Catching Up with the Joneses: Heterogeneous Preferences and the Dynamics of Asset Prices[1]

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Abstract

We examine how cross-sectional heterogeneity in preferences affects equilibrium behavior of asset prices. We obtain explicit characterization of the competitive equilbrium in an exchange economy in which individual agents have catching up with the Joneses preferences and differ only with respect to the curvature of their utility functions. We show that heterogeneity can have a drastic effect on the behavior of asset prices, in particular, on their conditional moments. Dynamic re-distribution of wealth among the agents in heterogeneous economies leads to time-variation in aggregate risk aversion and market price of risk, generating empirically observed negative relation between conditional return volatility and expected returns on one hand and the level of stock prices on the other hand. This stands in contrast with the behavior of homogeneous economies with the same preferences, in which such relation is positive. Quantitatively, the heterogeneous model is capable of replicating various empirical properties of asset prices.

Keywords: Equilibrium Asset Pricing, Heterogeneity, Catching Up with the Joneses, Time Varying Risk Aversion, Stochastic Volatility, Predictability

1 Introduction

This paper studies general equilibrium implications of cross-sectional heterogeneity in preferences on asset prices. Many classical dynamic models such as Lucas (1978) and Cox, Ingersoll and Ross (1985) use a representative investor framework to study the determination of asset prices. This approach renders the computation of equilibrium elegantly simple and contributes much to our understanding of how underlying economic structures such as preferences, endowments and production technologies, influence asset prices. However, heterogeneity among investors is a prevailing feature in capital markets. Literature on aggregation such as Rubinstein (1974) and Constantinides (1982) identifies conditions under which individual preferences can be aggregated and thus provides a theoretical justification for the representative agent framework. Despite these important aggregation results, a representative agent framework assumes away many interesting issues such as endogenous determination of the representative-agent preferences, the role of cross-sectional wealth distribution as a determinant of asset prices and trading patterns. To tackle these issues, heterogeneity must be modeled explicitly.

Perhaps the two most obvious forms of heterogeneity are heterogeneous information and preferences.[4] Information heterogeneity has long been recognized in the literature.[5] Indeed, Ross (1989) argues that the presence of diverse information is necessary in order to explain the observed level of trading in asset markets. While significant progress has been made in understanding the behavior of dynamic economies with asymmetric information (e.g., Kyle (1985), Wang (1993, 1994)), much less attention has been paid to asset pricing implications of cross-sectional heterogeneity in preferences.

Dumas (1989) and recently Wang (1996) make some progress in this direction. Dumas analyzes a two-person, production economy and relies purely on numerical analysis. Wang considers an exchange economy and is able to obtain closed-form expressions for certain asset prices. Our work differs from theirs in four important aspects. First, while Dumas and Wang emphasize the time-variation in interest rates and the term structure of bond prices, our main focus is on the dynamics of stock prices. Second, analysis in Dumas (1989) is completely numerical, while Wang (1996) obtains closed-form solutions only for a few rather special combinations of investors' risk aversion. In contrast, we allow for a continuum of risk aversion types and obtain explicit solutions for asset prices and other economic variables. Third, while the above two papers consider time-separable,

[4] Other forms of heterogeneity have also been considered in the literature. Mankiw (1986) and Constantinides and Duffie (1996) argue that differences in investors' non-insurable income process can help explain the equity premium puzzle of Mehra and Prescott (1985). See Brav, Constantinides and Geczy (1999) for some related empirical evidence. Moreover, in the context of portfolio insurance, Grossman and Zhou (1996) study a finite-horizon exchange economy with two types of agents.

[5] There exists an extensive literature on information asymmetry. See O'Hara (1998) for a textbook introduction and the references therein.

state-independent utility functions, preferences in our model exhibit the "catching up with the Joneses" feature. As a result, unlike the exchange economy in Wang (1996), the asymptotic cross-sectional distribution of wealth is not degenerate in our model, in the sense that no single type owns most of the wealth in the economy as aggregate wealth increases without bound. This result is important since it allows us to discuss long-run effects of heterogeneity on asset prices.[6] Finally, unlike the above-mentioned papers, we calibrate our model and assess its quantitative implications relative to historical data.

Our work is closely related to a recent paper by Campbell and Cochrane (1999), in which a particular representative-agent model with catching up with the Joneses preferences is shown to replicate numerous empirically observed features of stock returns. In their model of preferences, Campbell and Cochrane assume that the local curvature of the utility function is decreasing in the level of consumption, inducing counter-cyclical variation in Sharpe ratio. In addition, they use a carefully crafted nonlinear process for the social standard of living (the distinctive feature of catching up with the Joneses preferences, called *exogenous habit level* by Campbell and Cochrane) to control the volatility of interest rates. As we demonstrate in this paper, a heterogeneous economy can give rise to similar qualitative and quantitative properties of stock returns with far simpler assumptions about *individual* preferences: constant curvature of the utility function and linear process for the standard of living.

In a setting with homogeneous preferences, we provide a theoretical characterization of return volatility. We show that, except for the logarithmic case, the volatility of stock returns is stochastic and pro-cyclical[7]. While it is well known that return volatility varies over time (see, for example, Schwert (1989)), the variation is counter-cyclical in the data (Black (1976), Christie (1982), French, Schwert and Stambaugh (1987), Campbell and Hentschel (1992)). We show that heterogeneity in investors' risk preferences can give rise to such pattern of variation in return volatility. We argue that this can be understood by examining the evolution of the cross-sectional distribution of wealth in the economy. Since re-distribution of wealth among the investors affects the risk aversion of the representative investor, it also changes the properties of stock returns. Thus, changes in the value of the stock market cause time-variation in volatility via their impact on the wealth distribution in the economy.

In the homogeneous economy, time-varying volatility gives rise to slow-varying expected returns, which in turn leads to predictability of stock returns over long horizons. As we demonstrate, resulting patterns of predictability are counter-factual. Heterogeneous economies can have a different

[6] Dumas (1989) demonstrates that the cross-sectional wealth distribution in his model can be stationary under certain assumptions on model parameters. In our model the wealth distribution is stationary for all choices of parameters.

[7] Pro-cyclical behavior of volatility in this context means that high level of price-dividend ratio or a large increase in stock prices predicts higher levels of volatility. Alternatively, changes in volatility are positively correlated with stock returns.

sign of return correlations, which are broadly consistent with empirical observations. Quantitatively, we find that, while both homogeneous and heterogeneous economies can be calibrated to replicate basic unconditional moments of asset returns, they can have very different implications for the dynamics of conditional returns.

The paper is organized as follows. We formulate the model in Section 2. Section 3 characterizes the competitive equilibrium. In Section 4 we examine the dynamics of stock returns under homogeneous and heterogeneous preference structure: we establish general properties of the stock price-dividend ratio and return volatility in homogeneous economies; we carry out asymptotic analysis of the equilibrium, revealing the economic mechanism through which heterogeneity affects the asset prices; finally, we calibrate our model and analyze it numerically. Section 5 concludes.

2 The Model

We consider a continuous time, infinite horizon, complete markets exchange economy of Lucas (1978). In this economy, there exists a single perishable consumption good. There is only one source of uncertainty and investors trade in financial securities to share risk. There is a continuum of investors who differ from each other with respect to the curvature of their utility functions.

Aggregate Endowment

The aggregate endowment process Y_t is described by a geometric Brownian motion

$$dY_t = \mu Y_t dt + \sigma Y_t dB_t, \quad t \in [0, \infty) \tag{1}$$

where B_t is a standard Brownian motion. Both μ and σ are constants with $\mu > \sigma^2/2$ and $\sigma > 0$. Well known properties of this process include its conditional log-normality and non-negativity.

Capital Markets

There are two long-lived financial securities available for trading: a risky asset, *the stock,* and a locally riskless instrument, *the bond.* The stock price is denoted by P_t; the instantaneous interest rate is denoted by r_t. There is a single share of the stock outstanding, which entitles its holder to the dividend stream Y_t. The bond is available in zero net supply.

Preferences

In this economy, all investors maximize expected utility of the form

$$\mathrm{E}_0 \left[\int_0^\infty e^{-\rho t} U\left(C_t, X_t\right) dt \right],$$

where

$$U(C_t, X_t) = \frac{1}{1-\gamma}\left(\frac{C_t}{X_t}\right)^{1-\gamma}. \tag{2}$$

C_t is the consumption rate at time t, γ is the "relative risk aversion parameter", measuring the local curvature of the utility function[8], and X_t is an exogenous state variable, which will be given an interpretation of the average historical standard of living in the economy.[9] Thus, the utility of an investor is influenced not only directly by her own consumption, but also by the consumption of others. Abel (1990, 1999) refers to preferences of this type as "catching up with the Joneses".[10] The effect of this feature of preferences can be seen more clearly as follows: the marginal utility is given by

$$U_C(C_t, X_t) = C_t^{-\gamma} X_t^{\gamma-1}, \tag{3}$$

which implies

$$\frac{\partial U_C(C_t)}{\partial X_t} = (\gamma - 1) C_t^{-\gamma} X_t^{\gamma-2}. \tag{4}$$

Thus, for investors with $\gamma > 1$, a higher standard of living X_t provides a complementary effect on current consumption; for investors with $\gamma < 1$, X_t and C_t act as substitutes.

To interpret X_t as the average historical standard of living, we define it as a weighted geometric average of past realizations of the aggregate consumption process:

$$x_t = x_0 e^{-\lambda t} + \lambda \int_0^t e^{-\lambda(t-s)} y_s ds, \tag{5}$$

where $x_t \equiv \log(X_t)$ and $y_t \equiv \log(Y_t)$. It is convenient to describe the dynamics of X_t in terms of the state variable $\omega_t \equiv y_t - x_t$. Given the lognormal specification of the aggregate endowment process,

$$d\omega_t = -\lambda(\omega_t - \overline{\omega})\, dt + \sigma dB_t, \tag{6}$$

[8] As we demonstrate below, γ is the only preference parameter controlling the risk premium in the corresponding representative agent economy. This justifies its interpretation as the risk aversion parameter.

[9] This particular specification of the utility function is often called a "ratio model" (e.g., Campbell, Lo and MacKinlay (1997, Section 8.4.1)).

[10] This type of preferences is often refered to as "external habit formation". Various specifications of representative agent models with habit formation have been analyzed in the literature. Major contributions in continuous time setting include Ryder and Heal (1973), Sundaresan (1989), Constantinides (1990), Detemple and Zapatero (1991), Hindy and Huang (1992, 1993), Hindy, Huang and Zhu (1997), whereas Abel (1990, 1999) and Campbell and Cochrane (1999) consider discrete-time models. On the empirical side, Ferson and Constantinides (1991) and Heaton (1995) confront such models with data and find that they provide a better fit than standard models with time-separable, state-independent preferences.

where

$$\overline{\omega} \equiv \lim_{t\to\infty} \mathrm{E}\left(\omega_t | \omega_0\right) = \frac{\mu - \sigma^2/2}{\lambda}. \tag{7}$$

Thus, the state variable ω_t is a mean-reverting process with the long-run mean given by $\overline{\omega}$ and is conditionally normally distributed with the first and second moments given by

$$\begin{aligned} \mathrm{E}\left[\omega_t | \omega_0 = \omega\right] &= \overline{\omega} + (\omega - \overline{\omega})\, e^{-\lambda t}, \\ \mathrm{Var}\left[\omega_t | \omega_0 = \omega\right] &= \frac{\sigma^2}{2\lambda}\left(1 - e^{-2\lambda t}\right). \end{aligned} \tag{8}$$

One can see that the parameter λ governs the degree of history-dependence in X_t. When $\lambda \gg \sigma^2$, ω_t does not exhibit much variability and $x_t \approx y_t$, i.e., there is little history dependence. On the other hand, if $\lambda \approx 0$, $x_t \approx x_0$, i.e., X_t is influenced heavily by the past history of consumption.

Cross-Sectional Heterogeneity

All investors in the economy share the same time discount rate ρ, but differ with respect to their preference parameter γ. We assume that there is a continuum of preference types. For convenience, we parametrize these types by $b \equiv 1/\gamma$, $b \in (0, \infty)$. In terms of this new parameter, individual utility functions take the form

$$U\left(C_t, X_t; b\right) = \frac{1}{1 - \frac{1}{b}} \left(\frac{C_t}{X_t}\right)^{1-\frac{1}{b}}.$$

3 The Competitive Equilibrium

In this section we analyze the general properties of the competitive equilibrium in the heterogeneous-agent economy. We solve the model in three steps, as is standard in the literature (e.g., Wang (1996)). First, we analyze the social planner's problem in order to obtain the optimal consumption sharing rule. Then, we construct an Arrow-Debreu economy to support the optimal allocation found in the planner's problem. Finally, we implement the Arrow-Debreu equilibrium as a sequential-trade economy.

The Social Planner's Problem

The social planner distributes the aggregate endowment among the consumers so that the resulting allocation is Pareto optimal. This is the classical optimal allocation problem, studied by many such as Wilson (1968) and recently in a related context by Vanden (1998). Without loss of generality, we assume that there is only one investor for each type and $f(b)$ is the social weight attached by

the planner to type b.[11] Given the distribution of social weights $f(b)$, the objective of the central planner is

$$\sup_{\{C_t(\cdot;b)\}} \mathrm{E}_0 \int_0^\infty e^{-\rho t} \left[\int_0^\infty f(b)\, U\left(C_t(\cdot;b)\right) db \right] dt,$$

subject to the resource constraint,

$$\int_0^\infty C_t(\cdot;b)\, db \leq Y_t, \quad \forall t \in [0,\infty). \tag{9}$$

Since there is no intertemporal transfer of resources, this optimization reduces to a static problem: at each t, the planner solves

$$\sup_{\{C_t(\cdot;b)\}} \int_0^\infty f(b)\, U\left(C_t(\cdot;b)\right) db$$

subject to the same resources constraint (9).

