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Nonlinear Taxation, Tax Arbitrage and Equilibrium Asset Prices

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Abstract

This paper investigates the equilibrium implications of the presence of nonlinearly taxed, redundant securities, and of the resulting *tax arbitrage* opportunities. Heterogeneity in taxation leads to discrepancies in assets' pre-tax market prices of risk. We show that this *mispricing* is set so that agents effectively cooperate to minimize aggregate taxes, even though individually each agent may not minimize his own tax bill. Unlike the bulk of the existing tax arbitrage literature, but consistent with empirical evidence, equilibrium in our model allows discrepancy between agents' marginal tax rates. Equilibrium with a zero net supply derivative reveals financial innovation to alleviate taxation, in particular when the derivative is taxed linearly or is taxed less heterogeneously across agents than is the stock itself. In the presence of two redundant, positive supply securities, *clientele* effects arise, where one agent holds the aggregate supply of each risky security, and only the bond is traded across agents. Clientele effects are shown to arise when agents' tax rates are highly heterogeneous and when the aggregate wealth is divided fairly evenly across agents.

1. Introduction

The presence of heterogeneously taxed, redundant securities is a common occurrence in financial markets. For example: derivative securities may be subject to tax rules different from the underlying asset; some assets (e.g., municipal bonds) are exempt from some taxes (city and state taxes, vs. federal taxes); or the same asset may be treated differently depending on an agent's purpose in holding it (e.g., retirement investing). Financial economists have long recognized that heterogeneous taxation of redundant securities can provide agents with opportunities for *tax arbitrage*, where agents may be able to adjust their tax bill without affecting their risk exposure. There has been substantial work to determine conditions (on tax rules in particular) under which tax arbitrage is compatible with equilibrium (e.g., Schaefer (1982), Dammon and Green (1987)). This is the case when agents face *local* arbitrage opportunities (a notion formalized by Ross (1987)); once these have been exploited up to some (bounded) magnitude the tax treatment of the assets is symmetrized and the opportunity disappears, hence equilibrium is possible. Although its presence is well-recognized, little is known about the general equilibrium implications of tax arbitrage (though Schaefer (1982) emphasizes the importance of such analysis). Some intuition exists that agents will tend to "cooperate" to reduce taxes. For example, Samuelson (1964) states that "rich men will buy bonds at a discount from poor men [...]. Such a tax swap involves the purchase by the rich of the low tax base of the poor, at the expense of the government". However, this notion has hardly been formalized.

Our objective is to study the effects of tax arbitrage on prices and allocations, in a general equilibrium framework. A general analysis of tax arbitrage requires a fairly complex environment, leading to a formidable problem, which this paper starts to investigate. We do not incorporate all the nuances of a realistic tax code, and although explicit representation is taken as far as possible, endogeneous quantities subsist in some equilibrium expressions. Our emphasis is on the characterization and implications of equilibrium, while assuming the existence of optimal policies and equilibrium. Our environment is necessarily complex in several respects: redundant securities must be incorporated, to allow any notion of arbitrage; while nonlinear taxation must be assumed, to limit the arbitrage opportunities. Furthermore, a continuous-time framework is employed for several reasons: to provide tractability; to allow comparison with an abundance of benchmark models; and to circumvent (Basak and Gallmeyer (1998)) an issue identified by Dammon and Green (1987), that in a single-period finite-state setting, equilibrium is precluded when taxation is sufficiently heterogeneous across agents.

We work in a continuous time, pure exchange setting with two agents, possibly heterogeneous in their preferences, endowments and tax schedules. Agents trade in an untaxed riskless bond that does not pay dividends, and two redundant, taxed, dividend-paying risky securities. Agents' tax bills are a (generally nonlinear) function of their taxable income; taxable income in turn is a nonlinear function of the dividends received from each security, a generalization over the previous literature. To focus on the novel effects of this nonlinearity in dividends, much of the paper assumes tax bills linear in taxable income.[1] For tractability and clarity of our conclusions, we abstract away from capital gains taxation (a formidable problem in its own right, e.g., Constantinides (1983)). Tax treatment is generally taken to be heterogeneous across securities (via their contribution to taxable income). However, even homogeneous rules may have an asymmetric impact across securities, since a security's effective taxation depends on how important its dividends are relative to its capital gains and also on the price volatility.

Our agents' optimization problem is non-standard, because of the redundancy in the risky securities and the nonlinearity in taxation. The redundancy adds an extra step in an agent's optimization; once he has chosen his risk exposure, he must decide how to allocate that risk between the two securities. The problem is simplified by solving this step first, by offsetting *tax-arbitrage* and *price-arbitrage* opportunities. If (before-tax) the two risky securities were identically priced, an agent would allocate his risk between them so as to minimize his tax bill, indulging in tax-arbitrage activity until the effective marginal tax rates across the two securities become equated. However, under asymmetric taxation, a pre-tax *mispricing* between the securities (a divergence in their pre-tax market prices of risk) is allowed and the agent divides his risk between the two securities until the discrepancy in effective tax rates counterbalances the mispricing. He does not necessarily minimize his tax bill, because exploiting the mispricing (price-arbitrage) may make up for paying additional taxes. Then, a tax-arbitrage opportunity and a price-arbitrage opportunity both exist at the optimum, but the potential marginal profits exactly offset one another, which characterizes the allocation between the two securities.

The agent's remaining problem is one involving a single, "composite" risky asset whose drift, under nonlinear taxation, will generally be policy-dependent. We employ techniques of consumption-portfolio optimization in nonlinear financial markets (Cuoco and Cvitanic (1998)) to deal with this. These techniques consist in embedding the original (nonlinear) problem into a family of perfect (linear) "fictitious" markets, where security prices dynamics are modified and agents receive an additional stochastic "endowment" reflecting the nonlinearity in the market price of risk. The fictitious markets are designed in such a way that the optimal policy in one of them coincides with that in the actual, nonlinear market.

We examine in detail the case of the tax bill being linear in taxable income, while both securities contribute to taxable income according to a piecewise linear schedule with a single kink at zero. Here the family of fictitious markets can be explicitly solved for. Tractability obtains because the fictitious stochastic endowment is constantly zero. In particular, under logarithmic preferences and no stochastic endowment, fully explicit optimal policies are provided. This tax scheme captures the effect of the asymmetric treatment of long and short positions, a real-life feature of tax systems, but largely ignored by financial economists up to now. Relative to the short sales prohibition typically assumed, this tax structure constitutes a more realistic way to

[1]This can be interpreted as taking agents to lie in particular tax brackets due to some "background income" (e.g., wages) that is not explicitly incorporated into the model. The importance of such background income (which dominates dividend income for the overwhelming majority of people) is a further motivation for this simplifying assumption.

bound tax-arbitrage trades and thus allow for existence of equilibrium. Of independent interest, our analysis provides a valuable illustration of explicit use of the recently developed optimization techniques in nonlinear markets.

The assets' mispricing and all other quantities are solved for by appealing to general equilibrium restrictions. The pre-tax mispricing is set so that agents' allocation between the two securities is consistent both with price-arbitrage offsetting tax-arbitrage, and with market clearing. Under very general assumptions, it is shown that this is equivalent to minimization of the aggregate tax proceeds (for given risk sharing among agents). This is somewhat surprising because, individually, agents do not minimize their own tax bill.

The nonlinearity of taxable income in dividends, a novelty of our work with respect to the existing literature (Schaefer (1982), Dammon and Green (1987), Dybvig and Ross (1986)), is established to be rich in implications. In particular, unlike in earlier work, absence of arbitrage for the agents requires neither a no-short sales constraint nor the equalization of marginal tax rates across agents. So, equilibrium requires neither of these counterfactual restrictions. The significance of nonlinearity of taxable income is further emphasized by studying the equilibrium without this generalization (Section 7). Then, redundancy in the securities allows agents to fully circumvent the nonlinearity in their tax bill and so to face a linear problem. Only when the contributions to taxable income are also homogeneous across agents does equilibrium require agents' marginal tax rates to be equated. Individual-specific market prices of risk are then equated and the economy collapses to a frictionless one.

The analysis of some equilibrium examples (with logarithmic preferences and tax bills linear in taxable income) allows us to elaborate on the effects of tax arbitrage. First, we assume one risky security (the "derivative") to be in zero net supply, and explore how and when this derivative alleviates the tax burden. Such an analysis is of importance as much of the recent financial innovation may have been in response to features of the tax code, as pointed out, for example, by Allen and Gale (1994, pp. 349-350). In our first example, the derivative is assumed to contribute linearly to taxable income for both agents, while the stock is taxed nonlinearly. We show that agents share their risk exposure between the assets in such a way that they effectively face linear taxation. The linearly taxed derivative fully relieves the nonlinearity in the tax schedule on the stock. In our second, more general example, one agent is taxed piecewise linearly on both securities while the other is taxed linearly on both. We show, for given wealth distribution and risky security prices, that the agents' market prices of risk (and the interest rate) fall between those in the economy containing the stock alone and those in the economy with no taxation. Hence the presence of the derivative causes risk-sharing to tend towards the no-tax case and can be considered to relieve the frictional effects of the taxation. We can also look at the determinants of the use of the innovation in equilibrium. We find that the derivative is used only if: the nonlinearly taxed agent is wealthy enough (otherwise, he holds no risky security); and taxation on the derivative is more homogeneous across agents than is taxation on the stock.

A well-recognized notion in the literature (and observed empirically) is that of a (tax) clientele effect, which we define as agents trading in disjoint sets of risky securities. The most extreme type of clientele effect, with each agent holding the whole stock of a single risky security, has been denoted a "clientele effect in prices" by Dybvig and Ross (1986). In an example with both securities in positive supply we endogeneously generate such effects. We provide conditions for the occurrence of such situations and characterization of the resulting equilibrium. Endogeneous generation of a clientele effect is not trivial. A discontinuous kink in the taxation (at zero holding) on each asset for at least one agent is a minimal requirement; this is the only way to generate an extended price space over which a zero-holding is optimal. Moreover, equilibrium must allow for divergence in agents' marginal tax rates, generically unequal in a clientele situation. For simplicity, we assume piecewise linear tax schedules homogeneous across the two assets. Within our example, we show clientele effects to arise when: agents' tax rates are far enough apart; and the distribution of wealth is sufficiently even across agents. A distinctive feature of these clientele states is that agents do not exchange risky securities with each other so, unlike in a standard model, equilibrium is not driven by the ratio of marginal utilities. Rather, each risky security's price reflects the shadow cost of the single agent who is holding it. Which agent holds which risky security depends on assets' relative taxation, and agents' wealths relative to the value of the aggregate supply of each asset. For example, the higher taxed agent will hold the lower taxed risky asset, if any, in accordance with aggregate tax minimization, but only if he is wealthy enough.

This paper builds on the work of Schaefer (1982), who examines the effects of differential taxation (in a one period framework with no uncertainty), mainly using numerical examples. The payoffs from different securities are allowed to contribute differently (but linearly and homogeneously across agents) to taxable income. Except if tax rules satisfy some "neutrality" conditions that are not verified in practice, whenever agents occupy different tax brackets, equilibrium is possible only with imperfect capital markets. Schaefer assumes preclusion of short sales, but points out that asymmetries in the treatment of long and short positions (such as in our work) would suffice. Although, as in our paper, mispricing occurs in the equilbria he describes, he does not provide an analysis of its role. Dammon and Green (1987) extend Schaefer's work, also in a one period setting, and with linear, homogeneous (across agents) contributions of assets to taxable income, to provide a general condition for existence of prices that preclude unbounded arbitrage and existence of equilibrium (shown to be equivalent): the existence of a set of feasible trades such that agents equate marginal tax rates. This is a special case of our observation of agents minimizing aggregate tax proceeds; but, under our more general tax scheme, minimization of aggregate taxes does not require agents equating marginal tax rates. Jones and Milne (1992) employ a different approach to establish existence of equilibrium without assumptions on tax schedules and asset returns. They show that it may suffice for the government to be subject to a budget constraint precluding unbounded tax rebates and for agents to anticipate this. Ross (1987) is related in that he also aims at identifying no-arbitrage conditions. He extends the martingale approach to an environment with nonlinear taxation and finds that assets should be priced according to their marginal contribution to an agent's income, net of taxes, and not their actual cash-flows. A somewhat related paper is by Basak and Gallmeyer (1998), who solve for equilibrium when agents face linear differential dividend taxation from one risky security; there, agents face individual-specific market prices of risk as in the present work, but neither tax arbitrage nor mispricing arise with a single risky security.

Our analysis of clientele effects complements the work of Dybvig and Ross (1986). Assuming clientele effects arise, these authors derive pricing rules from agents' first-order conditions, under the assumption that short sales are precluded. Dybvig and Ross' objectives are different from ours, in that they solely aim to identify when linear pricing rules apply and when they do not. General conditions for the occurrence of clientele effects are not provided.

Section 2 describes the model, and Section 3 presents our technique for consumption-portfolio choice in the presence of tax-arbitrage. Section 4 establishes general necessary conditions for equilibrium and deduces a characterization of the mispricing and an analysis of its equilibrium role. Sections 5 and 6 present some equilibrium examples, in the presence of a zero net supply derivative security in the former section, and with redundant positive supply securities in the latter one. Section 7 examines the case where taxable income is linear in dividends received. Section 8 concludes and the Appendix provides all proofs.

2. The Economy

We consider a finite horizon, $[0, T]$, continuous-time, pure-exchange economy with a single consumption good (the numeraire). Uncertainty is represented by a filtered probability space $(-\ , \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$ on which is defined a one-dimensional Brownian motion W. The economy is populated by two agents, indexed by $i \in \{1, 2\}$, homogeneous in their (complete) information (represented by $\{\mathcal{F}_t\}$, the augmented filtration generated by W) and beliefs (represented by $\mathcal{P}$). All stochastic processes introduced are assumed adapted to $\{\mathcal{F}_t\}$, all (in)equalities involving random variables hold $\mathcal{P}$-a.s., and all stochastic differential equations are assumed to have a solution.

2.1. Investment Opportunities and Taxation

The agents trade in three securities: a locally riskless "bond" (money market account) with price B, earning the instantaneous interest rate r, in zero net supply and paying no dividends; and two risky securities with prices S and P, representing claims to the exogenously specified dividend processes δ_S and δ_P, with dynamics

$$d\delta_j(t) = \delta_j(t) \left[\mu_{\delta_j}(t)dt + \sigma_{\delta_j}(t)dW(t) \right] \ , \quad j \in \{S, P\}.$$

The security with price S is assumed to be in constant, positive net supply of one share and can thus be viewed as a "stock". The second risky security is in constant net supply of s_P share, with either $s_P = 0$ or $s_P = 1$ in our respective equilibrium examples. In the former case, which will allow us to study, in particular, the tax-relieving role of financial innovation, this security may be interpreted as a derivative. The latter case will allow us to endogenize clientele effects; an example fitting into this framework is that of long-lived, default-free bonds differing in their tax treatment (e.g., corporate and municipal). The security prices have dynamics

$$\begin{aligned} dB(t) &= B(t)r(t)dt \; ; \\ dS(t) + \delta_S(t)dt &= S(t)\left[\mu_S(t)dt + \sigma_S(t)dW(t)\right] \; ; \\ dP(t) + \delta_P(t)dt &= P(t)\left[\mu_P(t)dt + \sigma_P(t)dW(t)\right] \; , \end{aligned}$$

where the processes r, μ_S, μ_P, σ_S, σ_P are determined endogenously in equilibrium, with σ_S, σ_P assumed bounded above and below away from zero.[2] Agent i's holding (in units) of security j at time t is denoted by $\alpha_j^i(t)$, $j \in \{S, P\}$.

