FB} - p_L A_{HL}^{**} - (1 - p_L) A_{HH}^{**}.$$

We now show that the above payoff is smaller than 0. Since $p_L A_{HL}^{**} + (1 - p_L) A_{HH}^{**} = \theta_H u(q_H^*) - (\Delta\theta) u(q_L^*) - cq_H^*$, we need to prove that

$$\theta_L u(q_L^{FB}) - cq_L^{FB} + (\Delta\theta) u(q_L^*) \le \theta_H u(q_H^*) - cq_H^* \tag{22}$$

Notice that the right hand side of (22) would be smaller if q_H^* were replaced by $q_L^{FB} < q_H^*$. Therefore, (22) holds if it is satisfied when q_H^* is replaced by q_L^{FB}. When q_H^* is replaced by q_L^{FB}, (22) boils down to $(\Delta\theta) u(q_L^*) < (\Delta\theta) u(q_L^{FB})$, which holds since $q_L^* < q_L^{FB}$.

Proof that allowing coalition formation does not affect the outcome We now study the case with buyer coalition. We first define the third-party's program when the seller offers a menu of two-parts tariffs. The side-contract takes the form:

$$\{\phi(\widetilde{\theta}), q^i(\widetilde{\theta}, \widetilde{\phi}), x^i(\widetilde{\theta}, \widetilde{\phi}), y^i(\widetilde{\theta});\ i = 1, 2\},$$

where $\widetilde{\theta}^i$ is buyer i's report to the third-party and $\widetilde{\theta} = (\widetilde{\theta}^1, \widetilde{\theta}^2)$. Let $\phi(\cdot)$ be the tariff selection function that tells each buyer (possibly randomly) which tariff to select as a function of $\widetilde{\theta}$. Let $\widetilde{\phi} \in \Theta^2$ denote a realized outcome of $\phi(\cdot)$ and let $p^{\phi}(\widetilde{\theta}, \widetilde{\phi})$ denote the probability that the third party requests the buyers to choose the tariffs $\widetilde{\phi}$ when they report him $\widetilde{\theta}$. Let $q^i(\widetilde{\theta}, \widetilde{\phi})$ denote the quantity which the third party recommends buyer i to buy as a function of $(\widetilde{\theta},\widetilde{\phi})$. Let $x^i(\widetilde{\theta},\widetilde{\phi})$ denote the reallocation function which determines the quantity that buyer i receives from the third-party. Therefore, $q^i(\widetilde{\theta},\widetilde{\phi})+x^i(\widetilde{\theta},\widetilde{\phi})$ is the total quantity received by buyer i. $y^i(\cdot)$ is the monetary transfer from buyer i to the third party as defined in Section 2.2. We impose the usual ex post budget balance constraints for the reallocation of goods and for the side transfers. Finally, observe that the payment of buyer i to the seller depends both on the chosen tariffs $\widetilde{\phi}$ and the quantity bought $q^i(\widetilde{\theta},\widetilde{\phi})$. Hence, it is denoted by $t^i(q^i(\widetilde{\theta},\widetilde{\phi}),\widetilde{\phi})$. Let us define the null side-contract, denoted by S^0, as $S^0\equiv\{\phi(\cdot)=Id(\cdot),q^i(\cdot)=q^*(\widetilde{\theta}^i),x^1(\cdot)=x^2(\cdot)=0,y^1(\cdot)=y^2(\cdot)=0\}$ where $q^*(\theta^i)$ equals q^*_H (q^*_L) if $\widetilde{\theta}^i=\theta_H$ $(\widetilde{\theta}^i=\theta_L)$.

As in Definition 1, we say that a side-contract $S^*=\{\phi^*(\cdot),q^{i*}(\cdot),x^{i*}(\cdot),y^{i*}(\cdot)\}$ is coalition-interim-efficient with respect to an incentive compatible mechanism M providing the reservation utilities $\{U^M(\theta_L),U^M(\theta_H)\}$ if and only if it solves the following program:

$$\max_{\phi(\cdot),q^i(\cdot),x^i(\cdot),y^i(\cdot)}\sum_{(\theta^1,\theta^2)\in\Theta^2}p(\theta^1,\theta^2)[U^1(\theta^1)+U^2(\theta^2)]$$

subject to

$$\begin{aligned}U^i(\theta^i)=\textstyle\sum_{\theta^j\in\Theta}p(\theta^j)\Big\{\sum_{\widetilde{\phi}\in\Theta^2}p^\phi(\theta^i,\theta^j,\widetilde{\phi})\Big[\theta^iu(q^i(\theta^i,\theta^j,\widetilde{\phi})+x^i(\theta^i,\theta^j,\widetilde{\phi}))\\ -t^i(q^i(\theta^i,\theta^j,\widetilde{\phi}),\widetilde{\phi})\Big]-y^i(\theta^i,\theta^j)\Big\}\text{ for any }\theta^i\in\Theta;\end{aligned}$$

$$\begin{aligned}(BIC^S)\ U^i(\theta^i)\geq\textstyle\sum_{\theta^j\in\Theta}p(\theta^j)\Big\{\sum_{\widetilde{\phi}\in\Theta^2}p^\phi(\widetilde{\theta}^i,\theta^j,\widetilde{\phi})[\theta^iu(q^i(\widetilde{\theta}^i,\theta^j,\widetilde{\phi})+x^i(\widetilde{\theta}^i,\theta^j,\widetilde{\phi}))\\ -t^i(q^i(\widetilde{\theta}^i,\theta^j,\widetilde{\phi}),\widetilde{\phi})]-y^i(\widetilde{\theta}^i,\theta^j)\Big\}\text{ for any }(\theta^i,\widetilde{\theta}^i)\in\Theta^2;\end{aligned}$$

$$(BIR^S)\ U^i(\theta^i)\geq U^M(\theta^i),\ \text{for any }\theta^i\in\Theta;$$

$$\begin{aligned}&(BB\ :\ x)\ x^1(\theta^1,\theta^2,\widetilde{\phi})+x^2(\theta^1,\theta^2,\widetilde{\phi})=0,\ \text{for any }(\theta^1,\theta^2)\in\Theta^2\text{ and any }\widetilde{\phi}\in\Theta^2\\ &(BB\ :\ y)\ y^1(\theta^1,\theta^2)+y^2(\theta^1,\theta^2)=0,\ \text{for any }(\theta^1,\theta^2)\in\Theta^2.\end{aligned}$$

In what follows, for the sake of brevity we use $q^i_{jk,\widetilde{\phi}}$ instead of $q^i(\theta_j,\theta_k,\widetilde{\phi})$ with $jk\in\{HH,HL,LH,LL\}$. $p^\phi_{jk,\widetilde{\phi}}$ and $x^i_{jk,\widetilde{\phi}}$ are similarly defined. The third-party maximizes the following objective

$$\begin{aligned}&(1-p_L)^2\textstyle\sum_{\widetilde{\phi}\in\Theta^2}p^\phi_{HH,\widetilde{\phi}}\sum_{i=1}^2[\theta_Hu(q^i_{HH,\widetilde{\phi}}+x^i_{HH,\widetilde{\phi}})-t^i(q^i_{HH,\widetilde{\phi}},\widetilde{\phi})]\\ &+p_L(1-p_L)\textstyle\sum_{\widetilde{\phi}\in\Theta^2}p^\phi_{LH,\widetilde{\phi}}[\theta_Lu(q^1_{LH,\widetilde{\phi}}+x^1_{LH,\widetilde{\phi}})+\theta_Hu(q^2_{LH,\widetilde{\phi}}+x^2_{LH,\widetilde{\phi}})-\sum_{i=1}^2t^i(q^i_{LH,\widetilde{\phi}},\widetilde{\phi})]\\ &+p_L(1-p_L)\textstyle\sum_{\widetilde{\phi}\in\Theta^2}p^\phi_{HL,\widetilde{\phi}}[\theta_Hu(q^1_{HL,\widetilde{\phi}}+x^1_{HL,\widetilde{\phi}})+\theta_Lu(q^2_{HL,\widetilde{\phi}}+x^2_{HL,\widetilde{\phi}})-\sum_{i=1}^2t^i(q^i_{HL,\widetilde{\phi}},\widetilde{\phi})]\\ &+p_L^2\textstyle\sum_{\widetilde{\phi}\in\Theta^2}p^\phi_{LL,\widetilde{\phi}}\sum_{i=1}^2[\theta_Lu(q^i_{LL,\widetilde{\phi}}+x^i_{LL,\widetilde{\phi}})-t^i(q^i_{LL,\widetilde{\phi}},\widetilde{\phi})]\end{aligned}$$

subject to the usual budget balance constraints for the quantity reallocation and for the side transfers and, in addition,

- H-type's Bayesian incentive constraint for buyer 1:

$$\begin{aligned}
& p_L \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{HL,\widetilde{\phi}}[\theta_H u(q^1_{HL,\widetilde{\phi}} + x^1_{HL,\widetilde{\phi}}) - t^1(q^1_{HL,\widetilde{\phi}}, \widetilde{\phi}) - y^1_{HL}] + \\
& (1-p_L) \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{HH,\widetilde{\phi}}[\theta_H u(q^1_{HH,\widetilde{\phi}} + x^1_{HH,\widetilde{\phi}}) - t^1(q^1_{HH,\widetilde{\phi}}, \widetilde{\phi}) - y^1_{HH}] \\
\geq \; & p_L \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{LL,\widetilde{\phi}}[\theta_H u(q^1_{LL,\widetilde{\phi}} + x^1_{LL,\widetilde{\phi}}) - t^1(q^1_{LL,\widetilde{\phi}}, \widetilde{\phi}) - y^1_{LL}] + \\
& (1-p_L) \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{LH,\widetilde{\phi}}[\theta_H u(q^1_{LH,\widetilde{\phi}} + x^1_{LH,\widetilde{\phi}}) - t^1(q^1_{LH,\widetilde{\phi}}, \widetilde{\phi}) - y^1_{LH}],
\end{aligned}$$

- H-type's Bayesian incentive constraint for buyer 2 :

$$\begin{aligned}
& p_L \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{LH,\widetilde{\phi}}[\theta_H u(q^2_{LH,\widetilde{\phi}} + x^2_{LH,\widetilde{\phi}}) - t^2(q^2_{LH,\widetilde{\phi}}, \widetilde{\phi}) - y^2_{LH}] + \\
& (1-p_L) \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{HH,\widetilde{\phi}}[\theta_H u(q^2_{HH,\widetilde{\phi}} + x^2_{HH,\widetilde{\phi}}) - t^2(q^2_{HH,\widetilde{\phi}}, \widetilde{\phi}) - y^2_{HH}] \\
\geq \; & p_L \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{LL,\widetilde{\phi}}[\theta_H u(q^2_{LL,\widetilde{\phi}} + x^2_{LL,\widetilde{\phi}}) - t^2(q^2_{LL,\widetilde{\phi}}, \widetilde{\phi}) - y^2_{LL}] + \\
& (1-p_L) \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{HL,\widetilde{\phi}}[\theta_H u(q^2_{HL,\widetilde{\phi}} + x^2_{HL,\widetilde{\phi}}) - t^2(q^2_{HL,\widetilde{\phi}}, \widetilde{\phi}) - y^2_{HL}],
\end{aligned}$$

- H-type's acceptance constraint for buyer 1:

$$\begin{aligned}
& p_L \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{HL,\widetilde{\phi}}[\theta_H u(q^1_{HL,\widetilde{\phi}} + x^1_{HL,\widetilde{\phi}}) - t^1(q^1_{HL,\widetilde{\phi}}, \widetilde{\phi}) - y^1_{HL}] + \\
& (1-p_L) \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{HH,\widetilde{\phi}}[\theta_H u(q^1_{HH,\widetilde{\phi}} + x^1_{HH,\widetilde{\phi}}) - t^1(q^1_{HH,\widetilde{\phi}}, \widetilde{\phi}) - y^1_{HH}] \geq U^M(\theta_H)
\end{aligned}$$

- H-type's acceptance constraint for buyer 2:

$$\begin{aligned}
& p_L \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{LH,\widetilde{\phi}}[\theta_H u(q^2_{LH,\widetilde{\phi}} + x^2_{LH,\widetilde{\phi}}) - t^2(q^2_{LH,\widetilde{\phi}}, \widetilde{\phi}) - y^2_{LH}] + \\
& (1-p_L) \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{HH,\widetilde{\phi}}[\theta_H u(q^2_{HH,\widetilde{\phi}} + x^2_{HH,\widetilde{\phi}}) - t^2(q^2_{HH,\widetilde{\phi}}, \widetilde{\phi}) - y^2_{HH}] \geq U^M(\theta_H)
\end{aligned}$$

- L-type's acceptance constraint for buyer 1:

$$p_L \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{LL,\widetilde{\phi}}[\theta_L u(q^1_{LL,\widetilde{\phi}} + x^1_{LL,\widetilde{\phi}}) - t^1(q^1_{LL,\widetilde{\phi}}, \widetilde{\phi}) - y^1_{LL}] +$$
$$(1-p_L) \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{LH,\widetilde{\phi}}[\theta_L u(q^1_{LH,\widetilde{\phi}} + x^1_{LH,\widetilde{\phi}}) - t^1(q^1_{LH,\widetilde{\phi}}, \widetilde{\phi}) - y^1_{LH}] \geq U^M(\theta_L)$$

- L-type's acceptance constraint for buyer 2:

$$p_L \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{LL,\widetilde{\phi}}[\theta_L u(q^2_{LL,\widetilde{\phi}} + x^2_{LL,\widetilde{\phi}}) - t^2(q^2_{LL,\widetilde{\phi}}, \widetilde{\phi}) - y^2_{LL}] +$$
$$(1-p_L) \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{HL,\widetilde{\phi}}[\theta_L u(q^2_{HL,\widetilde{\phi}} + x^2_{HL,\widetilde{\phi}}) - t^2(q^2_{HL,\widetilde{\phi}}, \widetilde{\phi}) - y^2_{HL}] \geq U^M(\theta_L)$$

We introduce the same multipliers and Lagrangian function as in the proof of Proposition 4. Step 1 here is similar to the step 1 in the proof of Proposition 4: we obtain $v_L = \delta + \frac{p_L}{1-p_L}(\delta + v_H)$.

Step 2: Optimization with respect to $x^i_{jk,\widetilde{\phi}}$, given $(p^{\phi}_{jk,\widetilde{\phi}}, q^i_{jk,\widetilde{\phi}})$

After optimizing with respect to $x^i_{HH,\widetilde{\phi}}$:[43]

$$\rho^x_{HH,\widetilde{\phi}} + p^{\phi}_{HH,\widetilde{\phi}}(1 - p_L + \delta + v_H)(1-p_L)\theta_H u'(q^i_{HH,\widetilde{\phi}} + x^i_{HH,\widetilde{\phi}}) = 0, \text{ for } i = 1,2 \text{ and any } \widetilde{\phi} \in \Theta^2.$$

The above equations imply $q^1_{HH,\widetilde{\phi}} + x^1_{HH,\widetilde{\phi}} = q^2_{HH,\widetilde{\phi}} + x^2_{HH,\widetilde{\phi}}$. Since $x^1_{HH,\widetilde{\phi}} + x^2_{HH,\widetilde{\phi}} = 0$ from the budget balance constraint, we have $q^i_{HH,\widetilde{\phi}} + x^i_{HH,\widetilde{\phi}} = (q^1_{HH,\widetilde{\phi}} + q^2_{HH,\widetilde{\phi}})/2$ for each $\widetilde{\phi} \in \Theta^2$.

After optimizing with respect to $x^1_{HL,\widetilde{\phi}}$ and $x^2_{HL,\widetilde{\phi}}$ respectively, we have:

$$\begin{aligned} \rho^x_{HL,\widetilde{\phi}} + p^{\phi}_{HL,\widetilde{\phi}}(1 - p_L + \delta + v_H)p_L\theta_H u'(q^1_{HL,\widetilde{\phi}} + x^1_{HL,\widetilde{\phi}}) &= 0, \text{ for any } \widetilde{\phi} \in \Theta^2, \\ \rho^x_{HL,\widetilde{\phi}} + p^{\phi}_{HL,\widetilde{\phi}}(p_L\theta_L - \delta\theta_H + v_L\theta_L)(1-p_L)u'(q^2_{HL,\widetilde{\phi}} + x^2_{HL,\widetilde{\phi}}) &= 0, \text{ for any } \widetilde{\phi} \in \Theta^2. \end{aligned}$$

Since $v_L = \delta + \frac{p_L}{1-p_L}(\delta + v_H)$, we obtain from the two above equations:

$$\theta_H u'(q^1_{HL,\widetilde{\phi}} + x^1_{HL,\widetilde{\phi}}) = \theta^{\epsilon}_L u'(q^2_{HL,\widetilde{\phi}} + x^2_{HL,\widetilde{\phi}}), \text{ for any } \widetilde{\phi} \in \Theta^2. \tag{23}$$

Similarly, after optimizing with respect to $x^1_{LH,\widetilde{\phi}}$ and $x^2_{LH,\widetilde{\phi}}$ respectively, we obtain

$$\theta_H u'(q^2_{LH,\widetilde{\phi}} + x^2_{LH,\widetilde{\phi}}) = \theta^{\epsilon}_L u'(q^1_{LH,\widetilde{\phi}} + x^1_{LH,\widetilde{\phi}}), \text{ for any } \widetilde{\phi} \in \Theta^2.$$

[43] The remark about interior solutions for the reallocation problem which we made in the proof of proposition 4 applies here as well.

After optimizing with respect to $x^i_{LL,\widetilde{\phi}}$, we find

$$\rho^x_{LL,\widetilde{\phi}} + p^{\phi}_{LL,\widetilde{\phi}}(p_L\theta_L - \delta\theta_H + v_L\theta_L)p_L u'(q^i_{LL,\widetilde{\phi}} + x^i_{LL,\widetilde{\phi}}) = 0, \text{ for } i = 1,2 \text{ and any } \widetilde{\phi} \in \Theta^2.$$

The above equations imply $q^1_{LL,\widetilde{\phi}} + x^1_{LL,\widetilde{\phi}} = q^2_{LL,\widetilde{\phi}} + x^2_{LL,\widetilde{\phi}}$. Since $x^1_{LL,\widetilde{\phi}} + x^2_{LL,\widetilde{\phi}} = 0$ from the budget balance constraint, we have $q^i_{LL,\widetilde{\phi}} + x^i_{LL,\widetilde{\phi}} = (q^1_{LL,\widetilde{\phi}} + q^2_{LL,\widetilde{\phi}})/2$ for any $\widetilde{\phi} \in \Theta^2$.