The following lemma summarizes the optimal sharing rule.

Lemma 1 *The optimal consumption sharing rule is characterized by*

$$C_t^*(Y_t, X_t; b) = c_t^*(\omega_t; b)\, Y_t, \tag{10}$$

$$c_t^*(\omega_t; b) = \nu(b)\, e^{-b\mathcal{I}(\omega_t)-\omega_t}, \tag{11}$$

where $\nu(b) \equiv f(b)^b$ *and* $\mathcal{I}(\omega_t)$ *is defined by*

$$\int_0^\infty \nu(b)\, e^{-b\mathcal{I}(\omega_t)-\omega_t} db = 1. \tag{12}$$

The shadow price (the Lagrange multiplier) Z_t *of the resource constraint (9) equals*

$$\exp\left(\mathcal{I}(\omega_t) - x_t\right).$$

Proof. See the Appendix. ■

The Arrow-Debreu Economy

It is well known that the Pareto optimal allocation (10)–(12) can be supported as an equilibrium allocation in a particular Arrow-Debreu economy (e.g., Duffie and Huang (1985)). In this economy agents can trade in primitive state-contingent claims, paying off a unit of consumption in a particular state of the economy and zero otherwise. Let $\xi_{t,s}(\cdot)$ denote the price function in such economy: the time-zero price of an arbitrary payoff stream $\{F_s, s \in [0,\infty)\}$ is given by $\mathrm{E}_0\left[\int_0^\infty \xi_{0,s}(\cdot) F_s ds\right]$. In equilibrium, $\xi_{t,s}(\cdot)$ is determined by the marginal utility of the properly constructed representative investor, evaluated at the aggregate consumption. In particular, the utility function of the

[11] For technical reasons, it is convenient to assume that the distribution $f(b)$ has compact support.

representative investor is defined as

$$\mathrm{E}_0\left[\int_0^\infty e^{-\rho t}U^{(SP)}\left(Y_t,X_t\right)dt\right]$$

where

$$U^{(SP)}\left(Y_t,X_t\right)\equiv\int_0^\infty f\left(b\right)U\left(C_t^*\left(Y_{t,}X_t;b\right)\right)db$$

and

$$\xi_{t,s}\left(\cdot\right)=e^{-\rho(s-t)}\frac{\partial U^{(SP)}\left(Y_s,X_s\right)/\partial Y_s}{\partial U^{(SP)}\left(Y_t,X_t\right)/\partial Y_t}. \tag{13}$$

Since the marginal utility $\partial U^{(SP)}\left(Y_t,X_t\right)/\partial Y_t$ equals the Lagrange multiplier Z_t,

$$\xi_{t,s}=\exp\left(-\rho\left(s-t\right)+z_s-z_t\right),\quad t\leq s, \tag{14}$$

where $z_t\equiv\log\left(Z_t\right)$.[12]

The aggregate risk aversion in the economy is defined as the local curvature of the utility function of the representative investor

$$\gamma^{(SP)}=-\frac{Y_t\cdot U_{YY}^{(SP)}\left(Y_t,X_t\right)}{U_Y^{(SP)}\left(Y_t,X_t\right)}. \tag{15}$$

The preference specification in (2) and the optimal consumption sharing rule given in (10) imply that the utility function of the representative agent can be expressed as a function of a single state variable ω:

$$u^{(SP)}(\omega_t)\equiv U^{(SP)}(Y_t,X_t)=\int_0^\infty\frac{b}{b-1}\nu(b)e^{(1-b)\mathcal{I}(\omega_t)}db.$$

The aggregate risk aversion is also state-dependent and is characterized by the following lemma.

Lemma 2 *The aggregate risk aversion, as defined in (15), can be characterized as:*

$$\begin{aligned}\gamma^{(SP)}\left(\omega_t\right)&=1-\frac{u_{\omega\omega}^{(SP)}\left(\omega_t\right)}{u_\omega^{(SP)}\left(\omega_t\right)}\\&=1-\frac{\mathcal{I}''\left(\omega_t\right)}{\mathcal{I}'\left(\omega_t\right)}-\mathcal{I}'\left(\omega_t\right)\frac{\int_0^\infty\nu(b)e^{(1-b)\mathcal{I}(\omega_t)}b\left(1-b\right)db}{\int_0^\infty\nu\left(b\right)e^{(1-b)\mathcal{I}(\omega_t)}bdb},\end{aligned}$$

[12] Alternatively, $\xi_{t,s}$ can be computed in terms of the marginal utilities of consumption of the investor with logarithmic preferences. This is due to the fact that the ratios of marginal utilities are equalized across investors for Pareto optimal allocations. One can obtain this result formally by setting $b=1$ in (10)–(11) and using (49). In the dynamic implementation of the Arrow-Debrew equilibrium, the marginal utility of the log-investor equals the inverse of his optimally invested wealth (according to his envelope condition). Thus, the equivalence of the two definitions of $\xi_{t,s}$ reflects the relation between the state-price process and the value of the growth-optimal portfolio (see Merton (1990)).

where $\mathcal{I}(\omega_t)$ is defined in (12).

Proof. See the Appendix. ■

The Sequential-Trade Economy

Given the Arrow-Debreu equilibrium, a sequential-trade equilibrium can be constructed, in which investors trade continuously in a small number of long-lived securities. Duffie and Huang (1985) provide general analysis of such an implementation problem. Their results can be extended to our setting using arguments similar to those in Wang (1996, Lemma 3).

Prices of long-lived assets in equilibrium are determined by the prices of primitive Arrow-Debreu claims. In particular, the stock price satisfies

$$P_t = \mathrm{E}_t \int_t^\infty \xi_{t,s} Y_s ds = \mathrm{E}_t \int_t^\infty \exp\left(-\rho(s-t) + z_s - z_t\right) \cdot Y_s ds, \tag{16}$$

while the instantaneous interest rate is given by

$$r_t = \lim_{\Delta t \searrow 0} \frac{\mathrm{E}_t\left[\xi_{t,t+\Delta t}\right]}{\Delta t} = \lim_{\Delta t \searrow 0} \frac{\mathrm{E}_t\left[\exp\left(-\rho \Delta t + z_{t+\Delta t} - z_t\right)\right]}{\Delta t}. \tag{17}$$

4 Equilibrium Security Prices

In this section we study the behavior of security prices. We compute stock prices and the instantaneous interest rate analytically, using asymptotic analysis, and numerically. We point out qualitative differences in the behavior of stock returns in heterogeneous and homogeneous economies and argue that these differences can be understood in terms of the evolution of the cross-sectional distribution of wealth over time.

4.1 Stock Returns

While the general expression for the stock price is provided by (16), a more explicit characterization would facilitate further qualitative analysis and numerical computations. The following lemma provides two equivalent characterizations of the stock price.

Lemma 3 *(a) The equilibrium stock price is given by*[13]

$$P_t = Y_t \cdot \mathcal{F}(\omega_t), \tag{18}$$

[13]The stock price function $\mathcal{F}(\omega_t)$ is well defined: since $\nu(b)$ has compact support, the function $\mathcal{I}(\omega)$ is asymptotically linear as $|\omega| \to \infty$ and therefore the expectation of $\exp(\mathcal{I}(\omega_t))$ is finite.

where

$$\mathcal{F}(\omega_t) = \exp(-\mathcal{I}(\omega_t) - \omega_t) \cdot E\left[\int_t^\infty \exp(-\rho(s-t) + \mathcal{I}(\omega_s) + \omega_s)\, ds \,\middle|\, \omega_t\right].$$

(b) Alternatively, the equilibrium stock price can be computed as

$$P_t = Y_t \exp(-u_t - \log(\Phi(u_t))) \cdot E\left[\int_t^\infty \exp(-\rho(s-t) + u_s + \log(\Phi(u_s)))\, ds \,\middle|\, u_t\right], \tag{19}$$

where

$$\Phi(u) \equiv \int_0^\infty \nu(b)\, e^{-bu} db, \tag{20}$$

and the process u_s is defined as a solution of the stochastic differential equation

$$du_s = \left(\frac{\Phi(u_s)}{\Phi'(u_s)}\mu_\omega(\log(\Phi(u_s))) - \frac{\sigma^2}{2}\frac{\Phi''(u_s)(\Phi(u_s))^2}{(\Phi'(u_s))^3}\right) dt + \sigma\frac{\Phi(u_s)}{\Phi'(u_s)}\, dB_s. \tag{21}$$

The initial condition u_t is related to the state variable ω_t by

$$u_t = \mathcal{I}(\omega_t). \tag{22}$$

Proof. See the Appendix. ■

The expression (18) shows that the stock price is proportional to the aggregate endowment Y_t and the price-dividend ratio (P_t/Y_t) depends only on the state variable ω_t. The price-dividend ratio summarizes the conditional expectations of future discount rate and dividend growth rate, but given our specification of the aggregate endowment process, future dividend growth is independent of the current state of the economy. Thus, the price-dividend ratio is a sufficient statistic for capturing movements in future discount rate. The higher the expected future discount rates, the lower is the price-dividend ratio. To relate (18) to standard results, consider the special case in which all investors are of the logarithmic type. According to the definition (49), $\mathcal{I}(\omega_t) = -\omega_t$. As a result, the price-dividend ratio is constant:

$$\frac{P_t}{Y_t} = \mathcal{F}(\omega_t) = \mathrm{E}_t\left[\int_t^\infty \exp(-\rho s)\, ds\right] = \frac{1}{\rho},$$

which is the well-known solution.

The second characterization (19)–(21) is convenient for numerical computations when the function $\Phi(u)$ is known in closed form, since, unlike (18), it does not require solving the equation (49) to compute the expectation. Thus, one obtains an explicit representation for the stock price as a function of u_t. The drawback of this approach is that the variable u_t does not have an immediate economic interpretation. To express the stock price in terms of the state variable ω_t, the equation (49) must be solved, but only for the initial condition ω_t.

Stock returns can be decomposed into capital gains and dividend yield, therefore

$$\mu_{R,t} = \mu_{P,t} + \frac{Y_t}{P_t}, \tag{23}$$

where $\mu_{P,t}$ denotes the drift coefficient of the stock price process. With this representation the following result is immediate.

Lemma 4 *(a) The instantaneous expected return $\mu_{R,t}$ is given by*

$$\mu_{R,t}(\omega_t) = \mu + \frac{1}{\mathcal{F}(\omega_t)}\left(\mathcal{F}'(\omega)\,\mu_{\omega,t} + \frac{1}{2}\left(2\mathcal{F}'(\omega_t) + \mathcal{F}''(\omega_t)\right)\sigma^2 + 1\right), \tag{24}$$

where $\mu_{\omega,t} \equiv -\lambda(\omega_t - \overline{\omega})$.
(b) The return volatility (defined as the diffusion coefficient of the cumulative return process) is given by

$$\sigma_{R,t}(\omega_t) = \sigma + \frac{\mathcal{F}'(\omega_t)}{\mathcal{F}(\omega_t)}\sigma. \tag{25}$$

Proof. See the Appendix. ■

In a homogeneous economy with logarithmic preferences, $\mathcal{F} = 1/\rho$ and therefore expected stock returns and volatility are both constant. In general, changes in the current state of the economy affect expectations of state variable.

The return volatility in (25) consists of two components. The first component represents the volatility of dividend growth. The second component captures the volatility induced by changes in the discount rate. To the extent that the current state variable ω_t affects these expected discount rates, it affects the volatility of stock returns.

Finally, the instantaneous correlation between changes in volatility and returns is given by

$$\operatorname{sgn}\left(\frac{d\sigma_R(\omega_t)}{d\omega_t} \cdot \sigma_R(\omega_t)\right). \tag{26}$$

This correlation is either 0, +1 or −1 (since shocks to the aggregate endowment are the only source of uncertainty in this economy).

4.2 Interest Rates and the Price of Risk

In our complete-market economy, the stochastic discount factor $\xi_{t,s}$ in (14) is uniquely determined in equilibrium and can be used to analyze the instantaneous interest rate and the price of risk. Accordingly, define the price of risk process π_t as

$$\pi_t \equiv \frac{\mu_{R,t} - r_t}{\sigma_{R,t}}.$$

The price of risk is simply the instantaneous Sharpe ratio of stock returns. The following lemma characterizes the instantaneous interest rate and the price of risk in our economy. It is a simple application of standard relations between asset prices and the stochastic discount factor.

Lemma 5 *(a) The instantaneous interest rate r_t is given by*

$$r_t = \rho - \mu_{z,t} - \frac{1}{2}\sigma_{z,t}^2. \tag{27}$$

(b) The price of risk is given by

$$\pi_t = -\sigma_{z,t}, \tag{28}$$

where

$$\begin{aligned} \mu_{z,t} &\equiv \mathcal{I}'(\omega_t)\mu_{\omega,t} - \lambda\omega_t + \frac{1}{2}\sigma^2\mathcal{I}''(\omega_t), \\ \sigma_{z,t} &\equiv \sigma \cdot \mathcal{I}'(\omega_t), \end{aligned} \tag{29}$$

and $\mathcal{I}(\omega_t)$ is defined in (12).