Agent i is taxed on dividends received from the risky securities: at time t, he pays the instantaneous amount

$$T^i\left(t_S^i(\alpha_S^i(t)\delta_S(t)) + t_P^i(\alpha_P^i(t)\delta_P(t))\right), \tag{2.1}$$

where $T^i(\cdot)$, $t_S^i(\cdot)$ and $t_P^i(\cdot)$ are agent-specific, deterministic, time-independent, continuous, convex and increasing functions, with a derivative (if it exists) less than one. The argument of $T^i(\cdot)$ in (2.1) is understood as i's *taxable income*. For convenience, we sometimes denote agent i's time t tax bill by $T^i(t)$, and its derivative (i's *marginal tax rate*) by $T^{i\prime}(t)$. For much of the paper (Sections 3.3, 5, 6), to focus on the differential tax treatment across securities, we assume $T^i(\cdot)$ to be linear (w.l.o.g., equal to the identity): this can be interpreted as exogenously taking agents to lie in particular tax brackets, and amounts to separately taxing dividends received from each risky security. An important special case is that of a progressive piecewise linear contribution to taxable income, exhibiting a single kink at zero dividends and rates τ_{j+}^i for long positions and τ_{j-}^i for short positions:

$$t_j^i\left(\alpha_j^i\delta_j\right) = \left(\alpha_j^i\delta_j\right)^+\tau_{j+}^i - \left(\alpha_j^i\delta_j\right)^-\tau_{j-}^i, \tag{2.2}$$

with $0 \le \tau_{j-}^i \le \tau_{j+}^i < 1$, $j \in \{S, P\}$. In the special case of linear taxation, we have $\tau_{j-}^i = \tau_{j+}^i \equiv \tau_j^i$, so that $t_j^i\left(\alpha_j^i\delta_j\right) = \alpha_j^i\delta_j\tau_j^i$, $j \in \{S, P\}$. Beyond this special case, for clarity we assume all functions $T^i(\cdot)$, $t_j^i(\cdot)$ to be continuously differentiable. Our setup leaves aside many features of actual tax systems (e.g., consumption or capital gains taxes). However, we capture several real-life phenomena: different tax brackets across agents, differences in tax treatment across securities, and asymmetric tax treatment of long and short positions.

Without taxation, with all uncertainty generated by a one-dimensional Brownian motion, one of our risky securities would be redundant, and so no-arbitrage would enforce identical

[2] For simplicity in exposition, we focus on equilibria where σ_S, σ_P are both positive. Our other primitives do not rule out additional equilibria where one or both of σ_S, σ_P can be negative, but these equilibria have similar features to the ones we discuss. The analysis of the continuously differentiable taxation case (Sections 3.1-3.2, 4, 5.1 and 7) is fully independent of the signs of the volatilities, and conceptually all our main conclusions are equally valid for negative volatilities.

instantaneous market prices of risk. This motivates our definition of (pre-tax) *mispricing* as:

$$\Delta_{S,P}(t) \equiv \frac{\mu_S(t) - r(t)}{\sigma_S(t)} - \frac{\mu_P(t) - r(t)}{\sigma_P(t)} .$$

In the presence of heterogeneous taxation across securities, the securities may be "mispriced" (non-zero $\Delta_{S,P}$) when judged on a pre-tax basis, while not necessarily being mispriced after taxes are taken into account. Hence a non-zero "mispricing" may be compatible with no-arbitrage.

2.2. Agents' Endowments and Preferences

Agent i is endowed with e^i_j share of security $j \in \{S, P\}$, with: $e^i_S > 0$, $e^1_S + e^2_S = 1$; $e^i_P \geq 0$, $e^1_P + e^2_P = s_P$. In our later analyses, tax proceeds will either be assumed to be redistributed to the agents or taken out of the economy. In the former case, we assume agent i to be endowed with the stochastic process $\epsilon^i(t) \geq 0$, with $\epsilon^1(t) + \epsilon^2(t) = T^1(t) + T^2(t)$; in the latter case, $\epsilon^i(t) = 0$. Agent i chooses a consumption process c^i and a portfolio process $\alpha^i \equiv \left(\alpha^i_S, \alpha^i_P\right)^\top$, and pays taxes $T^i(t)$. A consumption-portfolio pair (c^i, α^i) is *admissible* if the associated wealth process X^i is bounded below, satisfies $X^i(T) \geq 0$ and obeys the dynamic budget constraint

$$\begin{aligned} dX^i(t) &= \left[X^i(t)r(t) + \epsilon^i(t) - c^i(t)\right] dt \\ &\quad +\alpha^i_S(t)S(t)\left\{[\mu_S(t) - r(t)]dt + \sigma_S(t)dW(t)\right\} + \alpha^i_P(t)P(t)\left\{[\mu_P(t) - r(t)]dt + \sigma_P(t)dW(t)\right\} \\ &\quad -T^i\left(t^i_S(\alpha^i_S(t)\delta_S(t)) + t^i_P(\alpha^i_P(t)\delta_P(t))\right) dt , \end{aligned} \tag{2.3}$$

with $X^i(0) = e^i_S S(0) + e^i_P P(0)$. The first and second line contain the standard terms, affine in portfolio holdings; the third, additional, line accounts for taxation and may exhibit non-linearity in portfolio holdings, adding substantial complexity. Each agent derives time-additive, state-independent utility $u_i\left(c^i(t)\right)$ from intertemporal consumption in $[0, T]$. The function $u_i(\cdot)$ is assumed three times continuously differentiable, strictly increasing, strictly concave, and to satisfy $\lim_{c\to 0} u'_i(c) = \infty$ and $\lim_{c\to\infty} u'_i(c) = 0$. Agent i's optimization problem is to maximize $E\left[\int_0^T u_i(c^i(t))dt\right]$ over all admissible (c^i, α^i) for which the expected integral is well-defined.

3. Agents' Optimization in the Presence of Tax Arbitrage

Redundancy in the risky securities adds an extra layer to the solution for optimality: once the agent has chosen his risk exposure, he must further decide how to allocate that risk between the two securities. It turns out that the optimal allocation between S and P, for a given risk exposure, obtains from non-satiation alone, independently of the preferences of the agent. Hence, the problem is simplified by solving "in reverse", first for the optimal allocation between S and P (Section 3.1), second for the optimal risk exposure (Section 3.2).

The first stage in the optimization introduces the concepts of *tax-arbitrage* and *price-arbitrage*. Under no taxation, any mispricing between S and P would induce an agent to take on an unbounded, costless, riskless position (long in one security and short in the other) yielding positive gain: a "price-arbitrage opportunity." Under no mispricing, differential tax treatment of S and P (due to differential contribution to taxable income) might induce an agent to prefer one security to the other and so take on a long position in one financed by a short position in the other, yielding positive gain (in the form of a tax rebate): a "tax-arbitrage opportunity." Under linear taxation, the opportunity would allow for an unbounded profitable position. Under nonlinear taxation, however, it may be that some bounded position equates the (marginal) tax treatments of S and P and so is optimal, in which case we will call this a "bounded" or "local tax-arbitrage opportunity" (as in Ross (1987)).

In equilibrium, agents' non-satiation requires no net arbitrage opportunity, but, under differential taxation, a price-arbitrage opportunity (mispricing) may be present in equilibrium, yet offset by a tax-arbitrage opportunity. Any gain from buying the underpriced security and selling the overpriced one must be offset by an increase in taxes (and conversely, any attempt by the agent to reduce his tax bill must be offset by the mispricing). Section 3.1 quantifies these notions.

We introduce the risk-weighted sum of holdings in the risky securities,

$$\Phi^i(t) \equiv \alpha_S^i(t)S(t) + \frac{\sigma_P(t)}{\sigma_S(t)}\alpha_P^i(t)P(t) \,, \tag{3.1}$$

interpreted as agent i's "composite" risk exposure, because all pairs of holdings in S and P that lead to the same value for $\Phi^i(t)$ yield the same diffusion for i's wealth, $\Phi^i(t)\sigma_S(t)$.

3.1. Non-Satiation and Tax-Arbitrage

Agent i will divide his composite risk exposure between S and P in such a way that he is either indifferent to marginal shifts from one security to the other, or any shift yields negative gain. Proposition 3.1 presents the resulting condition on rational holdings $(\hat{\alpha}_S^i, \hat{\alpha}_P^i)$.

Proposition 3.1. *Let $\Phi^i(t)$ and $\Delta_{S,P}(t)$ be given. (a) If taxation is continuously differentiable, agent i is indifferent between all pairs $(\hat{\alpha}_S^i(t), \hat{\alpha}_P^i(t)) \equiv (\hat{\alpha}_S^i\left(\Phi^i(t), \Delta_{S,P}(t), t\right), \hat{\alpha}_P^i\left(\Phi^i(t), \Delta_{S,P}(t), t\right))$ leading to the same value for his risk exposure $\Phi^i(t)$ and such that*[3]

$$\theta_S^i\left(\hat{\alpha}_S^i(t), \hat{\alpha}_P^i(t), t\right) = \theta_P^i\left(\hat{\alpha}_S^i(t), \hat{\alpha}_P^i(t), t\right) \,, \tag{3.2}$$

where, for $j \in \{S, P\}$,

$$\theta_j^i\left(\alpha_S, \alpha_P, t\right) \equiv \frac{\mu_j(t) - r(t)}{\sigma_j(t)} - d_j(t) t_j^{i\prime}\left(\alpha_j \delta_j(t)\right) T^{i\prime}\left(t_S^i\left(\alpha_S \delta_S(t)\right) + t_P^i\left(\alpha_P \delta_P(t)\right)\right) \,; \tag{3.3}$$

$$d_S(t) \equiv \frac{\delta_S(t)}{S(t)\sigma_S(t)} \,, \quad d_P(t) \equiv \frac{\delta_P(t)}{P(t)\sigma_P(t)} \,.$$

[3] Existence of a solution to (3.2) (or (3.5)) on a positive measure space for $\Delta_{S,P}(t)$ is guaranteed by the assumptions on tax schedules made in Remark 4.1.

Equivalently,

$$\Delta_{S,P}(t) = \left[d_S(t)t_S^{i\prime}(\hat{\alpha}_S^i(t)\delta_S(t)) - d_P(t)t_P^{i\prime}(\hat{\alpha}_P^i(t)\delta_P(t))\right] T^{i\prime}\left(t_S^i(\hat{\alpha}_S^i(t)\delta_S(t)) + t_P^i(\hat{\alpha}_P^i(t)\delta_P(t))\right) . \tag{3.4}$$

(b) If taxation is not continuously differentiable, agent i is indifferent between all pairs $(\hat{\alpha}_S^i(t), \hat{\alpha}_P^i(t))$ leading to the same $\Phi^i(t)$ and such that

$$\theta_{S[+]}^i\left(\hat{\alpha}_S^i(t), \hat{\alpha}_P^i(t), t\right) \leq \theta_{P[-]}^i\left(\hat{\alpha}_S^i(t), \hat{\alpha}_P^i(t), t\right) \text{ and } \theta_{P[+]}^i\left(\hat{\alpha}_S^i(t), \hat{\alpha}_P^i(t), t\right) \leq \theta_{S[-]}^i\left(\hat{\alpha}_S^i(t), \hat{\alpha}_P^i(t), t\right) , \tag{3.5}$$

where, for $j, k \in \{S, P\}$, $k \neq j$,

$$\theta_{j[-]}^i(\alpha_S, \alpha_P, t) \equiv \frac{\mu_j(t) - r(t)}{\sigma_j(t)} - d_j(t) \lim_{x \uparrow \alpha_j} t_j^{i\prime}(x\delta_j(t)) T^{i\prime}(x\delta_j(t) + \alpha_k\delta_k(t)) , \tag{3.6}$$

$$\theta_{j[+]}^i(\alpha_S, \alpha_P, t) \equiv \frac{\mu_j(t) - r(t)}{\sigma_j(t)} - d_j(t) \lim_{x \downarrow \alpha_j} t_j^{i\prime}(x\delta_j(t)) T^{i\prime}(x\delta_j(t) + \alpha_k\delta_k(t)) . \tag{3.7}$$

(3.3) or (3.6)-(3.7) can be identified as agent i's (subjective) after-tax *marginal* market prices of risk (or left and right limits thereof) for securities S and P, marginal in the sense of providing the (after-tax) risk premium per unit of risk on the additional dollar invested in a risky security. A security's relative favorability is, then, driven by its pre-tax risk-premium-to-risk ratio (price favorability), minus its "net" marginal tax rate (accounting for both its marginal contribution to taxable income and the agent's marginal tax rate) normalized by its risk and price-to-dividend ratio (tax unfavorability). Thus, a non-zero pre-tax mispricing $\Delta_{S,P}(t)$ is consistent with the existence of a solution to an agent's problem, and will offset any difference in marginal tax rates across assets. Whenever one security's after-tax marginal market price of risk dominates the other, i can increase the drift of his wealth by exchanging the less favorable security for the more favorable. In the differentiable taxation case, (3.2) states that at the optimum an agent must face no after-tax mispricing, i.e., no discrepancy across securities between after-tax market prices of risk, accounting for his individual-specific (generally policy-dependent) marginal tax rates. Equivalently, in (3.4), consistent with our intuition, any pre-tax mispricing must be exactly compensated by the differential taxation of the assets at the agent's optimum.

To facilitate understanding of Proposition 3.1, rewrite the budget constraint (2.3) as:

$$\begin{aligned} dX^i(t) = & \left(r(t)X^i(t) + \epsilon^i(t) - c^i(t)\right) dt + \Phi^i(t)\left[(\mu_S(t) - r(t))\, dt + \sigma_S(t)dW(t)\right] \\ & -\alpha_P^i(t)P(t)\sigma_P(t)\Delta_{S,P}(t)dt - T^i\left(t_S^i(\alpha_S^i(t)\delta_S(t)) + t_P^i(\alpha_P^i(t)\delta_P(t))\right) dt . \end{aligned} \tag{3.8}$$

(3.4) can easily be checked to be the first-order condition of the problem to maximize the drift of wealth, for given Φ^i. The sharing between S and P affects only the second line of (3.8). Increasing the value of the former term (the price-arbitrage profit) in this line amounts to buying the favorably mispriced security and selling the other, while decreasing the second term (the agent's tax bill) amounts to financing purchases of the lower taxed security by sales of the more heavily taxed one. Hence, (3.4) states that, at the optimum, the marginal profit/cost from the former activity (price-arbitrage) is offset by an increase/reduction in the tax bill (tax-arbitrage). A consequence is that an agent generally does not minimize the amount of his tax bill: he may be willing to pay more taxes and surrender tax-arbitrage profits in order to exploit the mispricing.

Remark 3.1. A tax-arbitrage opportunity arises from a divergence between the securities' product of net marginal tax rate with d_S or d_P. These products represent "effective" tax rates, accounting for the assets' dividend-to-price ratios, and how volatile, hence effective at adjusting risk exposure, they are: they state by how much agent i's tax bill will vary if he adjusts his risk exposure using one asset or the other. Hence, for the assets to be affected heterogeneously by taxation, it is not necessary that they be subject to explicitly different tax rules (in terms of contribution to taxable income); it suffices that d_S and d_P differ.

Proposition 3.1 implies (via substitution into the dynamic budget constraint (2.3)) that any non-satiated agent's optimal wealth will always follow:

$$\begin{aligned} dX^i(t) = & \left[X^i(t)r(t) + \epsilon^i(t) - c^i(t)\right] dt + \Phi^i(t)\left\{[\mu_S(t) - r(t)]\, dt + \sigma_S(t)dW(t)\right\} \\ & -\hat{\alpha}^i_P\left(\Phi^i(t), \Delta_{S,P}(t), t\right) P(t)\sigma_P(t)\Delta_{S,P}\, dt \\ & -T^i\left(t^i_S(\hat{\alpha}^i_S(\Phi^i(t), \Delta_{S,P}(t), t)\delta_S(t)) + t^i_P(\hat{\alpha}^i_P(\Phi^i(t), \Delta_{S,P}(t), t)\delta_P(t))\right) dt\,, \end{aligned} \tag{3.9}$$

where $\hat{\alpha}^i_S\left(\Phi^i(t), \Delta_{S,P}(t), t\right)$, $\hat{\alpha}^i_P\left(\Phi^i(t), \Delta_{S,P}(t), t\right)$ are as provided by Proposition 3.1. When the $\hat{\alpha}^i_j$'s are not unique, all possible choices lead to the same wealth dynamics. Because non-satiation suffices to pin down the choice between S and P, agents effectively face a (generally nonlinear) optimization problem involving a single, "composite" risky security with volatility σ_S. Φ^i can be interpreted as the weight in the composite and (3.9) as the corresponding budget constraint. The nonlinearity of the problem stems from both the nonlinearity in Φ^i of $\hat{\alpha}^i_S$ and $\hat{\alpha}^i_P$ and the nonlinearity of the tax schedules.