Step 3: Optimizing with respect to $q^i_{jk,\widetilde{\phi}}$ given $p^{\phi}_{jk,\widetilde{\phi}}$

Given $\widetilde{\phi}$, a realized outcome of $\phi(\cdot)$, let $\bar{q}^1_{\widetilde{\phi}}$ (resp. $\bar{q}^2_{\widetilde{\phi}}$) denote the quantity level at which the kink occurs for buyer 1 (resp. for buyer 2). For instance, if $\widetilde{\phi} = (\theta_H, \theta_H)$, then we set $\bar{q}^1_{\theta_H,\theta_H} = \bar{q}^2_{\theta_H,\theta_H} = q^*_H$. We consider the third party's optimization problem with respect to $(q^1_{jk,\widetilde{\phi}}, q^2_{jk,\widetilde{\phi}})$. Let $m \in \{0,1,2\}$ denote the number of buyers with H-type in the coalition with $(\theta^1, \theta^2) = (\theta_j, \theta_k)$. Then, the total surplus that the coalition derives from consuming quantity x is given by $V^{\epsilon}_m(x)$. Therefore, we can write the payoff of the coalition as a function of $(q^1_{jk,\widetilde{\phi}}, q^2_{jk,\widetilde{\phi}})$, without considering the fixed fees, as follows:

$$\begin{array}{rl} V^{\epsilon}_m(q^1_{jk,\widetilde{\phi}} + q^2_{jk,\widetilde{\phi}}) - c(q^1_{jk,\widetilde{\phi}} + q^2_{jk,\widetilde{\phi}}) & \text{if } q^1_{jk,\widetilde{\phi}} \le \bar{q}^1_{\widetilde{\phi}} \text{ and } q^2_{jk,\widetilde{\phi}} \le \bar{q}^2_{\widetilde{\phi}} \\ V^{\epsilon}_m(q^1_{jk,\widetilde{\phi}} + q^2_{jk,\widetilde{\phi}}) - \bar{p}(q^1_{jk,\widetilde{\phi}} - \bar{q}^1_{\widetilde{\phi}}) - c(\bar{q}^1_{\widetilde{\phi}} + q^2_{jk,\widetilde{\phi}}) & \text{if } q^1_{jk,\widetilde{\phi}} > \bar{q}^1_{\widetilde{\phi}} \text{ and } q^2_{jk,\widetilde{\phi}} \le \bar{q}^2_{\widetilde{\phi}} \\ V^{\epsilon}_m(q^1_{jk,\widetilde{\phi}} + q^2_{jk,\widetilde{\phi}}) - \bar{p}(q^2_{jk,\widetilde{\phi}} - \bar{q}^2_{\widetilde{\phi}}) - c(q^1_{jk,\widetilde{\phi}} + \bar{q}^2_{\widetilde{\phi}}) & \text{if } q^1_{jk,\widetilde{\phi}} \le \bar{q}^1_{\widetilde{\phi}} \text{ and } q^2_{jk,\widetilde{\phi}} > \bar{q}^2_{\widetilde{\phi}} \\ V^{\epsilon}_m(q^1_{jk,\widetilde{\phi}} + q^2_{jk,\widetilde{\phi}}) - \bar{p}(q^1_{jk,\widetilde{\phi}} - \bar{q}^1_{\widetilde{\phi}} + q^2_{jk,\widetilde{\phi}} - \bar{q}^2_{\widetilde{\phi}}) - c(\bar{q}^1_{\widetilde{\phi}} + \bar{q}^2_{\widetilde{\phi}}) & \text{if } q^1_{jk,\widetilde{\phi}} > \bar{q}^1_{\widetilde{\phi}} \text{ and } q^2_{jk,\widetilde{\phi}} > \bar{q}^2_{\widetilde{\phi}} \end{array} \tag{24}$$

For any $\widetilde{\phi}$, we find below the maximum value of the function defined in (24) with respect to $(q^1_{jk,\widetilde{\phi}}, q^2_{jk,\widetilde{\phi}}) \in \mathbb{R}^2_+$. If we denote that maximum value by $U_{jk}(\widetilde{\phi})$, then the net payoff which jk coalition receives if it reports $\widetilde{\phi}$ to the seller is $U_{jk}(\widetilde{\phi})$ minus the fixed fees associated with $\widetilde{\phi}$.

Step 4: Optimizing with respect to $\phi(\theta_1, \theta_2)$.

Instead of finding conditions such that coalitions report truthfully and then verifying that such conditions are satisfied by the null side mechanism S^0, we prove directly that S^0 is optimal for the third party when $\epsilon = 1$, given $\{T^{**}_H, T^{**}_L\}$. In other words, our menu of two part tariffs is weakly collusion-proof since we show that the third party will require each j-type to choose the tariff T^{**}_j and to buy quantity q^*_j.

HH **coalition** If HH coalition chooses (T^{**}_H, T^{**}_H) (which means $p^{\phi}_{HH,HH} = 1$) then it buys a total quantity $q > 0$ in order to maximize $V_2(q) - cq1_{[q \le 2q^*_H]} - [c2q^*_H + \bar{p}(q - 2q^*_H)]\,1_{[q > 2q^*_H]}$, where $V_2(q) = 2\theta_H u(\frac{q}{2})$. The solution is $q = 2q^*_H$, therefore choosing (T^{**}_H, T^{**}_H) yields the coalition the same payoff $2\theta_H u(q_H^*) - 2t_{HH}^*$ that is obtained by reporting HH in the mechanism M^{**}.

If it chooses (T_H^{**}, T_L^{**}), then it should select (q^H, q^L) (quantities which are bought by the buyer who chose T_H^{**} and T_L^{**}, respectively) in order to maximize $V_2(q^H + q^L) - cq^H 1_{[q^H \leq q_H^*]} - [cq_H^* + \bar{p}(q^H - q_H^*)] 1_{[q^H > q_H^*]} - cq^L 1_{[q^L \leq q_L^*]} - [cq_L^* + \bar{p}(q - q_L^*)] 1_{[q > q_L^*]}$. Certainly, $q^H \geq q_H^*$ and $q^L \geq q_L^*$ since marginal price is c for $q^H \leq q_H^*$ and $q^L \leq q_L^*$ while the marginal benefit for HH coalition from having an additional unit of good when it owns total quantity $q_H^* + q_L^*$ (or less) is (at least) $\frac{\partial V_2(q_H^* + q_L^*)}{\partial x} = \theta_H u'(\frac{q_H^* + q_L^*}{2}) > c$. However, it is not optimal to set $q^H > q_H^*$ and/or $q^L > q_L^*$ because then marginal price jumps to $\bar{p} > \theta_H u'(\frac{q_H^* + q_L^*}{2})$. This establishes that – conditional on choosing (T_H^{**}, T_L^{**}) – HH coalition will buy the total quantity $q_H^* + q_L^*$ and pay $t_{HL}^{**} + t_{LH}^{**}$ after taking into account the fixed fees. Therefore, choosing (T_H^{**}, T_L^{**}) yields the coalition the same payoff that is obtained by reporting HL in the mechanism M^{**}.

If it chooses (T_L^{**}, T_L^{**}), then it will buy the total quantity $2q_L^*$ – because $\frac{\partial V_2(2q_L^*)}{\partial x} = \bar{p} > c$ but $\frac{\partial V_2(q)}{\partial x} < \bar{p}$ for $q > 2q_L^*$ – and will pay $2t_{LL}^{**}$ overall. Hence, choosing (T_L^{**}, T_L^{**}) yields the same payoff that is obtained by reporting LL in the mechanism M^{**}.

Finally, since M^{**} satisfies $(CIC_{HH,HL})$ and $(CIC_{HH,LL})$, it is optimal for HH coalition to choose (T_H^{**}, T_H^{**}).

HL **coalition** If HL coalition selects (T_H^{**}, T_L^{**}), then it chooses $q^H \geq q_H^*$ and $q^L \geq q_L^*$ because the marginal price is c for $q^H < q_H^*$ and $q^L < q_L^*$, and the marginal benefit for HL coalition from having an additional unit of good when it owns total quantity $q_H^* + q_L^*$ (or less) is (at least) $\frac{\partial V_1^1(q_H^* + q_L^*)}{\partial x} = c$. Setting $q^H > q_H^*$ and/or $q^L > q_L^*$ is not profitable because $\bar{p} > c > \frac{\partial V_1^1(q^H + q^L)}{\partial x}$ if $q^H + q^L > q_H^* + q_L^*$.

If it chooses (T_H^{**}, T_H^{**}), it will buy the total quantity q in order to maximize

$$V_1^1(q) - cq 1_{[q \leq 2q_H^*]} - [c2q_H^* + \bar{p}(q - 2q_H^*)] 1_{[q > 2q_H^*]}$$

Since the above objective is maximized at $q = q_H^* + q_L^*$, taking into account the fixed fees we see that choosing (T_H^{**}, T_H^{**}) is not optimal if

$$V_1^1(q_H^* + q_L^*) - c(q_H^* + q_L^*) - 2A_{HH}^{**} \leq V_1^1(q_H^* + q_L^*) - c(q_H^* + q_L^*) - A_{HL}^{**} - A_{LH}^{**} \qquad (25)$$

By replacing the values of A_{HH}^{**}, A_{HL}^{**} and A_{LH}^{**} and recalling that $(CIC_{HH,HL})$ binds in M^{**}, we find that (25) reduces to

$$\theta_H[u(q_H^*) - u(\frac{q_H^* + q_L^*}{2})] \geq c(\frac{q_H^* - q_L^*}{2})$$

which can be proved by using the mean value theorem.

If the coalition chooses (T_L^{**}, T_L^{**}), then it buys the total quantity $2q_L^*$ because $\frac{\partial V_1^1(q)}{\partial x} > c$ if $q < 2q_L^*$, but $\frac{\partial V_1^1(q)}{\partial x} < \bar{p}$ if $q > 2q_L^*$. Recalling the value of A_{LL}^{**}, we find that choosing (T_L^{**}, T_L^{**}) yields HL coalition the same payoff that is obtained by reporting LL in M^{**}. Since $(CIC_{HL,LL})$ is satisfied in M^{**}, choosing (T_L^{**}, T_L^{**}) is not better than (T_H^{**}, T_L^{**}).

LL **coalition** It is straightforward to find that LL coalition buys total quantity $2q_L^*$ regardless of the choice of tariffs. Therefore, it will choose (T_L^{**}, T_L^{**}) if and only if the fixed fees are larger for (T_H^{**}, T_H^{**}) and (T_H^{**}, T_L^{**}) than for (T_L^{**}, T_L^{**}). Since we have already proved that $2A_{HH}^{**} \geq A_{HL}^{**} + A_{LH}^{**}$, we only need to show that $A_{HL}^{**} + A_{LH}^{**} \geq 2A_{LL}^{**}$. Recalling that $(CIC_{HL,LL})$ binds in M^{**}, we have:

$$\begin{aligned} A_{HL}^{**} + A_{LH}^{**} &\geq 2A_{LL}^{**} \Leftrightarrow t_{HL}^{**} + t_{LH}^{**} - 2t_{LL}^{**} - cq_H^* - cq_L^* \geq -2cq_L^* \\ &\Leftrightarrow V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*) \geq c(q_H^* - q_L^*). \end{aligned}$$

The latter inequality can be proved by using the mean value theorem.

Proof of Proposition 7

Let $x^i_{\theta,\widetilde{\phi}} = x^i(\theta, \widetilde{\phi})$ represent the reallocation function. For expositional convenience, let $\{\mathbf{q}(\theta), \mathbf{t}(\theta)\}$ represent the symmetric sale mechanism introduced in Subsection 6.1, i.e., $M_n = \{\mathbf{q}_n, \mathbf{t}_n\}$. Given $\{\mathbf{q}(\theta), \mathbf{t}(\theta)\}$, the third party wishes to maximize the following objective

$$\sum_{\theta \in \Theta^n} p(\theta) \left\{ \sum_{\widetilde{\phi} \in \Theta^n} p^\phi(\theta, \widetilde{\phi}) \sum_{i=1}^n \left[\theta^i u(q^i(\widetilde{\phi}) + x^i_{\theta,\widetilde{\phi}}) - t^i(\widetilde{\phi}) \right] \right\}$$

subject to the budget balance constraints for quantity reallocation and side transfers, (BIC^S) and (BIR^S)

$$(BIC_i^S(\theta_H)) \qquad \sum_{\theta^{-i} \in \Theta^{n-1}} p(\theta^{-i}) \left\{ \sum_{\widetilde{\phi} \in \Theta^n} p^\phi(\theta^{-i}, \theta_H, \widetilde{\phi}) \left[\theta_H u(q^i(\widetilde{\phi}) + x^i_{\theta^{-i},\theta_H,\widetilde{\phi}}) - t^i(\widetilde{\phi}) \right] - y^i(\theta^{-i}, \theta_H) \right\}$$

$$\geq \sum_{\theta^{-i} \in \Theta^{n-1}} p(\theta^{-i}) \left\{ \sum_{\widetilde{\phi} \in \Theta^n} p^\phi(\theta^{-i}, \theta_L, \widetilde{\phi}) \left[\theta_H u(q^i(\widetilde{\phi}) + x^i_{\theta^{-i},\theta_L,\widetilde{\phi}}) - t^i(\widetilde{\phi}) \right] - y^i(\theta^{-i}, \theta_L) \right\},$$

$$(BIR_i^S(\theta_H)) \sum_{\theta^{-i} \in \Theta^{n-1}} p(\theta^{-i}) \left\{ \sum_{\widetilde{\phi} \in \Theta^n} p^\phi(\theta^{-i}, \theta_H, \widetilde{\phi}) \left[\theta_H u(q^i(\widetilde{\phi}) + x^i_{\theta^{-i},\theta_H,\widetilde{\phi}}) - t^i(\widetilde{\phi}) \right] - y^i(\theta^{-i}, \theta_H) \right\}$$

$$\geq U^M(\theta_H),$$

$$(BIR_i^S(\theta_L)) \sum_{\theta^{-i}\in\Theta^{n-1}} p(\theta^{-i}) \left\{ \sum_{\widetilde{\phi}\in\Theta^n} p^{\phi}(\theta^{-i},\theta_L,\widetilde{\phi}) \left[\theta_L u(q^i(\widetilde{\phi}) + x^i_{\theta^{-i},\theta_L,\widetilde{\phi}}) - t^i(\widetilde{\phi})\right] - y^i(\theta^{-i},\theta_L) \right\}$$

$$\geq U^M(\theta_L).$$

After writing down the Lagrangian function we can find the conditions under which the null side mechanism solves this maximization problem. In particular, consider the m-coalition $\theta_{m,n-m} = (H...HL...L)$: Under which conditions is the third party reporting truthfully and not reallocating when $\widetilde{\theta} = \theta_{m,n-m}$?[44] We describe such conditions by considering multipliers such that $\delta^i = \delta$, $v^i_H = v_H$ and $v^i_L = v_L$ for any i.[45] The manipulation and reallocation which are chosen by the third party need to maximize

$$\begin{aligned}
&p_H^m p_L^{n-m} \sum_{\widetilde{\phi}\in\Theta^n} p^{\phi}(\theta_{m,n-m},\widetilde{\phi}) \left\{ \sum_{i=1}^m \left[\theta_H u(q^i(\widetilde{\phi}) + x^i_{\theta_{m,n-m},\widetilde{\phi}}) - t^i(\widetilde{\phi})\right] \right. \qquad (26)\\
&\left. + \sum_{i'=m+1}^n \left[\theta_L u(q^{i'}(\widetilde{\phi}) + x^{i'}_{\theta_{m,n-m},\widetilde{\phi}}) - t^{i'}(\widetilde{\phi})\right] \right\}\\
&+ p_H^{m-1} p_L^{n-m}(\delta + v_H) \left\{ \sum_{\widetilde{\phi}\in\Theta^n} p^{\phi}(\theta_{m,n-m},\widetilde{\phi}) \sum_{i=1}^m \left[\theta_H u(q^i(\widetilde{\phi}) + x^i_{\theta_{m,n-m},\widetilde{\phi}}) - t^i(\widetilde{\phi}) - y^i(\theta_{m,n-m})\right] \right\}\\
&+ p_H^m p_L^{n-m-1} v_L \left\{ \sum_{\widetilde{\phi}\in\Theta^n} p^{\phi}(\theta_{m,n-m},\widetilde{\phi}) \sum_{i'=m+1}^n \left[\theta_L u(q^{i'}(\widetilde{\phi}) + x^{i'}_{\theta_{m,n-m},\widetilde{\phi}}) - t^{i'}(\widetilde{\phi}) - y^{i'}(\theta_{m,n-m})\right] \right\}\\
&- p_H^m p_L^{n-m-1} \delta \left\{ \sum_{\widetilde{\phi}\in\Theta^n} p^{\phi}(\theta_{m,n-m},\widetilde{\phi}) \sum_{i'=m+1}^n \left[(\theta_L + \Delta\theta) u(q^{i'}(\widetilde{\phi}) + x^{i'}_{\theta_{m,n-m},\widetilde{\phi}}) - t^{i'}(\widetilde{\phi}) - y^{i'}(\theta_{m,n-m})\right] \right\}\\
&+ \sum_{\widetilde{\phi}\in\Theta^n} \rho^x_{\theta_{m,n-m},\widetilde{\phi}} \left(\sum_{i=1}^n x^i_{\theta_{m,n-m},\widetilde{\phi}} - 0 \right) + \rho^y_{\theta_{m,n-m}} \left(\sum_{i=1}^n y^i(\theta_{m,n-m}) - 0 \right),
\end{aligned}$$

[44] The results we obtain below extend to any other m-coalition - i.e., the ones in which the m buyers with H-type are not buyers 1 to m.

[45] Remember from the proof of Proposition 4 that δ^i , v^i_H and v^i_L represent the multipliers associated with (BIC^{iS}_H), (BIR^{iS}_H) and (BIR^{iS}_L).

where $p_H = 1 - p_L$ and $\rho^x_{\theta_{m,n-m},\widetilde{\phi}}$ and $\rho^y_{\theta_{m,n-m}}$ are the Lagrangian multiplier associated with the budget balance constraint for quantity reallocation and the side transfer.

Step 1: Optimizing with respect to side-transfers $y^i(\theta_{m,n-m})$

From the first order conditions with respect to side transfers, we obtain:

$$\begin{aligned}
\rho^y_{\theta_{m,n-m}} - p_H^{m-1} p_L^{n-m}(\delta + v_H) &= 0 \quad \text{for } y^i(\theta_{m,n-m}) \text{ with } 1 \leq i \leq m \\
\rho^y_{\theta_{m,n-m}} + p_H^{m} p_L^{n-m-1}(\delta - v_L) &= 0 \quad \text{for } y^{i'}(\theta_{m,n-m}) \text{ with } m+1 \leq i' \leq n
\end{aligned}$$

Therefore, we have

$$v_L = \delta + (\delta + v_H)\frac{p_L}{p_H}. \tag{27}$$

Step 2: Optimizing with respect to quantity reallocation $x^i_{\theta_{m,n-m},\widetilde{\phi}}$

From the first order conditions with respect to $x^i_{\theta_{m,n-m},\widetilde{\phi}}$, we have: for $1 \leq i \leq m$

$$\rho^x_{\theta_{m,n-m},\widetilde{\phi}} + p^{\phi}(\theta_{m,n-m},\widetilde{\phi})\left[p_H^m p_L^{n-m} + p_H^{m-1}p_L^{n-m}(\delta + v_H)\right]\theta_H u'(q^i(\widetilde{\phi}) + x^i_{\theta_{m,n-m},\widetilde{\phi}}) = 0;$$

for $m+1 \leq i' \leq n$,

$$\rho^x_{\theta_{m,n-m},\widetilde{\phi}} + p^{\phi}(\theta_{m,n-m},\widetilde{\phi})\left[p_H^m p_L^{n-m}\theta_L + p_H^m p_L^{n-m-1}(v_L\theta_L - \delta\theta_H)\right] u'(q^{i'}(\widetilde{\phi}) + x^{i'}_{\theta_{m,n-m},\widetilde{\phi}}) = 0.$$

Then, by using (27), we have

$$\begin{aligned}
\theta_H u'(q^i(\widetilde{\phi}) + x^i_{\theta_{m,n-m},\widetilde{\phi}}) &= \theta_L^{\epsilon} u'(q^{i'}(\widetilde{\phi}) + x^{i'}_{\theta_{m,n-m},\widetilde{\phi}}), \\
\text{for } (i,i') &\in \{1,\ldots,m\} \times \{m+1,\ldots,n\} \text{ for any } \widetilde{\phi} \in \Theta^2
\end{aligned} \tag{28}$$

Therefore, conditional on that there is no manipulation of report, the third party does not reallocate the goods in a $m-$coalition if there is $\epsilon \in [0,1)$ such that (17) in the paper holds

Step 3: Optimizing with respect to report manipulation $\phi(\theta_{m,n-m})$

The truthful (and deterministic) manipulation $\phi(\theta_{m,n-m}) = \theta_{m,n-m}$ is optimal if and only if [after substituting (27) into (26)]

$$\begin{aligned}
\theta_{m,n-m} \in \arg\max_{\widetilde{\phi}\in\Theta^n} &\left[p_H^m p_L^{n-m} + p_H^{m-1}p_L^{n-m}(\delta + v_H)\right]\left\{\sum_{i=1}^{m}\theta_H u(q^i(\widetilde{\phi}) + x^i_{\theta_{m,n-m},\widetilde{\phi}})\right. \\
&\left. + \sum_{i'=m+1}^{n}\theta_L^{\epsilon} u(q^{i'}(\widetilde{\phi}) + x^{i'}_{\theta_{m,n-m},\widetilde{\phi}}) - \sum_{i=1}^{n} t^i(\widetilde{\phi})\right\},
\end{aligned} \tag{29}$$

in which $x^i_{\theta_{m,n-m},\widetilde{\phi}}$ and $x^{i'}_{\theta_{m,n-m},\widetilde{\phi}}$ are such that $\theta_H u'(q^i(\widetilde{\phi})+x^i_{\theta_{m,n-m},\widetilde{\phi}}) = \theta^{\epsilon}_L u(q^{i'}(\widetilde{\phi})+x^{i'}_{\theta_{m,n-m},\widetilde{\phi}})$, for $(i,i') \in \{1,\dots,m\} \times \{m+1,\dots,n\}$. The latter fact implies that $\sum_{i=1}^m \theta_H u(q^i(\widetilde{\phi}) + x^i_{\theta_{m,n-m},\widetilde{\phi}}) + \sum_{i'=m+1}^n \theta^{\epsilon}_L u(q^{i'}(\widetilde{\phi})+x^{i'}_{\theta_{m,n-m},\widetilde{\phi}}) - \sum_{i=1}^n t^i(\widetilde{\phi})$ is actually equal to $V^{\varepsilon}_m[\sum_{i=1}^n q^i(\widetilde{\phi})][-\sum_{i=1}^n t^i(\widetilde{\phi})$. Hence, (29) is equivalent to condition (a) in proposition 7.