Proof. See the Appendix. ■

Using the definition of the price of risk process, the expected excess return on the stock is given by

$$\mu_{R,t} - r_t = \pi_t \cdot \sigma_{R,t}.$$

4.3 Benchmark Model: Homogeneous Preferences

We now establish some general properties of the stock price and return volatility in homogeneous economies. Before we characterize the economy with catching up with the Joneses preferences, we first discuss the case with standard time-separable preferences. The next proposition summarizes the result.

Proposition 1 *In a homogeneous economy with a standard isoelastic, time-separable utility function, the price-dividend ratio is constant and given by*

$$\frac{P_t}{Y_t} = \left(\rho - (1-\gamma)\left(\mu - \frac{\sigma^2}{2}\right) - \frac{(1-\gamma)^2}{2}\sigma^2\right)^{-1}.$$

Proof. See the Appendix. ■

When preferences are time-separable, our specification of the aggregate endowment process implies that changes in the marginal utility of the representative investor are i.i.d. over time and

are independent of the current state of the economy:

$$\frac{dU_C}{U_C} = \left(-\gamma\mu + \frac{\sigma^2\gamma}{2}(\gamma+1)\right)dt - \gamma\sigma dB_t.$$

Thus, all future expected discount rates are constant, resulting in a constant price-dividend ratio. Proposition 1 immediately implies that the return volatility is the same as the volatility of the dividend process. This is clearly at odds with empirical observations, the "volatility puzzle" (e.g., Campbell and Shiller (1988)): stock prices are far more volatile than dividends. The catching up with the Joneses feature of preferences overcomes this shortcoming of the standard model.

Corollary 1 *In a homogeneous economy with standard isoelastic, time-separable utility function, the conditional return volatility is constant and equals the conditional volatility of the aggregate endowment process.*

Having characterized the behavior of the stock price and return volatility in a world with time-separable preferences, we now contrast these results with those from the time-nonseparable economy. The next proposition establishes that the price-dividend ratio is a monotone, convex function of the state of the economy.

Proposition 2 *In a homogeneous economy with catching up with the Joneses preferences, the price-dividend ratio has the following properties:*
(a) It is increasing (decreasing) in the state variable ω_t for $\gamma > 1$ ($\gamma < 1$). Formally,

$$\frac{d(P_t/Y_t)}{d\omega_t} \begin{cases} > 0, & \gamma > 1, \\ = 0, & \gamma = 1, \\ < 0, & \gamma < 1; \end{cases}$$

(b) It is a convex function of ω_t, for $\gamma \neq 1$.

Proof. See the Appendix. ■

The price-dividend ratio is the ratio of the expected future marginal-utility-weighted dividends to the present marginal-utility-weighted dividend. Because of mean-reversion in the state of the economy, the former is less sensitive to the state than the latter. Thus, the qualitative properties of the ratio are determined to large extent by the properties of the inverse of the marginal-utility-weighted dividend as a function of the state, which, given the particular form of the utility function in this economy, equals $\exp((\gamma-1)\omega)$. This function is increasing for $\gamma > 1$ and decreasing for $\gamma < 1$. It is also convex in ω.

More formally, the marginal utility of the representative investor follows

$$\frac{dU_C}{U_C} = \left(\lambda(\gamma-1)\omega_t - \gamma\mu + \frac{\sigma^2\gamma}{2}(\gamma+1)\right)dt - \gamma\sigma dB_t. \tag{30}$$

Thus, a change in the state variable affects future marginal utility except when $\gamma = 1$. For $\gamma > 1$, an increase in ω raises the intertemporal marginal rate of substitution (the ratio of future to current marginal utility). In other words, the stock becomes more expensive relative to the current dividend, as the state prices for future dividend claims increase. This explains the positive relation between the price-dividend ratio and the state variable when $\gamma > 1$. The opposite occurs when $\gamma < 1$.

A comparison of Proposition 1 and 2 highlights the state-dependence property of the price-dividend ratio as derived from the history-dependence of preferences. We now turn to the equilibrium price of risk and the instantaneous interest rate.

Proposition 3 *In the homogeneous economy with catching up with the Joneses preferences,*
(a) the price of risk is constant and given by:

$$\pi_t = \gamma\sigma.$$

(b) The instantaneous interest rate is given by

$$r_t = \rho - \lambda(\gamma-1)(\omega_t - \overline{\omega}) + \lambda\overline{\omega} - \frac{1}{2}\gamma^2\sigma^2.$$

The proposition shows that the price of risk π_t is constant. This is not surprising since the standard consumption CAPM holds and the aggregate risk aversion and volatility of aggregate endowment are both constant. The instantaneous interest rate inherits the stochastic behavior of the state variable. Moreover, its variation is increasing in both λ and risk aversion.

As (30) indicates, the growth rate of the marginal utility is state-dependent for $\gamma \neq 1$. This implies that volatility also depends on the state of the economy. The next proposition formally shows that the conditional volatility of stock returns is also a monotone function of the state.

Proposition 4 *In an economy with homogeneous preferences, the following properties hold:*
(a) Return volatility is increasing in the state variable ω for all risk preferences other than the logarithmic type. Formally,

$$\frac{d\sigma_R(\omega_t)}{d\omega_t} \begin{cases} > 0, & \text{for } \gamma \neq 1, \\ = 0, & \text{for } \gamma = 1. \end{cases}$$

(b) The instantaneous correlation between changes in volatility and returns is positive for $\gamma \neq 1$ and equal to zero for $\gamma = 1$.

Proof. See the Appendix. ■

The pro-cyclical variation in stock return volatility contradicts empirical evidence (e.g., Black (1976), French, Schwert and Stambaugh (1987), Campbell and Hentchel (1992)). In the following sections we demonstrate that heterogeneity in preferences can qualitatively affect the properties of stock returns, generating counter-cyclical variations in volatility.

4.4 Stock Returns: Asymptotic Analysis

In this section we perform asymptotic analysis of the competitive equilibrium. Our asymptotic expansions are designed to study the equilibrium when the cross-sectional distribution of risk aversion is "local to unity", i.e., when most of the wealth in the economy is controlled by agents with risk aversion coefficient close to one. This approach is introduced in Kogan and Uppal (1999), where it is used to derive optimal portfolio policies in partial and general equilibrium. Here we use similar techniques to study the "centralized version" of the competitive equilibrium with complete markets, the social planner's problem. The great advantage of the asymptotic analysis is that it delivers closed-form expressions for the equilibrium prices and other endogenous variables and clarifies the qualitative effects of various structural parameters of the economy on the equilibrium prices and individual behavior of investors.

We first develop the framework in which the distribution of the social utility weights is concentrated around $b = 1$. Thus, heterogeneity is still present but investors with preferences close to the logarithmic are given more weight. We analyze homogeneous economies as a special case.

Formulation

We parameterize the cross-sectional distribution of social weights by a "small parameter" $\varepsilon > 0$:

$$\nu(b;\varepsilon) = \left[\frac{1}{\sigma_b \varepsilon} g\left(\frac{b - 1 + \mu_b \varepsilon}{\sigma_b \varepsilon}\right)\right],$$

where $g(\cdot)$ is an exogenously given density function with zero mean and unit variance. Thus, the mean of $\nu(b;\varepsilon)$ equals $1 - \mu_b \varepsilon$ and the variance is $\sigma_b^2 \varepsilon^2$. We also assume that the distribution $g(\cdot)$ (and $\nu(\cdot;\varepsilon)$) possess well defined moment generating functions

$$G_g(u) \equiv \int_0^\infty g(b)\, e^{ub} db,$$
$$G_\nu(u;\varepsilon) \equiv \int_0^\infty \nu(b;\varepsilon)\, e^{ub} db.$$

Then, by construction,

$$G_\nu(u;\varepsilon) = e^{(1-\varepsilon\mu_b)u} G_g(\varepsilon\sigma_b u)$$

and

$$\begin{aligned} G_\nu(u;0) &= e^u, \\ \frac{\partial G_\nu(u;0)}{\partial \varepsilon} &= -e^u \mu_b u, \\ \frac{\partial^2 G_\nu(u;0)}{\partial \varepsilon^2} &= e^u \left(\mu_b^2 + \sigma_b^2\right) u^2, \end{aligned}$$

since

$$G_g(0) = 1, \quad G_g'(0) = 0, \quad G_g''(0) = 1.$$

The stock price in (18) involves the inverse function $\mathcal{I}(\cdot)$, which in general cannot be obtained in closed form. Instead, we derive an explicit asymptotic expansion for this function. To proceed, the function $\Phi(u;\varepsilon)$, defined as

$$\Phi(u;\varepsilon) \equiv \int_0^\infty e^{-bu} \nu(b;\varepsilon)\, db,$$

can be expanded as[14]

$$\begin{aligned} \Phi(u;\varepsilon) &= G_\nu(-u;\varepsilon) \\ &= G_\nu(-u;0) + \frac{\partial G_\nu(-u;0)}{\partial \varepsilon}\varepsilon + \frac{1}{2}\frac{\partial^2 G_\nu(-u;0)}{\partial \varepsilon^2}\varepsilon^2 + O\left(\varepsilon^3\right) \\ &= e^{-u}\left[1 + \varepsilon\mu_b u + \frac{1}{2}\varepsilon^2\left(\mu_b^2 + \sigma_b^2\right)u^2\right] + O\left(\varepsilon^3\right) \end{aligned}$$

This allows one to compute $\mathcal{I}(\cdot) \equiv \Phi^{-1} \circ \exp$ as:

$$\mathcal{I}(\omega) = -\omega - \varepsilon\mu_b\omega + \varepsilon^2\left(-\mu_b^2\omega + \frac{\sigma_b^2}{2}\omega^2\right) + O\left(\varepsilon^3\right). \tag{31}$$

Thus, the function $\mathcal{I}(\omega)$ is asymptotically (to the first order) the same as in a homogeneous economy with risk tolerance parameter $1 - \mu_b\varepsilon$. Hence, we interpret $\mu_b\varepsilon$ as the *average* risk aversion in the economy. This should not be confused with the risk aversion of the representative investor, which is determined below. With the inverse function $\mathcal{I}(\omega)$ in place, analysis of the asymptotic behavior of asset prices is straightforward.

Homogeneous Preferences

We first consider the special case of the model when all agents in the economy have identical preferences. Formally, this corresponds to $\sigma_b = 0$. Although preferences are identical, they are not necessarily of the logarithmic type ($\mu_b \neq 0$). The next two propositions summarize the results on

[14] Because the distribution $\nu(b)$ has compact support, its moment generating function $G_\nu(u;\varepsilon)$ is analytic and can be represented as an infinite power series in a neighborhood of $\varepsilon = 0$ (see, e.g., Dybvig and Rogers (1997, Lemma 1)).

the price-dividend ratio, price of risk, excess returns and return volatility.

Proposition 5 *In the homogeneous economy, the price-dividend ratio is given by*

$$\frac{P_t}{Y_t} = \frac{1}{\rho} + \varepsilon \frac{\lambda \mu_b}{\rho(\lambda+\rho)}(\omega_t - \overline{\omega}) \tag{32}$$
$$+ \varepsilon^2 \mu_b^2 \left(\frac{\sigma^2}{2\rho\,(2\lambda+\rho)} + \frac{\lambda}{\rho\,(\lambda+\rho)}\,(\omega_t - \overline{\omega}) + \frac{\lambda^2}{\rho\,(\lambda+\rho)\,(2\lambda+\rho)}\,(\omega_t - \overline{\omega})^2 \right) + O(\varepsilon^3).$$

For sufficiently small values of ε, it is increasing (decreasing) in the state variable for $\varepsilon\mu_b > 0$ ($\varepsilon\mu_b < 0$) and convex.

Proof. See the Appendix. ■

The first term in the asymptotic expansion $(1/\rho)$ is simply the price-dividend ratio under homogeneous logarithmic preferences. The next two terms capture the dependence of the price-dividend ratio on the state of the economy when the risk aversion coefficient is different from one. Sensitivity of the price-dividend ratio to changes in the state variable is controlled by λ. Large values of λ imply that the price-dividend ratio is more responsive to changes in the state variable (see (30)) and therefore has higher conditional volatility. When investors have standard time-separable, state-independent preferences ($\lambda = 0$), the price-dividend ratio is constant, as we discussed earlier.

As we showed in Proposition 2, the price-dividend ratio is a convex function of the state variable in the homogeneous economy. It is easy to verify this property from (32):

$$\frac{d^2\,(P_t/Y_t)}{d\omega_t^2} = \varepsilon^2 \frac{2(\mu_b\lambda)^2}{\rho\,(\lambda+\rho)\,(2\lambda+\rho)} \begin{cases} > 0, & \text{for } \mu_b \neq 0 \\ = 0, & \text{for } \mu_b = 0. \end{cases} + O\left(\varepsilon^3\right). \tag{33}$$

Moreover, the approximate closed-form expressions allow us to clarify its dependence on the structural parameters. Equation (33) shows that the degree of convexity is inversely related to ρ and depends positively on λ and on how different the risk aversion is from the logarithmic type (i.e., on the absolute value of $\varepsilon\mu_b$).