3.2. Optimal Risk Exposure (Φ) and Consumption Choice

The methodology employed to deal with our nonlinear problem consists of embedding it into a family of "fictitious", unconstrained, perfect (linear) market problems (Cvitanic and Karatzas (1992), Cuoco and Cvitanic (1998)). The fictitious price dynamics and endowments are modified so that the solution to one of the perfect market problems coincides with that of the original problem. Each (individual-specific) fictitious market, parametrized by the process ν^i, has no taxation, one bond and one stock with price parameters (3.10)[4] and agents' endowments (3.11):

$$r_{\nu^i}(t) \equiv r(t)\,; \quad \mu_{\nu^i}(t) \equiv \mu_S(t) - \nu^i(t)\,, \quad \sigma_{\nu^i}(t) \equiv \sigma_S(t)\,; \tag{3.10}$$

$$\epsilon^i_{\nu^i}(t) \equiv \epsilon^i(t) + \tilde{g}^i\left(\nu^i(t), t\right)\,, \tag{3.11}$$

[4] In our application, it does not prove necessary to adjust the interest rate in the fictitious market, because only an agent's composite risk exposure, and not his total wealth, affects his individual-specific price parameters. This would not be the case if, for example, interest income from the bond were taxed nonlinearly.

where

$$\tilde{g}^i(\nu,t) \equiv \sup_{\Phi}\left\{g^i(\Phi,t)+\Phi\nu\right\} , \tag{3.12}$$

$$\begin{aligned} g^i(\Phi,t) \equiv& -\hat{\alpha}^i_P(\Phi,\Delta_{S,P}(t),t)\,P(t)\sigma_P(t)\Delta_{S,P}(t) \\ &-T^i\left(t^i_S(\hat{\alpha}^i_S(\Phi,\Delta_{S,P}(t),t)\delta_S(t))+t^i_P(\hat{\alpha}^i_P(\Phi,\Delta_{S,P}(t),t)\delta_P(t))\right) , \end{aligned} \tag{3.13}$$

and $\hat{\alpha}^i_S$, $\hat{\alpha}^i_P$ are as provided by Proposition 3.1.[5] We define the set

$$\mathcal{N}^i_t \equiv \left\{\nu : \tilde{g}^i(\nu,t)<\infty\right\} . \tag{3.14}$$

and $\mathcal{N}^i$ as the set of processes ν such that, for any t, $\nu(t)\in\mathcal{N}^i_t$. The market price of risk in the fictitious market is given by

$$\theta_{\nu^i}(t) \equiv \frac{\mu_{\nu^i}(t)-r_{\nu^i}(t)}{\sigma_{\nu^i}(t)} = \frac{\mu_S(t)-r(t)}{\sigma_S(t)} - \frac{\nu^i(t)}{\sigma_S(t)} ,$$

and the fictitious state price density process ξ_{ν^i} has dynamics

$$d\xi_{\nu^i}(t) = -\xi_{\nu^i}(t)\left[r_{\nu^i}(t)dt+\theta_{\nu^i}(t)dW(t)\right] .$$

Informally, ν^i reflects the potential policy-dependence of agent i's marginal after-tax market price of risk, while $\tilde{g}^i$ ensures agreement between the actual and fictitious wealth dynamics. At the supremum of the problem in (3.12), we have

$$g^i = -\Phi^i\nu^i+\tilde{g}^i = -\Phi^i\frac{\delta_S}{S}t^{i\prime}_S T^{i\prime} + \left[\left(\hat{\alpha}^i_S\delta_S t^{i\prime}_S+\hat{\alpha}^i_P\delta_P t^{i\prime}_P\right)T^{i\prime}-T^i\right] ,$$

revealing that the $\tilde{g}^i$ endowment term captures convexity in the tax schedules causing not all of agent i's dividend income to be taxed at the marginal rate ($t^{i\prime}_S T^{i\prime}$). Hence, part of his risk exposure in the actual economy is not rewarded at the fictitious market price of risk θ_{ν^i} but at a higher rate.

Standard martingale methods (Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987)) can be used to solve agent i's (linear) optimization problem in the fictitious market, for a given ν^i. The determination of i's optimal policy in the original market involves setting up a dual, "minimax" problem. Proposition 3.2 establishes that, if ν^i solves the dual, the optimum in the corresponding fictitious market is attainable and optimal in the actual economy. (At this minimax ν^i, $\xi_{\nu^i},\theta_{\nu^i}$ are henceforth denoted ξ^i,θ^i.)

Proposition 3.2. *Assume that there exists a solution ν^i to the problem*

$$\min_{\nu\in\mathcal{N}^i}\left\{\max_{c^i} E\left[\int_0^T u_i(c^i(t))dt\right] \text{ s.t. } E\left[\int_0^T \xi_\nu(t)\left(c^i(t)-\epsilon^i_\nu(t)\right)dt\right]\le\xi_\nu(0)X^i(0)\right\}. \tag{3.15}$$

Then, there exists a solution to agent i's optimization problem and his optimal consumption is

$$c^i(t) = I^i\left(y^i\xi^i(t)\right) , \tag{3.16}$$

[5] Noting that g^i equals the supremum of the second line in (3.8) maximized over $(\alpha_S(t),\alpha_P(t))$, and applying the enveolope theorem shows that g^i is concave in Φ^i, when taxation is differentiable. In our subsequent examples where taxation is not differentiable, concavity of g^i (determined explicitly) can be verified case-by-case.

where $I^i(\cdot)$ *is the inverse of* $u_i'(\cdot)$ *and* y^i *satisfies* $E\left[\int_0^T \xi^i(t)\left(I^i(y^i\xi^i(t)) - \epsilon^i_{\nu^i}(t)\right)dt\right] = \xi^i(0)X^i(0)$.

The optimal holding in the composite asset is given by

$$\Phi^i(t) = \frac{\theta^i(t)X^i(t)}{\sigma_S(t)} + \frac{\kappa^i(t)}{\sigma_S(t)\xi^i(t)}\,, \tag{3.17}$$

where κ^i *satisfies* $\xi^i(t)X^i(t) + \int_0^t \xi^i(s)\left(c^i(s) - \epsilon^i_{\nu^i}(s)\right)ds = \xi^i(0)X^i(0) + \int_0^t \kappa^i(s)dW(s)$.

Proposition 3.1 then yields the portfolio strategy in terms of the actual securities S and P.

3.3. Optimization under Progressive Piecewise Linear Taxation

We illustrate Sections 3.1 and 3.2 using a linear T^i and piecewise linear contributions to taxable income (2.2) for both securities. Proposition 3.1 explicitly provides agent i's optimal allocation between the two risky securities given Φ, as summarized by Table I Panel (a). Agent i is indifferent between all $(\hat{\alpha}^i_S, \hat{\alpha}^i_P)$ satisfying (3.1). Results are presented for the range of mispricings which do not allow net unbounded arbitrage.[6]

An agent is in one of thirteen cases (A-M), depending on the sign of his composite risk exposure and the value of the mispricing. There are four values of the mispricing (A, C, E, F, H, I, K, M) which exactly compensate the differential tax treatment of the two securities, as long as the holdings in each remain within the positive or the negative region. At these mispricings, the agent is indifferent between all such portfolio allocations between S and P. Additional price-arbitrage profits are exactly offset by an increase in his tax bill. At intermediate levels of mispricing (B, D, G, J, L), the agent rationally invests zero in one or both risky securities. He is, then, at the kink in the tax schedule for one or both securities, and any price-favorability and tax-favorability of one asset over the other is insufficient to induce him to deviate from that kink. For some values of the mispricing (e.g., those such that the agent can be in either B, G or L), he uses the same risky security for both signs of Φ^i; for other mispricings (e.g., when the agent is in one of B, G or J), he uses one risky security for one sign of Φ^i and the other for the other sign. In all cases, the agent never needs to hold more than one of the risky securities simultaneously; if he holds both, they are perfect substitutes locally, and he is indifferent between them. In the particular case of linear taxation on both securities ($\tau^i_{j+} = \tau^i_{j-} \equiv \tau^i_j$, $j \in \{S, P\}$), the mispricing is pegged at $\Delta_{S,P}(t) = d_S(t)\tau^i_S - d_P(t)\tau^i_P$, exactly offsetting the differential tax treatment.[7]

Applying (3.12)-(3.14), in this piecewise linear case with a kink at zero, an explicit determination of the space of fictitious market parameters $\mathcal{N}^i_t$ and the associated $\tilde{g}^i$ is possible, as

[6]From (3.5), existence of an optimal solution requires: $d_S(t)\tau^i_{S-} - d_P(t)\tau^i_{P+} \leq \Delta_{S,P}(t) \leq d_S(t)\tau^i_{S+} - d_P(t)\tau^i_{P-}$. For mispricings outside this region, a finite choice for α^i_S and α^i_P is never rational. For large enough mispricing, $\Delta_{S,P} > d_S\tau^i_{S+} - d_P\tau^i_{P-}$, the price-favorability of S more than compensates for any tax unfavorability so there is a net unbounded arbitrage opportunity, and analogously for $\Delta_{S,P} < d_S\tau^i_{S-} - d_P\tau^i_{P+}$.

[7]Similar results (involving many more cases) would obtain with more than one kink in the tax schedules: either the agent would be indifferent between all holdings such that dividend income from each security lies between two kinks in its tax schedule; or his holdings would be uniquely determined, the income from one security being fixed at a kink in its tax schedule, the holdings in the other security adjusting to attain Φ^i.

Table I:
Panel (a): Portfolio Holdings Consistent with Non-Satiation (General Preferences).
Panel (b): Fictitious Market Parameter for a Logarithmic Agent

		Panel (a)		Panel (b)	
	$\Delta_{S,P}$	Φ^i	$\hat{\alpha}^i_S, \hat{\alpha}^i_P$	$\frac{\mu_S - r}{\sigma_S}$	ν^i
A	$= d_S\tau^i_{S-} - d_P\tau^i_{P+}$	< 0	$< 0, \geq 0$	$< d_S\tau^i_{S-}$	$\sigma_S d_S\tau^i_{S-}$
B	$\in \left(d_S\tau^i_{S-} - d_P\tau^i_{P+}, d_S\tau^i_{S-} - d_P\tau^i_{P-}\right)$		$< 0, = 0$	$< d_S\tau^i_{S-}$	$\sigma_S d_S\tau^i_{S-}$
C	$= d_S\tau^i_{S-} - d_P\tau^i_{P-}$		$\leq 0, \leq 0$	$< d_S\tau^i_{S-}$	$\sigma_S d_S\tau^i_{S-}$
D	$\in \left(d_S\tau^i_{S-} - d_P\tau^i_{P-}, d_S\tau^i_{S+} - d_P\tau^i_{P-}\right)$		$= 0, < 0$	$< \Delta_{S,P} + d_P\tau^i_{P-}$	$\sigma_S\left(\Delta_{S,P} + d_P\tau^i_{P-}\right)$
E	$= d_S\tau^i_{S+} - d_P\tau^i_{P-}$		$\geq 0, < 0$	$< d_S\tau^i_{S+}$	$\sigma_S d_S\tau^i_{S+}$
F	$= d_S\tau^i_{S-} - d_P\tau^i_{P+}$	$= 0$	$\leq 0, \geq 0$	$= d_S\tau^i_{S-}$	$\sigma_S d_S\tau^i_{S-}$
G	$\in \left(d_S\tau^i_{S-} - d_P\tau^i_{P+}, d_S\tau^i_{S+} - d_P\tau^i_{P-}\right)$		$= 0, = 0$	$\in \left[\max\left\{d_S\tau^i_{S-}, \Delta_{S,P} + d_P\tau^i_{P-}\right\}, \min\left\{d_S\tau^i_{S+}, \Delta_{S,P} + d_P\tau^i_{P+}\right\}\right]$	$(\mu_S - r)$
H	$= d_S\tau^i_{S+} - d_P\tau^i_{P-}$		$\geq 0, \leq 0$	$= d_S\tau^i_{S+}$	$\sigma_S d_S\tau^i_{S+}$
I	$= d_S\tau^i_{S-} - d_P\tau^i_{P+}$	> 0	$\leq 0, > 0$	$> d_S\tau^i_{S-}$	$\sigma_S d_S\tau^i_{S-}$
J	$\in \left(d_S\tau^i_{S-} - d_P\tau^i_{P+}, d_S\tau^i_{S+} - d_P\tau^i_{P+}\right)$		$= 0, > 0$	$> \Delta_{S,P} + d_P\tau^i_{P+}$	$\sigma_S\left(\Delta_{S,P} + d_P\tau^i_{P+}\right)$
K	$= d_S\tau^i_{S+} - d_P\tau^i_{P+}$		$\geq 0, \geq 0$	$> d_S\tau^i_{S+}$	$\sigma_S d_S\tau^i_{S+}$
L	$\in \left(d_S\tau^i_{S+} - d_P\tau^i_{P+}, d_S\tau^i_{S+} - d_P\tau^i_{P-}\right)$		$> 0, = 0$	$> d_S\tau^i_{S+}$	$\sigma_S d_S\tau^i_{S+}$
M	$= d_S\tau^i_{S+} - d_P\tau^i_{P-}$		$> 0, \leq 0$	$> d_S\tau^i_{S+}$	$\sigma_S d_S\tau^i_{S+}$

reported in Proposition 3.3.