Proof of Proposition 8

Proof. The proof is very similar to the one provided for $n = 2$.
First, the seller can choose $\epsilon = 1$ such that the third-party has the same virtual valuation as she has; therefore, (17) holds at $\mathbf{q}_n = \mathbf{q}^*_n$.
Second, there exist transfers $\mathbf{t}^{**}_n$ satisfying with equality (BIC_H), (BIR_L) and $(CIC_{m+1,m})$ $(m = 0,1,...,n-1)$ bind written with $\mathbf{q}_n = \mathbf{q}^*_n$ and $\varepsilon = 1$. Let $C^n_m = \frac{n!}{m!(n-m)!}$, so that writing (BIR_L) and (BIC_H) with equality give us

$$\begin{cases} \theta_L u(q^*_L) - \sum_{m=0}^{n-1} C^{n-1}_m p_L^{n-1-m}(1-p_L)^m t_{Lm} = 0 \\ \theta_H u(q^*_H) - \sum_{m=1}^{n} C^{n-1}_{m-1} p_L^{n-m}(1-p_L)^{m-1} t_{Hm} = (\Delta\theta) u(q^*_L) \end{cases} \tag{30}$$

Let us define $\Delta V^1_m \equiv V^1_m[mq^*_H + (n-m)q^*_L] - V^1_m[(m-1)q^*_H + (n-m+1)q^*_L]$, $m = 1,...,n$. ΔV^1_m represents the difference between the gross payoff that $m-$coalition obtains by reporting truthfully and the one that it obtains by reporting $m-1$. Constraints $(CIC_{m+1,m})$ written with equality $(m = 0,1,...,n-1)$ yield $(t_{H1}, t_{H2}, ..., t_{Hn})$ as a function of $(t_{L0}, t_{L1}, ..., t_{Ln-1})$ and of the constants $(\Delta V^1_1, ..., \Delta V^1_n)$. More precisely,

$$\begin{cases} t_{H1} = V^1_1[q^*_H + (n-1)q^*_L] - V^1_1(nq^*_L) - (n-1)t_{L1} + nt_{L0} \\ \qquad = \Delta V^1_1 - (n-1)t_{L1} + nt_{L0} \\ t_{H2} = \frac{\Delta V^1_2 + \Delta V^1_1}{2} - \frac{n-2}{2} t_{L2} + \frac{n}{2} t_{L0} \\ \vdots \\ t_{Hn-1} = \frac{\Delta V^1_{n-1} + ... + \Delta V^1_1}{n-1} - \frac{1}{n-1} t_{Ln-1} + \frac{n}{n-1} t_{L0} \\ t_{Hn} = \frac{\Delta V^1_n + \Delta V^1_{n-1} + ... + \Delta V^1_1}{n} + t_{L0} \end{cases} \tag{31}$$

After setting $t_{L2} = t_{L3} = ... = t_{Ln-1} = 0$ (this is one of many possibilities) and substituting (31) into (30), we obtain the following linear system in (t_{L0}, t_{L1}) which admits a (unique) solution because the matrix of the unknowns is non-singular.

$$\begin{aligned} \theta_L u(q^*_L) &= p_L^{n-1} t_{L0} + (n-1)p_L^{n-2}(1-p_L)t_{L1} \\ \theta_H u(q^*_H) + const(\Delta V^1_1, ..., \Delta V^1_n) - (\Delta\theta)u(q^*_L) &= \frac{1-p_L^n}{1-p_L} t_{L0} - (n-1)p_L^{n-1} t_{L1} \end{aligned}$$

Third, the single crossing condition for coalitions holds: $\frac{\partial V^{\epsilon}_{m+1}(x)}{\partial x} > \frac{\partial V^{\epsilon}_{m}(x)}{\partial x}$ for $m = 0, 1..., n-1$. We consider for simplicity interior allocations because the seller optimally serves any type of buyer.[46] Then, $q^{\epsilon}_{Hm}(x)$ and $q^{\epsilon}_{Lm}(x)$ are such that $\theta_H u'[q^{\epsilon}_{Hm}(x)] = \theta^{\epsilon}_L u'[q^{\epsilon}_{Lm}(x)]$ and the envelope theorem implies $\frac{\partial V^{\epsilon}_{m}(x)}{\partial x} = \theta_H u'[q^{\epsilon}_{Hm}(x)]$. Likewise, $\theta_H u'[q^{\epsilon}_{Hm+1}(x)] = \theta^{\epsilon}_L u'[q^{\epsilon}_{Lm+1}(x)]$ and $\frac{\partial V^{\epsilon}_{m+1}(x)}{\partial x} = \theta_H u'[q^{\epsilon}_{Hm+1}(x)]$. We below show (by contradiction) that $q^{\epsilon}_{Hm}(x) > q^{\epsilon}_{Hm+1}(x)$; this implies $\frac{\partial V^{\epsilon}_{m+1}(x)}{\partial x} > \frac{\partial V^{\epsilon}_{m}(x)}{\partial x}$. Suppose $q^{\epsilon}_{Hm+1}(x) \geq q^{\epsilon}_{Hm}(x)$. Then (i) the marginal utility of each H-type is smaller in a $(m+1)$-coalition than in a m-coalition; (ii) we have $q^{\epsilon}_{Lm+1}(x) < q^{\epsilon}_{Lm}(x)$, which implies that the marginal utility of each L-type is higher in a $(m+1)$-coalition than in a m-coalition. As a consequence, starting from $\theta_H u'[q^{\epsilon}_{Hm}(x)] = \theta^{\epsilon}_L u'[q^{\epsilon}_{Lm}(x)]$, we obtain $\theta_H u'[q^{\epsilon}_{Hm+1}(x)] < \theta^{\epsilon}_L u'[q^{\epsilon}_{Lm+1}(x)]$, which is a contradiction.
Finally, since $(m+1)q^*_{Hm+1} + (n-m-1)q^*_{Lm+1} \geq mq^*_{Hm} + (n-m)q^*_{Lm}$ for $m = 0, ..., n-1$, we argue as in the proof to Proposition 5 to conclude that M^{**}_n satisfies all the coalition incentive constraints. ■

Proof of Proposition 9

As in the case of two types, when $\Theta = \{\theta_L, \theta_M, \theta_H\}$ the optimal mechanisms when there are two buyers are closely related to the optimal mechanism for the single-buyer model. In particular, the quantity each buyer receives is independent of the report of the other buyer and equal to the quantity he would obtain in the single-buyer setting, as the first part of the statement describes. The transfers are such that the binding constraints are as in the single-buyer model.

Proof of Proposition 10

For the sake of brevity, let $S_{jk} \equiv \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{jk,\widetilde{\phi}} \left[\theta_j u(q^1(\widetilde{\phi}) + x^1_{jk,\widetilde{\phi}}) + \theta_k u(q^2(\widetilde{\phi}) + x^2_{jk,\widetilde{\phi}}) - t^1(\widetilde{\phi}) - t^2(\widetilde{\phi})\right]$ for $j, k = L, M, H$. S_{jk} denotes the expected real surplus of coalition jk given manipulation of reports and reallocation of goods. Also, let $U^i_{jk} \equiv \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{jk,\widetilde{\phi}} u(q^i(\widetilde{\phi}) + x^i_{jk,\widetilde{\phi}})$, $T^i_{jk} = \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{jk,\widetilde{\phi}} t^i(\widetilde{\phi})$ and $y^i_{jk} = y^i(\theta_j, \theta_k)$. The third-party maximizes the following objective function:

$$\begin{aligned} &p^2_H S_{HH} + p_H p_M(S_{HM} + S_{MH}) + p_H p_L(S_{HL} + S_{LH}) \\ &+ p^2_M S_{MM} + p_M p_L(S_{ML} + S_{LM}) + p^2_L S_{LL} \end{aligned}$$

subject to the following constraints.

[46] The proof can be slightly modified in order to cover the non-interior case.

- Budget balance constraints: for the quantity reallocation

$$\sum_{i=1}^{2} x^i(\theta^1, \theta^2, \widetilde{\phi}) = 0, \text{ for any } (\theta^1, \theta^2) \in \Theta^2 \text{ and any } \widetilde{\phi} \in \Theta^2$$

for the side transfers

$$\sum_{i=1}^{2} y^i(\theta^1, \theta^2) = 0, \text{ for any } (\theta^1, \theta^2) \in \Theta^2,$$

- Bayesian incentive constraint HM for H-type of buyer 1:

$$\begin{aligned} & p_L[\theta_H U^1_{HL} - T^1_{HL} - y^1_{HL}] + p_M[\theta_H U^1_{HM} - T^1_{HM} - y^1_{HM}] + p_H[\theta_H U^1_{HH} - T^1_{HH} - y^1_{HH}] \\ \geq\ & p_L[\theta_H U^1_{ML} - T^1_{ML} - y^1_{ML}] + p_M[\theta_H U^1_{MM} - T^1_{MM} - y^1_{MM}] + p_H[\theta_H U^1_{MH} - T^1_{MH} - y^1_{MH}] \end{aligned}$$

- Bayesian incentive constraint HM for H-type of buyer 2

$$\begin{aligned} & p_L[\theta_H U^2_{LH} - T^2_{LH} - y^2_{LH}] + p_M[\theta_H U^2_{MH} - T^2_{MH} - y^2_{MH}] + p_H[\theta_H U^2_{HH} - T^2_{HH} - y^2_{HH}] \\ \geq\ & p_L[\theta_H U^2_{LM} - T^2_{LM} - y^2_{LM}] + p_M[\theta_H U^2_{MM} - T^2_{MM} - y^2_{MM}] + p_H[\theta_H U^2_{HM} - T^2_{HM} - y^2_{HM}] \end{aligned}$$

- Bayesian incentive constraint ML for M-type of buyer 1:

$$\begin{aligned} & p_L[\theta_M U^1_{ML} - T^1_{ML} - y^1_{ML}] + p_M[\theta_M U^1_{MM} - T^1_{MM} - y^1_{MM}] + p_H[\theta_M U^1_{MH} - T^1_{MH} - y^1_{MH}] \\ \geq\ & p_L[\theta_M U^1_{LL} - T^1_{LL} - y^1_{LL}] + p_M[\theta_M U^1_{LM} - T^1_{LM} - y^1_{LM}] + p_H[\theta_M U^1_{LH} - T^1_{LH} - y^1_{LH}] \end{aligned}$$

- Bayesian incentive constraint ML for M-type of buyer 2:

$$\begin{aligned} & p_L[\theta_M U^2_{LM} - T^2_{LM} - y^2_{LM}] + p_M[\theta_M U^2_{MM} - T^2_{MM} - y^2_{MM}] + p_H[\theta_M U^2_{HM} - T^2_{HM} - y^2_{HM}] \\ \geq\ & p_L[\theta_M U^2_{LL} - T^2_{LL} - y^2_{LL}] + p_M[\theta_M U^2_{ML} - T^2_{ML} - y^2_{ML}] + p_H[\theta_M U^2_{HL} - T^2_{HL} - y^2_{HL}] \end{aligned}$$

- H-type's acceptance constraint for buyer 1:

$$p_L[\theta_H U^1_{HL} - T^1_{HL} - y^1_{HL}] + p_M[\theta_H U^1_{HM} - T^1_{HM} - y^1_{HM}] + p_H[\theta_H U^1_{HH} - T^1_{HH} - y^1_{HH}] \geq U^M(\theta_H)$$

- H-type's acceptance constraint for buyer 2:

$$p_L[\theta_H U^2_{LH} - T^2_{LH} - y^2_{LH}] + p_M[\theta_H U^2_{MH} - T^2_{MH} - y^2_{MH}] + p_H[\theta_H U^2_{HH} - T^2_{HH} - y^2_{HH}] \geq U^M(\theta_H)$$

- M-type's acceptance constraint for buyer 1:

$$p_L[\theta_M U^1_{ML} - T^1_{ML} - y^1_{ML}] + p_M[\theta_M U^1_{MM} - T^1_{MM} - y^1_{MM}] + p_H[\theta_M U^1_{MH} - T^1_{MH} - y^1_{MH}] \geq U^M(\theta_M)$$

• M-type's acceptance constraint for buyer 2:

$$p_L[\theta_M U^2_{LM} - T^2_{LM} - y^2_{LM}] + p_M[\theta_M U^2_{MM} - T^2_{MM} - y^2_{MM}] + p_H[\theta_M U^2_{HM} - T^2_{HM} - y^2_{HM}] \geq U^M(\theta_M)$$

• L-type's acceptance constraint for buyer 1:

$$p_L[\theta_L U^1_{LL} - T^1_{LL} - y^1_{LL}] + p_M[\theta_L U^1_{LM} - T^1_{LM} - y^1_{LM}] + p_H[\theta_L U^1_{LH} - T^1_{LH} - y^1_{LH}] \geq U^M(\theta_L)$$

• L-type's acceptance constraint for buyer 2:

$$p_L[\theta_L U^2_{LL} - T^2_{LL} - y^2_{LL}] + p_M[\theta_L U^2_{ML} - T^2_{ML} - y^2_{ML}] + p_H[\theta_L U^2_{HL} - T^2_{HL} - y^2_{HL}] \geq U^M(\theta_L)$$

We introduce the following multipliers:

• $\rho^x(\theta^1, \theta^2, \widetilde{\phi})$ for the budget-balance constraint for the quantity reallocation in state $(\theta^1, \theta^2, \widetilde{\phi})$,

• $\rho^y(\theta^1, \theta^2)$ for the budget-balance constraint for the side-transfers in state (θ^1, θ^2),

• δ^i_{HM} for HM Bayesian incentive constraint concerning buyer i,

• δ^i_{ML} for ML Bayesian incentive constraint concerning buyer i,

• v^i_H for H-type's acceptance constraint concerning buyer i,

• v^i_M for M-type's acceptance constraint concerning buyer i,

• v^i_L for L-type's acceptance constraint concerning buyer i.

We define the Lagrangian as follows:

$$\mathcal{L} = p^2_H S_{HH} + p_H p_M(S_{HM} + S_{MH}) + p_H p_L(S_{HL} + S_{LH}) + p^2_M S_{MM} + p_M p_L(S_{ML} + S_{LM}) + p^2_L S_{LL}$$

$$+ \sum_{(\theta^1,\theta^2)\in\Theta^2} \sum_{\widetilde{\phi}\in\Theta^2} \rho^x(\theta^1, \theta^2, \widetilde{\phi})(BB : x)(\theta^1, \theta^2, \widetilde{\phi}) + \sum_{(\theta^1,\theta^2)\in\Theta^2} \rho^y(\theta^1, \theta^2)(BB : y)(\theta^1, \theta^2) +$$

$$\sum_{i=1,2} \delta^i_{HM}(BIC^S)^i_{HM} + \sum_{i=1,2} \delta^i_{ML}(BIC^S)^i_{ML} + \sum_{i=1,2} v^i_H(BIR^S)^i_H + \sum_{i=1,2} v^i_M(BIR^S)^i_M + \sum_{i=1,2} v^i_L(BIR^S)^i_L$$

Step 1: Optimization with respect to $y^i(\theta^1, \theta^2)$

After optimizing with respect to y^i_{HH}, we have:

$$\rho^y_{HH} - (\delta^i_{HM} + v^i_H)p_H = 0, \text{ for } i = 1, 2.$$

After optimizing with respect to y^1_{HM} and y^2_{HM} respectively, we have:

$$\begin{aligned}\rho^y_{HM} - (\delta^1_{HM} + v^1_H)p_M &= 0;\\ \rho^y_{HM} + (\delta^2_{HM} - \delta^2_{ML} - v^2_M)p_H &= 0.\end{aligned}$$

After optimizing with respect to y^1_{HL} and y^2_{HL} respectively, we have:

$$\begin{aligned}\rho^y_{HL} - (\delta^1_{HM} + v^1_H)p_L &= 0;\\ \rho^y_{HL} + (\delta^2_{ML} - v^2_L)p_H &= 0.\end{aligned}$$

After optimizing with respect to y^1_{MH} and y^2_{MH} respectively, we have:

$$\begin{aligned}\rho^y_{MH} + (\delta^1_{HM} - \delta^1_{ML} - v^1_M)p_H &= 0;\\ \rho^y_{MH} - (\delta^2_{HM} + v^2_H)p_M &= 0.\end{aligned}$$

After optimizing with respect to y^1_{MM} and y^2_{MM} respectively, we have:

$$\rho^y_{MM} + (\delta^i_{HM} - \delta^i_{ML} - v^i_M)p_M = 0 \text{ for } i = 1, 2.$$

After optimizing with respect to y^1_{ML} and y^2_{ML} respectively, we have:

$$\begin{aligned}\rho^y_{ML} + (\delta^1_{HM} - \delta^1_{ML} - v^1_M)p_L &= 0;\\ \rho^y_{ML} + (\delta^2_{ML} - v^2_L)p_M &= 0.\end{aligned}$$

After optimizing with respect to y^1_{LH} and y^2_{LH} respectively, we have:

$$\begin{aligned}\rho^y_{LH} + (\delta^1_{ML} - v^1_L)p_H &= 0;\\ \rho^y_{LH} - (\delta^2_{HM} + v^2_H)p_L &= 0.\end{aligned}$$

After optimizing with respect to y^1_{LM} and y^2_{LM} respectively, we have:

$$\begin{aligned}\rho^y_{LM} + (\delta^1_{ML} - v^1_L)p_M &= 0;\\ \rho^y_{LM} + (\delta^2_{HM} - \delta^2_{ML} - v^2_M)p_L &= 0.\end{aligned}$$

After optimizing with respect to y^i_{LL}, we have:

$$\rho^y_{LL} + (\delta^i_{ML} - v^i_L)p_L = 0, \text{ for } i = 1, 2.$$

In what follows, we restrict our attention to symmetric multipliers (this is without loss of generality, as Proposition 11 establishes[47]):

$$\begin{aligned}\delta_{HM} &\equiv \delta^1_{HM} = \delta^2_{HM}, \quad \delta_{ML} \equiv \delta^1_{ML} = \delta^2_{ML}\\ v_H &\equiv v^1_H = v^2_H, \quad v_M \equiv v^1_M = v^2_M, \quad v_L \equiv v^1_L = v^2_L.\end{aligned}$$

[47] Indeed, we just need to show that there exists a system of multipliers such that the third-party finds it optimal to offer the null-side contract.