Proposition 3 establishes that the price of risk is constant in the homogeneous economy. To facilitate comparison with the heterogeneous economy, the following proposition summarizes our findings on the price of risk, expected excess returns and return volatility. The intuition is given in the discussion of Proposition 3. It is commonplace in the empirical literature to use log excess returns. We follow this convention in the coming discussion on stock returns.

Proposition 6 *In the homogeneous economy,*
(a) the price of risk is constant and is given by

$$\pi_t = \pi = \sigma + \varepsilon \cdot \sigma\mu_b + \varepsilon^2 \cdot \sigma\mu_b^2 + O\left(\varepsilon^3\right). \tag{34}$$

(b) The expected log excess stock return is

$$\mu_{R,t} - r_t - \frac{1}{2}\sigma_{R,t}^2 = \frac{\sigma^2}{2} + \varepsilon \cdot \sigma^2 \mu_b + \varepsilon^2 \cdot \sigma^2 \left[\frac{\left(3\lambda^2 + 6\lambda\rho + 2\rho^2\right)\mu_b^2}{2\left(\lambda+\rho\right)^2} \right] + O\left(\varepsilon^3\right). \tag{35}$$

(c) The conditional volatility of stock returns is given by

$$\sigma_R\left(\omega_t\right) = \sigma + \varepsilon \frac{\lambda\mu_b}{\lambda+\rho}\sigma + \varepsilon^2 \lambda \mu_b^2 \sigma \left(\frac{1}{\lambda+\rho} + \frac{\lambda\rho}{\left(\lambda+\rho\right)^2\left(2\lambda+\rho\right)} \left(\omega_t - \overline{\omega}\right) \right) + O\left(\varepsilon^3\right). \tag{36}$$

It is increasing in the risk aversion of the representative agent. Changes in conditional volatility are positively correlated with market returns.

Proof. See Appendix. ■

Consistent with the results in Section 4.3, volatility is increasing in the state variable. It is easy to see that the conditional correlation between changes in conditional volatility and stock returns is constant and equal to one. Thus, positive stock returns are accompanied by increased volatility of returns. Another important implication of (36) is that volatility is increasing in the risk aversion of the representative agent. This observation helps understand the following results on heterogeneous economies.

Heterogeneous Preferences

The previous section provides a benchmark against which we evaluate the effects of heterogeneity. We now consider the general case: $\sigma_b > 0$.

Proposition 7 *In the heterogeneous economy, the price-dividend ratio is characterized by*

$$\begin{aligned} \frac{P_t}{Y_t} = {} & \frac{1}{\rho} + \varepsilon \frac{\lambda\mu_b}{\rho\left(\lambda+\rho\right)}\left(\omega_t - \overline{\omega}\right) \\ & + \varepsilon^2 \left(\frac{\left(\mu_b^2 + \sigma_b^2\right)\sigma^2}{2\rho\left(2\lambda+\rho\right)} + \frac{\lambda\mu_b^2 - \left(\mu - \sigma^2\right)\sigma_b^2}{\rho\left(\lambda+\rho\right)}\left(\omega_t - \overline{\omega}\right) + \frac{\lambda\left(\lambda\mu_b^2 - \left(\lambda+\rho\right)\sigma_b^2\right)}{\rho\left(\lambda+\rho\right)\left(2\lambda+\rho\right)}\left(\omega_t - \overline{\omega}\right)^2 \right) \\ & + O\left(\varepsilon^3\right). \end{aligned} \tag{37}$$

Proof. See the Appendix. ■

Compared to (32), heterogeneity manifests its effect on the price-dividend ratio via the second-order term. In particular, it affects the curvature of the price-dividend ratio as a function of the

state variable. From (37),

$$\frac{d^2\left(P_t/Y_t\right)}{d\omega_t^2} = \varepsilon^2 \frac{2\lambda\left(\lambda\mu_b^2 - (\lambda+\rho)\,\sigma_b^2\right)}{\rho\,(\lambda+\rho)\,(2\lambda+\rho)} + O\left(\varepsilon^3\right). \tag{38}$$

When there is sufficient heterogeneity in the economy (i.e., when σ_b^2 is large relative to μ_b^2), the price-dividend ratio is a concave function (it is convex in homogeneous economies, see Proposition 5). Concavity of the price-dividend ratio can lead to a negative relation between conditional volatility and stock returns, as stated in the following proposition.

Proposition 8 *In the heterogeneous economy,*
(a) the instantaneous volatility of stock returns is given by

$$\begin{aligned}\sigma_R\left(\omega_t\right) = \sigma &+ \varepsilon\frac{\lambda\mu_b}{\lambda+\rho}\sigma \\ &+ \varepsilon^2\left(\frac{\lambda\mu_b^2 - \left(\mu-\sigma^2\right)\sigma_b^2}{\lambda+\rho} + \frac{\lambda^2\rho\left(\mu_b^2 - \frac{2(\lambda+\rho)^2}{\lambda\rho}\sigma_b^2\right)}{(\lambda+\rho)^2\,(2\lambda+\rho)}\left(\omega_t - \overline{\omega}\right)\right)\sigma + O\left(\varepsilon^3\right).\end{aligned} \tag{39}$$

(b) Changes in conditional volatility are negatively correlated with stock returns when

$$\sigma_b^2 > \mu_b^2 \frac{\lambda\rho}{2\,(\lambda+\rho)^2}.$$

Proof. See the Appendix. ■

The conditional correlation between changes in return volatility and stock returns is constant and is given by

$$\operatorname{sgn}\left(\mu_b^2 - \frac{2\,(\lambda+\rho)^2}{\lambda\rho}\sigma_b^2\right). \tag{40}$$

When σ_b^2 is small relative to μ_b^2, there is little heterogeneity in the population. In this case, the correlation is equal to one, just as in the homogeneous case. However, when σ_b^2 is sufficiently large, the correlation is negative. Below we argue that the intuition behind these findings is derived from the behavior of the cross-sectional distribution of wealth.

Proposition 9 *In the heterogeneous economy,*
(a) the price of risk process is decreasing in the state variable and is given by

$$\pi_t = \sigma + \varepsilon\cdot\sigma\mu_b + \varepsilon^2\cdot\sigma\left(\mu_b^2 - \sigma_b^2\omega_t\right) + O\left(\varepsilon^3\right). \tag{41}$$

(b) The expected log excess return is decreasing in the state variable and is given by

$$\begin{aligned}&\mu_{R,t} - r_t - \frac{1}{2}\sigma_{R,t}^2 \\ &= \frac{\sigma^2}{2} + \varepsilon \cdot \sigma^2 \mu_b + \varepsilon^2 \cdot \sigma^2 \left[\frac{\lambda\left(3\lambda^2 + 6\lambda\rho + 2\rho^2\right)\mu_b^2 - (\lambda+\rho)^2 (2\mu - \sigma^2)\sigma_b^2}{2\lambda(\lambda+\rho)^2} - \sigma_b^2 (\omega_t - \overline{\omega}) \right] \\ &+ O\left(\varepsilon^3\right).\end{aligned} \tag{42}$$

Proof. As in the proof of Proposition 6. ■

In sharp contrast to the homogeneous economy, both the price of risk and the expected excess return vary over time. As one would expect, both are lower in good times (high ω) and higher in bad times (low ω), and their sensitivity to the state variable is captured by the degree of heterogeneity, σ_b^2. This state-dependence is more pronounced when heterogeneity is more prominent. The standard consumption CAPM establishes that the price of risk is proportional to the aggregate risk aversion. As shown below in Proposition 12, the response of the aggregate risk aversion to changes in the state variable is proportional to σ_b^2. This explains the behavior of the price of risk. Since both the volatility of stock returns and the aggregate risk aversion are counter-cyclical, changes in log excess stock returns are negatively correlated with the stock market.

Our model gives rise to slow-varying expected stock returns and is therefore capable of generating predictability of stock returns. We focus on the effect of preference heterogeneity on the autocorrelation of log excess returns. Let $\mathcal{R}_{t,t+\Delta}$ denote the cumulative log excess return over a period of length Δ:

$$\mathcal{R}_{t,t+\Delta} = \int_t^{t+\Delta} \left(\mu_{R,s} - \frac{1}{2}\sigma_{R,s}^2 - r_s \right) ds + \sigma_{R,s} dB_s.$$

The following proposition presents the results for the homogeneous and heterogeneous economies.

Proposition 10 *(a) In the heterogeneous economy, the unconditional auto-correlation of return is negative (asymptotically). Specifically,*

$$Cov\left(\mathcal{R}_{t,t+\Delta_1}, \mathcal{R}_{t+\Delta_1,t+\Delta_1+\Delta_2}\right) = -\varepsilon^2 \cdot \sigma_b^2 \frac{\sigma^4}{\lambda^2}\left(1 - e^{-\lambda\Delta_1}\right)\left(1 - e^{-\lambda\Delta_2}\right) + O\left(\varepsilon^3\right) < 0. \tag{43}$$

And,

$$Corr\left(\mathcal{R}_{t,t+\Delta_1}, \mathcal{R}_{t+\Delta_1,t+\Delta_1+\Delta_2}\right) = -\varepsilon^2 \cdot \sigma_b^2 \frac{\sigma^2}{\lambda^2\sqrt{\Delta_1\Delta_2}}\left(1 - e^{-\lambda\Delta_1}\right)\left(1 - e^{-\lambda\Delta_2}\right) + O\left(\varepsilon^3\right) < 0. \tag{44}$$

(b) In the homogeneous economy, autocorrelation is equal to zero, up to $O\left(\varepsilon^3\right)$ terms.

Proof. See the Appendix. ■

Figure 1 presents the (normalized) autocorrelation of excess stock returns over a range of return horizons for the case $\Delta_1 = \Delta_2$. The magnitude of autocorrelation increases initially for short horizons and slowly reverts to zero as return horizon tends to infinity. At short horizons, autocorrelation is proportional to Δ and hence predictability is weak. Similarly, stationarity of the state variable forces autocorrelation to zero at very long horizons.

Cross-Sectional Distribution of Wealth and Individual Consumption-Portfolio Choice

The asymptotic framework allows us to obtain explicit expressions for individual consumption-portfolio policies. Since most of the weight in the social utility function is placed on the investors with risk aversion close to one, we parameterize an individual investor's type as $b = 1 - b_1\varepsilon$, $\varepsilon > 0$. The next proposition characterizes the individual wealth, consumption-wealth ratio and stock holdings as functions of the state of the economy.

Proposition 11 *An investor of type $b = 1 - b_1\varepsilon$ adopts the following policies:*
(a) The optimal consumption-wealth ratio is given by

$$\frac{C_t^*}{W_t} = \rho\left[1 - \varepsilon\frac{\lambda b_1}{\lambda+\rho}\left(\omega_t - \overline{\omega}\right)\right] + O\left(\varepsilon^2\right); \tag{45}$$

(b) The optimal position in the stock as a fraction of individual wealth is given by

$$\alpha_t^* = 1 + \varepsilon\frac{\rho}{\lambda+\rho}\left(\mu_b - b_1\right) + O\left(\varepsilon^2\right); \tag{46}$$

(c) The wealth process is given by

$$\frac{W_t\left(b\right)}{Y_t} = \frac{1}{\sigma_b\varepsilon}g\left(\frac{\mu_b - b_1}{\sigma_b}\right)\left[\frac{1}{\rho} + \varepsilon\frac{\mu_b - b_1}{\rho}\overline{\omega} + \varepsilon\left(\frac{\mu_b}{\rho} - \frac{b_1}{\lambda+\rho}\right)\left(\omega_t - \overline{\omega}\right) + O\left(\varepsilon^2\right)\right]. \tag{47}$$

Proof. See the Appendix. ■

Equation (46) shows that investors' positions in the risky asset (as a fraction of their wealth) are approximately constant over time. Moreover, investors who are less risk averse than the average investor, i.e., those with $b_1 < \mu_b$, borrow at the risk-free rate and invest more than 100% of their wealth in the stock market; investors with higher-than-average risk aversion ($b_1 > \mu_b$) invest part of their wealth in the risk-free asset. As a result, less risk averse investors have larger exposure to the stock market: their wealth responds more both to positive and negative market moves. Since the risky asset earns positive risk premium in equilibrium, this implies that less risk averse investors could eventually come to dominate the economy, as their portfolios grow at higher than average

rate.[15] While the latter observation is correct, it ignores the fact that different investors follow different consumption policies. In particular, as equation (45) shows, less risk averse investors (lower b_1) consume at a higher rate than more risk averse investors (higher b_1) in "good" states of the economy ($\omega_t > \overline{\omega}$) and vice versa in "bad" states ($\omega_t < \overline{\omega}$). Thus, while positive stock returns (positive increments in ω_t) increase the relative value of the portfolios of low-risk-aversion investors, they also cause them to raise their consumption rate, which tends to dissipate the wealth-inequality over time. As a result of this trade-off, wealth of various investors grows at the same average rate in equilibrium and their relative wealth is stationary, as given by (47).

It is useful to contrast the results in Proposition 11 with the standard case of $\lambda = 0$. The following corollary states the parallel results.