Proposition 3.3. *If taxation is piecewise linear with a single kink at zero, for any t,* $\tilde{g}^i\left(\nu^i(t), t\right) = 0$ *for any* $\nu^i(t)$ *in* $\mathcal{N}^i_t$. $\mathcal{N}^i_t$ *is a closed interval, denoted by* $[\underline{\nu}^i(t), \overline{\nu}^i(t)]$, *where the lower and upper limits depend on* $\Delta_{S,P}(t)$ *and are given in Table II.*

Depending on the mispricing value and the relative taxation of securities, Table II reveals the agent to face three types of optimization problems. In the first type, whenever $\mathcal{N}^i_t$ is a

Table II: Fictitious Market Parameter Space under Piecewise Linear Taxation

$\Delta_{S,P}$	$\mathcal{N}_t^i = [\underline{\nu}^i, \overline{\nu}^i]$	Cases (from Table I)
(i) "Taxation of P closer to linear": $d_P\left(\tau^i_{P+} - \tau^i_{P-}\right) < d_S\left(\tau^i_{S+} - \tau^i_{S-}\right)$		
$= d_S\tau^i_{S-} - d_P\tau^i_{P+}$	$\left\{\sigma_S d_S\tau^i_{S-}\right\}$	AFI
$\in \left(d_S\tau^i_{S-} - d_P\tau^i_{P+}, d_S\tau^i_{S-} - d_P\tau^i_{P-}\right)$	$\left[\sigma_S d_S\tau^i_{S-}, \sigma_S\left(\Delta_{S,P} + d_P\tau^i_{P+}\right)\right]$	BGJ
$= d_S\tau^i_{S-} - d_P\tau^i_{P-}$	$\left[\sigma_S d_S\tau^i_{S-}, \sigma_S d_S\tau^i_{S-} + d_P\sigma_S\left(\tau^i_{P+} - \tau^i_{P-}\right)\right]$	CGJ
$\in \left(d_S\tau^i_{S-} - d_P\tau^i_{P-}, d_S\tau^i_{S+} - d_P\tau^i_{P+}\right)$	$\left[\sigma_S\left(\Delta_{S,P} + d_P\tau^i_{P-}\right), \sigma_S\left(\Delta_{S,P} + d_P\tau^i_{P+}\right)\right]$	DGJ
$= d_S\tau^i_{S+} - d_P\tau^i_{P+}$	$\left[\sigma_S d_S\tau^i_{S+} - d_P\sigma_S\left(\tau^i_{P+} - \tau^i_{P-}\right), \sigma_S d_S\tau^i_{S+}\right]$	DGK
$\in \left(d_S\tau^i_{S+} - d_P\tau^i_{P+}, d_S\tau^i_{S+} - d_P\tau^i_{P-}\right)$	$\left[\sigma_S\left(\Delta_{S,P} + d_P\tau^i_{P-}\right), \sigma_S d_S\tau^i_{S+}\right]$	DGL
$= d_S\tau^i_{S+} - d_P\tau^i_{P-}$	$\left\{\sigma_S d_S\tau^i_{S+}\right\}$	EHM
(ii) $d_P\left(\tau^i_{P+} - \tau^i_{P-}\right) = d_S\left(\tau^i_{S+} - \tau^i_{S-}\right)$		
$= d_S\tau^i_{S-} - d_P\tau^i_{P+}$	$\left\{\sigma_S d_S\tau^i_{S-}\right\}$	AFI
$\in \left(d_S\tau^i_{S-} - d_P\tau^i_{P+}, d_S\tau^i_{S-} - d_P\tau^i_{P-}\right)$	$\left[\sigma_S d_S\tau^i_{S-}, \sigma_S\left(\Delta_{S,P} + d_P\tau^i_{P+}\right)\right]$	BGJ
$= d_S\tau^i_{S-} - d_P\tau^i_{P-}$	$\left[\sigma_S d_S\tau^i_{S-}, \sigma_S d_S\tau^i_{S+}\right]$	CGK
$\in \left(d_S\tau^i_{S-} - d_P\tau^i_{P-}, d_S\tau^i_{S+} - d_P\tau^i_{P-}\right)$	$\left[\sigma_S\left(\Delta_{S,P} + d_P\tau^i_{P-}\right), \sigma_S d_S\tau^i_{S+}\right]$	DGL
$= d_S\tau^i_{S+} - d_P\tau^i_{P-}$	$\left\{\sigma_S d_S\tau^i_{S+}\right\}$	EHM
(iii) Taxation of S "closer to linear": $d_P\left(\tau^i_{P+} - \tau^i_{P-}\right) > d_S\left(\tau^i_{S+} - \tau^i_{S-}\right)$		
$= d_S\tau^i_{S-} - d_P\tau^i_{P+}$	$\left\{\sigma_S d_S\tau^i_{S-}\right\}$	AFI
$\in \left(d_S\tau^i_{S-} - d_P\tau^i_{P+}, d_S\tau^i_{S+} - d_P\tau^i_{P+}\right)$	$\left[\sigma_S d_S\tau^i_{S-}, \sigma_S\left(\Delta_{S,P} + d_P\tau^i_{P+}\right)\right]$	BGJ
$= d_S\tau^i_{S+} - d_P\tau^i_{P+}$	$\left[\sigma_S d_S\tau^i_{S-}, \sigma_S d_S\tau^i_{S+}\right]$	BGK
$\in \left(d_S\tau^i_{S+} - d_P\tau^i_{P+}, d_S\tau^i_{S-} - d_P\tau^i_{P-}\right)$	$\left[\sigma_S d_S\tau^i_{S-}, \sigma_S d_S\tau^i_{S+}\right]$	BGL
$= d_S\tau^i_{S-} - d_P\tau^i_{P-}$	$\left[\sigma_S d_S\tau^i_{S-}, \sigma_S d_S\tau^i_{S+}\right]$	CGL
$\in \left(d_S\tau^i_{S-} - d_P\tau^i_{P-}, d_S\tau^i_{S+} - d_P\tau^i_{P-}\right)$	$\left[\sigma_S\left(\Delta_{S,P} + d_P\tau^i_{P-}\right), \sigma_S d_S\tau^i_{S+}\right]$	DGL
$= d_S\tau^i_{S+} - d_P\tau^i_{P-}$	$\left\{\sigma_S d_S\tau^i_{S+}\right\}$	EHM

singleton (AFI, EHM), the agent faces a linear optimization problem. The value of ν^i reveals the effective tax rate to be either τ^i_{S-} or τ^i_{S+}. In the former case (AFI), security S becomes the more favorable for short positions, while P is more favorable for long positions, and vice versa in the latter case (EHM). In either case, the mispricing is such that the market price of risk provided by the more favorable security for long positions coincides with that provided by the more favorable security for short positions, effectively symmetrizing the tax treatment of long and short composite holdings. The agent is able to use security P to fully circumvent either the higher tax rate for long positions in S (AFI), or the lower tax rate for short positions in S (EHM).

Otherwise, whenever $\mathcal{N}^i_t$ is a closed interval, the agent faces a nonlinear problem. From the proof of Proposition 3.3, we may also deduce that the optimal composite holding, which solves (3.12) (Cuoco and Cvitanic (1998), Theorem 2), satisfies

$$\Phi^i_{\text{sup}}(t) \begin{cases} \leq 0 & \text{when } \nu^i(t) = \underline{\nu}^i(t) \\ = 0 & \text{when } \nu^i(t) \in (\underline{\nu}^i(t), \overline{\nu}^i(t)) \\ \geq 0 & \text{when } \nu^i(t) = \overline{\nu}^i(t) \end{cases} .$$

When the agent's risk exposure is zero, he remains at the kink in the tax schedule for both risky securities; for a region of market prices of risk he has no incentive to move away from the kink. He faces an effective tax rate $(\underline{\nu}^i/\sigma_S d_S)$, between τ^i_{S-} and τ^i_{S+}, for short positions, and a higher effective tax rate $(\overline{\nu}^i/\sigma_S d_S)$, again between τ^i_{S-} and τ^i_{S+}, for long positions. The effective tax rates are determined by which risky security is favorable to use to attain composite holdings of that sign; the agent's (individual-specific) fictitious market price of risk (θ_{ν^i}) then coincides with the after-tax market price of risk on the security to be used.

If security P were not present, the agent would face a nonlinear optimization problem with tax rate τ^i_{S-} for short positions and τ^i_{S+} for long positions. In the presence of P, since he could always choose not to use P, the tax rate for short positions must always be at least τ^i_{S-}, and that for long positions at most τ^i_{S+}. In our second type of optimization problem, it is never optimal for the agent to use security P, so he uses the stock S for both negative and positive holdings, facing tax rates τ^i_{S-} and τ^i_{S+} respectively (when $\mathcal{N}^i_t = \left[\sigma_S d_S \tau^i_{S-}, \sigma_S d_S \tau^i_{S+}\right]$). In the third type of optimization problem, the mispricing makes security P favorable for short positions, long positions or both. Use of security P does not allow full linearization of the tax schedule, but yields a tax rate for short positions higher than τ^i_{S-} and /or yields a tax rate for long positions less than τ^i_{S+}. Both of these latter two types resemble the problem with only one security taxed differently on long versus short positions (Basak et al. (1998)).

Under piecewise linear taxation with a single kink at zero, the market price of risk (net of taxation) provided by a security depends only on the sign of the holding therein, not on its magnitude. It is never necessary for an agent to hold more than one risky security, and the whole of his risk exposure is taxed at a constant rate. Hence, in spite of the nonlinearity, no fictitious "endowment" term is needed to capture any risk exposure taxed at a higher rate, and $\tilde{g}^i = 0$.

For the remainder of the section, we assume logarithmic preferences: $u_i(c) \equiv \log c$, and no actual stochastic endowment: $\epsilon^i \equiv 0$. Proposition 3.4 reports agent i's explicit optimal policy.

Proposition 3.4. *For $u_i(c) \equiv \log c$ and $\epsilon^i \equiv 0$, agent i's optimal consumption is given by*

$$c^i(t) = \frac{X^i(t)}{T-t} , \tag{3.18}$$

and his optimal investment in the composite asset by

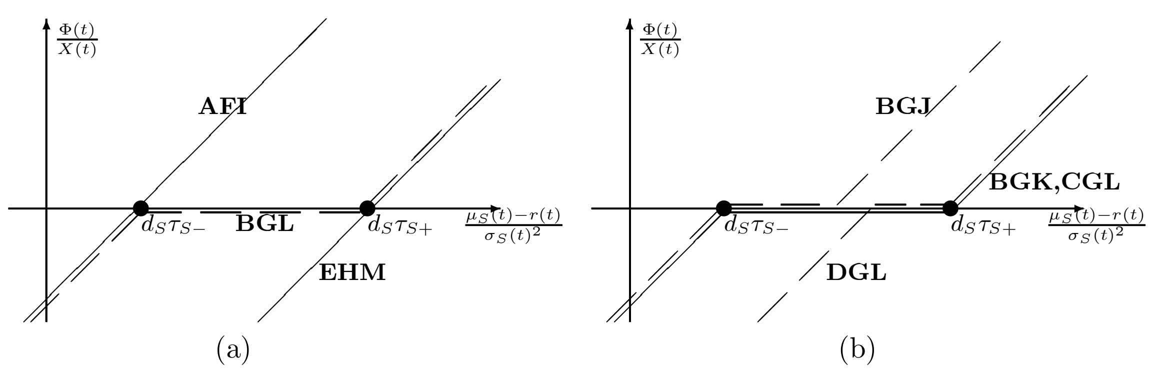
$$\Phi^i(t) = \frac{\theta_{\nu^i}(t)}{\sigma_S(t)} X^i(t) = \left(\frac{\mu_S(t) - r(t)}{\sigma_S(t)^2} - \frac{\nu^i(t)}{\sigma_S(t)^2} \right) X^i(t) , \tag{3.19}$$

$$\textit{where } \nu^i(t) = \begin{cases} \underline{\nu}^i(t) & \textit{if } \mu_S(t) - r(t) < \underline{\nu}^i(t) \\ (\mu_S(t) - r(t)) & \textit{if } \underline{\nu}^i(t) \le \mu_S(t) - r(t) \le \overline{\nu}^i(t) \\ \overline{\nu}^i(t) & \textit{if } \mu_S(t) - r(t) > \overline{\nu}^i(t) \end{cases} , \tag{3.20}$$

where $\underline{\nu}^i(t)$ and $\overline{\nu}^i(t)$ are as specified in Table II.

In a standard model with only security S and no taxation, the logarithmic agent would invest $\Phi^i(t) = (\mu_S(t) - r(t)) X^i(t)/\sigma_S(t)^2$ in the risky asset. Under nonlinear taxation, we always have $\nu^i(t) > 0$, implying the agent invests less, as a proportion of his wealth, in the risky assets. The taxation discourages him from positive investment in the dividend-paying securities. His strategy may remain linear in the pre-tax market price of risk, but shifted downwards; this occurs when he can use security P to fully symmetrize the tax schedule, when $\mathcal{N}_t^i$ is a singleton. When $\mathcal{N}_t^i$ is a closed interval, the agent's optimal strategy becomes nonlinear in the pre-tax market price of risk (Basak et al. (1998)). His long positions are reduced (as a fraction of his wealth) relative to the untaxed case, while his short positions are increased, but not by as much. As an example, Figure 1 plots the various optimal strategies of Panel (iii) in Table II. (Qualitatively the cases of Panels (i) and (ii) are similar.)

Figure 1: Optimal Strategies of a Logarithmic Agent for Several Values of the Mispricing when $d_P\,(\tau_{P+} - \tau_{P-}) > d_S\,(\tau_{S+} - \tau_{S-})$.

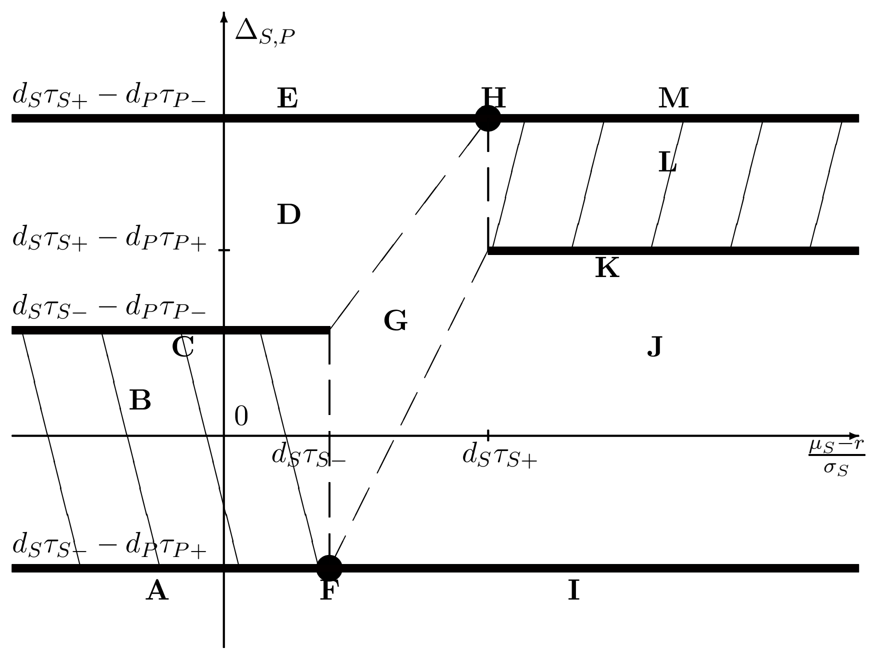


Panel (a) in the figure corresponds to the first two types of optimization problem discussed earlier (where either use of both securities effectively linearizes the problem or where S only is used), while Panel (b) corresponds to the third type (where security P is used but does not allow linearization of the problem). For example, DGL is followed when security P is favorable for short holdings (D), while S is favorable for long holdings (L).

For our future analysis of equilibrium, it will prove convenient to combine Table II with Table I Panel (a) and provide (necessary and sufficient) price conditions for our 13 original cases A-M. These, together with the value of the fictitious market parameter $\nu^i(t)$, are reported in Table I Panel (b), and are plotted in Figure 2. We note that the cases are mutually exclusive and cover the whole space of price parameters consistent with no unbounded arbitrage.

Figure 2 and Table I (last column) show that for low enough pre-tax market price of risk and low enough mispricing $\Delta_{S,P}$ (A, B, C, F, I), the effective tax rate is τ^i_{S-}. For high enough pre-tax market price of risk and $\Delta_{S,P}$ (E, H, K, L, M), the effective tax rate is τ^i_{S+}. For intermediate levels of both, the after-tax market price of risk is zero, and the agent holds no risky assets (G). Finally, when the mispricing is low and the pre-tax market price of risk is high (J) or vice-versa (D), the effective tax rate is that provided by P and lies intermediate between τ^i_{S-} and τ^i_{S+}. Ceteris paribus, a higher pre-tax market price of risk makes long positions preferred, and conversely. A higher $\Delta_{S,P}$ makes S preferable to P for long positions (and P to S for short positions), and a lower $\Delta_{S,P}$ makes it preferable for short positions.

Figure 2: Cases over the Price Parameter Space for a Logarithmic Agent



Note: - - - boundaries all belong to G.

$\nu^i = \sigma_S d_S \tau^i_{S-}$; $\nu^i = \sigma_S d_S \tau^i_{S+}$; $\nu^i \in \left[\sigma_S d_S \tau^i_{S-}, \sigma_S d_S \tau^i_{S+}\right]$.