From the above conditions about side transfers we have:

$$\frac{p_H}{p_L}(v_L - \delta_{ML}) = \delta_{HM} + v_H = \frac{p_H}{p_M}(\delta_{ML} + v_M - \delta_{HM})$$

Step 2: Optimizing with respect to $x^i(\theta^1, \theta^2, \widetilde{\phi})$

As in the proof of Proposition 4, the side-contract may specify a stochastic manipulation. However, the reallocation occurs after the outcome of the stochastic manipulation has been observed. Here, we optimize the third party's payoff with respect to $x^i_{jk,\widetilde{\phi}}$ for any $\widetilde{\phi} \in \Theta^2$.

After optimizing with respect to $x^i_{HH,\widetilde{\phi}}$, we have:[48]

$$\rho^x_{HH,\widetilde{\phi}} + p^\phi_{HH,\widetilde{\phi}}(p_H + \delta_{HM} + v_H)p_H\theta_H u'(q^i(\widetilde{\phi}) + x^i_{HH,\widetilde{\phi}}) = 0, \text{ for } i = 1, 2$$

The above equations imply $q^1(\widetilde{\phi}) + x^1_{HH,\widetilde{\phi}} = q^2(\widetilde{\phi}) + x^2_{HH,\widetilde{\phi}}$. Since $x^1_{HH,\widetilde{\phi}} + x^2_{HH,\widetilde{\phi}} = 0$ from the budget balance constraint, we have $q^i(\widetilde{\phi}) + x^i_{HH,\widetilde{\phi}} = \frac{q^1(\widetilde{\phi})+q^2(\widetilde{\phi})}{2}$, $i = 1, 2$, for any $\widetilde{\phi} \in \Theta^2$.

After optimizing with respect to $x^1_{HM,\widetilde{\phi}}$ and $x^2_{HM,\widetilde{\phi}}$ respectively, we have:

$$\begin{aligned} \rho^x_{HM,\widetilde{\phi}} + p^\phi_{HM,\widetilde{\phi}} p_M(p_H + \delta_{HM} + v_H)\theta_H u'(q^1(\widetilde{\phi}) + x^1_{HM,\widetilde{\phi}}) &= 0, \\ \rho^x_{HM,\widetilde{\phi}} + p^\phi_{HM,\widetilde{\phi}} p_H[\theta_M p_M + \theta_M(\delta_{ML} + v_M - \delta_{HM}) - \Delta_H\delta_{HM}]u'(q^2(\widetilde{\phi}) + x^2_{HM,\widetilde{\phi}}) &= 0. \end{aligned}$$

Define $\epsilon_{HM} \equiv \frac{\delta_{HM}}{p_H+\delta_{HM}+v_H} \in [0, 1)$ and recall that $\theta^\epsilon_M \equiv \theta_M - \frac{p_H}{p_M}\Delta_H\epsilon_{HM}$. Since $(\delta_{HM} + v_H)p_M = p_H(\delta_{ML} + v_M - \delta_{HM})$, we obtain from the two above equations:

$$\begin{aligned} & p_M(p_H + \delta_{HM} + v_H)\theta_H u'(q^1(\widetilde{\phi}) + x^1_{HM,\widetilde{\phi}}) \\ = \; & [p_M\theta_M(p_H + \delta_{HM} + v_H) - p_H\Delta_H\delta_{HM}]u'(q^2(\widetilde{\phi}) + x^2_{HM,\widetilde{\phi}}). \end{aligned}$$

Hence,

$$\theta_H u'(q^1(\widetilde{\phi}) + x^1_{HM,\widetilde{\phi}}) = \theta^\epsilon_M u'(q^2(\widetilde{\phi}) + x^2_{HM,\widetilde{\phi}}) \text{ for any } \widetilde{\phi} \in \Theta^2$$

After optimizing with respect to $x^1_{HL,\widetilde{\phi}}$ and $x^2_{HL,\widetilde{\phi}}$ respectively, we have:

$$\begin{aligned} \rho^x_{HL,\widetilde{\phi}} + p^\phi_{HL,\widetilde{\phi}} p_L(p_H + \delta_{HM} + v_H)\theta_H u'(q^1(\widetilde{\phi}) + x^1_{HL,\widetilde{\phi}}) &= 0, \\ \rho^x_{HL,\widetilde{\phi}} + p^\phi_{HL,\widetilde{\phi}}[\theta_L p_H(p_L + v_L - \delta_{ML}) - p_H\Delta_M\delta_{ML}]u'(q^2(\widetilde{\phi}) + x^2_{HL,\widetilde{\phi}}) &= 0. \end{aligned}$$

[48] In homogeneous coalitions – HH, MM and LL – the reallocation cannot lead to corner solutions. In HM, HL and ML coalitions instead, this is conceivable but is not going to occur when the seller designs the sale mechanism optimally. Hence, we only consider interior solutions for the reallocation problem.

Define $\epsilon_{ML} \equiv \frac{\delta_{ML}}{p_H+\delta_{HM}+v_H} = \epsilon_{HM}\frac{\delta_{ML}}{\delta_{HM}} \geq 0$ and recall that $\theta_L^{\epsilon} \equiv \theta_L - \frac{p_H}{p_L}\Delta_M\epsilon_{ML}$. Since $p_H(v_L-\delta_{ML}) = p_L(\delta_{HM}+v_H)$, from the two above equations we obtain:

$$\begin{aligned} & p_L(p_H+\delta_{HM}+v_H)\theta_H u'(q^1(\widetilde{\phi})+x^1_{HL,\widetilde{\phi}}) \\ = \;& [\theta_L p_L(p_H+\delta_{HM}+v_H) - p_H\Delta_M\delta_{ML})]u'(q^2(\widetilde{\phi})+x^2_{HL,\widetilde{\phi}}). \end{aligned}$$

Hence,

$$\theta_H u'(q^1(\widetilde{\phi})+x^1_{HL,\widetilde{\phi}}) = \theta_L^{\epsilon} u'(q^2(\widetilde{\phi})+x^2_{HL,\widetilde{\phi}}) \text{ for any } \widetilde{\phi}\in\Theta^2$$

After optimizing with respect to $x^i_{MM,\widetilde{\phi}}$, we have:

$$\rho^x_{MM,\widetilde{\phi}} + p^{\phi}_{MM,\widetilde{\phi}}[\theta_M p_M - \Delta_H\delta_{HM} + \theta_M(\delta_{ML}+v_M-\delta_{HM})]p_M u'(q^i(\widetilde{\phi})+x^i_{MM,\widetilde{\phi}}) = 0, \text{ for } i=1,2.$$

The above equations imply $q^1(\widetilde{\phi})+x^1_{MM,\widetilde{\phi}} = q^2(\widetilde{\phi})+x^2_{MM,\widetilde{\phi}}$. Since $x^1_{MM,\widetilde{\phi}}+x^2_{MM,\widetilde{\phi}}=0$ from the budget balance constraint, we have $q^i(\widetilde{\phi})+x^i_{MM,\widetilde{\phi}} = \frac{q^1(\widetilde{\phi})+q^2(\widetilde{\phi})}{2}$, $i=1,2$ for any $\widetilde{\phi}\in\Theta^2$.

After optimizing with respect to $x^1_{ML,\widetilde{\phi}}$ and $x^2_{ML,\widetilde{\phi}}$ respectively, we have:

$$\begin{aligned} \rho^x_{ML,\widetilde{\phi}} + p^{\phi}_{ML,\widetilde{\phi}} p_L[p_M\theta_M - \Delta_H\delta_{HM} + \theta_M(\delta_{ML}+v_M-\delta_{HM})]u'(q^1(\widetilde{\phi})+x^1_{ML,\widetilde{\phi}}) &= 0, \\ \rho^x_{ML,\widetilde{\phi}} + p^{\phi}_{ML,\widetilde{\phi}} p_M[\theta_L(p_L-\delta_{ML}+v_L) - \Delta_M\delta_{ML})]u'(q^2(\widetilde{\phi})+x^2_{ML,\widetilde{\phi}}) &= 0. \end{aligned}$$

Since $v_L-\delta_{ML} = \frac{p_L}{p_H}(\delta_{HM}+v_H)$ and $\delta_{ML}+v_M-\delta_{HM} = \frac{p_M}{p_H}(\delta_{HM}+v_H)$, from the two above equations we obtain:

$$\begin{aligned} & [\theta_M p_L p_M(p_H+v_H+\delta_{HM}) - \Delta_H p_L p_H\delta_{HM}]u'(q^1(\widetilde{\phi})+x^1_{ML,\widetilde{\phi}}) \\ = \;& [\theta_L p_L p_M(p_H+v_H+\delta_{HM}) - p_H p_M\Delta_M\delta_{ML}]u'(q^2(\widetilde{\phi})+x^2_{ML,\widetilde{\phi}}). \end{aligned}$$

Hence $\theta_M^{\epsilon} u'(q^1(\widetilde{\phi})+x^1_{ML,\widetilde{\phi}}) = \theta_L^{\epsilon} u'(q^2(\widetilde{\phi})+x^2_{ML,\widetilde{\phi}})$ for any $\widetilde{\phi}\in\Theta^2$.

After optimizing with respect to $x^i_{LL,\widetilde{\phi}}$, we have:

$$\rho^x_{LL,\widetilde{\phi}} + p^{\phi}_{LL,\widetilde{\phi}}[\theta_L p_L - \Delta_M\delta_{ML} + \theta_L(v_L-\delta_{ML})]p_L u'(q^i(\widetilde{\phi})+x^i_{LL,\widetilde{\phi}}) = 0, \text{ for } i=1,2.$$

The above equations imply $q^1(\widetilde{\phi})+x^1_{LL,\widetilde{\phi}} = q^2(\widetilde{\phi})+x^2_{LL,\widetilde{\phi}}$. Since $x^1_{LL,\widetilde{\phi}}+x^2_{LL,\widetilde{\phi}}=0$ from the budget balance constraint, we have $q^i(\widetilde{\phi})+x^i_{LL,\widetilde{\phi}} = \frac{q^1(\widetilde{\phi})+q^2(\widetilde{\phi})}{2}$, $i=1,2$, for any $\widetilde{\phi}\in\Theta^2$.

Since we are considering symmetric multipliers, we can infer that
(i) after optimizing with respect to $x^1_{MH,\widetilde{\phi}}$ and $x^2_{MH,\widetilde{\phi}}$ respectively, we have:

$\theta^{\epsilon}_M u'(q^1(\widetilde{\phi}) + x^1_{MH,\widetilde{\phi}}) = \theta_H u'(q^2(\widetilde{\phi}) + x^2_{MH,\widetilde{\phi}})$ for any $\widetilde{\phi} \in \Theta^2$;

(ii) after optimizing with respect to $x^1_{LH,\widetilde{\phi}}$ and $x^2_{LH,\widetilde{\phi}}$ respectively, we have:

$\theta^{\epsilon}_L u'(q^1(\widetilde{\phi}) + x^1_{LH,\widetilde{\phi}}) = \theta_H u'(q^2(\widetilde{\phi}) + x^2_{LH,\widetilde{\phi}})$ for any $\widetilde{\phi} \in \Theta^2$;

(iii) after optimizing with respect to $x^1_{LM,\widetilde{\phi}}$ and $x^2_{LM,\widetilde{\phi}}$ respectively, we have:

$\theta^{\epsilon}_L u'(q^1(\widetilde{\phi}) + x^1_{LM,\widetilde{\phi}}) = \theta^{\epsilon}_M u'(q^2(\widetilde{\phi}) + x^2_{LM,\widetilde{\phi}})$ for any $\widetilde{\phi} \in \Theta^2$.

Therefore, conditional on that there is no manipulation of report, i.e., $\phi(\theta_j, \theta_k) = (\theta_j, \theta_k)$ for any jk, the third party will not reallocate the goods among the buyers if there is an $\epsilon \in [0, 1) \times [0, +\infty)$ such that:

$$\theta^{\epsilon}_j u'(q^1(\theta_j, \theta_k)) = \theta^{\epsilon}_k u'(q^2(\theta_j, \theta_k)) \text{ for } j, k \in \{H, M, L\}.$$

For non-homogeneous coalitions this is equivalent to condition (19).

Step 3: Optimizing with respect to $\phi(\theta_1, \theta_2)$

Let $S_{jk}(\widetilde{\phi}) \equiv \theta_j u(q^1(\widetilde{\phi}) + x^1_{jk,\widetilde{\phi}}) + \theta_k u(q^2(\widetilde{\phi}) + x^2_{jk,\widetilde{\phi}}) - t^1(\widetilde{\phi}) - t^2(\widetilde{\phi})$ denote the real surplus of jk coalition when reports to the seller are manipulated into $\widetilde{\phi}$. Likewise, $S^1_{jk}(\widetilde{\phi}) \equiv \theta_j u(q^1(\widetilde{\phi}) + x^1_{jk,\widetilde{\phi}}) - t^1(\widetilde{\phi})$, $S^2_{jk}(\widetilde{\phi}) \equiv \theta_k u(q^2(\widetilde{\phi}) + x^2_{jk,\widetilde{\phi}}) - t^2(\widetilde{\phi})$ and $u^i_{jk}(\widetilde{\phi}) \equiv u(q^i(\widetilde{\phi}) + x^i_{jk,\widetilde{\phi}})$. Using this notation we find conditions under which the third party optimally requires any coalition with $(\theta^1, \theta^2) = (\theta_j, \theta_k)$ to report (θ_j, θ_k).

- HH coalition

$$\begin{aligned}
(\theta_H, \theta_H) &\in \arg\max_{\widetilde{\phi}\in\Theta^2} [p^2_H + p_H(v_H + \delta_{HM})] S_{HH}(\widetilde{\phi}) \\
&= \arg\max_{\widetilde{\phi}\in\Theta^2} \left\{ 2\theta_H u(\frac{q^1(\widetilde{\phi}) + q^2(\widetilde{\phi})}{2}) - t^1(\widetilde{\phi}) - t^2(\widetilde{\phi}) \right\}.
\end{aligned}$$

- HM coalition:

$$\begin{aligned}
(\theta_H, \theta_M) &\in \arg\max_{\widetilde{\phi}\in\Theta^2} \left\{ \begin{array}{c} p_H p_M S_{HM}(\widetilde{\phi}) + p_M(v_H + \delta_{HM}) S^1_{HM}(\widetilde{\phi}) \\ -p_H \Delta_H \delta_{HM} u^2_{HM}(\widetilde{\phi}) + p_H(v_M + \delta_{ML} - \delta_{HM}) S^2_{HM}(\widetilde{\phi}) \end{array} \right\} \\
&= \arg\max_{\widetilde{\phi}\in\Theta^2} \left\{ p_M(p_H + v_H + \delta_{HM}) S_{HM}(\widetilde{\phi}) - p_H \Delta_H \delta_{HM} u^2_{HM}(\widetilde{\phi}) \right\} \\
&= \arg\max_{\widetilde{\phi}\in\Theta^2} \left\{ S_{HM}(\widetilde{\phi}) - \frac{p_H}{p_M} \Delta_H \epsilon_{HM} u^2_{HM}(\widetilde{\phi}) \right\} \\
&= \arg\max_{\widetilde{\phi}\in\Theta^2} \left\{ \theta_H u^1_{HM}(\widetilde{\phi}) + \theta^{\epsilon}_M u^2_{HM}(\widetilde{\phi}) - t^1(\widetilde{\phi}) - t^2(\widetilde{\phi}) \right\}
\end{aligned}$$

where

$$\theta_H u'(q^1(\widetilde{\phi}) + x^1_{HM,\widetilde{\phi}}) = \theta^{\epsilon}_M u'(q^2(\widetilde{\phi}) + x^2_{HM,\widetilde{\phi}}) \text{ holds.}$$

- HL coalition:

$$\begin{aligned}
(\theta_H, \theta_L) &\in \arg\max_{\widetilde{\phi}\in\Theta^2} \left\{ \begin{array}{c} p_H p_L S_{HL}(\widetilde{\phi}) + p_L(v_H + \delta_{HM}) S^1_{HL}(\widetilde{\phi}) \\ + p_H(v_L - \delta_{ML}) S^2_{HL}(\widetilde{\phi}) - p_H \Delta_M \delta_{ML} u^2_{HL}(\widetilde{\phi}) \end{array} \right\} \\
&= \arg\max_{\widetilde{\phi}\in\Theta^2} \left\{ S_{HL}(\widetilde{\phi}) - \frac{p_H}{p_L} \Delta_M \epsilon_{ML} u^2_{HL}(\widetilde{\phi}) \right\} \\
&= \arg\max_{\widetilde{\phi}\in\Theta^2} \left\{ \theta_H u^1_{HL}(\widetilde{\phi}) + \theta^\epsilon_L u^2_{HL}(\widetilde{\phi}) - t^1(\widetilde{\phi}) - t^2(\widetilde{\phi}) \right\},
\end{aligned}$$

where

$$\theta_H u'(q^1(\widetilde{\phi}) + x^1_{HL,\widetilde{\phi}}) = \theta^\epsilon_L u'(q^2(\widetilde{\phi}) + x^2_{HL,\widetilde{\phi}}) \text{ holds.}$$

- MM coalition:

$$\begin{aligned}
(\theta_M, \theta_M) &\in \arg\max_{\widetilde{\phi}\in\Theta^2} \left\{ \begin{array}{c} p^2_M S_{MM}(\widetilde{\phi}) + p_M(\delta_{ML} + v_M) S_{MM}(\widetilde{\phi}) \\ - p_M \delta_{HM} \left[S_{MM}(\widetilde{\phi}) + \Delta_H u^1_{MM}(\widetilde{\phi}) + \Delta_H u^2_{MM}(\widetilde{\phi}) \right] \end{array} \right\} \\
&= \arg\max_{\widetilde{\phi}\in\Theta^2} \left\{ \begin{array}{c} p_M(p_M - \delta_{HM} + \delta_{ML} + v_M) S_{MM}(\widetilde{\phi}) \\ - p_M \delta_{HM} \Delta_H (u^1_{MM}(\widetilde{\phi}) + u^2_{MM}(\widetilde{\phi})) \end{array} \right\} \\
&= \arg\max_{\widetilde{\phi}\in\Theta^2} \left\{ S_{MM}(\widetilde{\phi}) - \frac{p_H}{p_M} \Delta_H \epsilon_{HM} (u^1_{MM}(\widetilde{\phi}) + u^2_{MM}(\widetilde{\phi})) \right\} \\
&= \arg\max_{\widetilde{\phi}\in\Theta^2} \left\{ 2\theta^\epsilon_M u(\frac{q^1(\widetilde{\phi}) + q^2(\widetilde{\phi})}{2}) - t^1(\widetilde{\phi}) - t^2(\widetilde{\phi}) \right\}
\end{aligned}$$

- ML coalition:

$$\begin{aligned}
(\theta_M, \theta_L) &\in \arg\max_{\widetilde{\phi}\in\Theta^2} \left\{ \begin{array}{c} p_M p_L S_{ML}(\widetilde{\phi}) - \delta_{HM} p_L \Delta_H u^1_{ML}(\widetilde{\phi}) + (v_M + \delta_{ML} - \delta_{HM}) p_L S^1_{ML}(\widetilde{\phi}) \\ - p_M \delta_{ML} (S^2_{ML}(\widetilde{\phi}) + \Delta_M u^2_{ML}(\widetilde{\phi})) + v_L p_M S^2_{ML}(\widetilde{\phi}) \end{array} \right\} \\
&= \arg\max_{\widetilde{\phi}\in\Theta^2} \left\{ \begin{array}{c} p_L(p_M + \delta_{ML} + v_M - \delta_{HM}) S_{ML}(\widetilde{\phi}) - \delta_{HM} p_L \Delta_H u^1_{ML}(\widetilde{\phi}) \\ - \delta_{ML} p_M \Delta_M u^2_{ML}(\widetilde{\phi}) \end{array} \right\} \\
&= \arg\max_{\widetilde{\phi}\in\Theta^2} \left\{ \theta^\epsilon_M u^1_{ML}(\widetilde{\phi}) + \theta^\epsilon_L u^2_{ML}(\widetilde{\phi}) - t^1(\widetilde{\phi}) - t^2(\widetilde{\phi}) \right\},
\end{aligned}$$

where

$$\theta^\epsilon_M u'(q^1(\widetilde{\phi}) + x^1_{ML,\widetilde{\phi}}) = \theta^\epsilon_L u'(q^2(\widetilde{\phi}) + x^2_{ML,\widetilde{\phi}}) \text{ holds.}$$

• LL coalition:

$$\begin{aligned}(\theta_L,\theta_L) &\in \arg\max_{\widetilde{\phi}\in\Theta^2}\left\{\begin{array}{c} p_L^2 S_{LL}(\widetilde{\phi}) - p_L\delta_{ML}(S_{LL}^1(\widetilde{\phi}) + \Delta_M u_{LL}^1(\widetilde{\phi})) \\ -p_L\delta_{ML}(S_{LL}^2(\widetilde{\phi}) + \Delta_M u_{LL}^2(\widetilde{\phi})) + v_L p_L S_{LL}(\widetilde{\phi}) \end{array}\right\} \\ &= \arg\max_{\widetilde{\phi}\in\Theta^2}\left\{S_{LL}(\widetilde{\phi}) - \frac{p_H}{p_L}\Delta_M\epsilon_{ML}(u_{LL}^1(\widetilde{\phi}) + u_{LL}^2(\widetilde{\phi}))\right\} \\ &= \arg\max_{\widetilde{\phi}\in\Theta^2}\left\{2\theta_L^\epsilon u(\frac{q^1(\widetilde{\phi}) + q^2(\widetilde{\phi})}{2}) - t^1(\widetilde{\phi}) - t^2(\widetilde{\phi})\right\}\end{aligned}$$

For HM coalition, for instance, we have $\theta_H u_{HM}^1(\widetilde{\phi}) + \theta_M^\epsilon u_{HM}^2(\widetilde{\phi}) - t^1(\widetilde{\phi}) - t^2(\widetilde{\phi}) = V_{HM}^\varepsilon(q^1(\widetilde{\phi}) + q^2(\widetilde{\phi})) - t^1(\widetilde{\phi}) - t^2(\widetilde{\phi})$ since $\theta_H u'(q^1(\widetilde{\phi}) + x_{HM,\widetilde{\phi}}^1) = \theta_M^\epsilon u'(q^2(\widetilde{\phi}) + x_{HM,\widetilde{\phi}}^2)$ holds. Hence, the condition $(\theta_H,\theta_M) \in \arg\max_{\widetilde{\phi}\in\Theta^2}\left\{\theta_H u_{HM}^1(\widetilde{\phi}) + \theta_M^\epsilon u_{HM}^2(\widetilde{\phi}) - t^1(\widetilde{\phi}) - t^2(\widetilde{\phi})\right\}$ is equivalent to (18) with $jk = HM$. The same remark applies to any other coalition and justifies the whole condition (18).