Corollary 2 *When preferences are described by the standard time-separable utility function with constant relative risk aversion, an investor of type $b = 1 - b_1\varepsilon$ adopts the following policies:*
(a) The optimal consumption-wealth ratio is given by

$$\frac{C_t^*}{W_t} = \rho\left[1 + \varepsilon\frac{b_1}{\rho}\left(\mu - \frac{\sigma^2}{2}\right)\right] + O\left(\varepsilon^2\right);$$

(b) The optimal position in the stock as a fraction of individual wealth is given by

$$\alpha_t^* = 1 + \varepsilon\left(\mu_b - b_1\right) + O\left(\varepsilon^2\right);$$

(c) The wealth process is given by

$$\frac{W_t\left(b\right)}{Y_t} = \frac{1}{\sigma_b\varepsilon}g\left(\frac{\mu_b - b_1}{\sigma_b}\right)\left[\frac{1}{\rho} - \varepsilon\frac{b_1}{\rho^2}\left(\mu - \frac{\sigma^2}{2}\right) + \varepsilon\left(\frac{\mu_b - b_1}{\rho}\right)\left(y_t - x_0\right)\right] + O\left(\varepsilon^2\right).$$

When the economy is populated by heterogeneous investors with time-separable preferences, the optimal consumption and investment policies are both approximately state-independent. Investors consume a constant fraction of their wealth, regardless of the state of the economy. Coupled with the fact that less risk averse investors invest relatively more in the stock, their wealth grows faster than that of more risk averse investors and this leads to a non-existent or degenerate steady-state distribution of cross-sectional wealth.

We have shown that in heterogeneous economies positive market returns give a relative "boost" to portfolios of less risk averse agents and temporarily increase their relative wealth in the economy (the relative wealth distribution in the economy is stationary, but it responds to moves of the stock market). This implies that, following positive market moves, the aggregate risk aversion in the

[15] This is the property of the exchange economy in Wang (1996). In his model, preferences are described by standard time-separable isoelastic utility functions, and the less risk averse investor dominates as the aggregate endowment tends to infinity.

economy (the risk aversion of the representative agent) temporarily falls and, similarly, the risk aversion rises in response to negative market moves. This intuition is formalized by the following proposition.

Proposition 12 *The relative risk aversion coefficient of the representative agent is given by*

$$\gamma^{(SP)}(\omega_t) = 1 + \varepsilon\mu_b + \varepsilon^2\left(\mu_b^2 - \sigma_b^2(1+2\omega_t)\right) + O\left(\varepsilon^3\right).$$

Proof. The result follows from Lemma 2 and (31). ■

Thus, changes in aggregate risk aversion are negatively correlated with stock returns. Sensitivity of $\gamma^{(SP)}$ to the state of the economy depends exclusively on the degree of heterogeneity. This result is intuitive, since changes in the aggregate risk aversion are due to variations in the cross-sectional wealth distribution. It also helps understand the behavior of stock return volatility. Recall from Proposition 6 that conditional volatility is increasing in both the state variable and risk aversion in homogeneous economies. Changing aggregate risk aversion in a heterogeneous economy can be interpreted informally as an effective shift across the homogeneous economies. As a result, a drop in the stock market has two effects: the direct effect is to reduce the conditional volatility (positive slope, $d\sigma_R/d\omega > 0$, in (36)); the indirect effect is to raise the volatility (increased level, $\partial\sigma_R/\partial(\varepsilon\mu_b) > 0$, in (36)). The second effect dominates when cross-sectional heterogeneity is sufficiently high, i.e., when $\sigma_b^2 > \mu_b^2\lambda\rho/\left(2(\lambda+\rho)^2\right)$, and leads to counter-cyclical variations in conditional volatility.

4.5 Calibration and Numerical Analysis

The previous sections have shown qualitatively that preference heterogeneity can have significant influence on asset behavior. In this section, we attempt to quantify these effects via numerical analysis, complementing our asymptotic results. As in Section 4.4, we consider both homogeneous and heterogeneous economies. We calibrate both types of models to match the same set of unconditional moments of asset returns and then investigate the dynamics of conditional moments.

Homogeneous Preferences

In simulating the economy, we chose model parameters to match several moments of the US data: the mean and standard deviation of consumption growth and excess stock returns, and the mean of the risk-free rate. Our choice of model parameters is summarized in Table 1.

In Table 2 we summarize the historical estimates reported in Campbell and Cochrane (1999) and Campbell, Lo and MacKinlay (1997)[16] as well as the corresponding moments of the model

[16]We use the set of estimates reported in Table 2 in Campbell and Cochrane (1999). Their estimates are based

economy. Overall, the model appears to be successful in replicating unconditional moments of stock and bond returns. The average level of the price-dividend ratio produced by the model is consistent with historical data, while the volatility of the risk-free rate and the volatility of the price-dividend ratio are somewhat higher than historical levels. A value of 9.27 for the risk aversion coefficient is used; while this value is still relatively large, it is consistent with the range of values considered in Mehra and Prescott (1985) and is much smaller than the local curvature of the utility function used by Campbell and Cochrane (1999). Interestingly, the model does not encounter the "risk-free rate puzzle" (Weil (1989)). This appealing feature of the "catching up with the Joneses" model is first discussed in Abel (1990). However, it is known that catching up with the Joneses preferences can induce additional volatility in the risk-free rate.[17] Table 2 confirms this result. Note, however, that the volatility of the risk-free rate in our continuous-time model is much lower than in its discrete-time analog, i.e., Abel (1990). The difference is due to the definition of the state variable X_t: while Abel assumes that it equals the last-period's aggregate consumption, we allow X_t to be a weighted average of lagged realizations of aggregate consumption. As a result, X_t is varying slowly over time and the interest rate is less volatile. In the next section we show how accounting for preference heterogeneity can help reduce the volatility of the risk-free rate even further.

As a first step in analyzing the dynamics of stock returns, we plot conditional moments of returns against the state variable ω. Figure 2 presents conditional moments for values of the state variable within three standard deviations of its long-run mean (under our choice of parameters, $\mathrm{E}[\omega] = 0.29$, $\sigma[\omega] = 0.10$). Panel (a) shows that the price-dividend ratio is an increasing function of the state variable. This property is driven to a large extent by state-dependence of the short-term interest rate, which is a decreasing function of ω. The expected log excess return (panel (c)) is an increasing function of ω, i.e., it is positively related to the price-dividend ratio. Instantaneous volatility of stock returns (panel (b)) is increasing in ω and the level of stock prices (pro-cyclical), and therefore it is positively correlated with stock returns. These properties of conditional moments appear to be inconsistent with empirical observations, i.e., the "leverage effect" (see Black (1976), Campbell and Hentschel (1992), Schwert (1989)). According to panel (d), the Sharpe ratio of stock returns is constant.[18] Thus, the only source of time-variation in stock returns is changing return volatility.

To analyze patterns of predictability (autocorrelations and cross-correlations) of stock returns,

on Standard & Poors 500 stock and commerical paper returns between 1871 and 1993 and per capita consumption between 1889 and 1992. The estimate of standard deviation of expected risk free rate is taken from Campbell, Lo and MacKinlay (1997, p.329). Based on Tables 8.1 and 8.2, they argue that the volatility of expected risk-free rate is approximately 3%. Their estimates are consistent with estimates constructed in Siegel (1992).

[17] See, for example, Campbell, Lo and MacKinlay (1997).

[18] In this one-factor economy, the Sharpe ratio is given by the product of the consumption growth volatility and the risk aversion coefficient of the representative agent.

we simulate 123 years of annual stock returns (the length of the historical data set) and compute several commonly used statistics. We repeat this experiment $5,000$ times and compare the empirical distribution of our statistics with estimates of Campbell and Cochrane (1999, Table 3,4), based on historical data over the 1871-1993 period. In particular, we record the statistics for each of the $5,000$ artificial time series and compute the mean as well as 5'th and 95'th percentiles (90% confidence interval) of the resulting empirical distribution of these statistics. This procedure captures finite-sample properties (and biases, e.g., Nelson and Kim (1993)) of the involved estimators, conditional on the underlying data being generated by our economic model. Our findings are summarized in Table 3.

Given the amount of historical data available, evidence on autocorrelation in annual excess returns is weak. Because of lack of precision in estimates of autocorrelation, the model is not rejected by the data, except possibly at a two-year lag[19]. Our analysis of partial sums of correlation coefficients reveals that even this apparent discrepancy may be attributed to estimation error: none of the partial sums appear to be inconsistent with the model.

Historical evidence on cross-correlation between price-dividend ratios and excess stock returns appears to be at odds with the model: three out of five estimates fall to the left of the 5'th percentile of the empirical distribution based on the model. Note that while the price-dividend ratio is positively related to the instantaneous expected excess return (Figure 2), estimates of cross-correlation are negative on average. This is due to a negative bias exhibited by these estimates of cross-correlation.

While historical cross-correlations between absolute excess returns and price-dividend ratios tend to be negative, the model produces positive estimates. At a one-year lag the historical estimate falls well outside the 90% confidence interval. The same is true for cross-correlations between absolute excess returns and one-year lagged returns.

Overall, the homogeneous economy model produces a rather poor fit of historical correlations and cross-correlations of stock returns and price-dividend ratios. However, the model is not rejected strongly due to substantial estimation errors.

Heterogeneous Preferences

We now analyze the general case in which the economy is populated with heterogeneous agents. For simplicity, we assume that there are only two types of agents in the economy, with risk aversion coefficients of 1 and 20 respectively. We normalize social weights of these agents so that $\nu(1) = 1$ and treat $\nu(1/20)$ as a free parameter.[20] We chose model parameters (see Table 4) to match the

[19] We base our discussion on univariate confidence intervals. Formal statistical inference would have to deal with the multivariate nature of the problem explicitly.

[20] Function $\nu(b)$ is defined in Lemma 1. Formally, with only two types present in the economy, $\nu(b) = \nu(1)\,\delta_1(b) + \nu(1/20)\,\delta_{1/20}(b)$, where $\delta_x(\cdot)$ denotes a delta function with a mass at x.

same set of moments as with the homogeneous economy. Table 5 relates moments of asset returns in the model economy to historical estimates. The fit of the heterogeneous-agent model is better than that of the homogeneous economy (Table 2). In particular, volatility of the risk-free rate is now consistent with the data and volatility of the price-dividend ratio is closer to the historical estimate.

Figure 3 illustrates the distribution of wealth across the agents in the economy and individual portfolio strategies. Panel (a) shows that only a relatively small fraction of wealth is controlled by the less risk-averse type of agents. This fraction is increasing in the state variables. The reason for that is made clear by panel (b): less risk-averse agents invest a much larger fraction of their wealth in stocks and hence benefit more from positive shocks to the stock price (which is positively related to the state variable). Our analysis below shows that, even though most of wealth in this economy is controlled by the more risk-averse type of agents, asset prices are significantly affected by agent heterogeneity.

Next, we plot conditional moments of returns against the state variable ω (the stock price-dividend ratio and the volatility of stock returns are computed according to (19)–(22) and (25)). Figure 4 presents conditional moments for values of the state variable within three standard deviations of its long-run mean ($\mathrm{E}[\omega] = 0.19$, $\sigma[\omega] = 0.08$). Panel (b) shows that return volatility is a decreasing function of the state variable over most of the range considered. As a result, the instantaneous correlation between changes in volatility and stock returns is negative. This stands in sharp contrast with the behavior of volatility in homogeneous economies analyzed in Section 4.3: as we have established in Proposition 4 and illustrated in Figure 2, the volatility of returns is always increasing in the state variable and positively correlated with stock returns. Proposition 8 demonstrates (using asymptotic analysis) the ability of the model with heterogeneous preferences to produce counter-cyclical behavior of return volatility. Figure 4 verifies this result numerically.

In a heterogeneous economy, the curvature of the utility function of the representative agent changes over time as a function of the state variable. Panel (d) shows that the Sharpe ratio of stock returns, which is proportional to the utility curvature, decreases in the state variable.[21] This reflects the fact that negative stock returns cause the distribution of wealth to shift towards more risk averse agents; positive realizations have the opposite effect. The resulting pattern of changing market price of risk is missing in the homogeneous model, which has been cited as a drawback of "ratio models" (e.g., Campbell, Lo and MacKinlay (1997, Section 8.4.1)) and an argument for using "difference models", in which utility curvature is *assumed* to be a function of the state, e.g., Campbell and Cochrane (1999). Clearly, our *heterogeneous* ratio model is capable of endogenously generating sizable counter-cyclical variation in aggregate risk aversion and market price of risk.

[21] Ferson and Harvey (1991) and Harvey (1991) provide empirical evidence on counter-cyclical variation in the market price of risk: it is negatively related to the price-dividend ratio and is higher during business cycle troughs than during peaks.

Given sufficient degree of heterogeneity, over a reasonable range of values of the state variable, the Sharpe ratio can deviate from its long-run average by a factor of two.

Finally, due to a counter-cyclical nature of both the return volatility and the Sharpe ratio, expected excess returns are negatively related to the price-dividend ratio, as shown in panel (c).

Table 6 is analogous to Table 3 and reports estimates of auto- and cross- correlations of returns. In general, the heterogeneous model performs better than its homogeneous counterpart. Excess stock returns are now negatively correlated with lagged price-dividend ratios. Historical estimates no longer fall below the 5'th percentile of the empirical distribution. Cross-correlations between absolute returns and lagged returns and price-dividend ratios are on average negative, which is consistent with historical data.