Remark 3.2. Recall, in the case of linear taxation on both risky securities ($\tau^i_{j+} = \tau^i_{j-} \equiv \tau^i_j$, $j \in \{S, P\}$), the mispricing is uniquely pinned down: $\Delta_{S,P}(t) = d_S(t)\tau^i_S - d_P(t)\tau^i_P$. Hence, the fictitious market parameter also is pegged: for any t, $\nu^i(t) = \sigma_S(t)d_S(t)\tau^i_S$, and i faces a linear problem. S and P become perfect substitutes. Our thirteen cases collapse to three (A-E, F-H, I-M), whose occurrence depends on the sign of the tax-adjusted market price of risk on S. Given $\Phi^i(t)$, the agent is indifferent to all allocations between S and P.[8]

4. Equilibrium and the Role of Mispricing

Definition 4.1. *An* equilibrium *is a price system* $(r, \mu_S, \mu_P, \sigma_S, \sigma_P)$ *and admissible consumption-portfolio processes* (c^i, α^i), $i = 1, 2$, *such that: (i)* (c^i, α^i) *solves agent* i*'s optimization problem; (ii) security markets clear:*

$$\alpha^1_S(t) + \alpha^2_S(t) = 1 \ , \quad \alpha^1_P(t) + \alpha^2_P(t) = s_P \ , \quad X^1(t) + X^2(t) = S(t) + s_P P(t) \ ;$$

and (iii) the good market clears:

$$c^1(t) + c^2(t) = \delta_S(t) + s_P\delta_P(t) + \epsilon^1(t) + \epsilon^2(t) - T^1(t) - T^2(t) \equiv \mathcal{D}(t) \ . \tag{4.1}$$

4.1. Determination of the Mispricing

This subsection assumes that an equilibrium exists, with given sharing of risk across agents (Φ^1, Φ^2), and characterizes the mispricing. This will allow us to provide intuition on the economic role of mispricing, and the properties of equilibrium. For clarity, most analyses assume the taxation functions to be continuously differentiable, but the intuition is not limited to this case. For given Φ^1, Φ^2, the mispricing has to be set so that agents' portfolio holdings α^i_S, α^i_P determined from non-satiation, clear the financial markets. This yields the expression for the mispricing reported in Proposition 4.1.

Proposition 4.1. *Assume that equilibrium exists, and that agents' composite risk exposure choices are given by* $\Phi^1(t)$ *and* $\Phi^2(t)$*. Then, the mispricing is given by*

$$\begin{aligned}\Delta_{S,P}(t) \ = \ & \Big[d_S(t)t^{1\prime}_S\left(\tilde{\alpha}^1_S(t)\delta_S(t)\right) - d_P(t)t^{1\prime}_P\left((\Phi^1(t) - \tilde{\alpha}^1_S(t)S(t))\sigma_S(t)d_P(t)\right)\Big] \\ & * T^{1\prime}\left(t^1_S(\tilde{\alpha}^1_S(t)\delta_S(t)) + t^1_P((\Phi^1(t) - \tilde{\alpha}^1_S(t)S(t))\sigma_S(t)d_P(t))\right) \ ,\end{aligned} \tag{4.2}$$

[8] If the tax schedule has more than one kink, we still expect optimal risk exposure to be a piecewise linear function of the market price of risk on S, with positive measure regions where an agent has no incentive to leave the kinks in the tax schedules. However, the analysis becomes considerably more complicated due to a non-zero fictitious endowment $\tilde{g}^i$ in some states. This occurs when Φ^i is high enough for part of i's risk exposure to be rewarded at a rate other than the marginal rate: either he is holding more than one risky security, or he is holding enough of one security for his taxable income therein to lie beyond a non-zero kink in the tax schedule. The associated fictitious endowment is perceived by the agent as an additional (spanned) risky income stream and has ambiguous effects on his behavior.

where $\tilde{\alpha}_S^1(t) = \tilde{\alpha}_S^1\left(\Phi^1(t), \Phi^2(t), t\right)$ *satisfies:*

$$\begin{aligned} & \left[d_S(t)t_S^{1\prime}\left(\tilde{\alpha}_S^1(t)\delta_S(t)\right) - d_P(t)t_P^{1\prime}\left((\Phi^1(t) - \tilde{\alpha}_S^1(t)S(t))\sigma_S(t)d_P(t)\right)\right] \\ & *T^{1\prime}\left(t_S^1(\tilde{\alpha}_S^1(t)\delta_S(t)) + t_P^1((\Phi^1(t) - \tilde{\alpha}_S^1(t)S(t))\sigma_S(t)d_P(t)\right) \\ = & \left[d_S(t)t_S^{2\prime}\left((1-\tilde{\alpha}_S^1(t))\delta_S(t)\right) - d_P(t)t_P^{2\prime}\left((\Phi^2(t) - (1-\tilde{\alpha}_S^1(t))S(t))\sigma_S(t)d_P(t)\right)\right] \\ & *T^{2\prime}\left(t_S^2((1-\tilde{\alpha}_S^1(t))\delta_S(t)) + t_P^2((\Phi^2(t) - (1-\tilde{\alpha}_S^1(t))S(t))\sigma_S(t)d_P(t))\right) . \end{aligned} \quad (4.3)$$

At the individual level, the mispricing compensates each agent for differences in taxation across securities: he will adjust his portfolio holdings until the compensation is exact.[9] As seen in Section 3, by doing so he will in general not choose the portfolio strategy that minimizes the amount of taxes paid. In spite of this, the mispricing is set so that agents effectively "cooperate" and jointly minimize the amount of aggregate taxes paid, as reported in Corollary 4.1.

Corollary 4.1. *In equilibrium, the mispricing adjusts so that, among all pairs of portfolio holdings* $(\alpha_S^i(t), \alpha_P^i(t))$, $i = 1, 2$, *that implement agents' composite risk exposure choices* $\Phi^i(t)$ *and clear financial markets, agents choose the one that minimizes aggregate taxes* $T^1(t) + T^2(t)$.

Recall from Section 3.1 that for a given risk exposure, each individual chooses an allocation between S and P so as to maximize the (instantaneous) mean growth of his wealth, which amounts to maximizing his (instantaneous) price-arbitrage profit $-\alpha_P^i P \sigma_P \Delta_{S,P}$ minus the associated (instantaneous) tax bill T^i. Hence in aggregate, the sum of these terms across agents should be maximized. However, in equilibrium, aggregate holdings in P are constrained by clearing, so for given prices agents cannot manipulate the aggregate price-arbitrage profit. This leaves the total tax bill alone to be minimized. Unlike the tax-arbitrage profit, at the aggregate level any extra price-arbitrage profit earned by one agent is exactly offset by a loss to the other agent. Therefore, such profits, hence the mispricing itself, can be viewed as a "fee" paid by one agent to the other for agreeing to hold more of the (marginally) higher-taxed security. From (4.2), the more heavily marginally taxed security is indeed the one that is favorably mispriced. Our equilibrium examples in Sections 5 and 6 reveal that agents jointly minimize aggregate tax proceeds by the more heavily taxed agent holding more of the less taxed security, and conversely.

Equation (4.3) also reveals that, in equilibrium, agents' marginal tax rates ($T^{1\prime}$, $T^{2\prime}$) will generically *not* be equated. This is at odds with the bulk of existing models, and follows from our assuming nonlinear contributions of dividends to taxable income. This result is a driving force behind many of our main results, as the properties of equilibrium would be quite different were

[9] This can be compared to the work of Basak and Croitoru (1998), who also endogenize mispricing between redundant securities, using another type of market imperfection, portfolio constraints rather than taxes. As in the current model, mispricing plays a role in market clearing: limiting agents' heterogeneity in portfolio demands, in keeping with the constraints. The mispricing reflects heterogeneity across agents (rather than heterogeneity across assets as here). The mispricing does not preclude equilibrium, because the portfolio constraints keep arbitrage trades bounded.

equilibrium to require equalization of marginal tax rates: for example, in our setup, aggregate taxes could not be minimized, nor would clientele effects (Section 6) be possible.

Remark 4.1. Sufficient conditions for the existence of $\tilde{\alpha}_S^1(t)$ satisfying (4.3) are:

$$\lim_{\alpha_S\delta_S\to\infty} d_S t_S^{i\prime}(\alpha_S\delta_S) > \lim_{\alpha_P\delta_P\to-\infty} d_P t_P^{i\prime}(\alpha_P\delta_P)\,, \qquad \lim_{\alpha_P\delta_P\to\infty} d_P t_P^{i\prime}(\alpha_P\delta_P) > \lim_{\alpha_S\delta_S\to-\infty} d_S t_S^{i\prime}(\alpha_S\delta_S)\,,$$

for $i = 1, 2$. This requirement on tax schedules is intuitive, ensuring that agents do not "pay" an unbounded negative tax without taking on an unbounded exposure to risk.

4.2. Determination of Equilibrium

Although not our main focus, in this section we briefly lay out the elements to fully solve for the equilibrium. For convenience, we introduce a representative agent with possibly state-dependent utility function (Cuoco and He (1994)):

$$U(c;\lambda) \equiv \max_{c^1+c^2=c} u_1\left(c^1\right) + \lambda u_2\left(c^2\right)\,,$$

where agent 2's weight λ is allowed to be stochastic. Identifying $\lambda(t) \equiv u_1'(c^1(t))/u_2'(c^2(t))$, using agents' first order condition (3.16) and good market clearing, we obtain the following equilibrium conditions on the agent-specific state price densities, the stochastic weighting and the stock price:

$$\begin{aligned}
\xi^1(t) &= \frac{U'(\mathcal{D}(t);\lambda(t))}{U'(\mathcal{D}(0);\lambda(0))}\,, \quad \xi^2(t) = \frac{\lambda(0)}{\lambda(t)}\frac{U'(\mathcal{D}(t);\lambda(t))}{U'(\mathcal{D}(0);\lambda(0))}\,,\\
\frac{d\lambda(t)}{\lambda(t)} &= \left(\theta^2(t)-\theta^1(t)\right)\left[\theta^2(t)dt + dW(t)\right]\\
&= \left(\frac{\nu^1(t)-\nu^2(t)}{\sigma_S(t)}\right)\left[\left(\frac{\mu_S(t)-r(t)}{\sigma_S(t)} - \frac{\nu^2(t)}{\sigma_S(t)}\right)dt + dW(t)\right]\,,
\end{aligned}$$

where ν^i are determined from (3.15), $\mathcal{D}(t)$ is defined in (4.1) and $\lambda(0)$ solves either agent's static budget constraint, i.e.,[10]

$$E\left[\int_0^T U'(\mathcal{D}(t);\lambda(t))I^1\left(U'(\mathcal{D}(t);\lambda(t))\right)\,dt\right] = E\left[\int_0^T U'(\mathcal{D}(t);\lambda(t))\left(e_S^1\delta_S(t) + e_P^1\delta_P(t)\right)\,dt\right].$$

Under standard regularity, equilibrium asset prices are given by

$$\begin{aligned}
S(t) &= \frac{1}{\xi^i(t)}E\left[\int_t^T \xi^i(s)\left[1 - t_S^{i\prime}\left(\alpha_S^i(s)\delta_S(s)\right)T^{i\prime}(s)\right]\delta_S(s)ds|\mathcal{F}_t\right]\\
&= \frac{1}{U'(\mathcal{D}(t);\lambda(t))}E\left[\int_t^T U'(\mathcal{D}(s);\lambda(s))\left[1 - t_S^{i\prime}\left(\alpha_S^i(s)\delta_S(s)\right)T^{i\prime}(s)\right]\delta_S(s)ds|\mathcal{F}_t\right] \quad (4.4)
\end{aligned}$$

[10] The two agents' budget constraints are equivalent, and only determine the ratio y^1/y^2. We set $y^1 = U'(\delta(0);\lambda(0))$ without loss of generality so that $\xi^1(0) = \xi^2(0) = 1$.

and an analogous expression for P, where the composite risk exposure Φ^i and the holdings α^i_S, α^i_P are determined from (3.17) and (4.3). As in Ross (1987), dividends are priced based not on their actual cash-flows (neither pre-tax nor after-tax ones), but on their marginal contribution to agents' net after-tax income.

The evolution of the equilibrium distribution of consumption across agents (for which λ proxies) is driven primarily by the difference in their fictitious market parameters, hence by the differential in their effective tax rates. With homogeneous (or zero) taxation, λ would be constant, and the equilibrium allocation Pareto optimal. Heterogeneous taxation perturbs risk-sharing and typically drives the equilibrium away from a Pareto optimum.

In models without taxation (e.g., Karatzas, Lehoczky and Shreve (1987), Basak and Cuoco (1998)), equilibrium quantities can be expressed in terms of the weighting λ, which itself satisfies an equation where it is the only unknown. The solution of equilibrium is thus reduced to the determination of λ. In the presence of taxation, however, as in Basak and Gallmeyer (1998), d_S and d_P appear in the expression for the stochastic weighting dynamics. This is because, as was seen in Section 3, these quantities affect the effective taxation of assets. Hence, the solution of equilibrium has extra layers of complexity and it is necessary to solve for S, σ_S, P, σ_P. (4.4) reveals that it is also, in general, necessary to jointly solve for each agent's portfolio holdings to close the model: under nonlinear taxation, tax rates depend on agents' holdings. Hence, yet another additional layer of complexity appears over the case of linear taxation (Basak and Gallmeyer (1998)).

5. Equilibrium with a Zero Net Supply Security: The Tax-Relieving Role of Financial Innovation

This section shows how, in equilibrium, the presence of a zero net supply "derivative" security may allow some relief of the frictional taxation. Hence we derive a motivation for agents to introduce such a security. It is fairly intuitive that the tax relief will depend on the taxation of the derivative relative to the original stock. We additionally show that the use of the derivative is more likely and the tax relief is larger when the derivative is taxed more homogeneously across agents. As we have discussed earlier, the effective tax treatment of a security is driven not only by the tax schedule thereon but also by the dividends paid out. Hence, the design of the derivative security impacts the effectiveness of the tax relief; the more volatile its price and the lower its dividends, the more effective the tax relief.

For clarity, we assume agents' tax bills to be linear in taxable income. Then, if both securities are subject to linear taxation, adjustment of the mispricing between the securities to offset the tax-arbitrage will make them perfect substitutes, and so the derivative will play no role. Hence we require at least one of the securities to effectively be taxed nonlinearly. We will discuss two particular logarithmic utility examples, labeled Economies I and II , to illustrate our points. The former example consists of, for both agents, the stock S being subject to continuously differentiable taxation, while the derivative P is taxed linearly. Both securities remain heterogeneously taxed across agents. This example represents the most extreme illustration of relief of a nonlinear tax schedule via a derivative security. This is because we show that, despite the nonlinear taxation of the stock, agents share the risk in such a way that they effectively face linear taxation. The presence of a linearly taxed security fully linearizes each agent's problem. Taxes are redistributed to both agents in the economy.

The second example is less extreme, but illustrates the point more generally. In this case, we allow both securities to be taxed nonlinearly for one agent, so as to maintain nonlinearity in the equilibrium, but for simplicity we let the other agent pay linear taxes on both securities. To further aid in tractability, we assume the nonlinearly-taxed agent to be subject to piecewise linear taxation ((2.2)) on both securities. Here we assume that taxes are redistributed to the linearly taxed agent.

To address the tax-relieving role of the derivative, we introduce two benchmark economies, labeled a and b, for comparison, one with only the stock present, and one with no taxation. The economies, then, are summarized as follows:

Economy I: $u_i(c) = \log c$, $s_P = 0$;
$T^i(t) = t_S^i\left(\alpha_S^i(t)\delta_S(t)\right) + \tau_P^i \alpha_P^i(t)\delta_P(t)$, $i = 1, 2$, $t_S^1(\cdot) \neq t_S^2(\cdot)$.

Economy II: $u_i(c) = \log c$, $s_P = 0$;
$T^1(t) = \tau_S^1 \alpha_S^1(t)\delta_S(t) + \tau_P^1 \alpha_P^1(t)\delta_P(t)$, $\epsilon^1(t) = T^1(t) + T^2(t)$;
$T^2(t) = \sum_{j \in \{S,P\}} \left(\alpha_j^2(t)\delta_j(t)\right)^+ \tau_{j+}^2 - \left(\alpha_j^2(t)\delta_j(t)\right)^- \tau_{j-}^2$, $\epsilon^2(t) = 0$.