Proof of Proposition 11

The proof of Proposition 11 depends on whether $q_H^* + q_L^* \geq 2q_M^*$ or the reverse inequality holds (notice that $q_H^* + q_L^* > 2q_M^*$ under non-monotone virtual values). Here we assume that $q_H^* + q_L^* \geq 2q_M^*$, but our argument below can be adapted to the case of $q_H^* + q_L^* < 2q_M^*$ – see the end of the proof. The proof consists of three steps.

Claim 1 If $\theta_M^v \geq \theta_L^v$, then there exists ϵ^* such that $\theta_M^{\epsilon^*} = \theta_M^v$ and $\theta_L^{\epsilon^*} = \theta_L^v$. If $\theta_M^v < \theta_L^v$, then there exists ϵ^* such that $\theta_M^{\epsilon^*} = \theta_L^{\epsilon^*} = \bar{\theta}_{ML}^v$. In both cases, the no arbitrage constraints (19) are satisfied at $\mathbf{q} = \mathbf{q}^*$ and $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^*$.

Claim 2 Let $\alpha \equiv V_{MM}^{\epsilon^*}(2q_M^*) - V_{MM}^{\epsilon^*}(q_M^* + q_L^*) - [V_{ML}^{\epsilon^*}(2q_M^*) - V_{ML}^{\epsilon^*}(q_M^* + q_L^*)] \geq 0$. If $\mathbf{t}$ is such that the following local downward coalition incentive constraints bind, then all the coalition incentive constraints are satisfied by mechanism $\{\mathbf{q}^*, \mathbf{t}\}$ when $\epsilon = \epsilon^*$:[49]

$$\begin{aligned} V_{HH}^{\epsilon^*}(2q_H^*) - 2t_{HH} &\geq V_{HH}^{\epsilon^*}(q_H^* + q_M^*) - t_{HM} - t_{MH} \ (CIC_{HH,HM}) \\ V_{HM}^{\epsilon^*}(q_H^* + q_M^*) - t_{HM} - t_{MH} &\geq V_{HM}^{\epsilon^*}(q_H^* + q_L^*) - t_{HL} - t_{LH} \ (CIC_{HM,HL}) \\ V_{HL}^{\epsilon^*}(q_H^* + q_L^*) - t_{HL} - t_{LH} &\geq V_{HL}^{\epsilon^*}(2q_M^*) - 2t_{MM} \ (CIC_{HL,MM}) \\ V_{MM}^{\epsilon^*}(2q_M^*) - 2t_{MM} &\geq V_{MM}^{\epsilon^*}(q_M^* + q_L^*) - t_{ML} - t_{LM} + \alpha \ (CIC_{MM,ML}^{\text{modified}}) \\ V_{ML}^{\epsilon^*}(q_M^* + q_L^*) - t_{ML} - t_{LM} &\geq V_{ML}^{\epsilon^*}(2q_L^*) - 2t_{LL} \ (CIC_{ML,LL}) \end{aligned}$$

[49] We note that $(CIC_{MM,ML})$ is modified with respect to the true constraint $(CIC_{MM,ML})$. In order to understand why α is introduced in $(CIC_{MM,ML}^{\text{modified}})$, see the proof of Claim 2 when dealing with $(CIC_{HL,ML})$ and $(CIC_{HL,LL})$.

Claim 3 We can find $\mathbf{t}^{**}$ such that (BIC_{HM}), (BIC_{ML}), (BIR_L) and coalition incentive constraints mentioned in Claim 2 are satisfied with equality at $\mathbf{q} = \mathbf{q}^*$ and $\varepsilon = \varepsilon^*$.

Once Claims 1-3 are proved, we conclude that $\{\mathbf{q}^*, \mathbf{t}^{**}\}$ is optimal under no coalition - since it satisfies the conditions of Proposition 9 - and is also weakly collusion-proof.

Proof of Claim 1

If $\theta_M^v \geq \theta_L^v$, then take $\epsilon^* = (1, \frac{p_H + p_M}{p_H})$.[50] If $\theta_M^v < \theta_L^v$, then take $\epsilon_{HM}^* = \frac{(\theta_M - \bar{\theta}_{ML}^v)p_M}{p_H \Delta_H} \in (0,1)$ and $\epsilon_{ML}^* = \frac{(\theta_L - \bar{\theta}_{ML}^v)p_L}{p_H \Delta_M} > 0$. In this way the virtual valuations for the third party in the side-contract are equal to the virtual valuations for the seller when there is no coalition.

Proof of Claim 2

The following lemma establishes single crossing properties (when $\epsilon = \epsilon^*$) which are useful to prove claim 2.

Lemma 2 *(i) $V_{jk}^{\epsilon^*}$ is strictly concave, $j, k = L, M, H$.*

(ii) For any $x > 0$, if $\theta_M^v \geq \theta_L^v$ then we have

$$\frac{\partial V_{HH}^{\epsilon^*}(x)}{\partial x} > \frac{\partial V_{HM}^{\epsilon^*}(x)}{\partial x} \geq \max\left\{\frac{\partial V_{HL}^{\epsilon^*}(x)}{\partial x}, \frac{\partial V_{MM}^{\epsilon^*}(x)}{\partial x}\right\} \tag{32}$$

$$\min\left\{\frac{\partial V_{HL}^{\epsilon^*}(x)}{\partial x}, \frac{\partial V_{MM}^{\epsilon^*}(x)}{\partial x}\right\} \geq \frac{\partial V_{ML}^{\epsilon^*}(x)}{\partial x} \geq \frac{\partial V_{LL}^{\epsilon^*}(x)}{\partial x} \tag{33}$$

If $\theta_M^v < \theta_L^v$, then

$$\frac{\partial V_{HH}^{\epsilon^*}(x)}{\partial x} > \frac{\partial V_{HM}^{\epsilon^*}(x)}{\partial x} = \frac{\partial V_{HL}^{\epsilon^*}(x)}{\partial x} > \frac{\partial V_{MM}^{\epsilon^*}(x)}{\partial x} = \frac{\partial V_{ML}^{\epsilon^*}(x)}{\partial x} = \frac{\partial V_{LL}^{\epsilon^*}(x)}{\partial x} \tag{34}$$

Proof. (i) The result is obvious for $V_{HH}^{\epsilon^*}$, $V_{MM}^{\epsilon^*}$ and $V_{LL}^{\epsilon^*}$, since $V_{jj}^{\epsilon^*}(x) = 2\theta_j^{\epsilon^*} u(\frac{x}{2})$, $j = L, M, H$. About $V_{jk}^{\epsilon^*}$ with $j \neq k$, observe that $\frac{\partial V_{jk}^{\epsilon^*}(x)}{\partial x} = \theta_j^{\epsilon^*} u'[q_j^{\epsilon^*}(x; jk)]$ and $\frac{\partial^2 V_{jk}^{\epsilon^*}(x)}{\partial x^2} = \theta_j^{\epsilon^*} u''[q_j^{\epsilon^*}(x; jk)] \frac{\partial q_j^{\epsilon^*}(x;jk)}{\partial x} < 0$ since $\frac{\partial q_j^{\epsilon^*}(x;jk)}{\partial x} = \frac{\theta_k^{\epsilon^*} u''[q_k^{\epsilon^*}(x;jk)]}{\theta_j^{\epsilon^*} u''[q_j^{\epsilon^*}(x;jk)] + \theta_k^{\epsilon^*} u''[q_k^{\epsilon^*}(x;jk)]} > 0$.

(ii) It is straightforward to find $\frac{\partial V_{HH}^{\epsilon^*}(x)}{\partial x} = \theta_H u'(\frac{x}{2})$, $\frac{\partial V_{HM}^{\epsilon^*}(x)}{\partial x} = \theta_H u'[q_H^{\epsilon^*}(x; HM)] = \theta_M^{\epsilon^*} u'[q_M^{\epsilon^*}(x; HM)]$, $\frac{\partial V_{HL}^{\epsilon^*}(x)}{\partial x} = \theta_H u'[q_H^{\epsilon^*}(x; HL)]$ and $\frac{\partial V_{MM}^{\epsilon^*}(x)}{\partial x} = \theta_M^{\epsilon^*} u'(\frac{x}{2})$. Since $\frac{x}{2} < q_H^{\epsilon^*}(x; HM) \leq q_H^{\epsilon^*}(x; HL)$ and $q_M^{\epsilon^*}(x; HM) < \frac{x}{2}$, we obtain (32). The proofs of (33) and (34) are very similar to the proof of (32), hence they are omitted. ■

In order to prove claim 2, we consider transfers such that the inequalities in the claim bind and recall that (i) $\alpha \geq 0$, so that the true (CIC$_{MM,ML}$) is satisfied; (ii) $q_H^* > q_M^* \geq q_L^*$. For

[50] In fact, $\epsilon_{HM} \in [0, 1)$. However, since we are interested in finding the Sup of the seller's payoff, we take $\epsilon_{HM} = 1$.

expositional simplicity, we introduce the following notation:

$$\begin{aligned}\Delta V^{\epsilon^*}_{HH} &\equiv V^{\epsilon^*}_{HH}(2q^*_H) - V^{\epsilon^*}_{HH}(q^*_H + q^*_M) \qquad \Delta V^{\epsilon^*}_{HM} \equiv V^{\epsilon^*}_{HM}(q^*_H + q^*_M) - V^{\epsilon^*}_{HM}(q^*_H + q^*_L)\\ \Delta V^{\epsilon^*}_{HL} &\equiv V^{\epsilon^*}_{HL}(q^*_H + q^*_L) - V^{\epsilon^*}_{HL}(2q^*_M) \qquad \Delta V^{\epsilon^*}_{MM} \equiv V^{\epsilon^*}_{MM}(2q^*_M) - V^{\epsilon^*}_{MM}(q^*_M + q^*_L)\\ \Delta V^{\epsilon^*}_{ML} &\equiv V^{\epsilon^*}_{ML}(q^*_M + q^*_L) - V^{\epsilon^*}_{ML}(2q^*_L)\end{aligned}$$

We first prove that all downward coalition incentive constraints are satisfied and then we deal with upward coalition incentive constraints.

Downward coalition incentive constraints

We start with downward coalition incentive constraints for HH coalition

CIC$_{HH}$ The payoff of HH coalition is $V^{\epsilon^*}_{HH}(q^*_H + q^*_M) - t_{HM} - t_{MH}$.

$$(\text{CIC}_{HH,HL})\ V^{\epsilon^*}_{HH}(q^*_H + q^*_M) - t_{HM} - t_{MH} \geq V^{\epsilon^*}_{HH}(q^*_H + q^*_L) - t_{HL} - t_{LH}$$

but $-t_{HM} - t_{MH} = -\Delta V^{\epsilon^*}_{HM} - t_{HL} - t_{LH}$, hence $(\text{CIC}_{HH,HL})$ reduces to

$$V^{\epsilon^*}_{HH}(q^*_H + q^*_M) - V^{\epsilon^*}_{HH}(q^*_H + q^*_L) \geq V^{\epsilon^*}_{HM}(q^*_H + q^*_M) - V^{\epsilon^*}_{HM}(q^*_H + q^*_L)$$

which is satisfied because $q^*_M \geq q^*_L$ and (32).

$$(\text{CIC}_{HH,MM})\ V^{\epsilon^*}_{HH}(q^*_H + q^*_M) - t_{HM} - t_{MH} \geq V^{\epsilon^*}_{HH}(2q^*_M) - 2t_{MM}$$

but $-t_{HM} - t_{MH} = -\Delta V^{\epsilon^*}_{HM} - \Delta V^{\epsilon^*}_{HL} - 2t_{MM}$, hence $(\text{CIC}_{HH,MM})$ reduces to

$$V^{\epsilon^*}_{HH}(q^*_H + q^*_M) - V^{\epsilon^*}_{HH}(2q^*_M) \geq V^{\epsilon^*}_{HM}(q^*_H + q^*_M) - V^{\epsilon^*}_{HM}(q^*_H + q^*_L) + V^{\epsilon^*}_{HL}(q^*_H + q^*_L) - V^{\epsilon^*}_{HL}(2q^*_M)$$

This inequality is satisfied because $q^*_M \geq q^*_L$, $q^*_H + q^*_L \geq 2q^*_M$ and (32).

$$(\text{CIC}_{HH,ML})\ V^{\epsilon^*}_{HH}(q^*_H + q^*_M) - t_{HM} - t_{MH} \geq V^{\epsilon^*}_{HH}(q^*_M + q^*_L) - t_{ML} - t_{LM}$$

but $-t_{HM} - t_{MH} = -\Delta V^{\epsilon^*}_{HM} - \Delta V^{\epsilon^*}_{HL} - \Delta V^{\epsilon^*}_{MM} - t_{ML} - t_{LM} + \alpha$, hence $(\text{CIC}_{HH,ML})$ reduces to

$$\begin{aligned}& V^{\epsilon^*}_{HH}(q^*_H + q^*_M) - V^{\epsilon^*}_{HH}(q^*_M + q^*_L) + \alpha\\ \geq\ & V^{\epsilon^*}_{HM}(q^*_H + q^*_M) - V^{\epsilon^*}_{HM}(q^*_H + q^*_L) + V^{\epsilon^*}_{HL}(q^*_H + q^*_L) - V^{\epsilon^*}_{HL}(2q^*_M) + V^{\epsilon^*}_{MM}(2q^*_M) - V^{\epsilon^*}_{MM}(q^*_M + q^*_L)\end{aligned}$$

This inequality is satisfied because $\alpha \geq 0$, $q^*_M \geq q^*_L$, $q^*_H + q^*_L \geq 2q^*_M$ and (32).

$$(\text{CIC}_{HH,LL})\ V^{\epsilon^*}_{HH}(q^*_H + q^*_M) - t_{HM} - t_{MH} \geq V^{\epsilon^*}_{HH}(2q^*_L) - 2t_{LL}$$

but $-t_{HM}-t_{MH}=-\Delta V^{\epsilon^*}_{HM}-\Delta V^{\epsilon^*}_{HL}-\Delta V^{\epsilon^*}_{MM}-\Delta V^{\epsilon^*}_{ML}-2t_{LL}+\alpha$, hence (CIC$_{HH,LL}$) reduces to

$$\begin{aligned} & V^{\epsilon^*}_{HH}(q^*_H+q^*_M)-V^{\epsilon^*}_{HH}(2q^*_L)+\alpha \\ \geq\ & V^{\epsilon^*}_{HM}(q^*_H+q^*_M)-V^{\epsilon^*}_{HM}(q^*_H+q^*_L)+V^{\epsilon^*}_{HL}(q^*_H+q^*_L)-V^{\epsilon^*}_{HL}(2q^*_M) \\ & +V^{\epsilon^*}_{MM}(2q^*_M)-V^{\epsilon^*}_{MM}(q^*_M+q^*_L)+V^{\epsilon^*}_{ML}(q^*_L+q^*_M)-V^{\epsilon^*}_{ML}(2q^*_L) \end{aligned}$$

This inequality is satisfied because $\alpha \geq 0$, $q^*_M \geq q^*_L$, $q^*_H+q^*_L \geq 2q^*_M$, (32) and (33).