5 Conclusion

Representative-agent models identify economic mechanisms that are sufficient to generate empirically-observed features of asset prices. One such mechanism is counter-cyclical variation in aggregate risk aversion and market price of risk, as shown by Campbell and Cochrane (1999). Such variation, in turn, may arise as a result of interaction among agents with heterogeneous risk preferences. In this paper we have characterized some of the general equilibrium implications of cross-sectional heterogeneity in risk aversion on asset prices. We have shown in the context of a standard representative agent model that preferences with catching up with the Joneses feature can be used to replicate unconditional moments of asset returns. However, homogeneous economies display patterns of time-variation in expected stock returns and volatility that are qualitatively inconsistent with empirical observations. In particular, such models generate pro-cyclical variation in return volatility and positive cross-correlation between excess returns and price-dividend ratios. We demonstrate that by accounting for heterogeneity in risk aversion one can overcome this shortcoming of the basic model. The economic mechanism delivering this result is the evolution of the cross-sectional wealth distribution among the investors. Changes in the distribution of wealth, triggered by moves in the stock market, cause time-variation in aggregate risk aversion and market price of risk. The level of return volatility and risk premium increase with the degree of aggregate risk aversion, giving rise to negative correlation between changes in conditional volatility and risk premium on one side and stock returns on the other side.

The ability of our heterogeneous agent model to replicate various empirical phenomena is encouraging. It points to the possibility that many salient features of the data can be a result of interaction of rational investors with different risk preferences.

6 Appendix

Proof of Lemma 1

Let

$$c_t\left(Y_t,X_t;b\right)\equiv\ \frac{C_t\left(\omega_t;b\right)}{Y_t}.$$

The sharing rule in (10) and (11) is simply the first order condition for consumption in the social planner's static optimization problem:

$$\sup_{c_t(\cdot;b)}\int_0^\infty f\left(b\right)\frac{1}{1-\frac{1}{b}}\left(\frac{c_t\left(\cdot;b\right)Y_t}{X_t}\right)^{1-\frac{1}{b}}db,\quad s.t.\quad\int_0^\infty c_t\left(\cdot;b\right)Y_tdb=Y_t$$

$$\Longleftrightarrow\ \inf_{Z_t\geq0}\sup_{c_t(\cdot;b)}\int_0^\infty f\left(b\right)\frac{1}{1-\frac{1}{b}}\left(c_t\left(\cdot;b\right)\right)^{1-\frac{1}{b}}e^{\left(1-\frac{1}{b}\right)\omega_t}db-Z_t\cdot Y_t\cdot\left(\int_0^\infty c_t\left(\cdot;b\right)db-1\right)$$

Thus,

$$c_t^*\left(\omega_t;b\right)=\nu\left(b\right)e^{-b(z_t+x_t)-\omega_t},$$
$$\int_0^\infty\nu\left(b\right)e^{-b(z_t+x_t)-\omega_t}db=1,$$

where

$$\nu\left(b\right)\equiv f\left(b\right)^b,$$
$$z_t\equiv\log\left(Z_t\right).\tag{48}$$

The result of the Lemma follows by identifying

$$\mathcal{I}\left(\omega_t\right)\equiv z_t+x_t.\tag{49}$$

■

Proof of Lemma 2

Using the definition of ω_t, we immediately have

$$U_Y^{(SP)}\left(Y_t,X_t\right)=\frac{u_\omega^{(SP)}\left(\omega_t\right)}{Y_t},\quad U_{YY}^{(SP)}\left(Y_t,X_t\right)=\frac{1}{Y_t^2}\left[u_{\omega\omega}^{(SP)}\left(\omega_t\right)-u_\omega^{(SP)}\left(\omega_t\right)\right].$$

Substituting these expressions into (15), we obtain the expression after the first equality. The expression after the second equality follows from simple differentiation of $u^{(SP)}\left(\omega_t\right)$ with respect to ω_t.■

Proof of Lemma 3

(a) Given the general expression for the stock price,

$$P_t=\mathrm{E}_t\int_t^\infty\exp\left(-\rho\left(s-t\right)+z_s-z_t\right)\cdot Y_sds,$$

and the relation

$$z_t=\mathcal{I}(\omega_t)-x_t,$$

we find

$$z_t + y_t = \mathcal{I}(\omega_t) + \omega_t$$

and

$$\begin{aligned} P_t &= Y_t \exp\left(-z_t - y_t\right) \cdot \mathrm{E}_t \int_t^\infty \exp\left(-\rho\left(s-t\right) + z_s + y_s - z_t - y_t\right) ds \\ &= Y_t \exp\left(-\mathcal{I}\left(\omega_t\right) - \omega_t\right) \cdot \mathrm{E}\left[\int_t^\infty \exp\left(-\rho\left(s-t\right) + \mathcal{I}\left(\omega_s\right) + \omega_s\right) ds \middle| \omega_t\right]. \end{aligned}$$

(b) Define the process $u_t = z_t + x_t = \mathcal{I}\left(\omega_t\right)$. Then, using (12) and the definition (20),

$$\Phi\left(u_t\right) = e^{\omega_t}$$

and the expression (19) is obtained from (18) by substituting the relations

$$\begin{aligned} \mathcal{I}\left(\omega_t\right) &= u_t, \\ \omega_t &= \log\left(\Phi\left(u_t\right)\right). \end{aligned}$$

Applying Itô's lemma to both sides of this equality yields

$$\begin{gathered} \left(\Phi'\left(u_t\right)\mu_u\left(u_t\right) + \tfrac{1}{2}\Phi''\left(u_t\right)\sigma_u\left(u_t\right)^2\right) dt + \Phi'\left(u_t\right)\sigma_u\left(u_t\right) dB_t = \\ \Phi\left(u_t\right)\left(\mu_\omega\left(\log\left(\Phi\left(u_t\right)\right)\right) dt + \sigma\, dB_t\right), \end{gathered}$$

where $\mu_u\left(u_t\right)$ and $\sigma_u\left(u_t\right)$ are the drift and the diffusion coefficients of u_t respectively and $\mu_\omega\left(\cdot\right)$ is the drift of ω_t defined in (6). Matching the deterministic and stochastic terms yields

$$\begin{aligned} \sigma_u\left(u_t\right) &= \sigma \frac{\Phi\left(u_s\right)}{\Phi'\left(u_s\right)}, \\ \mu_u\left(u_t\right) &= \frac{\Phi\left(u_t\right)}{\Phi'\left(u_t\right)}\mu_\omega\left(\log\left(\Phi\left(u_t\right)\right)\right) - \frac{1}{2}\frac{\Phi''\left(u_t\right)}{\Phi'\left(u_t\right)}\sigma_u\left(u_t\right)^2, \end{aligned}$$

from which (21) follows.■

Proof of Lemma 4

Simple application of Ito's Lemma to stock price in (18) yields

$$\frac{dP_t}{P_t} = \mu_{P,t}dt + \sigma_{P,t}dB_t,$$

where

$$\begin{aligned} \mu_{P,t} &\equiv \frac{\mathcal{F}'\left(\omega_t\right)}{\mathcal{F}\left(\omega_t\right)}\mu_{\omega,t} + \mu + \frac{1}{2\mathcal{F}\left(\omega_t\right)}\left(2\mathcal{F}'\left(\omega_t\right) + \mathcal{F}''\left(\omega_t\right)\right)\sigma^2, \\ \sigma_{P,t} &\equiv \sigma + \frac{\mathcal{F}'\left(\omega_t\right)}{\mathcal{F}\left(\omega_t\right)}\sigma, \end{aligned}$$

and $\mu_{\omega,t} \equiv -\lambda(\omega_t - \overline{\omega})$. The result then follows from the definition of stock returns in (23).■

Proof of Lemma 5

¿From the relation $z_t = \mathcal{I}(\omega_t) - x_t$ and the joint dynamics of ω_t and x_t,

$$\begin{pmatrix} d\omega_t \\ dx_t \end{pmatrix} = \begin{pmatrix} \mu_{\omega,t} \\ \lambda\omega_t \end{pmatrix} dt + \begin{pmatrix} \sigma \\ 0 \end{pmatrix} dB_t,$$

we immediately have

$$dz_t = \mu_{z,t}dt + \sigma_{z,t}dB_t,$$

where

$$\mu_{z,t} \equiv \mathcal{I}'(\omega_t)\mu_{\omega,t} - \lambda\omega_t + \frac{1}{2}\sigma^2\mathcal{I}''(\omega_t),$$
$$\sigma_{z,t} \equiv \sigma \cdot \mathcal{I}'(\omega_t).$$

Applying Ito's Lemma to $\xi_{0,t}$,

$$\frac{d\xi_{0,t}}{\xi_{0,t}} = \left(\mu_{z,t} - \rho + \frac{1}{2}\sigma_{z,t}^2\right)dt + \sigma_{z,t}dB_t.$$

Since the instantaneous interest rate is equal to the (negative) drift coefficient and the price of risk is the (negative) diffusion coefficient in the differential form of $\xi_{0,t}$, we immediately obtain the desired result.■

Proof of Proposition 1

With standard isoelastic, time-separable preferences, the stock price is given by

$$\begin{aligned}
P_t &= \mathrm{E}_t \int_t^\infty e^{-\rho(s-t)} \left(\frac{Y_{t+s}}{Y_t}\right)^{-\gamma} Y_{t+s}ds \\
&= Y_t \cdot \mathrm{E}_t \int_t^\infty e^{-\rho(s-t)} e^{(1-\gamma)(y_{t+s}-y_t)}ds \\
&= Y_t \cdot \int_t^\infty e^{-\rho(s-t)} \exp\left[(1-\gamma)\left(\mu - \frac{\sigma^2}{2}\right)(s-t) + \frac{(1-\gamma)^2}{2}\sigma^2(s-t)\right]ds \\
&= Y_t \cdot \left(\rho - (1-\gamma)\left(\mu - \frac{\sigma^2}{2}\right) - \frac{(1-\gamma)^2}{2}\sigma^2\right)^{-1}.
\end{aligned}$$

■

Proof of Proposition 2

With homogeneous agents, from the definitions of Φ and $\mathcal{I}(\cdot)$, we immediately have $\mathcal{I}(\omega_t) = -\omega_t/b$. For notational simplicity, define $\overline{P} \equiv P/Y$. Then, the price function in (18) becomes

$$\begin{aligned}
\overline{P} &= e^{(\gamma-1)\omega}\mathrm{E}\left[\int_0^\infty e^{-\rho t + (1-\gamma)\omega_t}dt \,\middle|\, \omega_0 = \omega\right] \\
&= e^{(\gamma-1)\omega}\left[\int_0^\infty e^{-\rho t + (1-\gamma)\mathrm{E}_0(\omega_t) + \frac{(1-\gamma)^2}{2}\mathrm{Var}_0(\omega_t)}dt\right] \\
&= e^{(\gamma-1)\omega} \cdot \int_0^\infty \kappa_t e^{(1-\gamma)e^{-\lambda t}\omega}dt,
\end{aligned}$$

where

$$\kappa_t \equiv \exp\left\{-\rho t + \left[(1-\gamma)(1-e^{-\lambda t})\overline{\omega} + \frac{\sigma^2(1-\gamma)^2}{4\lambda}\left(1-e^{-2\lambda t}\right)\right]\right\} > 0.$$

Differentiating with respect to ω and rearranging terms, we obtain

$$\frac{d\overline{P}}{d\omega} = (\gamma-1)\left(\overline{P} - \overline{P}_1\right), \tag{50}$$

where

$$\overline{P}_1 \equiv e^{(\gamma-1)\omega} \cdot \int_0^\infty \kappa_t e^{(1-\gamma)e^{-\lambda t}\omega} e^{-\lambda t} dt.$$

It is obvious that $\overline{P} > \overline{P}_1$. Thus, we have established part (a).

(b) Differentiating (50) again with respect to ω gives

$$\frac{d^2\overline{P}}{d\omega^2} = (\gamma-1)\left[\frac{d\overline{P}}{d\omega}\right] + (\gamma-1)\left[\frac{d\overline{P}}{d\omega} - (\gamma-1)\,\overline{P}\right] + (\gamma-1)^2\,\overline{P}_2,$$

where

$$\overline{P}_2 \equiv e^{(\gamma-1)\omega} \cdot \int_0^\infty \kappa_t e^{(1-\gamma)e^{-\lambda t}\omega} e^{-2\lambda t} dt.$$

Again, it is obvious that $\overline{P} > \overline{P}_1 > \overline{P}_2$. Using (50) and rearranging terms,

$$\begin{aligned}\frac{d^2\overline{P}}{d\omega^2} &= 2(\gamma-1)^2\left(\overline{P} - \overline{P}_1\right) - (\gamma-1)^2\left[\overline{P} - \overline{P}_2\right] \\ &= (\gamma-1)^2\left(\overline{P} + \overline{P}_2 - 2\overline{P}_1\right).\end{aligned}$$

Thus, to complete the proof, it remains to show that $\overline{P} + \overline{P}_2 - 2\overline{P}_1 > 0$. But,

$$\begin{aligned}&\overline{P} + \overline{P}_2 - 2\overline{P}_1 \\ &= e^{(\gamma-1)\omega} \cdot \int_0^\infty \kappa_t e^{(1-\gamma)e^{-\lambda t}\omega}\left(1 + e^{-2\lambda t} - 2e^{-\lambda t}\right) dt \\ &= e^{(\gamma-1)\omega} \cdot \int_0^\infty \kappa_t e^{(1-\gamma)e^{-\lambda t}\omega}\left(1 - e^{-\lambda t}\right)^2 dt.\end{aligned}$$

Since all the terms involved are positive, we have $\overline{P} + \overline{P}_2 - 2\overline{P}_1 > 0$. This completes the proof.■

Proof of Proposition 4

(a) From the expression for return volatility,

$$\begin{aligned}\frac{d\sigma_R(\omega)}{d\omega} &= \sigma\left[\frac{d^2\overline{P}/d\omega^2}{\overline{P}} - \left(\frac{d\overline{P}/d\omega}{\overline{P}}\right)^2\right] \\ &= \sigma(\gamma-1)^2\left[\frac{\overline{P}_2}{\overline{P}} - \left(\frac{\overline{P}_1}{\overline{P}}\right)^2\right],\end{aligned}$$

where the second equality follows from part (b) of Proposition 2. Showing the expression in brackets is positive is equivalent to showing

$$\left(\int_0^\infty \overline{\kappa}_t dt\right)\left(\int_0^\infty \overline{\kappa}_t\left(1 - e^{-\lambda t}\right)^2 dt\right) > \left(\int_0^\infty \overline{\kappa}_t\left(1 - e^{-\lambda t}\right) dt\right)^2, \tag{51}$$

where $\overline{\kappa}_t \equiv \kappa_t e^{(1-\gamma)e^{-\lambda t}\omega}$. (51) follows from Schwartz's inequality.