Benchmark a: No derivative P; S taxed as in the appropriate example.

Benchmark b: No derivative P; S untaxed.

5.1. Economy I: Both Agents Taxed Linearly on the Derivative

The analysis in this subsection generalizes readily to general preferences, but we specialize to log utility for comparison with Sections 5.2 and 6. Proposition 3.1 ((3.4)) specializes to

$$\Delta_{S,P}(t) = d_S(t) t_S^{i\prime}\left(\hat{\alpha}_S^i(t)\delta_S(t)\right) - d_P(t)\tau_P^i \,, \tag{5.1}$$

yielding $\hat{\alpha}_S^i\left(\Delta_{S,P}(t), t\right)$, independent of $\Phi^i(t)$. Substituting into (3.9), we find each agent to face a budget constraint linear in Φ^i:

$$\begin{aligned} dX^i(t) = &\left[X^i(t)r(t) + \epsilon^i(t) - c^i(t)\right]dt + \Phi^i(t)\sigma_S(t)\left\{\left[\frac{\mu_P(t) - r(t)}{\sigma_P(t)} - d_P(t)\tau_P^i\right]dt + dW(t)\right\} \\ &+ \left\{\hat{\alpha}_S^i\left(\Delta_{S,P}(t), t\right)\delta_S(t) t_S^{i\prime}\left(\hat{\alpha}_S^i\left(\Delta_{S,P}(t), t\right)\delta_S(t)\right) - t_S^i\left(\hat{\alpha}_S^i\left(\Delta_{S,P}(t), t\right)\delta_S(t)\right)\right\}dt\,. \end{aligned} \tag{5.2}$$

Each agent faces an effectively linear taxation driven by the tax rate on the linearly taxed security, P. An explanation is that, to prevent arbitrage, the mispricing must adjust to locally make the derivative a perfect substitute for the stock, and the derivative's local tax rate is always τ_P^i.

Then, agent i only uses the linearly taxed security to implement adjustments in Φ^i and hence faces a linear problem. The nonlinearity in the stock's taxation is taken care of by the extra (risk-exposure independent) "endowment" term in the dynamic budget constraint; some part of agent i's existing holding is taxed at a rate other than the marginal rate τ_P^i, according to the tax schedule on S.

The equilibrium in Economy I is, then, similar to the case of heterogeneous linear taxation for both agents (Basak and Gallmeyer (1998)), only adjusted for the fictitious endowments. Proposition 4.2 immediately yields the mispricing and holdings in S, independently of Φ^i. Proposition 5.1 summarizes the characterization of equilibrium.

Proposition 5.1. *If equilibrium exists in Economy I, the mispricing, individual-specific (after-tax) market prices of risk, interest rate and stochastic weighting dynamics are given by*

$$\Delta_{S,P}(t) = d_S(t) t_S^{1\prime}\left(\tilde{\alpha}_S^1(t)\delta_S(t)\right) - d_P(t)\tau_P^1 \ , \tag{5.3}$$

where $\tilde{\alpha}_S^1(t)$ is determined from aggregate tax minimization, or

$$d_S(t) t_S^{1\prime}\left(\tilde{\alpha}_S^1(t)\delta_S(t)\right) - d_P(t)\tau_P^1 = d_S(t) t_S^{2\prime}\left(\left(1-\tilde{\alpha}_S^1(t)\right)\delta_S(t)\right) - d_P(t)\tau_P^2 \ ; \tag{5.4}$$

$$\theta^1(t) = \sigma_{\delta_S}(t) - \frac{\lambda(t)}{1+\lambda(t)} d_P(t)\left(\tau_P^1 - \tau_P^2\right) \ , \tag{5.5}$$

$$\theta^2(t) = \sigma_{\delta_S}(t) - \frac{1}{1+\lambda(t)} d_P(t)\left(\tau_P^2 - \tau_P^1\right) \ ; \tag{5.6}$$

$$r(t) = \mu_{\delta_S}(t) - \frac{1}{2}\sigma_{\delta_S}(t)^2 - \frac{\lambda(t)}{(1+\lambda(t))^2} d_P(t)^2\left(\tau_P^1 - \tau_P^2\right)^2 \ ; \tag{5.7}$$

$$d\lambda(t) = \lambda(t) d_P(t)\left(\tau_P^1 - \tau_P^2\right)\left\{\left[\sigma_{\delta_S}(t) + \frac{1}{1+\lambda(t)} d_P(t)\left(\tau_P^1 - \tau_P^2\right)\right] dt + dW(t)\right\} \ . \tag{5.8}$$

The expressions (except for $\Delta_{S,P}$) are identical to an economy with a single security taxed as P, linearly but differentially across agents. The tax differential, appropriately normalized, acts as an additional driving factor in the interest rate and the market prices of risk (and hence in the CCAPM). Interestingly enough, the taxation on S does not appear in the expressions. (It impacts the values of the equilibrium quantities, however, because it affects agents' wealths via the endowment term in their budget constraint, hence the value of λ.) A comparison with benchmark economy b (no taxation) reveals the higher taxed agent to have a reduced market price of risk and the lower taxed to have an increased market price of risk in Economy I, and the interest rate to be reduced by the differential taxation. Finally, at the individual level, each agent's holding in the stock depends only on the mispricing and the tax schedules (and d_S and d_P). Holdings in P, on the other hand, also reflect the optimal risk exposure.

5.2. Economy II: One Logarithmic Agent Linearly Taxed on Both Securities; One Logarithmic Agent Progressively Piecewise Linearly Taxed on Both

Since agent 1 is linearly taxed on both securities, (3.4) uniquely pins down the mispricing as

$$\Delta_{S,P}(t) = d_S(t)\tau_S^1 - d_P(t)\tau_P^1 \,, \tag{5.9}$$

independently of agents' holdings. The role of the mispricing is immediate from no-arbitrage, preventing the linearly taxed agent from taking on an infinite tax-arbitrage position due to the heterogeneous taxation of S and P. The linearly taxed agent is made indifferent between the two securities. Then, to prevent unbounded arbitrage for the nonlinearly taxed agent for all values of d_S and d_P, equilibrium requires

$$\tau_{S-}^2 \leq \tau_S^1 \leq \tau_{S+}^2 \quad \text{and} \quad \tau_{P-}^2 \leq \tau_P^1 \leq \tau_{P+}^2 \,.$$

The nonlinearly taxed agent must everywhere face a worse tax schedule in both assets and hence also in the composite risky asset than the linearly taxed agent: for long positions, he is taxed at a higher rate and for short positions he is "rebated" at a lower tax rate.

Since agent 2 is the only one taxed nonlinearly, we will distinguish equilibria only by his situation from Table I. Analysis reveals that the only (non-zero measure) equilibria (consistent with a nonnegative σ_S) are cases G, J, L, having $\Phi^2 \geq 0$. The intuition is that since agent 2 is unfavorably taxed (relative to agent 1) on negative holdings of the composite, if his holding were to be negative, agent 1 would also choose a negative holding, inconsistent with the positive supply of the composite. The nonlinearly taxed agent either: holds neither risky security (G); invests all his composite holdings in the derivative (J); or all in the stock (L). Accordingly, he never holds any risky security short.

Proposition 5.2 summarizes the conditions for each of these three equilibria to occur.

Proposition 5.2. *Assume that equilibrium exists in Economy II. Then:*
Case (G) exhibits $\alpha_S^1(t) = 1$, $\alpha_P^1(t) = 0$, $\alpha_S^2(t) = 0$, $\alpha_P^2(t) = 0$ *and occurs if and only if*

$$(1+\lambda(t))\sigma_{\delta_S}(t) \leq \min\left\{ d_S(t)\left(\tau_{S+}^2 - \tau_S^1\right), d_P(t)\left(\tau_{P+}^2 - \tau_P^1\right)\right\} \,.$$

Case (J) exhibits $\alpha_S^1(t) = 1$, $\alpha_P^1(t) < 0$, $\alpha_S^2(t) = 0$, $\alpha_P^2(t) > 0$ *and occurs if and only if*

$$d_P(t)\left(\tau_{P+}^2 - \tau_P^1\right) < d_S(t)\left(\tau_{S+}^2 - \tau_S^1\right) \;\; \textit{and} \;\; d_P(t)\left(\tau_{P+}^2 - \tau_P^1\right) < (1+\lambda(t))\sigma_{\delta_S}(t) \,.$$

Case (L) exhibits $\alpha_S^1(t) < 1$, $\alpha_P^1(t) = 0$, $\alpha_S^2(t) > 0$, $\alpha_P^2(t) = 0$ *and occurs if and only if*

$$d_S(t)\left(\tau_{S+}^2 - \tau_S^1\right) < d_P(t)\left(\tau_{P+}^2 - \tau_P^1\right) \;\; \textit{and} \;\; d_S(t)\left(\tau_{S+}^2 - \tau_S^1\right) < (1+\lambda(t))\sigma_{\delta_S}(t) \,.$$ [11]

[11] To show that all three cases are consistent with "reasonable" parameter values, take for example: $S = 100$, $P = 200$, $\delta_S = 4$, $\delta_P = 16$, $\sigma_S = 0.1$, $\sigma_P = 0.12$, $\sigma_{\delta_S} = 0.05$, $\tau_S^1 = \tau_P^1 = 0.2$, $\tau_{S+}^2 = \tau_{P+}^2 = 0.4$, $\tau_{S-}^2 = \tau_{P-}^2 = 0.1$. Then, the condition for (G) is: $\lambda \leq 0.6$; (L) obtains if this is not the case, since $d_S\left(\tau_{S+}^2 - \tau_S^1\right) = 0.08 < d_P\left(\tau_{P+}^2 - \tau_P^1\right) = 0.13$, but only a slight modification of the parameters would yield case (J).

Which case arises is driven by which security is taxed more heterogeneously across agents and also by the distribution of wealth across these. Consistent with aggregate tax minimization, the former decides which risky securities are traded (S only or both S and P); the latter decides which agents hold risky securities (1 only or both 1 and 2). The linearly taxed agent is indifferent between the two securities, so it is the nonlinearly taxed agent who determines whether P is used or not. When agent 2 is relatively poor (G: low λ), he (as a CRRA agent) desires low risk exposure and his piecewise linear taxation drives him to its kink, so he holds nothing in the composite, nor in either S or P. When agent 2 is relatively wealthy (high λ), his holding can move away from the kink to a positive holding in the composite. Since agent 2 is taxed higher than 1, he will choose to put all his composite holding in the security that is taxed more homogeneously across agents. In L it is the stock and, in J, the derivative. This choice also agrees with aggregate tax minimization: for given risk sharing, the lower the heterogeneity in taxation, the more effectively the tax rebate to the short agent (or the agent who reduces an existing long position) offsets the tax bill of the long agent and thus reduces the aggregate tax bill. A higher volatility for the derivative (hence a lower d_P) is also desirable.

The three types of equilibria have properties similar either to the case of one agent restricted from participating in the risky markets (Basak and Cuoco (1998)) (G), or to the case of heterogeneous linear taxation in one security (Basak and Gallmeyer (1998)) (J, L). In the latter case the equilibrium is as if the nonlinear agent faced the more favorable asset only (P in J and S in L). Proposition 5.3 characterizes the three equilibria.

Proposition 5.3. *If equilibrium exists and the following cases occur, the individual-specific (after-tax) market prices of risk, the interest rate, and the dynamics of the stochastic weighting are as follows:*

In case (G):
$$\theta^1(t) = (1+\lambda(t))\sigma_{\delta_S}(t)\ , \quad \theta^2(t) = 0\ ,$$
$$r(t) = \mu_{\delta_S}(t) - (1+\lambda(t))\sigma_{\delta_S}(t)^2\ ,$$
$$d\lambda(t) = -\lambda(t)(1+\lambda(t))\sigma_{\delta_S}(t)dW(t)\ .$$

In case (J):
$$\theta^1(t) = \sigma_{\delta_S}(t) + \frac{\lambda(t)}{1+\lambda(t)} d_P(t)\left(\tau^2_{P+} - \tau^1_P\right)\ ,$$
$$\theta^2(t) = \sigma_{\delta_S}(t) - \frac{1}{1+\lambda(t)} d_P(t)\left(\tau^2_{P+} - \tau^1_P\right)\ ,$$
$$r(t) = \mu_{\delta_S}(t) - \sigma_{\delta_S}(t)^2 - \frac{\lambda(t)}{(1+\lambda(t))^2} d_P(t)^2\left(\tau^2_{P+} - \tau^1_P\right)^2\ ,$$
$$d\lambda(t) = -\lambda(t)d_P(t)\left(\tau^2_{P+} - \tau^1_P\right)\left\{\left[\sigma_{\delta_S}(t) - \frac{1}{1+\lambda(t)} d_P(t)\left(\tau^2_{P+} - \tau^1_P\right)\right]dt + dW(t)\right\}\ .$$

In case (L):
$$\theta^1(t) = \sigma_{\delta_S}(t) + \frac{\lambda(t)}{1+\lambda(t)} d_S(t)\left(\tau^2_{S+} - \tau^1_S\right)\ ,$$
$$\theta^2(t) = \sigma_{\delta_S}(t) - \frac{1}{1+\lambda(t)} d_S(t)\left(\tau^2_{S+} - \tau^1_S\right)\ ,$$

$$r(t) = \mu_{\delta_S}(t) - \sigma_{\delta_S}(t)^2 - \frac{\lambda(t)}{(1+\lambda(t))^2} d_S(t)^2 \left(\tau^2_{S+} - \tau^1_S\right)^2 \ ,$$

$$d\lambda(t) = -\lambda(t) d_S(t) \left(\tau^2_{S+} - \tau^1_S\right) \left\{ \left[\sigma_{\delta_S}(t) - \frac{1}{1+\lambda(t)} d_S(t) \left(\tau^2_{S+} - \tau^1_S\right)\right] dt + dW(t) \right\} \ .$$

Agent 2 always has the lower market price of risk; this is because for long positions he faces the higher tax rate.

Comparing with the two benchmark economies, for given wealth distribution $\lambda(t)$, security prices $S(t)$, $P(t)$, and volatilities $\sigma_S(t)$, $\sigma_P(t)$, we have for any of the equilibria:

$$\theta^2_a(t) \le \theta^2_{II}(t) \le \theta_b(t) \le \theta^1_{II}(t) \le \theta^1_a(t) \ ,$$

$$r_a(t) \le r_{II}(t) \le r_b(t) \ .$$

The frictional taxation adversely affects risk-sharing across agents so that their market prices of risk diverge from the common value taken on in a frictionless market (b). As a result, the interest rate decreases to compensate for the increased tendency towards precautionary saving. However, the divergence in risk-sharing and corresponding drop in the interest rate are not as pronounced as in the economy with only the stock present (a). The derivative provides a means for the agents to partially circumvent the frictional taxation.

6. Equilibrium with Two Positive Net Supply Securities: Clientele Effects

This section shows how clientele effects may arise in equilibrium and examines the properties of the resulting equilibrium. We define a *clientele* effect as a situation where agents trade in disjoint sets of risky securities. This naturally requires the assumption that both risky securities are in positive net supply ($s_P = 1$), retained for the whole of this section. (If a zero net supply security is present, either both agents trade in it or it is not traded at all.) This definition includes both clientele effects "in prices" as studied by Dybvig and Ross (1986), with each agent holding the whole stock of a single risky security, as well as an "endogenous restricted participation" situation where one agent holds the whole stock of both risky securities, and so the other agent does not participate in the markets for these. The former situation is of more interest to us here.