CIC$_{HM}$ The payoff of HM coalition is $V^{\epsilon^*}_{HM}(q^*_H+q^*_L)-t_{HL}-t_{LH}$.

$$(\text{CIC}_{HM,MM})\ V^{\epsilon^*}_{HM}(q^*_H+q^*_L)-t_{HL}-t_{LH} \geq V^{\epsilon^*}_{HM}(2q^*_M)-2t_{MM}$$

but $-t_{HL}-t_{LH}=-\Delta V^{\epsilon^*}_{HL}-2t_{MM}$, hence (CIC$_{HM,MM}$) reduces to

$$V^{\epsilon^*}_{HM}(q^*_H+q^*_L)-V^{\epsilon^*}_{HM}(2q^*_M) \geq V^{\epsilon^*}_{HL}(q^*_H+q^*_L)-V^{\epsilon^*}_{HL}(2q^*_M).$$

$$(\text{CIC}_{HM,ML})\ V^{\epsilon^*}_{HM}(q^*_H+q^*_L)-t_{HL}-t_{LH} \geq V^{\epsilon^*}_{HM}(q^*_M+q^*_L)-t_{ML}-t_{LM}$$

but $-t_{HL}-t_{LH}=-\Delta V^{\epsilon^*}_{HL}-\Delta V^{\epsilon^*}_{MM}-t_{ML}-t_{LM}+\alpha$, hence (CIC$_{HM,ML}$) reduces to

$$V^{\epsilon^*}_{HM}(q^*_H+q^*_L)-V^{\epsilon^*}_{HM}(q^*_M+q^*_L)+\alpha \geq V^{\epsilon^*}_{HL}(q^*_H+q^*_L)-V^{\epsilon^*}_{HL}(2q^*_M)+V^{\epsilon^*}_{MM}(2q^*_M)-V^{\epsilon^*}_{MM}(q^*_M+q^*_L).$$

$$(\text{CIC}_{HM,LL})\ V^{\epsilon^*}_{HM}(q^*_H+q^*_L)-t_{HL}-t_{LH} \geq V^{\epsilon^*}_{HM}(2q^*_L)-2t_{LL}$$

but $-t_{HL}-t_{LH}=-\Delta V^{\epsilon^*}_{HL}-\Delta V^{\epsilon^*}_{MM}-\Delta V^{\epsilon^*}_{ML}-2t_{LL}+\alpha$, hence (CIC$_{HM,LL}$) reduces to

$$\begin{aligned} V^{\epsilon^*}_{HM}(q^*_H+q^*_L)-V^{\epsilon^*}_{HM}(2q^*_L)+\alpha \quad \geq \quad & V^{\epsilon^*}_{HL}(q^*_H+q^*_L)-V^{\epsilon^*}_{HL}(2q^*_M)+ \\ & V^{\epsilon^*}_{MM}(2q^*_M)-V^{\epsilon^*}_{MM}(q^*_M+q^*_L)+V^{\epsilon^*}_{ML}(q^*_L+q^*_M)-V^{\epsilon^*}_{ML}(2q^*_L). \end{aligned}$$

CIC$_{HL}$ The payoff of HL coalition is $V^{\epsilon^*}_{HL}(2q^*_M)-2t_{MM}$.

$$(\text{CIC}_{HL,ML})\ V^{\epsilon^*}_{HL}(2q^*_M)-2t_{MM} \geq V^{\epsilon^*}_{HL}(q^*_M+q^*_L)-t_{ML}-t_{LM}$$

but $-2t_{MM} = -\Delta V^{\epsilon^*}_{MM} - t_{ML} - t_{LM} + \alpha$, hence ($\text{CIC}_{HL,ML}$) reduces to

$$V^{\epsilon^*}_{HL}(2q^*_M) - V^{\epsilon^*}_{HL}(q^*_M + q^*_L) + \alpha \geq V^{\epsilon^*}_{MM}(2q^*_M) - V^{\epsilon^*}_{MM}(q^*_M + q^*_L). \tag{35}$$

This inequality holds because the right hand side minus α reduces to $V^{\epsilon^*}_{ML}(2q^*_M) - V^{\epsilon^*}_{ML}(q^*_M + q^*_L)$ and then we can use (33) and $q^*_M \geq q^*_L$. Observe that (35) is the reason why α is introduced in ($\text{CIC}^{\text{modified}}_{MM\ ML}$): If $\alpha = 0$, then we do not know whether (35) is satisfied or not. A similar argument applies to (36) below.

$$(\text{CIC}_{HL,LL})\ V^{\epsilon^*}_{HL}(2q^*_M) - 2t_{MM} \geq V^{\epsilon^*}_{HL}(2q^*_L) - 2t_{LL}$$

but $-2t_{MM} = -\Delta V^{\epsilon^*}_{MM} - \Delta V^{\epsilon^*}_{ML} - 2t_{LL} + \alpha$, hence ($\text{CIC}_{HL,LL}$) reduces to

$$V^{\epsilon^*}_{HL}(2q^*_M) - V^{\epsilon^*}_{HL}(2q^*_L) + \alpha \geq V^{\epsilon^*}_{MM}(2q^*_M) - V^{\epsilon^*}_{MM}(q^*_M + q^*_L) + V^{\epsilon^*}_{ML}(q^*_L + q^*_M) - V^{\epsilon^*}_{ML}(2q^*_L) \tag{36}$$

This inequality holds because the right hand side minus α reduces to $V^{\epsilon^*}_{ML}(2q^*_M) - V^{\epsilon^*}_{ML}(2q^*_L)$.

CIC$_{MM}$ The payoff of MM coalition is $V^{\epsilon^*}_{MM}(q^*_M + q^*_L) - t_{ML} - t_{LM} + \alpha$.

$$(\text{CIC}_{MM,LL})\ V^{\epsilon^*}_{MM}(q^*_M + q^*_L) - t_{ML} - t_{LM} + \alpha \geq V^{\epsilon^*}_{MM}(2q^*_L) - 2t_{LL}$$

but $-t_{ML} - t_{LM} = -\Delta V^{\epsilon^*}_{ML} - 2t_{LL}$, hence ($\text{CIC}_{MM,LL}$) reduces to

$$V^{\epsilon^*}_{MM}(2q^*_M) - V^{\epsilon^*}_{MM}(2q^*_L) \geq V^{\epsilon^*}_{ML}(2q^*_M) - V^{\epsilon^*}_{ML}(2q^*_L).$$

Upward CIC

CIC$_{HM}$ The payoff of HM coalition is $V^{\epsilon^*}_{HM}(q^*_H + q^*_M) - t_{HM} - t_{MH}$.

$$(CIC_{HM,HH})\ V^{\epsilon^*}_{HM}(q^*_H + q^*_M) - t_{HM} - t_{MH} \geq V^{\epsilon^*}_{HM}(2q^*_H) - 2t_{HH}$$

but $-2t_{HH} = -\Delta V^{\epsilon^*}_{HH} - t_{HM} - t_{MH}$, hence ($CIC_{HM,HH}$) reduces to

$$V^{\epsilon^*}_{HH}(2q^*_H) - V^{\epsilon^*}_{HH}(q^*_H + q^*_M) \geq V^{\epsilon^*}_{HM}(2q^*_H) - V^{\epsilon^*}_{HM}(q^*_H + q^*_M).$$

CIC$_{HL}$ The payoff of HL coalition is $V_{HL}^{\epsilon^*}(q_H^* + q_L^*) - t_{HL} - t_{LH}$.

$$(CIC_{HL,HM})\ V_{HL}^{\epsilon^*}(q_H^* + q_L^*) - t_{HL} - t_{LH} \geq V_{HL}^{\epsilon^*}(q_H^* + q_M^*) - t_{HM} - t_{MH}$$

but $-t_{HM} - t_{MH} = -\Delta V_{HM}^{\epsilon^*} - t_{HL} - t_{LH}$, hence $(CIC_{HL,HM})$ reduces to

$$V_{HM}^{\epsilon^*}(q_H^* + q_M^*) - V_{HM}^{\epsilon^*}(q_H^* + q_L^*) \geq V_{HL}^{\epsilon^*}(q_H^* + q_M^*) - V_{HL}^{\epsilon^*}(q_H^* + q_L^*).$$

$$(CIC_{HL,HH})\ V_{HL}^{\epsilon^*}(q_H^* + q_L^*) - t_{HL} - t_{LH} \geq V_{HL}^{\epsilon^*}(2q_H^*) - 2t_{HH}$$

but $-2t_{HH} = -\Delta V_{HH}^{\epsilon^*} - \Delta V_{HM}^{\epsilon^*} - t_{HL} - t_{LH}$, hence $(CIC_{HL,HH})$ reduces to

$$V_{HH}^{\epsilon^*}(2q_H^*) - V_{HH}^{\epsilon^*}(q_H^* + q_M^*) + V_{HM}^{\epsilon^*}(q_H^* + q_M^*) - V_{HM}^{\epsilon^*}(q_H^* + q_L^*) \geq V_{HL}^{\epsilon^*}(2q_H^*) - V_{HL}^{\epsilon^*}(q_H^* + q_L^*).$$

CIC$_{MM}$ The payoff of MM coalition is $V_{MM}^{\epsilon^*}(2q_M^*) - 2t_{MM}$.

$$(CIC_{MM,HL})\ V_{MM}^{\epsilon^*}(2q_M^*) - 2t_{MM} \geq V_{MM}^{\epsilon^*}(q_H^* + q_L^*) - t_{HL} - t_{LH}$$

but $-t_{HL} - t_{LH} = -\Delta V_{HL}^{\epsilon^*} - 2t_{MM}$, hence $(CIC_{MM,HL})$ reduces to

$$V_{HL}^{\epsilon^*}(q_H^* + q_L^*) - V_{HL}^{\epsilon^*}(2q_M^*) \geq V_{MM}^{\epsilon^*}(q_H^* + q_L^*) - V_{MM}^{\epsilon^*}(2q_M^*). \tag{37}$$

In order to prove that (37) is satisfied, recall that $q_H^* + q_L^* \geq 2q_M^*$ and observe that lemma 2(ii) is not helpful if $\theta_M^v > \theta_L^v$. We start by noticing that $\frac{\partial V_{HL}^{\epsilon^*}(x)}{\partial x} = \theta_H u'[q_H^{\epsilon^*}(x; HL)]$, $\frac{\partial V_{MM}^{\epsilon^*}(x)}{\partial x} = \theta_M^{\epsilon^*} u'(\frac{x}{2})$ and $q_H^{\epsilon^*}(q_H^* + q_L^*; HL) = q_H^*$. Hence, $\frac{\partial V_{HL}^{\epsilon^*}(q_H^* + q_L^*)}{\partial x} = \theta_H u'(q_H^*) = c$ and $\frac{\partial V_{MM}^{\epsilon^*}(2q_M^*)}{\partial x} = \theta_M^v u'(q_M^*) = c$. Moreover, both $\frac{\partial V_{HL}^{\epsilon^*}}{\partial x}$ and $\frac{\partial V_{MM}^{\epsilon^*}}{\partial x}$ are strictly decreasing (because $V_{HL}^{\epsilon^*}$ and $V_{MM}^{\epsilon^*}$ are strictly concave), thus $\frac{\partial V_{HL}^{\epsilon^*}(x)}{\partial x} > c > \frac{\partial V_{MM}^{\epsilon^*}(x)}{\partial x}$ for any $x \in [2q_M^*, q_H^* + q_L^*]$. After integrating these inequalities in $[2q_M^*, q_H^* + q_L^*]$ we conclude that (37) is satisfied.

$$(CIC_{MM,HM})\ V_{MM}^{\epsilon^*}(2q_M^*) - 2t_{MM} \geq V_{MM}^{\epsilon^*}(q_H^* + q_M^*) - t_{HM} - t_{MH}$$

but $-t_{HM} - t_{MH} = -\Delta V_{HM}^{\epsilon^*} - \Delta V_{HL}^{\epsilon^*} - 2t_{MM}$, hence $(CIC_{MM,HM})$ reduces to

$$V_{HM}^{\epsilon^*}(q_H^* + q_M^*) - V_{HM}^{\epsilon^*}(q_H^* + q_L^*) + V_{HL}^{\epsilon^*}(q_H^* + q_L^*) - V_{HL}^{\epsilon^*}(2q_M^*) \geq V_{MM}^{\epsilon^*}(q_H^* + q_M^*) - V_{MM}^{\epsilon^*}(2q_M^*).$$

This inequality holds because $q_M^* \geq q_L^*$, $q_H^* + q_L^* \geq 2q_M^*$, (32) and (37).

$$(CIC_{MM,HH})\ V_{MM}^{\epsilon^*}(2q_M^*) - 2t_{MM} \geq V_{MM}^{\epsilon^*}(2q_H^*) - 2t_{HH}$$

but $-2t_{HH} = -\Delta V_{HH}^{\epsilon^*} - \Delta V_{HM}^{\epsilon^*} - \Delta V_{HL}^{\epsilon^*} - 2t_{MM}$, hence $(CIC_{MM,HH})$ reduces to

$$V_{HH}^{\epsilon^*}(2q_H^*) - V_{HH}^{\epsilon^*}(q_H^* + q_M^*) + V_{HM}^{\epsilon^*}(q_H^* + q_M^*) - V_{HM}^{\epsilon^*}(q_H^* + q_L^*)$$
$$+ V_{HL}^{\epsilon^*}(q_H^* + q_L^*) - V_{HL}^{\epsilon^*}(2q_M^*) \geq V_{MM}^{\epsilon^*}(2q_H^*) - V_{MM}^{\epsilon^*}(2q_M^*).$$

This inequality holds because $q_H^* > q_M^* \geq q_L^*, q_H^* + q_L^* \geq 2q_M^*$, (32) and (37).

CIC$_{ML}$ The payoff of ML coalition is $V^{\epsilon^*}_{ML}(q^*_M + q^*_L) - t_{ML} - t_{LM}$.

$$(\text{CIC}_{ML,MM})\ V^{\epsilon^*}_{ML}(q^*_M + q^*_L) - t_{ML} - t_{LM} \geq V^{\epsilon^*}_{ML}(2q^*_M) - 2t_{MM}$$

but $-2t_{MM} = -\Delta V^{\epsilon^*}_{MM} - t_{ML} - t_{LM} + \alpha$, hence (CIC$_{ML,MM}$) reduces to

$$V^{\epsilon^*}_{MM}(2q^*_M) - V^{\epsilon^*}_{MM}(q^*_M + q^*_L) \geq V^{\epsilon^*}_{ML}(2q^*_M) - V^{\epsilon^*}_{ML}(q^*_M + q^*_L) + \alpha.$$

This inequality holds with equality.

$$(CIC_{ML,HL})\ V^{\epsilon^*}_{ML}(q^*_M + q^*_L) - t_{ML} - t_{LM} \geq V^{\epsilon^*}_{ML}(q^*_H + q^*_L) - t_{HL} - t_{LH}$$

but $-t_{HL} - t_{LH} = -\Delta V^{\epsilon^*}_{HL} - \Delta V^{\epsilon^*}_{MM} - t_{ML} - t_{LM} + \alpha$, hence (CIC$_{ML,HL}$) reduces to

$$V^{\epsilon^*}_{HL}(q^*_H + q^*_L) - V^{\epsilon^*}_{HL}(2q^*_M) + V^{\epsilon^*}_{MM}(2q^*_M) - V^{\epsilon^*}_{MM}(q^*_M + q^*_L) \geq V^{\epsilon^*}_{ML}(q^*_H + q^*_L) - V^{\epsilon^*}_{ML}(q^*_M + q^*_L) + \alpha,$$

which reduces to

$$V^{\epsilon^*}_{HL}(q^*_H + q^*_L) - V^{\epsilon^*}_{HL}(2q^*_M) \geq V^{\epsilon^*}_{ML}(q^*_H + q^*_L) - V^{\epsilon^*}_{ML}(2q^*_M).$$

$$(CIC_{ML,HM})\ V^{\epsilon^*}_{ML}(q^*_M + q^*_L) - t_{ML} - t_{LM} \geq V^{\epsilon^*}_{ML}(q^*_H + q^*_M) - t_{HM} - t_{MH}$$

but $-t_{HM} - t_{MH} = -\Delta V^{\epsilon^*}_{HM} - \Delta V^{\epsilon^*}_{HL} - \Delta V^{\epsilon^*}_{MM} - t_{ML} - t_{LM} + \alpha$, hence (CIC$_{ML,HM}$) reduces to

$$\begin{aligned} & V^{\epsilon^*}_{HM}(q^*_H + q^*_M) - V^{\epsilon^*}_{HM}(q^*_H + q^*_L) + V^{\epsilon^*}_{HL}(q^*_H + q^*_L) - V^{\epsilon^*}_{HL}(2q^*_M) + V^{\epsilon^*}_{MM}(2q^*_M) - V^{\epsilon^*}_{MM}(q^*_M + q^*_L) \\ \geq\ & V^{\epsilon^*}_{ML}(q^*_H + q^*_M) - V^{\epsilon^*}_{ML}(q^*_M + q^*_L) + \alpha, \end{aligned}$$

which reduces to

$$V^{\epsilon^*}_{HM}(q^*_H + q^*_M) - V^{\epsilon^*}_{HM}(q^*_H + q^*_L) + V^{\epsilon^*}_{HL}(q^*_H + q^*_L) - V^{\epsilon^*}_{HL}(2q^*_M) \geq V^{\epsilon^*}_{ML}(q^*_H + q^*_M) - V^{\epsilon^*}_{ML}(2q^*_M).$$

$$(CIC_{ML,HH})\ V^{\epsilon^*}_{ML}(q^*_M + q^*_L) - t_{ML} - t_{LM} \geq V^{\epsilon^*}_{ML}(2q^*_H) - 2t_{HH}$$

but $-2t_{HH} = -\Delta V^{\epsilon^*}_{HH} - \Delta V^{\epsilon^*}_{HM} - \Delta V^{\epsilon^*}_{HL} - \Delta V^{\epsilon^*}_{MM} - t_{ML} - t_{LM} + \alpha$, hence (CIC$_{ML,HH}$) reduces to

$$\begin{aligned} & V^{\epsilon^*}_{HH}(2q^*_H) - V^{\epsilon^*}_{HH}(q^*_H + q^*_M) + V^{\epsilon^*}_{HM}(q^*_H + q^*_M) - V^{\epsilon^*}_{HM}(q^*_H + q^*_L) \\ & + V^{\epsilon^*}_{HL}(q^*_H + q^*_L) - V^{\epsilon^*}_{HL}(2q^*_M) + V^{\epsilon^*}_{MM}(2q^*_M) - V^{\epsilon^*}_{MM}(q^*_M + q^*_L) \\ \geq\ & V^{\epsilon^*}_{ML}(2q^*_H) - V^{\epsilon^*}_{ML}(q^*_M + q^*_L) + \alpha, \end{aligned}$$

which reduces to

$$\begin{aligned} & V^{\epsilon^*}_{HH}(2q^*_H) - V^{\epsilon^*}_{HH}(q^*_H + q^*_M) + V^{\epsilon^*}_{HM}(q^*_H + q^*_M) - V^{\epsilon^*}_{HM}(q^*_H + q^*_L) + V^{\epsilon^*}_{HL}(q^*_H + q^*_L) - V^{\epsilon^*}_{HL}(2q^*_M) \\ \geq\ & V^{\epsilon^*}_{ML}(2q^*_H) - V^{\epsilon^*}_{ML}(2q^*_M). \end{aligned}$$

CIC$_{LL}$ The payoff of LL coalition is $V_{LL}^{\epsilon^*}(2q_L^*) - 2t_{LL}$.