(b) The instantaneous correlation between changes in volatility and returns is given by

$$\operatorname{sgn}\left(\frac{d\sigma_R}{d\omega} \cdot \sigma_R\right) = \operatorname{sgn}\left(\frac{d\sigma_R}{d\omega} \cdot \left(\sigma + \sigma\frac{d(P/Y)/d\omega}{(P/Y)}\right)\right).$$

As we have shown in Proposition 2, $\text{sgn}(d\sigma_R/d\omega) \geq 0$. In addition, according to (50),

$$\frac{d\left(P/Y\right)/d\omega}{\left(P/Y\right)} = \left(\gamma - 1\right)\left(1 - \frac{\overline{P}_1}{\overline{P}}\right) \geq -1,$$

therefore

$$\text{sgn}\left(\sigma\left(1 + \frac{d\left(P/Y\right)/d\omega}{\left(P/Y\right)}\right)\right) \geq 0.$$

This establishes the result of part (b). ■

Proof of Proposition 5 and 6

Proposition 5 and 6 (c). The results follow from Proposition 7 and 8 with $\sigma_b = 0$, respectively.

Proposition 5 (a,b). Using the asymptotic expression for $\mathcal{I}\left(\omega_t\right)$ in (31), σ_z in (29) can be easily computed. Then, expected excess returns can be recovered from π_t and the desired results follow from expanding $\sigma_{R,t}^2$. ■

Proof of Proposition 7

The stock price can be written as

$$P_t = Y_t \cdot e^{-\mathcal{I}(\omega_t)-\omega_t}\,\text{E}\left[\int_t^\infty e^{-\rho(s-t)+\mathcal{I}(\omega_s)+\omega_s}\,ds\,\middle|\,\omega_t\right].$$

Using the expression for $\mathcal{I}(\omega_t)$ in (31) and applying a second-order expansion to the exponential terms,

$$\begin{aligned} P_t/Y_t = &\left(1 + \varepsilon\cdot\mu_b\omega_t + \varepsilon^2\left(\mu_b^2\omega_t + \frac{1}{2}\left(\mu_b^2 - \sigma_b^2\right)\omega_t^2\right)\right) \\ &\times \text{E}\left[\int_t^\infty e^{-\rho(s-t)}\left(1 - \varepsilon\cdot\mu_b\omega_s - \varepsilon^2\left(\mu_b^2\omega_s - \frac{1}{2}\left(\mu_b^2 + \sigma_b^2\right)\omega_s^2\right)\right)ds\,\middle|\,\omega_t\right] + O\left(\varepsilon^3\right). \end{aligned}$$

The first and second conditional moments of ω can be easily calculated from the conditional normal distribution given in (8). The result of the proposition then follows immediately from integrating over time and expanding the resulting expression in powers of ε.■

Proof of Proposition 8

Return volatility is given by (25). Hence, we can obtain the desired expression by using the price-dividend ratio in Proposition 7 and applying a second-order expansion to

$$\frac{d\left(P_t/Y_t\right)/d\omega_t}{P_t/Y_t}.$$

■

Proof of Proposition 10

For notational convenience, define the following coefficients:

$$\begin{aligned} \Lambda_{R,1} &\equiv \sigma^2\left(\frac{1}{2} + \frac{2\varepsilon\lambda\left(\lambda+\rho\right)^2\mu_b + \varepsilon^2\left[\lambda\left(3\lambda^2 + 6\lambda\rho + 2\rho^2\right)\mu_b^2 + \left(\lambda+\rho\right)^2\left(\sigma^2 - 2\mu\right)\sigma_b^2\right]}{2\lambda\left(\lambda+\rho\right)^2}\right), \\ \Lambda_{R,2} &\equiv -\varepsilon^2\sigma^2\sigma_b^2. \end{aligned}$$

Then, the cumulative log excess return between t and $t+\Delta$ can be written as

$$\mathcal{R}_{t,t+\Delta} \equiv \int_t^{t+\Delta}\left(\Lambda_{R,1} + \Lambda_{R,2}\left(\omega_s - \overline{\omega}\right)\right)ds + \int_t^{t+\Delta}\sigma_{R,s}dB_s.$$

Note that,

$$\begin{aligned}&\operatorname{Cov}\left(\mathcal{R}_{t,t+\Delta_1},\mathcal{R}_{t+\Delta_1,t+\Delta_1+\Delta_2}\right)\\&=\mathrm{E}\left(\mathrm{E}_t\left(\mathcal{R}_{t,t+\Delta_1}\mathcal{R}_{t+\Delta_1,t+\Delta_1+\Delta_2}\right)\right)-\mathrm{E}\left(\mathrm{E}_t\left(\mathcal{R}_{t,t+\Delta_1}\right)\right)\cdot\mathrm{E}\left(\mathrm{E}_t\left(\mathcal{R}_{t+\Delta_1,t+\Delta_1+\Delta_2}\right)\right).\end{aligned}$$

Thus, we need to first calculate the conditional moments. For any Δ,

$$\begin{aligned}\mathrm{E}_t\left(\mathcal{R}_{t,t+\Delta}\right)&=\mathrm{E}_t\left(\int_t^{t+\Delta}\left(\Lambda_{R,1}+\Lambda_{R,2}\left(\omega_s-\overline{\omega}\right)\right)ds\right)\\&=\Lambda_{R,1}\Delta+\frac{1}{\lambda}\left(1-e^{-\lambda\Delta}\right)\Lambda_{R,2}\left(\omega_t-\overline{\omega}\right)\end{aligned}\tag{52}$$

And

$$\mathrm{E}_t\left(\mathcal{R}_{t+\Delta_1,t+\Delta_1+\Delta_2}\right)=\Lambda_{R,1}\Delta_2+\frac{1}{\lambda}e^{-\lambda\Delta_1}\left(1-e^{-\lambda\Delta_2}\right)\Lambda_{R,2}\left(\omega_t-\overline{\omega}\right)$$

The first conditional expectation can be written as

$$\begin{aligned}&\mathrm{E}_t\left(\mathcal{R}_{t,t+\Delta_1}\mathcal{R}_{t+\Delta_1,t+\Delta_1+\Delta_2}\right)\\&=\mathrm{E}_t\left[\mathcal{R}_{t,t+\Delta_1}\cdot E_{t+\Delta_1}\left(\mathcal{R}_{t+\Delta_1,t+\Delta_1+\Delta_2}\right)\right]\\&=\Lambda_{R,1}\Delta_2\cdot\mathrm{E}_t\left(\mathcal{R}_{t,t+\Delta_1}\right)+\frac{1}{\lambda}\left(1-e^{-\lambda\Delta_2}\right)\Lambda_{R,2}\cdot\mathrm{E}_t\left(\mathcal{R}_{t,t+\Delta_1}\left(\omega_{t+\Delta_1}-\overline{\omega}\right)\right).\end{aligned}$$

But,

$$\begin{aligned}&\mathrm{E}_t\left(\mathcal{R}_{t,t+\Delta_1}\left(\omega_{t+\Delta_1}-\overline{\omega}\right)\right)\\&=\mathrm{E}_t\left[\left(\int_t^{t+\Delta_1}\left(\Lambda_{R,1}+\Lambda_{R,2}\left(\omega_s-\overline{\omega}\right)\right)ds+\int_t^{t+\Delta_1}\sigma_{R,s}dB_s\right)\cdot\left(\omega_{t+\Delta_1}-\overline{\omega}\right)\right],\end{aligned}$$

The first integral is easy to compute. By the Law of Iterated Expectations,

$$\begin{aligned}&\mathrm{E}_t\left[\int_t^{t+\Delta_1}\left(\Lambda_{R,1}+\Lambda_{R,2}\left(\omega_s-\overline{\omega}\right)\right)ds\cdot\left(\omega_{t+\Delta_1}-\overline{\omega}\right)\right]\\&=\mathrm{E}_t\left[\int_t^{t+\Delta_1}E_s\left[\left(\omega_{t+\Delta_1}-\overline{\omega}\right)\right]\cdot\left(\Lambda_{R,1}+\Lambda_{R,2}\left(\omega_s-\overline{\omega}\right)\right)ds\right]\\&=\mathrm{E}_t\left[\int_t^{t+\Delta_1}\left(\left(\omega_s-\overline{\omega}\right)e^{-\lambda(t+\Delta_1-s)}\right)\left(\Lambda_{R,1}+\Lambda_{R,2}\left(\omega_s-\overline{\omega}\right)\right)ds\right]\\&=\Lambda_{R,2}\left(\frac{\sigma^2}{2\lambda^2}\right)\left(1-e^{-\lambda\Delta_1}\right)^2+\Lambda_{R,1}e^{-\lambda\Delta_1}\Delta_1\left(\omega_t-\overline{\omega}\right).\end{aligned}$$

The next integral is more subtle.

$$\begin{aligned}
&\mathrm{E}_t\left[\left(\int_t^{t+\Delta_1}\sigma_{R,s}dB_s\right)\cdot\left(\omega_{t+\Delta_1}-\overline{\omega}\right)\right]\\
&=\mathrm{E}_t\left[\left(\int_t^{t+\Delta_1}\left(\Lambda_{\sigma,1}+\Lambda_{\sigma,2}\left(\omega_s-\overline{\omega}\right)\right)dB_s\right)\cdot\left(\omega_{t+\Delta_1}-\overline{\omega}\right)\right]\\
&=\Lambda_{\sigma,1}\mathrm{E}_t\left[\left(\int_t^{t+\Delta_1}dB_s\right)\left(\omega_{t+\Delta_1}-\overline{\omega}\right)\right]+\Lambda_{\sigma,2}\mathrm{E}_t\left[\left(\int_t^{t+\Delta_1}\left(\omega_s-\overline{\omega}\right)dB_s\right)\cdot\left(\omega_{t+\Delta_1}-\overline{\omega}\right)\right]
\end{aligned}$$

where

$$\Lambda_{\sigma,1}\equiv\sigma+\varepsilon\frac{\lambda\mu_b}{\lambda+\rho}\sigma+\varepsilon^2\frac{\lambda\mu_b^2-\left(\mu-\sigma^2\right)\sigma_b^2}{\lambda+\rho}\sigma,\quad\Lambda_{\sigma,2}\equiv\varepsilon^2\frac{\lambda^2\rho\left(\mu_b^2-\frac{2(\lambda+\rho)^2}{\lambda\rho}\sigma_b^2\right)}{\left(\lambda+\rho\right)^2\left(2\lambda+\rho\right)}\sigma.$$

The first term becomes

$$\begin{aligned}
&\mathrm{E}_t\left[\left(\int_t^{t+\Delta_1}dB_s\right)\left(\omega_{t+\Delta_1}-\overline{\omega}\right)\right]\\
&=\mathrm{E}_t\left[\left(\int_t^{t+\Delta_1}dB_s\right)\left(\left(\omega_t-\overline{\omega}\right)e^{-\lambda\Delta_1}+\sigma\int_t^{t+\Delta_1}e^{-\lambda(t+\Delta_1-s)}dB_s\right)\right]\\
&=\sigma\cdot\mathrm{E}_t\int_t^{t+\Delta_1}e^{-\lambda(t+\Delta_1-s)}ds\\
&=\frac{\sigma}{\lambda}\left(1-e^{-\lambda\Delta_1}\right)
\end{aligned}$$

where we have used the fact that

$$\mathrm{E}_t\left[\left(\int_t^{t+\Delta_1}dB_s\right)\cdot\left(\int_t^{t+\Delta_1}e^{-\lambda(t+\Delta_1-s)}dB_s\right)\right]=\mathrm{E}_t\left[\left(\int_t^{t+\Delta_1}e^{-\lambda(t+\Delta_1-s)}ds\right)\right]$$