As in the previous section, for clarity we assume tax bills to be linear in taxable income, which amounts to agents lying in exogenously specified, heterogeneous tax brackets. Then, for any clientele situation to occur, it is necessary for each asset to contribute nonlinearly (with a kink at zero) to taxable income for at least one agent. Otherwise, for example if either asset is linearly taxed for both agents, both agents will generically trade in that asset. For symmetry and in order to simplify the expressions while retaining sufficiently rich a model, we introduce Economy III, where each agent faces a piecewise linear tax schedule with a single kink at zero

((2.2)) on each asset, and zero tax on negative holdings, on each security, under the assumption of log utility and the tax proceeds taken out of the economy:[12]

$$\begin{aligned} \textit{Economy III}: \quad & u_i(c) \equiv \log c,\ \epsilon^i \equiv 0,\ s_P = 1; \\ & T^i(t) = \left[\left(\alpha_S^i(t)\delta_S(t) \right)^+ + \left(\alpha_P^i(t)\delta_P(t) \right)^+ \right] \tau_+^i;\ \tau_+^2 > \tau_+^1. \end{aligned}$$

With respect to the "general" case (with $\tau_{j-}^i \neq 0$, $\tau_{S+}^i \neq \tau_{P+}^i$), these assumptions simplify notation but should entail little loss of interest. In particular, heterogeneity is preserved across assets (because effective tax rates also depend on d_S and d_P), as well as across agents (because, in equilibrium, both agents cannot take on short positions, so they will never face the zero tax rate simultaneously).

As before, it will prove useful to distinguish several cases. We will denote by (A,B) the case where agent 1 is in A and 2 in B, etc.. From now on, we will focus only on the following three cases, which are the only ones possible in equilibrium, not of measure zero and exhibiting clientele effects:[13] (K,G), (J,L), (L,J).[14] In (J,L) and (L,J), each agent holds all of the aggregate stock of one risky asset. Whether the economy is in (J,L) or (L,J), i.e., which agent holds which asset, depends on whether $d_S(t)/d_P(t)$ is greater or less than one, i.e., which asset is effectively more taxed: the lower taxed agent holds the more taxed asset, and conversely. This is consistent with the minimization of aggregate tax proceeds. In (K,G) the higher-taxed agent (2) does not trade at all in the risky securities and the lower-taxed (1) holds the whole supply of both.

Proposition 6.1 provides necessary and sufficient conditions for each of the three cases:

Proposition 6.1. *Assume that equilibrium exists in Economy III. Then the equilibria exhibiting clientele effects arise as follows.*
Case (K,G) exhibits $\alpha_S^1(t) = \alpha_P^1(t) = 1$, $\alpha_S^2(t) = \alpha_P^2(t) = 0$ *and occurs if and only if*

$$1 + \lambda(t) \leq \frac{\min\{d_S(t), d_P(t)\}\left(\tau_+^2 - \tau_+^1\right)}{\sigma_{\mathcal{D}}(t)}. \tag{6.1}$$

Case (J,L) exhibits $\alpha_S^1(t) = 0$, $\alpha_P^1(t) = 1$, $\alpha_S^2(t) = 1$, $\alpha_P^2(t) = 0$ *and occurs if and only if* $d_S(t) < d_P(t)$ *and*

$$\frac{d_S(t)\left(\tau_+^2 - \tau_+^1\right)}{\frac{P(t)\sigma_P(t)}{X^1(t)} - \sigma_{\mathcal{D}}(t)} < \frac{1 + \lambda(t)}{\lambda(t)} < \frac{\min\{d_S(t)\tau_+^2, d_P(t)\left(\tau_+^2 - \tau_+^1\right)\}}{\frac{P(t)\sigma_P(t)}{X^1(t)} - \sigma_{\mathcal{D}}(t)}. \tag{6.2}$$

Case (L,J) exhibits $\alpha_S^1(t) = 1$, $\alpha_P^1(t) = 0$, $\alpha_S^2(t) = 0$, $\alpha_P^2(t) = 1$ *and occurs if and only if*

[12] The latter assumption is made only for tractability. Our intuition for the occurrence of clientele effects is unrelated to whether or not tax proceeds are taken out of the economy.

[13] Equilibria are eliminated using Table I based on: market clearing, agreement of agents on the mispricing and agreement on the assets' prices. Nine cases (not of measure zero) remain possible in equilibrium: (I,L), (J,K), (J,L), (K,G), (K,J), (K,L), (L,J), (L,K), (M,J).

[14] Note that in these three cases both agents' composite risk exposures are non-negative. This is because market clearing requires an agent who desires a short position to have a counterparty: hence, if one agent desires a short position, both agents have to trade together in at least one asset and no clientele effect occurs.

$d_S(t) > d_P(t)$ *and*

$$\frac{d_P(t)\left(\tau_+^2 - \tau_+^1\right)}{\frac{S(t)\sigma_S(t)}{X^1(t)} - \sigma_{\mathcal{D}}(t)} < \frac{1+\lambda(t)}{\lambda(t)} < \frac{\min\left\{d_P(t)\tau_+^2, d_S(t)\left(\tau_+^2 - \tau_+^1\right)\right\}}{\frac{S(t)\sigma_S(t)}{X^1(t)} - \sigma_{\mathcal{D}}(t)} . \tag{6.3}$$

The after-tax aggregate dividend dynamics satisfy $d\mathcal{D}(t) = \mathcal{D}(t)\left[\mu_{\mathcal{D}}(t)dt + \sigma_{\mathcal{D}}(t)dW(t)\right]$ *with, respectively,* $\mathcal{D}(t) = \left(\delta_S(t) + \delta_P(t)\right)\left(1 - \tau_+^1\right)$, $\delta_S(t)\left(1-\tau_+^2\right) + \delta_P(t)\left(1-\tau_+^1\right)$ *or* $\delta_S(t)\left(1-\tau_+^1\right) + \delta_P(t)\left(1-\tau_+^2\right)$, *in the three cases.*[15]

The case the economy falls into is driven by the distribution of wealth across agents ($\lambda = X^2/X^1$), the relative tax rates of the agents and the relative dividends (d_j) paid out by the assets. Conditions (6.1)-(6.3) all reveal clientele situations to be more likely when agents' tax rates are further apart, which makes it easier for the agents to perceive securities differently.

(K,G) occurs if the lower-taxed agent (1) is rich enough. From the viewpoint of aggregate tax minimization, it is always better for him to hold more of the taxed risky securities. For him to hold all of both risky securities requires him to be rich enough because, under logarithmic preferences, the amount he invests in risky securities is proportional to his wealth. Specifically, occurrence of (K,G) is driven by the lower taxed agent's wealth relative to the value of the aggregate stock of risky securities and the aggregate risk in the economy; the higher either is, the more difficult it is to entice agent 1 to take on the whole stock, and the less likely (K,G). Prices must be able to adjust enough to make agent 1 want to hold the whole supply of both S and P, while not enticing the higher-taxed agent to hold any risky securities. This is facilitated by a higher tax differential between the two agents. In contrast to (K,G), situations (J,L) and (L,J), where each agent holds the whole supply of a different risky security, occur when agents' wealths are relatively close to each other. The optimum from the aggregate tax minimization viewpoint is for the lower-taxed agent to hold the (effectively) higher-taxed security, and conversely. Accordingly, which one of the two cases occurs depends on which asset is the lower taxed (via a lower d_j); this decides which agent holds which asset.

Proposition 6.2 characterizes the three equilibria with clientele effects.

Proposition 6.2. *If equilibrium exists in Economy III and the following clientele cases occur, the pre-tax mispricing, individual-specific market prices of risk, interest rate and stochastic weighting dynamics are as follows:*

$$\text{In (K,G):}\quad \Delta_{S,P}(t) = \left(d_S(t) - d_P(t)\right)\tau_+^1 ,$$
$$\theta^1(t) = (1+\lambda(t))\sigma_{\mathcal{D}}(t) , \quad \theta^2(t) = 0 ,$$

[15]To show that these conditions are consistent with "reasonable" parameter values, take for example: $S = 100$, $P = 200$, $\delta_S = 4$, $\delta_P = 16$, $\sigma_S = 0.1$, $\sigma_P = 0.12$, $\tau_+^1 = 0.1$, $\tau_+^2 = 0.45$, $X^1 = 130$, implying $\sigma_{\mathcal{D}} = 0.11$, $X^2 = 170$, $(1+\lambda)/\lambda = 2.31$. Then, the condition for (J,L) is satisfied since $1.96 < (1+\lambda)/\lambda < 2.53$, (the condition for (L,J) can analogously be satisfied since it is a mirror image of (J,L), obtained by swapping agents). The condition for (K,G) is then $1 + \lambda \leq 1.24$, also feasible, for different agents' wealths.

$$
\begin{aligned}
r(t) &= \mu_{\mathcal{D}}(t) - (1+\lambda(t))\sigma_{\mathcal{D}}(t)^2 \ , \\
d\lambda(t) &= -(1+\lambda(t))\sigma_{\mathcal{D}}(t) dW(t) \ .
\end{aligned}
$$

$$
\begin{aligned}
\textit{In (J,L):}\quad \Delta_{S,P}(t) &= \frac{1+\lambda(t)}{\lambda(t)}\left(\sigma_{\mathcal{D}}(t) - \frac{P(t)\sigma_P(t)}{X^1(t)}\right) + d_S(t)\tau_+^2 - d_P(t)\tau_+^1 < 0 \ , \\
\theta^1(t) &= \frac{P(t)\sigma_P(t)}{X^1(t)} \ , \quad \theta^2(t) = \frac{S(t)\sigma_S(t)}{\lambda(t)X^1(t)} \ , \\
r(t) &= \mu_{\mathcal{D}}(t) - \sigma_{\mathcal{D}}(t)^2 - \frac{1}{\lambda(t)}\left(\sigma_{\mathcal{D}}(t) - \frac{P(t)\sigma_P(t)}{X^1(t)}\right)^2 \ , \\
d\lambda(t) &= (1+\lambda(t))\left(\sigma_{\mathcal{D}}(t) - \frac{P(t)\sigma_P(t)}{X^1(t)}\right)\left(\frac{S(t)\sigma_S(t)}{\lambda(t)X^1(t)}dt + dW(t)\right) \ .
\end{aligned}
\tag{6.4}
$$

$$
\begin{aligned}
\textit{In (L,J):}\quad \Delta_{S,P}(t) &= \frac{1+\lambda(t)}{\lambda(t)}\left(\frac{S(t)\sigma_S(t)}{X^1(t)} - \sigma_{\mathcal{D}}(t)\right) + d_S(t)\tau_+^1 - d_P(t)\tau_+^2 > 0 \ , \\
\theta^1(t) &= \frac{S(t)\sigma_S(t)}{X^1(t)} \ , \quad \theta^2(t) = \frac{P(t)\sigma_P(t)}{\lambda(t)X^1(t)} \ , \\
r(t) &= \mu_{\mathcal{D}}(t) - \sigma_{\mathcal{D}}(t)^2 - \frac{1}{\lambda(t)}\left(\frac{S(t)\sigma_S(t)}{X^1(t)} - \sigma_{\mathcal{D}}(t)\right)^2 \ , \\
d\lambda(t) &= (1+\lambda(t))\left(\sigma_{\mathcal{D}}(t) - \frac{S(t)\sigma_S(t)}{X^1(t)}\right)\left(\frac{P(t)\sigma_P(t)}{\lambda(t)X^1(t)}dt + dW(t)\right) \ .
\end{aligned}
\tag{6.5}
$$

In case (K,G), characterizations are similar to the restricted participation model of Basak and Cuoco (1998). Cases (J,L) and (L,J), however, illustrate the extra level of complexity in the solution of equilibrium, as we discussed in Section 4. Equilibrium quantities are functions of not only the weighting λ and the risky security prices S and P, but also of either agent's wealth process. In some sense, the stochastic weighting (or agents' relative wealths) does not play a central role in these clientele "in prices" situations. This is because the agents do not trade risky securities with each other. The equilibrium is instead driven by each agent's wealth relative to the value of the security whose aggregate supply he is holding.

The individual-specific market prices of risk are set so that the appropriate agent exactly demands the aggregate supply of one or both risky securities. We may deduce that, in all three cases: $\theta^2(t) < \sigma_{\mathcal{D}}(t) < \theta^1(t)$. Noting that $\sigma_{\mathcal{D}}(t)$ equals the market price of risk in a frictionless, benchmark economy with identical aggregate consumption ($\mathcal{D}$), we see that, with respect to such an economy, the lower taxed agent's market price of risk is increased and the higher taxed agent's one is decreased (as in an economy with a single stock and differential linear taxation (Basak and Gallmeyer (1998))). Even though, in (J,L) and (L,J), the lower taxed agent (1) holds the effectively higher taxed security, he remains subject to an effectively lower taxation than agent 2, via the mispricing favoring the security he is holding. The stochastic weighting dynamics reveal the lower taxed agent to tend to become relatively wealthier over time, on average, at the expense of the higher taxed agent. In all three cases, the interest rate is unambiguously reduced with respect to a frictionless economy with identical aggregate consumption ($r(t) = \mu_{\mathcal{D}}(t) - \sigma_{\mathcal{D}}(t)^2$).

In case (K,G), the role of the mispricing is to make agent 1 indifferent between the two risky securities S and P, so its value reflects the differential in their effective taxation for agent 1. In cases (J,L) and (L,J), the mispricing must prevent both agents from deviating from their zero-holding in one security, while clearing both markets. Hence, the mispricing reflects the differential in the premia required by each agent to hold S or P. In these regions, the price of each security is set so that one agent demands the aggregate supply therein and hence solely reflects that agent's shadow cost. This is reflected in equations (6.4) and (6.5).

7. The Case of Linear Taxable Income

We now make the assumption, typical in earlier work, that taxable income is linear in the dividends received from the risky securities. Then, agent i's tax bill can be written as:

$$T^i(t) = T^i\left(\tau_S^i \alpha_S^i(t)\delta_S(t) + \tau_P^i \alpha_P^i(t)\delta_P(t)\right) .$$

(A particular case is that of taxable income being equal to total dividend income, where $\tau_S^i = \tau_P^i = 1$.) Proposition 3.1 ((3.4)) then yields the mispricing as satisfying:

$$\Delta_{S,P}(t) = \left(d_S(t)\tau_S^i - d_P(t)\tau_P^i\right) T^{i\prime}(t) , \tag{7.1}$$

revealing agent i's marginal tax rate $T^{i\prime}$ (and hence taxable income and tax bill) to be equal to $\Delta_{S,P}(t)/\left(\tau_S^i d_S(t) - \tau_P^i d_P(t)\right)$ and hence independent of $\Phi^i(t)$. Hence, agents face a linear problem. This can be verified using the optimal portfolio holdings, that now obtain (using (3.1) and (7.1)) as explicit functions of Φ^i and $\Delta_{S,P}$:

$$\begin{aligned} \hat{\alpha}_S^i(t) &= \frac{\left(T^{i\prime}\right)^{-1}\left(\frac{\Delta_{S,P}(t)}{\tau_S^i d_S(t) - \tau_P^i d_P(t)}\right) - \sigma_S(t)\tau_P^i(t) d_P(t)\Phi^i(t)}{S(t)\sigma_S(t)\left(\tau_S^i d_S(t) - \tau_P^i d_P(t)\right)} , & (7.2) \\ \hat{\alpha}_P^i(t) &= \frac{\sigma_S(t)}{P(t)\sigma_P(t)}\left(\Phi^i(t) - \hat{\alpha}_S^i(t)S(t)\right) . & (7.3) \end{aligned}$$

Substitution into the dynamic budget constraint (2.3) yields

$$\begin{aligned} dX^i(t) &= \left[X^i(t)r(t) + \epsilon^i(t) - c^i(t)\right]dt + \Phi^i(t)\sigma_S(t)dW(t) \\ &+ \frac{\Phi^i(t)}{\tau_S^i d_S(t) - \tau_P^i d_P(t)}\left[\frac{\sigma_S(t)}{\sigma_P(t)}\tau_S^i d_S(t)\left(\mu_P(t) - r(t)\right) - \tau_P^i d_P(t)\left(\mu_S(t) - r(t)\right)\right] dt \qquad (7.4) \\ &+ \frac{\Delta_{S,P}(t)}{\tau_S^i d_S(t) - \tau_P^i d_P(t)}\left(T^{i\prime}\right)^{-1}\left(\frac{\Delta_{S,P}(t)}{\tau_S^i d_S(t) - \tau_P^i d_P(t)}\right) dt - T^i\left(\left(T^{i\prime}\right)^{-1}\left(\frac{\Delta_{S,P}(t)}{\tau_S^i d_S(t) - \tau_P^i d_P(t)}\right)\right) dt , \end{aligned}$$

which is always linear in the composite holding $\Phi^i(t)$, notwithstanding the nature of the tax function $T^i(\cdot)$. The nonlinear taxation, however, does add a stochastic "endowment" term to the budget constraint. Since each agent effectively faces a linear problem, the equilibrium is similar to the case of heterogeneous linear taxation (Basak and Gallmeyer (1998)), but with stochastic, endogenous effective tax rates.