$$(\text{CIC}_{LL,ML})\ V_{LL}^{\epsilon^*}(2q_L^*) - 2t_{LL} \geq V_{LL}^{\epsilon^*}(q_M^* + q_L^*) - t_{ML} - t_{LM}$$

but $-t_{ML} - t_{LM} = -\Delta V_{ML}^{\epsilon^*} - 2t_{LL}$, hence (CIC$_{LL,ML}$) reduces to

$$V_{ML}^{\epsilon^*}(q_M^* + q_L^*) - V_{ML}^{\epsilon^*}(2q_L^*) \geq V_{LL}^{\epsilon^*}(q_M^* + q_L^*) - V_{LL}^{\epsilon^*}(2q_L^*)$$

$$(CIC_{LL,MM})\ V_{LL}^{\epsilon^*}(2q_L^*) - 2t_{LL} \geq V_{LL}^{\epsilon^*}(2q_M^*) - 2t_{MM}$$

but $-2t_{MM} = -\Delta V_{MM}^{\epsilon^*} - \Delta V_{ML}^{\epsilon^*} - 2t_{LL} + \alpha$, hence (CIC$_{LL,MM}$) reduces to

$$V_{MM}^{\epsilon^*}(2q_M^*) - V_{MM}^{\epsilon^*}(q_M^* + q_L^*) + V_{ML}^{\epsilon^*}(q_M^* + q_L^*) - V_{ML}^{\epsilon^*}(2q_L^*) \geq V_{LL}^{\epsilon^*}(2q_M^*) - V_{LL}^{\epsilon^*}(2q_L^*) + \alpha,$$

which reduces to

$$V_{ML}^{\epsilon^*}(2q_M^*) - V_{ML}^{\epsilon^*}(2q_L^*) \geq V_{LL}^{\epsilon^*}(2q_M^*) - V_{LL}^{\epsilon^*}(2q_L^*).$$

$$(CIC_{LL,HL})\ V_{LL}^{\epsilon^*}(2q_L^*) - 2t_{LL} \geq V_{LL}^{\epsilon^*}(q_H^* + q_L^*) - t_{HL} - t_{LH}$$

but $-t_{HL} - t_{LH} = -\Delta V_{HL}^{\epsilon^*} - \Delta V_{MM}^{\epsilon^*} - \Delta V_{ML}^{\epsilon^*} - 2t_{LL} + \alpha$, hence (CIC$_{LL,HL}$) reduces to

$$\begin{aligned} & V_{HL}^{\epsilon^*}(q_H^* + q_L^*) - V_{HL}^{\epsilon^*}(2q_M^*) + V_{MM}^{\epsilon^*}(2q_M^*) - V_{MM}^{\epsilon^*}(q_M^* + q_L^*) + V_{ML}^{\epsilon^*}(q_M^* + q_L^*) - V_{ML}^{\epsilon^*}(2q_L^*) \\ \geq\ & V_{LL}^{\epsilon^*}(q_H^* + q_L^*) - V_{LL}^{\epsilon^*}(2q_L^*) + \alpha, \end{aligned}$$

which reduces to

$$V_{HL}^{\epsilon^*}(q_H^* + q_L^*) - V_{HL}^{\epsilon^*}(2q_M^*) + V_{ML}^{\epsilon^*}(2q_M^*) - V_{ML}^{\epsilon^*}(2q_L^*) \geq V_{LL}^{\epsilon^*}(q_H^* + q_L^*) - V_{LL}^{\epsilon^*}(2q_L^*).$$

$$(CIC_{LL,HM})\ V_{LL}^{\epsilon^*}(2q_L^*) - 2t_{LL} \geq V_{LL}^{\epsilon^*}(q_H^* + q_M^*) - t_{HM} - t_{MH}$$

but $-t_{HM} - t_{MH} = -\Delta V_{HM}^{\epsilon^*} - \Delta V_{HL}^{\epsilon^*} - \Delta V_{MM}^{\epsilon^*} - \Delta V_{ML}^{\epsilon^*} - 2t_{LL} + \alpha$, hence (CIC$_{LL,HM}$) reduces to

$$\begin{aligned} & V_{HM}^{\epsilon^*}(q_H^* + q_M^*) - V_{HM}^{\epsilon^*}(q_H^* + q_L^*) + V_{HL}^{\epsilon^*}(q_H^* + q_L^*) - V_{HL}^{\epsilon^*}(2q_M^*) \\ & + V_{MM}^{\epsilon^*}(2q_M^*) - V_{MM}^{\epsilon^*}(q_M^* + q_L^*) + V_{ML}^{\epsilon^*}(q_M^* + q_L^*) - V_{ML}^{\epsilon^*}(2q_L^*) \\ \geq\ & V_{LL}^{\epsilon^*}(q_H^* + q_M^*) - V_{LL}^{\epsilon^*}(2q_L^*) + \alpha, \end{aligned}$$

which reduces to

$$\begin{aligned}&V^{\epsilon^*}_{HM}(q^*_H+q^*_M)-V^{\epsilon^*}_{HM}(q^*_H+q^*_L)+V^{\epsilon^*}_{HL}(q^*_H+q^*_L)-V^{\epsilon^*}_{HL}(2q^*_M)+V^{\epsilon^*}_{ML}(2q^*_M)-V^{\epsilon^*}_{ML}(2q^*_L)\\ \geq\ &V^{\epsilon^*}_{LL}(q^*_H+q^*_M)-V^{\epsilon^*}_{LL}(2q^*_L).\end{aligned}$$

$$(CIC_{LL,HH})\ V^{\epsilon^*}_{LL}(2q^*_L)-2t_{LL}\geq V^{\epsilon^*}_{LL}(2q^*_H)-2t_{HH}$$

but $-2t_{HH}=-\Delta V^{\epsilon^*}_{HH}-\Delta V^{\epsilon^*}_{HM}-\Delta V^{\epsilon^*}_{HL}-\Delta V^{\epsilon^*}_{MM}-\Delta V^{\epsilon^*}_{ML}-2t_{LL}+\alpha$, hence (CIC$_{LL,HH}$) reduces to

$$\begin{aligned}&V^{\epsilon^*}_{HH}(2q^*_H)-V^{\epsilon^*}_{HH}(q^*_H+q^*_M)+V^{\epsilon^*}_{HM}(q^*_H+q^*_M)-V^{\epsilon^*}_{HM}(q^*_H+q^*_L)+V^{\epsilon^*}_{HL}(q^*_H+q^*_L)-V^{\epsilon^*}_{HL}(2q^*_M)\\ &+V^{\epsilon^*}_{MM}(2q^*_M)-V^{\epsilon^*}_{MM}(q^*_M+q^*_L)+V^{\epsilon^*}_{ML}(q^*_M+q^*_L)-V^{\epsilon^*}_{ML}(2q^*_L)\\ \geq\ &V^{\epsilon^*}_{LL}(2q^*_H)-V^{\epsilon^*}_{LL}(2q^*_L)+\alpha,\end{aligned}$$

which reduces to

$$\begin{aligned}&V^{\epsilon^*}_{HH}(2q^*_H)-V^{\epsilon^*}_{HH}(q^*_H+q^*_M)+V^{\epsilon^*}_{HM}(q^*_H+q^*_M)-V^{\epsilon^*}_{HM}(q^*_H+q^*_L)+V^{\epsilon^*}_{HL}(q^*_H+q^*_L)-V^{\epsilon^*}_{HL}(2q^*_M)\\ &+V^{\epsilon^*}_{ML}(2q^*_M)-V^{\epsilon^*}_{ML}(2q^*_L)\geq V^{\epsilon^*}_{LL}(2q^*_H)-V^{\epsilon^*}_{LL}(2q^*_L).\end{aligned}$$

Proof of Claim 3

Consider the linear system in $\mathbf{t}$ made of $(CIC_{HH,HM})$, $(CIC_{HM,HL})$, $(CIC_{HL,MM})$, $(CIC^{\text{modified}}_{MM,ML})$, $(CIC_{ML,LL})$, (BIC_{HM}), (BIC_{ML}) and (BIR_L) written with equality with $\mathbf{q}=\mathbf{q}^*$. We show that this system admits at least one solution in $\mathbf{t}$. In order to prove this claim, it is not sufficient to observe that the system has eight equations and nine variables. However, we report below the 8×9 matrix A of the unknowns and we can show that – for any probability distribution (p_L,p_M,p_H) – its rank is 8. This claim is proved by finding an 8×8 submatrix of A with nonvanishing determinant. For instance, we can take A after deleting its second column, the one corresponding to t_{HM} and obtain an 8×8 matrix with determinant equal to $-4p_L$.

equation\variable	t_{HH}	t_{HM}	t_{HL}	t_{MH}	t_{MM}	t_{ML}	t_{LH}	t_{LM}	t_{LL}
$(CIC_{HH,MM})$	-2	1	0	1	0	0	0	0	0
$(CIC_{HM,HL})$	0	-1	1	-1	0	0	1	0	0
$(CIC_{HL,MM})$	0	0	-1	0	2	0	-1	0	0
$(CIC^{\text{modified}}_{MM,ML})$	0	0	0	0	-2	1	0	1	0
$(CIC_{ML,LL})$	0	0	0	0	0	-1	0	-1	2
(BIC_{HM})	$-p_H$	$-p_M$	$-p_L$	p_H	p_M	p_L	0	0	0
(BIC_{ML})	0	0	0	$-p_H$	$-p_M$	$-p_L$	p_H	p_M	p_L
(BIR_L)	0	0	0	0	0	0	$-p_H$	$-p_M$	$-p_L$

Since the rank of A is 8, the range (or image) of the function $f(\mathbf{t}) = A\mathbf{t}$ is $\mathbb{R}^8$; hence for any $b \in \mathbb{R}^8$ the linear system $A\mathbf{t} = b$ admits a solution (actually, infinitely many solutions exist). In particular, there exists a solution for our specific linear system.

How to modify Claim 2 if $q_H^* + q_L^* < 2q_M^*$

Claim 2 Let $\alpha \equiv V_{HL}^{\epsilon^*}(q_H^* + q_L^*) - V_{HL}^{\epsilon^*}(q_M^* + q_L^*) - [V_{ML}^{\epsilon^*}(q_H^* + q_L^*) - V_{ML}^{\epsilon^*}(q_M^* + q_L^*)] > 0$. If $\mathbf{t}^{**}$ is such that the following local downward coalition incentive constraints bind, then all the coalition incentive constraints are satisfied by mechanism $\{\mathbf{q}^*, \mathbf{t}^{**}\}$ when $\epsilon = \epsilon^*$.

$$\begin{aligned}
V_{HH}^{\epsilon^*}(2q_H^*) - 2t_{HH} &\geq V_{HH}^{\epsilon^*}(q_H^* + q_M^*) - t_{HM} - t_{MH} \ (CIC_{HH,HM}) \\
V_{HM}^{\epsilon^*}(q_H^* + q_M^*) - t_{HM} - t_{MH} &\geq V_{HM}^{\epsilon^*}(2q_M^*) - 2t_{MM} \ (CIC_{HM,MM}) \\
V_{MM}^{\epsilon^*}(2q_M^*) - 2t_{MM} &\geq V_{MM}^{\epsilon^*}(q_H^* + q_L^*) - t_{HL} - t_{LH} \ (CIC_{MM,HL}) \\
V_{HL}^{\epsilon^*}(q_H^* + q_L^*) - t_{HL} - t_{LH} &\geq V_{HL}^{\epsilon^*}(q_M^* + q_L^*) - t_{ML} - t_{LM} + \alpha \ (CIC_{HL,ML}^{\text{modified}}) \\
V_{ML}^{\epsilon^*}(q_M^* + q_L^*) - t_{ML} - t_{LM} &\geq V_{ML}^{\epsilon^*}(2q_L^*) - 2t_{LL} \ (CIC_{ML,LL})
\end{aligned}$$

The proof is very similar to the previous one and therefore it is omitted.

Proof of Proposition 12

Step 1 No reallocation occurs if $\varepsilon = 1$, conditional on no manipulation of reports.

Proof. In this setting, $q_H^\varepsilon(x)$ is not defined as in (5) but as follows: $q_H^\varepsilon(x) \equiv \arg\max_{z\in[0,x]} \mathcal{U}(z, \theta_H) + \mathcal{U}(x-z, \theta_L) - \frac{1-p_L}{p_L}\varepsilon[\mathcal{U}(x-z, \theta_H) - \mathcal{U}(x-z, \theta_L)]$. When $\varepsilon = 1$ and $x = q_{HL}^* + q_{LH}^*$, this function is maximized at $z = q_{HL}^*$ since, by definition, (q_{HL}^*, q_{LH}^*) maximize $\mathcal{U}(q_{HL}, \theta_H) + \mathcal{U}(q_{LH}, \theta_L) - \frac{1-p_L}{p_L}[\mathcal{U}(q_{LH}, \theta_H) - \mathcal{U}(q_{LH}, \theta_L)]$ under the constraint $q_{HL} + q_{LH} = q_{HL}^* + q_{LH}^*$; hence, no reallocation occurs when $\varepsilon = 1$ if there is no manipulation of reports. ■

In order to deal with the coalition incentive constraints we define $V_2(x) \equiv 2\mathcal{U}(\frac{x}{2}, \theta_H)$, $V_1^\varepsilon(x) \equiv \max_{z\in[0,x]} \mathcal{U}(z, \theta_H) + \mathcal{U}(x-z, \theta_L) - \frac{1-p_L}{p_L}\varepsilon[\mathcal{U}(x-z, \theta_H) - \mathcal{U}(x-z, \theta_L)]$ and $V_0^\varepsilon(x) \equiv 2[\mathcal{U}(\frac{x}{2}, \theta_L) - \frac{(1-p_L)\varepsilon[\mathcal{U}(\frac{x}{2},\theta_H) - \mathcal{U}(\frac{x}{2},\theta_L)]}{p_L}]$.

Step 2 $\frac{\partial V_2(x)}{\partial x} > \max\{\frac{\partial V_1^\epsilon(x)}{\partial x}, \frac{\partial V_0^\epsilon(x)}{\partial x}\}$ and $2q_{HH}^* > \max\{q_{HL}^* + q_{LH}^*, 2q_{LL}^*\}$.

Proof. We find $\frac{\partial V_2(x)}{\partial x} = \mathcal{U}_1(\frac{x}{2}, \theta_H)$, $\frac{\partial V_1^\epsilon(x)}{\partial x} = \mathcal{U}_1[q_H^\varepsilon(x), \theta_H] = \mathcal{U}_1(q_L^\varepsilon(x), \theta_L) - \frac{1-p_L}{p_L}\varepsilon[\mathcal{U}_1(q_L^\varepsilon(x), \theta_H) - \mathcal{U}_1(q_L^\varepsilon(x), \theta_L)]$ and $\frac{\partial V_0^\epsilon(x)}{\partial x} = \mathcal{U}_1(\frac{x}{2}, \theta_L) - \frac{1-p_L}{p_L}\varepsilon[\mathcal{U}_1(\frac{x}{2}, \theta_H) - \mathcal{U}_1(\frac{x}{2}, \theta_L)]$. Furthermore, $q_H^\varepsilon(x) > \frac{x}{2} > q_L^\epsilon(x)$ because the function $\mathcal{U}(z, \theta_H) + \mathcal{U}(x-z, \theta_L) - \frac{1-p_L}{p_L}\varepsilon[\mathcal{U}(x-z, \theta_H) - \mathcal{U}(x-z, \theta_L)]$ is strictly increasing in z for $z \in [0, \frac{x}{2}]$; this implies $\frac{\partial V_2(x)}{\partial x} > \max\{\frac{\partial V_1^\epsilon(x)}{\partial x}, \frac{\partial V_0^\epsilon(x)}{\partial x}\}$. We have $2q_{HH}^* > q_{HL}^* + q_{LH}^*$ because (i) from the first order conditions for q_{HL}^* and q_{LH}^* it is straightforward to see that $q_{HL}^* > q_{LH}^*$; (ii) the first order conditions for q_{HH}^* and q_{HL}^* are $\mathcal{U}_1(q_{HH}^*, \theta_H) = C'(2q_{HH}^*)$ and $\mathcal{U}_1(q_{HL}^*, \theta_H) = C'(q_{HL}^* + q_{LH}^*)$, respectively; thus $2q_{HH}^* \leq q_{HL}^* + q_{LH}^*$ leads to the contradiction $q^*_{HH} \geq q^*_{HL} > q^*_{LH}$. From the first order conditions for q^*_{HH} and q^*_{LL} it is straightforward to see that $q^*_{HH} > q^*_{LL}$. ■

Step 3 Suppose $q^*_{HL}+q^*_{LH} \geq 2q^*_{LL}$ and let $\mathbf{t}^{**}$ be such that (BIR_L), (BIC_H), $(CIC_{HH,HL})$ and $(CIC_{HL,LL})$ bind when $\mathbf{q} = \mathbf{q}^*$ and $\varepsilon = 1$. Then $\{\mathbf{q}^*, \mathbf{t}^{**}\}$ satisfies all the coalition incentive constraints.

Proof. Given that $(CIC_{HH,HL})$ and $(CIC_{HL,LL})$ bind, $(CIC_{HL,HH})$ reduces to $V_2(2q^*_{HH}) - V_2(q^*_{HL} + q^*_{LH}) \geq V_1^1(2q^*_{HH}) - V_1^1(q^*_{HL} + q^*_{LH})$, which holds since $2q^*_{HH} > q^*_{HL} + q^*_{LH}$ and $\frac{\partial V_2(x)}{\partial x} > \frac{\partial V_1^1(x)}{\partial x}$. $(CIC_{LL,HL})$ reduces to

$$V_1^1(q^*_{HL} + q^*_{LH}) - V_1^1(2q^*_{LL}) \geq V_0^1(q^*_{HL} + q^*_{LH}) - V_0^1(2q^*_{LL}) \tag{38}$$

we now prove that this inequality holds. Examining the seller's profit function after using (BIR_L) and (BIC_H) written with equality we see that (q^*_{HL}, q^*_{LH}) maximize $\mathcal{U}(q_{HL}, \theta_H) + \frac{\mathcal{U}(q_{LH},\theta_L)}{p_L} - \frac{1-p_L}{p_L}\mathcal{U}(q_{LH}, \theta_H) - C(q_{HL}+q_{LH})$ and q^*_{LL} maximizes $2\frac{\mathcal{U}(q_{LL},\theta_L)}{p_L} - \frac{2(1-p_L)}{p_L}\mathcal{U}(q_{LL}, \theta_H) - C(2q_{LL})$; the maximized values are equal to $V_1^1(q^*_{HL} + q^*_{LH}) - C(q^*_{HL} + q^*_{LH})$ and $V_0^1(2q^*_{LL}) - C(2q^*_{LL})$, respectively. Revealed preferences imply $V_1^1(q^*_{HL}+q^*_{LH}) - C(q^*_{HL}+q^*_{LH}) \geq V_1^1(2q^*_{LL}) - C(2q^*_{LL})$ and $V_0^1(2q^*_{LL}) - C(2q^*_{LL}) \geq V_0^1(q^*_{HL} + q^*_{LH}) - C(q^*_{HL} + q^*_{LH})$; thus, (38) is satisfied. $(CIC_{HH,LL})$ reduces to $V_2(q^*_{HL} + q^*_{LH}) - V_2(2q^*_{LL}) \geq V_1^1(q^*_{HL} + q^*_{LH}) - V_1^1(2q^*_{LL})$, which holds since $q^*_{HL} + q^*_{LH} \geq 2q^*_{LL}$ and $\frac{\partial V_2(x)}{\partial x} > \frac{\partial V_1^1(x)}{\partial x}$. $(CIC_{LL,HH})$ reduces to $V_2(2q^*_{HH}) - V_2(q^*_{HL} + q^*_{LH}) + V_1^1(q^*_{HL} + q^*_{LH}) - V_1^1(2q^*_{LL}) \geq V_0^1(2q^*_{HH}) - V_0^1(2q^*_{LL})$, which holds in view of $2q^*_{HH} > q^*_{HL} + q^*_{LH} \geq 2q^*_{LL}$, $\frac{\partial V_2(x)}{\partial x} > \frac{\partial V_0^1(x)}{\partial x}$ and (38). ■

Step 4 Suppose $q^*_{HL}+q^*_{LH} < 2q^*_{LL}$ and let $\mathbf{t}^{***}$ be such that (BIR_L), (BIC_H), $(CIC_{HH,LL})$ and $(CIC_{LL,HL})$ bind when $\mathbf{q} = \mathbf{q}^*$ and $\varepsilon = 1$. Then $\{\mathbf{q}^*, \mathbf{t}^{***}\}$ satisfies all the coalition incentive constraints.