Important simplifications occur under the additional assumption that the contributions of securities to taxable income are homogeneous across agents: $\tau_j^1 = \tau_j^2 \equiv \tau_j$, $i = 1, 2$, $j \in \{S, P\}$. This is the case in the earlier work of Schaefer (1982), Dammon and Green (1987), Dybvig and Ross (1988). The determination of the mispricing is simplified with respect to Proposition 4.1, as rearranging (7.1), summing across agents and using clearing yield

$$\left(T^{1\prime}\right)^{-1}\left(\frac{\Delta_{S,P}(t)}{\tau_S d_S(t) - \tau_P d_P(t)}\right) + \left(T^{2\prime}\right)^{-1}\left(\frac{\Delta_{S,P}(t)}{\tau_S d_S(t) - \tau_P d_P(t)}\right) = \tau_S \delta_S(t) + s_P \tau_P \delta_P(t)\ . \quad (7.5)$$

Unlike in the general case, there is no direct dependence of the mispricing on agents' composite risk exposure choices. In addition, from (7.1), agents' agreement on the mispricing implies

$$T^{1\prime}(t) = T^{2\prime}(t)\ .$$

This explicitly reveals the often-quoted notion that in the presence of redundant securities agents will adjust holdings to equate marginal tax rates (e.g., Schaefer (1982)), notwithstanding heterogeneity in taxation, wealth, and risk aversion. Our work, however, shows this finding to depend crucially on taxable income being linear in dividends, and computed homogeneously across agents. This result implies that agents face equal marginal market prices of risk. Hence, we may introduce a representative agent with constant weighting to deal with the equilibrium, and equilibrium quantities will appear similar to an economy with no taxation. For example, when the tax proceeds are returned to the economy, agents have logarithmic preferences, and P is in zero net supply, the after-tax market prices of risk and interest rate are as in the no-tax benchmark:

$$\begin{aligned} \theta^1(t) &= \theta^2(t) = \sigma_{\delta_S}(t)\,, \\ r(t) &= \mu_{\delta_S}(t) - \sigma_{\delta_S}(t)^2\,. \end{aligned}$$

In this sense, the presence of the derivative allows agents to fully circumvent the frictional taxation.

8. Conclusion

This paper develops a general equilibrium, continuous time model with two heterogeneous agents where the presence of redundant, nonlinearly taxed securities provides opportunities for tax arbitrage. Tax arbitrage is shown to have important equilibrium implications. In the presence of redundant securities, for given risk sharing, agents may choose between an infinity of portfolio holdings leading to different tax bills. We show that they always effectively "cooperate" and pick the one that minimizes aggregate tax proceeds. Security prices reflect this, via the pre-tax mispricing, as do portfolio holdings. For example, in the presence of a positive net supply "stock" and a zero net supply "derivative" both subject to piecewise linear taxation, trade in the latter will occur only if it is taxed more homogeneously across agents, so that trade therein allows a reduction of the aggregate tax bill. Hence, not only do we exhibit how a derivative may alleviate taxation, but we can sketch the conditions under which the tax alleviation is most effective. Furthermore, in the presence of two redundant, positive net supply risky securities also subject to piecewise linear taxation, in some states each agent holds the whole supply of one security (so that the agents only exchange the riskless bond), the higher taxed agent holding the lower taxed security, and conversely: our model endogenizes clientele effects.

The nonlinearity of taxable income in dividends received (not to be confused with the nonlinearity of tax bills in taxable income), a novelty of our work, is revealed to be a non-trivial generalization as, under perfect markets, taxable income being linear allows agents to fully circumvent the nonlinear taxation and face a linear problem with homogeneous marginal tax rates. In contrast, our setup allows for an equilibrium where agents' marginal tax rates are not equated, an innovation with respect to the existing literature. Our main results can be seen as an offspring of this discrepancy in tax rates. Natural extensions of the present work would include studying more general taxation schemes in greater detail, although explicit solutions are unlikely; or incorporating taxation on capital gains, which would add significantly to the complexity of the problem.

Appendix: Proofs

Proof of Proposition 3.1: We show (b), as (a) is a special case (where $\theta^i_{j[-]} = \theta^i_{j[+]}$ everywhere). Assume that, say, the first inequality in (3.5) is violated. Then, our assumptions on T^i, t^i_j (implying they are continuously differentiable except on a countable set) imply that there exists $\overline{x} > 0$ such that, $\forall 0 < x \leq \overline{x}$,

$$\frac{\mu_S - r}{\sigma_S} - d_S t^{i\prime}_S\left(\left(\hat{\alpha}^i_S + x\right)\delta_S\right)T^{i\prime}\left(t^i_S\left(\left(\hat{\alpha}^i_S + x\right)\delta_S\right) + t^i_P\Big(\Big(\hat{\alpha}^i_P - x\frac{S\sigma_S}{P\sigma_P}\Big)\delta_P\Big)\right) >$$
$$\frac{\mu_P - r}{\sigma_P} - d_P t^{i\prime}_P\left(\left(\hat{\alpha}^i_P - x\frac{S\sigma_S}{P\sigma_P}\right)\delta_P\right)T^{i\prime}\left(t^i_S\left(\left(\hat{\alpha}^i_S + x\right)\delta_S\right) + t^i_P\Big(\Big(\hat{\alpha}^i_P - x\frac{S\sigma_S}{P\sigma_P}\Big)\delta_P\Big)\right). \quad \text{(A.1)}$$

Adding the riskless, costless position consisting of x share(s) of S, $-xS\sigma_S/P\sigma_P$ share(s) of P and $(S\sigma_S/P\sigma_P - 1)\,x$ share(s) of the bond to i's portfolio increases his wealth's drift by

$$\begin{aligned}
&\overline{x}\left[S\left(\mu_S - r\right) - S\frac{\sigma_S}{\sigma_P}\left(\mu_P - r\right)\right] + T^i\left(t^i_S(\hat{\alpha}^i_S\delta_S) + t^i_P(\hat{\alpha}^i_P\delta_P)\right)\\
&-T^i\left(t^i_S\left(\left(\hat{\alpha}^i_S + \overline{x}\right)\delta_S\right) + t^i_P\Big(\Big(\hat{\alpha}^i_P - \overline{x}\frac{S\sigma_S}{P\sigma_P}\Big)\delta_P\Big)\right)\\
= &\int_0^{\overline{x}}\Bigg\{S\left(\mu_S - r\right) - S\frac{\sigma_S}{\sigma_P}\left(\mu_P - r\right) - \left[\delta_S t^{i\prime}_S\left(\left(\hat{\alpha}^i_S + x\right)\delta_S\right) - \frac{S\sigma_S}{P\sigma_P}\delta_P t^{i\prime}_P\Big(\Big(\hat{\alpha}^i_P - x\frac{S\sigma_S}{P\sigma_P}\Big)\delta_P\Big)\right]\\
&* T^{i\prime}\left(t^i_S\left(\left(\hat{\alpha}^i_S + x\right)\delta_S\right) + t^i_P\Big(\Big(\hat{\alpha}^i_P - x\frac{S\sigma_S}{P\sigma_P}\Big)\delta_P\Big)\right)\Bigg\}dx\ ,
\end{aligned}$$

positive from (A.1). The argument is similar, with only the signs being reversed, when the second inequality in (3.5) fails. Hence, whenever (3.5) fails on a subset of - $\times [0, T]$ with positive measure, from i's non-satiation there exists a portfolio strategy that is strictly prefered to $(\hat{\alpha}^i_S, \hat{\alpha}^i_P)$. All portfolio strategies such that this is not the case yield the same wealth dynamics and so i is indifferent between them. *Q.E.D.*

Proof of Proposition 3.2: Follows from Theorem 2 in Cuoco and Cvitanic (1998), with only obvious changes in notations. *Q.E.D.*

Proof of Proposition 3.3: Follows from the straightforward region-by-region solution of the optimization problem in (3.12). Since very lengthy, details are omitted. *Q.E.D.*

Proof of Proposition 3.4: Substituting logarithmic utility and rearranging show that, for given ν, the solution of the maximization problem in (3.15) verifies $c^i(t) = \xi_\nu(0)X^i(0)/\xi_\nu(t)T$. Substitution into i's wealth process $X^i(t) = E\left[\int_t^T \xi^i(s)c^i(s)ds|\mathcal{F}_t\right]/\xi^i(t)$ leads to (3.18), into the definition of κ^i shows that $\kappa^i \equiv 0$, hence (3.19), and substitution into (3.15) shows that, at any time t, $\nu^i(t)$ solves $\min_{\nu(t)\in[\underline{\nu}^i(t),\overline{\nu}^i(t)]}|\mu_S(t) - \nu(t) - r(t)|$, hence (3.20). *Q.E.D.*

Proof of Proposition 4.1: In equilibrium, (3.4) has to hold for both agents, hence portfolio holdings satisfy

$$\begin{aligned} & \left[d_S(t)t_S^{1\prime}(\alpha_S^1(t)\delta_S(t)) - d_P(t)t_P^{1\prime}(\alpha_P^1(t)\delta_P(t))\right] * T^{1\prime}\left(t_S^1(\alpha_S^1(t)\delta_S(t)) + t_P^1(\alpha_P^1(t)\delta_P(t))\right) \\ = & \left[d_S(t)t_S^{2\prime}(\alpha_S^2(t)\delta_S(t)) - d_P(t)t_P^{2\prime}(\alpha_P^2(t)\delta_P(t))\right] * T^{2\prime}\left(t_S^2(\alpha_S^2(t)\delta_S(t)) + t_P^2(\alpha_P^2(t)\delta_P(t))\right) . \end{aligned}$$

Substitution of clearing in S (implying $\alpha_S^2(t) = 1 - \alpha_S^1(t)$) and the definition of Φ^i (implying $\alpha_P^i(t) = (\Phi^i(t) - \alpha_S^i(t)S(t))\,\sigma_S(t)/P(t)\sigma_P(t)$) leads to (4.3). (3.4) then yields (4.2). *Q.E.D.*

Proof of Corollary 4.1: In equilibrium, aggregate taxes are given by

$$\begin{aligned} & T^1\left(t_S^1(\alpha_S^1(t)\delta_S(t)) + t_P^1((\Phi^1(t) - \alpha_S^1(t)S(t))\sigma_S(t)d_P(t))\right) \\ + \; & T^2\left(t_S^2((1-\alpha_S^1(t))\delta_S(t)) + t_P^2((\Phi^2(t) - (1-\alpha_S^1(t))S(t))\sigma_S(t)d_P(t))\right) . \end{aligned}$$

The first-order condition of the problem consisting in minimizing this expression with respect to $\alpha_S^1(t)$ is (4.3). *Q.E.D.*

Proof of Equation (4.4): We first show that the solution ν^i to (3.15) and the optimal (α_S^i, α_P^i) are related by

$$\nu^i(t) = \frac{\delta_S(t)}{S(t)} t_S^{i\prime}\left(\alpha_S^i(t)\delta_S(t)\right) T^{i\prime}(t) . \tag{A.2}$$

Making use of (3.4), we may express (3.13) alternatively as

$$g^i(\Phi(t), t) = \max_{\alpha_S^i, \alpha_P^i}\left\{-\alpha_P^i(t)P(t)\sigma_P(t)\Delta_{S,P}(t) - T^i\left(t_S^i(\alpha_S^i(t)\delta_S(t)) + t_P^i(\alpha_P^i(t)\delta_P(t))\right)\right\} \text{ s.t. (3.1).}$$

Substituting α_S^i and applying the envelope theorem yields $g^{i\prime}(\Phi, t) = -\delta_S(t)t_S^{i\prime}\left(\hat{\alpha}_S^i(t)\delta_S(t)\right)T^{i\prime}(t)/S(t)$. On the other hand, we have $\Phi^{\sup}$ solves $g^{i\prime}\left(\Phi^{\sup}, t\right) = -\nu^i(t)$. Since the optimal Φ coincides with the supremal Φ (Cuoco and Cvitanic (1998), Theorem 2), we have (A.2). Applying Itô's Lemma and using $\theta^i = \left(\mu_S - r - \nu^i\right)/\sigma_S$, we deduce that $\xi^i(t)S(t) + \int_0^t \xi^i(s)\left[1 - t_S^{i\prime}\left(\alpha_S^i(s)\delta_S(s)\right)T^{i\prime}(s)\right]\delta_S(s)ds$ is a martingale, under appropriate regularity conditions. Hence (4.4) follows. *Q.E.D.*

Proof of Proposition 5.1: (5.3) and (5.4) follow from Propositions 3.1 and 4.1. Noting that, from (5.4), $\hat{\alpha}_S^i(t)$ is independent of $\Phi^i(t)$, (5.2) reveals i to face the policy-independent market price of risk $\left(\mu_P(t) - r(t)\right)/\sigma_P(t) - d_P(t)\tau_P^i$, and so $\theta^1(t) - \theta^2(t) = d_P(t)\left(\tau_P^2 - \tau_P^1\right)$. Then, applying Itô's lemma to agents' first-order conditions (3.16), matching drift and diffusion terms and using clearing in the consumption good market (here, $c^1(t) + c^2(t) = \delta_S(t)$) leads to (5.5)-(5.7). (5.8) follows from applying Itô's lemma to $\lambda(t) = y^1\xi^1(t)/y^2\xi^2(t)$. *Q.E.D.*

Proof of Proposition 5.2: The conditions for the equilibrium cases follow from substituting the mispricing value (5.9) and the θ^i expressions of Proposition 5.3 into the $\Delta_{S,P}$ and $\left(\mu_S - r\right)/\sigma_S$ conditions in Table I. The portfolio holdings follow from Table I and security market clearing. *Q.E.D.*

Proof of Proposition 5.3: The expressions for the market prices of risk and interest rate obtain by applying Itô's lemma to both sides of agents' first-order conditions (3.16), matching drifts and diffusions and using good market clearing. The stochastic weighting dynamics follow from applying Itô's lemma to $\lambda(t) = y^1\xi^1(t)/y^2\xi^2(t)$. *Q.E.D.*

Proof of Proposition 6.1: The conditions for the cases follow from agreement between agents on $\Delta_{S,P}(t)$ (yielding the conditions relating $d_S(t)$ and $d_P(t)$) and from substituting the $\Delta_{S,P}$ and θ^i expressions in Proposition 6.2 into the $\Delta_{S,P}$ and $(\mu_S - r)/\sigma_S$ conditions in Table I. The portfolio holdings, and hence the after-tax aggregate dividend $\mathcal{D}$, follow from Table I and security market clearing. *Q.E.D.*

Proof of Proposition 6.2: In (K,G), the mispricing value obtains from Table I, and the expressions for the market prices of risk, interest rate and stochastic weighting dynamics follow from applying Itô's lemma to agents' first-order conditions (3.16) and $\lambda(t) = y^1\xi^1(t)/y^2\xi^2(t)$, and using good market clearing. In (J,L) and (L,J), the market prices of risk follow from clearing in the risky securities and the fact that for a logarithmic agent without stochastic endowment, $\Phi^i(t) = \theta^i(t)/\sigma_S(t)$. The other expressions then obtain by applying Itô's lemma to (3.16) and $\lambda(t) = y^1\xi^1(t)/y^2\xi^2(t)$, and using good market clearing. *Q.E.D.*

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