Proof. Given that $(CIC_{HH,LL})$ and $(CIC_{LL,HL})$ bind, $(CIC_{LL,HH})$ reduces to $V_2(2q^*_{HH}) - V_2(2q^*_{LL}) \geq V_0^1(2q^*_{HH}) - V_0^1(2q^*_{LL})$, which holds since $q^*_{HH} > q^*_{LL}$ and $\frac{\partial V_2(x)}{\partial x} > \frac{\partial V_0^1(x)}{\partial x}$. $(CIC_{HL,LL})$ is equivalent to (38), whose proof does not depend on whether $q^*_{HL}+q^*_{LH} \geq 2q^*_{LL}$ or $q^*_{HL}+q^*_{LH} < 2q^*_{LL}$. $(CIC_{HH,HL})$ reduces to $V_2(2q^*_{LL}) - V_2(q^*_{HL}+q^*_{LH}) \geq V_0^1(2q^*_{LL}) - V_0^1(q^*_{HL} + q^*_{LH})$, which holds since $q^*_{HL} + q^*_{LH} < 2q^*_{LL}$ and $\frac{\partial V_2(x)}{\partial x} > \frac{\partial V_0^1(x)}{\partial x}$. Finally, $(CIC_{HL,HH})$ reduces to $V_2(2q^*_{HH}) - V_2(2q^*_{LL}) + V_0^1(2q^*_{LL}) - V_0^1(q^*_{HL} + q^*_{LH}) \geq V_1^1(2q^*_{HH}) - V_1^1(q^*_{HL} + q^*_{LH})$, which is satisfied in view of $q^*_{HH} > q^*_{LL}$, $\frac{\partial V_2(x)}{\partial x} > \frac{\partial V_1^1(x)}{\partial x}$ and (38). ■

Proof of Proposition 13

We start by establishing the following inequality

$$t^{**}_{LH} - t^{**}_{LL} > t^{**}_{HH} - t^{**}_{HL} \tag{39}$$

Recalling that $(CIC_{HH,HL})$ and $(CIC_{HL,LL})$ bind in the transfer scheme $\mathbf{t}^{**}$, we find

$$\begin{aligned} t_{HH}^{**} &= \frac{V_2(2q_H^*) - V_2(q_H^* + q_L^*)}{2} + \frac{t_{LH}^{**} + t_{HL}^{**}}{2} \\ t_{LL}^{**} &= \frac{V_1^1(2q_L^*) - V_1^1(q_H^* + q_L^*)}{2} + \frac{t_{LH}^{**} + t_{HL}^{**}}{2} \end{aligned}$$

hence (39) is equivalent to

$$V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*) > V_2(2q_H^*) - V_2(q_H^* + q_L^*) \tag{40}$$

Define $g(z) \equiv V_1^1(2q_L^* + z) - V_1^1(2q_L^*) - [V_2(q_H^* + q_L^* + z) - V_2(q_H^* + q_L^*)]$ and notice that $g(0) = 0$ while (40) is equivalent to $g(q_H^* - q_L^*) > 0$. Now we prove that $g'(z) > 0$ if $z \in (0, q_H^* - q_L^*)$, hence $g(q_H^* - q_L^*) > 0$. We find

$$g'(z) = \theta_H u'[q_H^1(2q_L^* + z)] - \theta_H u'(\frac{q_H^* + q_L^* + z}{2})$$

Thus, $g'(z) > 0$ if and only if $q_H^1(2q_L^* + z) < \frac{q_H^* + q_L^* + z}{2}$. In order to establish the latter inequality we apply the implicit function theorem to $\theta_H u'[q_H^1(x)] = \theta_L^1 u'[x - q_H^1(x)]$ [recall that $q_L^1(x) = x - q_H^1(x)$] and obtain

$$\frac{dq_H^1(x)}{dx} = \frac{\theta_L^1 u''[q_L^1(x)]}{\theta_H u''[q_H^1(x)] + \theta_L^1 u''[q_L^1(x)]} = \frac{1}{\frac{\theta_H u''[q_H^1(x)]}{\theta_L^1 u''[q_L^1(x)]} + 1}$$

Moreover, using again $\theta_H u'[q_H^1(x)] = \theta_L^1 u'[q_L^1(x)]$, we find $\frac{\theta_H u''[q_H^1(x)]}{\theta_L^1 u''[q_L^1(x)]} = \frac{u'[q_L^1(x)]u''[q_H^1(x)]}{u'[q_H^1(x)]u''[q_L^1(x)]}$ and the assumption that $\frac{u''}{u'}$ is strictly increasing implies $\frac{u'[q_L^1(x)]u''[q_H^1(x)]}{u'[q_H^1(x)]u''[q_L^1(x)]} < 1$. Hence, $\frac{dq_H^1(x)}{dx} > \frac{1}{2}$ and $q_H^1(q_H^* + q_L^*) - q_H^1(2q_L^* + z) > \frac{q_H^* - q_L^* - z}{2}$. Since $q_H^1(q_H^* + q_L^*) = q_H^*$, the latter condition is equivalent to $\frac{q_H^* + q_L^* + z}{2} > q_H^1(2q_L^* + z)$, the inequality which implies $g'(z) > 0$ for $z \in (0, q_H^* - q_L^*)$.

(a)-1. The proof of "reporting L is strictly dominant for each L-type".

It is useful to write down the payoff matrices in M^{**} for L-type and H-type, respectively. For example, $\theta_L u(q_H^*) - t_{HL}^{**}$, the entry in the left table below corresponding to row H and column L, is the payoff to L-type if he claims H and his opponent reports L.

L-type	L	H
L	$\theta_L u(q_L^*) - t_{LL}^{**}$	$\theta_L u(q_L^*) - t_{LH}^{**}$
H	$\theta_L u(q_H^*) - t_{HL}^{**}$	$\theta_L u(q_H^*) - t_{HH}^{**}$

H-type	L	H
L	$\theta_H u(q_L^*) - t_{LL}^{**}$	$\theta_H u(q_L^*) - t_{LH}^{**}$
H	$\theta_H u(q_H^*) - t_{HL}^{**}$	$\theta_H u(q_H^*) - t_{HH}^{**}$

"Reporting L is strictly dominant for each L-type" is equivalent to

$$\theta_L u(q_L^*) - t_{LL}^{**} > \theta_L u(q_H^*) - t_{HL}^{**} \quad (41)$$

$$\theta_L u(q_L^*) - t_{LH}^{**} > \theta_L u(q_H^*) - t_{HH}^{**} \quad (42)$$

We first show that (42) holds and then prove (41). In view of the expressions for $\mathbf{t}^{**}$, (42) is equivalent to

$$[\theta_L - (1+p_L)\theta_H]u(q_L^*) + [(2+p_L)\theta_H - 2\theta_L]u(q_H^*) > 2p_L\theta_H u(\frac{q_H^* + q_L^*}{2}) - p_L V_1^1(2q_L^*)$$

The definition of V_1^1 and the strict concavity of u imply that $V_1^1(2q_L^*) > (\theta_H + \theta_L^1)u(q_L^*)$ and $u(\frac{q_H^*+q_L^*}{2}) < u(q_H^*) - \frac{q_H^*-q_L^*}{2}u'(q_H^*) = u(q_H^*) - \frac{q_H^*-q_L^*}{2}\frac{c}{\theta_H}$. Hence, it is sufficient to prove that

$$[2\theta_L - (2-p_L)\theta_H]u(q_L^*) + [(2+p_L)\theta_H - 2\theta_L]u(q_H^*) \geq 2p_L\theta_H[u(q_H^*) - \frac{q_H^* - q_L^*}{2}\frac{c}{\theta_H}],$$

which reduces to

$$[(2-p_L)\theta_H - 2\theta_L][u(q_H^*) - u(q_L^*)] + p_L c(q_H^* - q_L^*) \geq 0$$

If $(2-p_L)\theta_H - 2\theta_L \geq 0$, then we are done. If instead $(2-p_L)\theta_H - 2\theta_L < 0$, then we use again the strict concavity of u to write $u(q_H^*) - u(q_L^*) < u'(q_L^*)(q_H^* - q_L^*) = \frac{c}{\theta_L^1}(q_H^* - q_L^*)$. We obtain $[(2-p_L)\theta_H - 2\theta_L]\frac{c}{\theta_L^1}(q_H^* - q_L^*) + p_L c(q_H^* - q_L^*) > 0$, which is easy to verify.

In order to prove (41), simply observe that it is obtained by adding $t_{LH}^{**} - t_{LL}^{**}$ and $t_{HH}^{**} - t_{HL}^{**}$ to the left and the right hand side of (42), respectively. Since the latter holds, (39) implies that (41) is satisfied as well.

(a)-2. The proof of "each H-type strictly prefers to report H (L) if his opponent plays H (L)".

By observing the right payoff matrix above we find that "each H-type strictly prefers to report H (L) if his opponent plays H (L)" is equivalent to

$$\theta_H u(q_L^*) - t_{LL}^{**} > \theta_H u(q_H^*) - t_{HL}^{**} \quad \text{and} \quad \theta_H u(q_L^*) - t_{LH}^{**} < \theta_H u(q_H^*) - t_{HH}^{**} \quad (43)$$

These inequalities are proved as follows. If we had $\theta_H u(q_L^*) - t_{LH}^{**} \geq \theta_H u(q_H^*) - t_{HH}^{**}$, then (39) would imply $\theta_H u(q_L^*) - t_{LL}^{**} > \theta_H u(q_H^*) - t_{HL}^{**}$ and (BIC_H) would be violated: contradiction. Hence, $\theta_H u(q_L^*) - t_{LH}^{**} < \theta_H u(q_H^*) - t_{HH}^{**}$; since (BIC_H) binds, we infer $\theta_H u(q_L^*) - t_{LL}^{**} > \theta_H u(q_H^*) - t_{HL}^{**}$.

(b) Since $t_{LL}^{**} > t_{LH}^{**}$ and $t_{HL}^{**} > t_{HH}^{**}$, buyer 1, for instance (regardless of his type), has a chance to be better off with respect to the truthtelling equilibrium only if his opponent plays

H more often than under truthtelling. However, this cannot occur in any equilibrium of M^{**} – regardless of buyer 2's beliefs about θ^1 - since reporting L is strictly dominant for L-type of buyer 2. Hence, in any equilibrium of M^{**} the probability that 2 reports H is at most equal to the probability that 2 reports H under truthtelling.

(c) Each L-type reports L in any equilibrium of M^{**}. Consider the payoff bimatrix (in Section 7) of the game played by 1_H and 2_H (actually, t_{jk} should be replaced by t^{**}_{jk}, for any jk). That game has two equilibria, since report L weakly dominates H both for 1_H and for 2_H. In one of them, both 1_H and 2_H play H; in the other one, both 1_H and 2_H play L. In the latter equilibrium, the payoff of $j-$type is $\theta_j u(q^*_L) - t^{**}_{LL}$, $j = L, H$. From (15) we know that $\theta_L u(q^*_L) - t^{**}_{LL} < 0$ and $\theta_H u(q^*_L) - t^{**}_{LL} < (\Delta\theta)u(q^*_L)$ because $p_L[\theta_H u(q^*_L) - t^{**}_{LL}] + (1 - p_L)[\theta_H u(q^*_L) - t^{**}_{LH}] = (\Delta\theta)u(q^*_L)$ and $t^{**}_{LL} > t^{**}_{LH}$. Thus, the untruthful equilibrium is strictly Pareto dominated by truthtelling.

Proof of Proposition 14

(a) M^R is optimal under no coalition formation since $\mathbf{q}^R = \mathbf{q}^*$ and (BIC_H) and (BIR_L) bind. In order to show that M^R is weakly collusion-proof, notice that no reallocation occurs if $\epsilon = 1$ since $\mathbf{q}^R = \mathbf{q}^*$, hence we need to prove that all coalition incentive constraints are satisfied by M^R when $\epsilon = 1$.

First observe that we need to take care only of local (upward and downward) coalition incentive constraints. Indeed, both $(CIC_{HH,LL})$ and $(CIC_{LL,HH})$ are automatically satisfied if all the other coalition incentive constraints hold, thanks to the single crossing condition. To prove this claim, suppose that $(CIC_{HH,HL})$, $(CIC_{HL,HH})$, $(CIC_{HL,LL})$ and $(CIC_{LL,HL})$ are all satisfied. Then, add up $(CIC_{HH,HL})$ and $(CIC_{HL,LL})$ to find $V_2(2q^*_H) - 2t_{HH} \geq V_2(q^*_H+q^*_L)-V^1_1(q^*_H+q^*_L)+V^1_1(2q^*_L)-2t_{LL}$; since $V_2(q^*_H+q^*_L)-V^1_1(q^*_H+q^*_L)+V^1_1(2q^*_L) > V_2(2q^*_L)$ by single crossing, we obtain $V_2(2q^*_H)-2t_{HH} > V_2(2q^*_L)-2t_{LL}$. Thus, $(CIC_{HH,LL})$ is satisfied. About $(CIC_{LL,HH})$, add up $(CIC_{LL,HL})$ and $(CIC_{HL,HH})$ to obtain $V^1_0(2q^*_L)-2t_{LL} \geq V^1_0(q^*_H+q^*_L) - V^1_1(q^*_H+q^*_L)+V^1_1(2q^*_H)-2t_{HH} > V^1_0(2q^*_H)-2t_{HH}$ by single crossing; hence $(CIC_{LL,HH})$ is satisfied. Therefore, we take care only of $(CIC_{HH,HL})$, $(CIC_{HL,HH})$, $(CIC_{HL,LL})$ and $(CIC_{LL,HL})$.

From (BIR_L), (BIC_H) and (20) written with equality we obtain

$$t_{LL} = \theta_H[u(q^*_L) - u(q^*_H)] + \alpha + t_{HL} \qquad t_{HH} = \frac{\theta_H u(q^*_H) - (\Delta\theta)u(q^*_L) - p_L t_{HL}}{1-p_L}$$

$$t_{LH} = \frac{\theta_L u(q^*_L) + p_L\theta_H[u(q^*_H) - u(q^*_L)] - p_L\alpha - p_L t_{HL}}{1-p_L}$$

We substitute these expressions into the local coalition incentive constraints – after letting $K \equiv (2-p_L)\theta_H u(q^*_H) + [\theta_L - (2-p_L)\theta_H]u(q^*_L)$ – to find that $(CIC_{HH,HL})$ and $(CIC_{HL,HH})$

are equivalent to

$$\begin{aligned} & K + p_L\alpha - (1-p_L)[V_2(2q_H^*) - V_2(q_H^* + q_L^*)] \\ & \leq t_{HL} \leq K + p_L\alpha - (1-p_L)[V_1^1(2q_H^*) - V_1^1(q_H^* + q_L^*)] \end{aligned} \tag{44}$$

while $(CIC_{HL,LL})$ and $(CIC_{LL,HL})$ are equivalent to

$$\begin{aligned} & K - (2-p_L)\alpha - (1-p_L)[V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)] \\ & \leq t_{HL} \leq K - (2-p_L)\alpha - (1-p_L)[V_0^1(q_H^* + q_L^*) - V_0^1(2q_L^*)] \end{aligned} \tag{45}$$

If $V_2(2q_H^*) - V_2(q_H^* + q_L^*) < V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)$, then we set t_{HL} so that $(CIC_{HH,HL})$ binds. We want to prove that the other local coalition incentive constraints hold if $\alpha > 0$ is small. For this purpose, first we show that they are strictly satisfied when $\alpha = 0$ and then argue by continuity. $(CIC_{HL,HH})$ is strictly satisfied because of single crossing [see (44)], while $(CIC_{HL,LL})$ is equivalent to $V_2(2q_H^*) - V_2(q_H^* + q_L^*) \leq V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)$ – which strictly holds by hypothesis – and $(CIC_{LL,HL})$ reduces to $V_2(2q_H^*) - V_2(q_H^* + q_L^*) - [V_0^1(q_H^* + q_L^*) - V_0^1(2q_L^*)] \geq 0$. In order to establish that the latter inequality holds strictly, define $g(z) \equiv V_2(q_H^* + q_L^* + z) - V_2(q_H^* + q_L^*) - [V_0^1(2q_L^* + z) - V_0^1(2q_L^*)]$; we want to prove that $g(q_H^* - q_L^*) > 0$. Observe that $g(0) = 0$ and $g'(z) = \theta_H u'(\frac{q_H^*+q_L^*+z}{2}) - \theta_L^1 u'(q_L^* + \frac{z}{2}) > 0$ because $\theta_H u'(\frac{q_H^*+q_L^*+z}{2}) > c > \theta_L^1 u'(q_L^* + \frac{z}{2})$ for any $z \in [0, q_H^* - q_L^*)$. Here transfers are found by solving the linear system made up of (BIR_L), (BIC_H), $(CIC_{HH,HL})$ and (20), all written with equality:

$$\begin{aligned} t_{HL}^R &= (p_L\theta_L^1 - \theta_H)u(q_L^*) + p_L\theta_H u(q_H^*) + 2(1-p_L)\theta_H u(\frac{q_H^* + q_L^*}{2}) + p_L\alpha \\ t_{LH}^R &= (\theta_L + p_L\theta_H)u(q_L^*) + p_L\theta_H u(q_H^*) - 2p_L\theta_H u(\frac{q_H^* + q_L^*}{2}) - \frac{p_L(1+p_L)}{1-p_L}\alpha \\ t_{HH}^R &= p_L\theta_L^1 u(q_L^*) + (1+p_L)\theta_H u(q_H^*) - 2p_L\theta_H u(\frac{q_H^* + q_L^*}{2}) - \frac{p_L^2}{1-p_L}\alpha \\ t_{LL}^R &= p_L\theta_L^1 u(q_L^*) - (1-p_L)\theta_H u(q_H^*) + 2(1-p_L)\theta_H u(\frac{q_H^* + q_L^*}{2}) + (1+p_L)\alpha \end{aligned}$$

If $V_2(2q_H^*) - V_2(q_H^* + q_L^*) \geq V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)$, then we set

$$t_{HL}^R = K - (2-p_L)\alpha - (1-p_L)[V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)] + \beta$$

with $\beta > 0$ and small so that $(CIC_{HL,LL})$ is slightly slack. We now show that the other local coalition incentive constraints are strictly satisfied when $\alpha = 0$, hence they are still so if $\alpha > 0$ is small. $(CIC_{LL,HL})$ is strictly satisfied because of single crossing [see (45)], while $(CIC_{HH,HL})$ is equivalent to $V_2(2q_H^*) - V_2(q_H^* + q_L^*) + \frac{\beta}{1-p_L} \geq V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*) -$

which holds strictly by assumption – and $(CIC_{HL\ HH})$ reduces to $V_1^1(q_H^*+q_L^*)-V_1^1(2q_L^*)-[V_1^1(2q_H^*)-V_1^1(q_H^*+q_L^*)]-\frac{\beta}{1-p_L}\geq 0$. The latter inequality holds strictly because of the following argument. Define $g(z)\equiv V_1^1(2q_L^*+z)-V_1^1(2q_L^*)-[V_1^1(q_H^*+q_L^*+z)-V_1^1(q_H^*+q_L^*)]$ and notice that $g(0)=0$. Moreover, $g'(z)=\theta_H u'[q_H^1(2q_L^*+z)]-\theta_H u'[q_H^1(q_H^*+q_L^*+z)]>0$ because $q_H^1(2q_L^*+z)<q_H^1(q_H^*+q_L^*+z)$ for any $z\in[0,q_H^*-q_L^*]$. Hence $g(q_H^*-q_L^*)>\frac{\beta}{1-p_L}$ since $\beta>0$ is small. In this case transfers are found by solving the linear system made up of (BIR_L), (BIC_H), $(CIC_{HL,LL}^{\beta})$ and (20), all written with equality:

$$
\begin{aligned}
t_{HL}^R &= (2-p_L)\theta_H u(q_H^*)+[\theta_L-(2-p_L)\theta_H]u(q_L^*)-(1-p_L)[V_1^1(q_H^*+q_L^*)-V_1^1(2q_L^*)] \\
&\quad -(2-p_L)\alpha+\beta \\
t_{LL}^R &= (1-p_L)\theta_H u(q_H^*)+[\theta_L-(1-p_L)\theta_H]u(q_L^*)-(1-p_L)[V_1^1(q_H^*+q_L^*)-V_1^1(2q_L^*)] \\
&\quad -(1-p_L)\alpha+\beta \\
t_{HH}^R &= p_L\theta_L^1 u(q_L^*)+(1-p_L)\theta_H u(q_H^*)+p_L[V_1^1(q_H^*+q_L^*)-V_1^1(2q_L^*)] \\
&\quad +\frac{p_L(2-p_L)}{1-p_L}\alpha-\frac{p_L}{1-p_L}\beta \\
t_{LH}^R &= (\theta_L+p_L\theta_H)u(q_L^*)-p_L\theta_H u(q_H^*)+p_L[V_1^1(q_H^*+q_L^*)-V_1^1(2q_L^*)] \\
&\quad +p_L\alpha-\frac{p_L}{1-p_L}\beta
\end{aligned}
$$

(b) Since $t_{LL}^R>t_{LH}^R$ and $t_{HL}^R>t_{HH}^R$ by (15), we can apply exactly the same arguments of the proof of Proposition 13(b).

(c) Consider $\mathbf{t}^R$ with $\alpha=0$. Then, by (20) and since (BIC_H) binds, each H-type is indifferent between reporting H or L, regardless of the report of the opponent. If $\alpha>0$ is small, then from (20) we infer that H-type strictly prefers reporting H if his opponent plays L; since (BIC_H) binds, he strictly prefers reporting L when his opponent plays H. About L-type, he strictly prefers reporting L when his opponent plays H because $\theta_H u(q_L^*)-t_{LH}^R>\theta_H u(q_H^*)-t_{HH}^R$ implies $\theta_L u(q_L^*)-t_{LH}^R>\theta_L u(q_H^*)-t_{HH}^R$. Furthermore, he strictly prefers reporting L when his opponent plays L because (20) implies $\theta_L u(q_L^*)-t_{LL}^R>\theta_L u(q_H^*)-t_{HL}^R$ when $\alpha=0$ or $\alpha>0$ is small.

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