Similarly, the second term becomes

$$\begin{aligned}
&\mathrm{E}_t\left[\left(\int_t^{t+\Delta_1}\left(\omega_s-\overline{\omega}\right)dB_s\right)\cdot\left(\omega_{t+\Delta_1}-\overline{\omega}\right)\right]\\
&=\mathrm{E}_t\left[\left(\int_t^{t+\Delta_1}\left(\omega_s-\overline{\omega}\right)dB_s\right)\cdot\left(\left(\omega_t-\overline{\omega}\right)e^{-\lambda\Delta_1}+\sigma\int_t^{t+\Delta_1}e^{-\lambda(t+\Delta_1-s)}dB_s\right)\right]\\
&=\left(\omega_t-\overline{\omega}\right)e^{-\lambda\Delta_1}\cdot\mathrm{E}_t\left[\int_t^{t+\Delta_1}\left(\omega_s-\overline{\omega}\right)dB_s\right]+\sigma\cdot\mathrm{E}_t\left[\int_t^{t+\Delta_1}\left(\omega_s-\overline{\omega}\right)e^{-\lambda(t+\Delta_1-s)}ds\right]\\
&=\sigma e^{-\lambda\Delta_1}\Delta_1\left(\omega_t-\overline{\omega}\right)
\end{aligned}$$

since

$$\begin{aligned}
\mathrm{E}_t\left[\int_t^{t+\Delta_1}\left(\omega_s-\overline{\omega}\right)dB_s\right]&=\mathrm{E}_t\left[\int_t^{t+\Delta_1}\left(\left(\omega_t-\overline{\omega}\right)e^{-\lambda(s-t)}+\sigma\int_t^s e^{-\lambda(s-u)}dB_u\right)dB_s\right]\\
&=0
\end{aligned}$$

Collecting these expressions,

$$\begin{aligned}&\mathrm{E}_t\left(\mathcal{R}_{t,t+\Delta_1}\left(\omega_{t+\Delta_1}-\overline{\omega}\right)\right) \\ &=\frac{\sigma}{\lambda}\left(1-e^{-\lambda\Delta_1}\right)\left[\Lambda_{\sigma,1}+\frac{\sigma}{2\lambda}\Lambda_{R,2}\left(1-e^{-\lambda\Delta_1}\right)\right]+e^{-\lambda\Delta_1}\Delta_1\left(\Lambda_{R,1}+\Lambda_{\sigma,2}\sigma\right)\left(\omega_t-\overline{\omega}\right)\end{aligned} \tag{53}$$

Together with the results in (52), we obtain the desired expression in (43) by applying the unconditional expectation operator.

Let $\overline{\mathcal{R}}(u) \equiv \mathcal{R}_{t,t+u}-\mathrm{E}(\mathcal{R}_{t,t+u})$. Then, the unconditional variance of log excess returns is given by $\mathrm{E}\left(\overline{\mathcal{R}}(t+\Delta)\right)^2$. Since, via Ito's lemma,

$$\overline{\mathcal{R}}(t+\Delta)^2=\overline{\mathcal{R}}(t)^2+\int_t^{t+\Delta} d\left(\overline{\mathcal{R}}(u)^2\right),$$

and

$$d\left(\overline{\mathcal{R}}(u)^2\right)=2\overline{\mathcal{R}}(u)\cdot d\overline{\mathcal{R}}+d\overline{\mathcal{R}}\cdot d\overline{\mathcal{R}},$$

we have

$$\begin{aligned}&\mathrm{E}\left(\overline{\mathcal{R}}(t+\Delta)\right)^2 \\ &=\int_t^{t+\Delta}\mathrm{E}\left(d\left(\overline{\mathcal{R}}(u)^2\right)\right) \\ &=\int_t^{t+\Delta}\left[2\Lambda_{R,2}\cdot\mathrm{E}\left(\overline{\mathcal{R}}(u)\cdot\left(\omega_u-\overline{\omega}\right)\right)+\mathrm{E}\left(\sigma_{R,u}^2\right)\right]du.\end{aligned}$$

The expression $\mathrm{E}\left(\overline{\mathcal{R}}(u)\cdot\left(\omega_u-\overline{\omega}\right)\right)$ can be computed by taking the unconditional expectation of (53). The unconditional autocorrelation follows immediately from these results.■

Proof of Proposition 11

(c) The optimal wealth for an investor of type b is simply the present value of future optimal consumption (see, for example, Lemma 2.4 in Cox and Huang (1989)). Formally,

$$\begin{aligned}W_t(b)&=\mathrm{E}_t\int_t^{\infty}C_s^*\left(Y_s,X_s;b\right)\xi_{t,s}ds \\ &=\nu(b)\,e^{-z_t}\mathrm{E}_t\int_t^{\infty}e^{-\rho(s-t)-(b-1)\mathcal{I}(\omega_s)}ds,\end{aligned}$$

since C_s^* is given by the optimal sharing rule (10,11). Using the parameterization $b=1-b_1\varepsilon$ and (31) and ignoring terms of order ε^2 and higher,

$$\begin{aligned}W_t(b)&=\nu(b)\,e^{-z_t}\mathrm{E}_t\int_t^{\infty}e^{-\rho(s-t)-b_1\varepsilon(\omega_s+\varepsilon\mu_b\omega_s)}ds \\ &=\nu(b)\,Y_t e^{-\mathcal{I}(\omega_t)-\omega_t}\cdot\mathrm{E}_t\int_t^{\infty}e^{-\rho(s-t)}\left(1-b_1\varepsilon\omega_s\right)ds \\ &=\nu(b)\,Y_t\left(1+\varepsilon\mu_b\omega_t\right)\int_t^{\infty}e^{-\rho(s-t)}\left(1-\varepsilon b_1\left[\overline{\omega}+\left(\omega_t-\overline{\omega}\right)e^{-\lambda(s-t)}\right]\right)ds \\ &=\nu(b)\,Y_t\left(1+\varepsilon\mu_b\omega_t\right)\left[\frac{1}{\rho}-\varepsilon\left(\frac{b_1\overline{\omega}}{\rho}+\frac{b_1}{\lambda+\rho}\left(\omega_t-\overline{\omega}\right)\right)\right] \\ &=\nu(b)\,Y_t\left\{\frac{1}{\rho}+\varepsilon\left[\frac{-b_1+\mu_b}{\rho}\overline{\omega}+\left(-\frac{b_1}{\lambda+\rho}+\frac{\mu_b}{\rho}\right)\left(\omega_t-\overline{\omega}\right)\right]\right\}.\end{aligned}$$

This establishes the result of part (c).

(a) Substituting (31) into the consumption sharing rule (10),

$$\begin{aligned} C_t^*\left(Y_t, X_t; b\right) &= \nu\left(b\right) e^{-b\mathcal{I}(\omega_t)-\omega_t} Y_t \\ &= \nu\left(b\right) e^{\varepsilon(\mu_b - b_1)\omega_t} Y_t \\ &= \nu\left(b\right)\left(1+\varepsilon\left(\mu_b - b_1\right)\omega_t\right) Y_t. \end{aligned}$$

Then,

$$\begin{aligned} \frac{C_t^*\left(Y_t, X_t; b\right)}{W_t(b)} &= \frac{1+\varepsilon\left(-b_1+\mu_b\right)\omega_t}{\frac{1}{\rho}+\varepsilon\frac{-b_1+\mu_b}{\rho}\overline{\omega}+\varepsilon\left(-\frac{b_1}{\lambda+\rho}+\frac{\mu_b}{\rho}\right)\left(\omega_t-\overline{\omega}\right)} \\ &= \rho\left(1+\varepsilon\left(-b_1+\mu_b\right)\omega_t\right)\left(1-\varepsilon\rho\left(\frac{-b_1+\mu_b}{\rho}\overline{\omega}+\left(-\frac{b_1}{\lambda+\rho}+\frac{\mu_b}{\rho}\right)\left(\omega_t-\overline{\omega}\right)\right)\right). \end{aligned}$$

The result follows from expanding the final expression in ε.

(b) Applying Itô's Lemma to (47),

$$\frac{dW_t}{W_t}\cdot dB_t = \sigma\left(1+\varepsilon\nu\left(b\right)\left(-\frac{b_1}{\lambda+\rho}+\frac{\mu_b}{\rho}\right)\frac{Y_t}{W_t}\right)dt.$$

On the other hand, in the sequential trade economy, each investor faces a budget constraint

$$dW_t = \left(\alpha_t\left(\mu_R - r_t\right)W_t + r_t W_t - C_t\right)dt + \alpha_t\sigma_R dB_t.$$

Combining the two expressions, we obtain

$$\begin{aligned} \alpha_t &= \frac{\sigma}{\sigma_R\left(\omega_t\right)}\left(1+\varepsilon\nu\left(b\right)\left(-\frac{b_1}{\lambda+\rho}+\frac{\mu_b}{\rho}\right)\frac{Y_t}{W_t}\right)+O\left(\varepsilon^2\right) \\ &= \frac{\sigma}{\sigma_R\left(\omega_t\right)}\left[1+\frac{\varepsilon\left(-\frac{b_1}{\lambda+\rho}+\frac{\mu_b}{\rho}\right)}{\frac{1}{\rho}+O\left(\varepsilon\right)}\right]+O\left(\varepsilon^2\right) \\ &= \frac{\sigma}{\sigma_R\left(\omega_t\right)}\left[1+\varepsilon\rho\left(-\frac{b_1}{\lambda+\rho}+\frac{\mu_b}{\rho}\right)\right]+O\left(\varepsilon^2\right) \\ &= \frac{1}{1+\varepsilon\frac{\lambda}{\lambda+\rho}\mu_b}\left[1+\varepsilon\rho\left(-\frac{b_1}{\lambda+\rho}+\frac{\mu_b}{\rho}\right)\right]+O\left(\varepsilon^2\right) \\ &= 1+\varepsilon\frac{\rho}{\lambda+\rho}\left(\mu_b - b_1\right)+O\left(\varepsilon^2\right), \end{aligned}$$

where we have used (39).■

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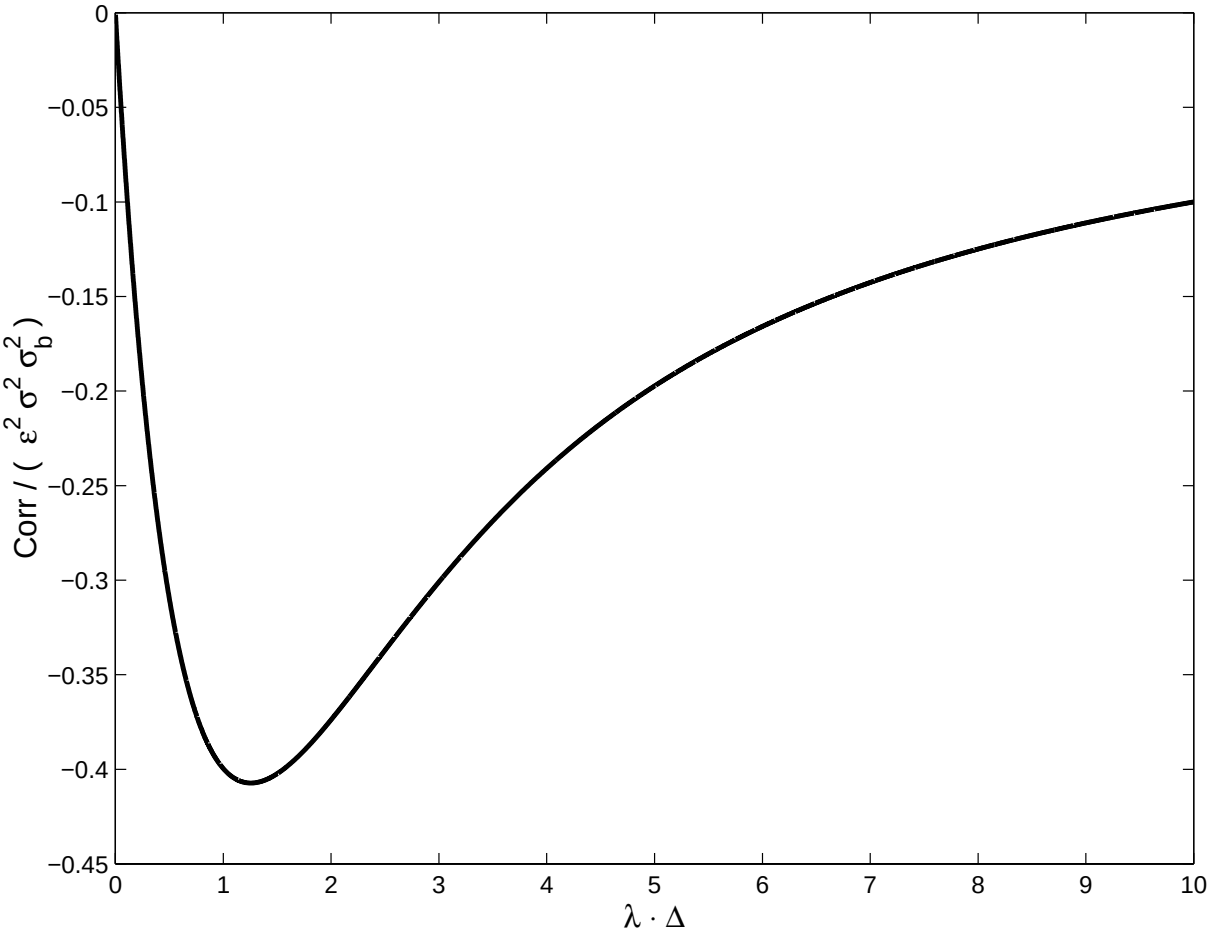

Figure 1: Auto-correlation of Excess Stock Returns in the Heterogeneous Economy.

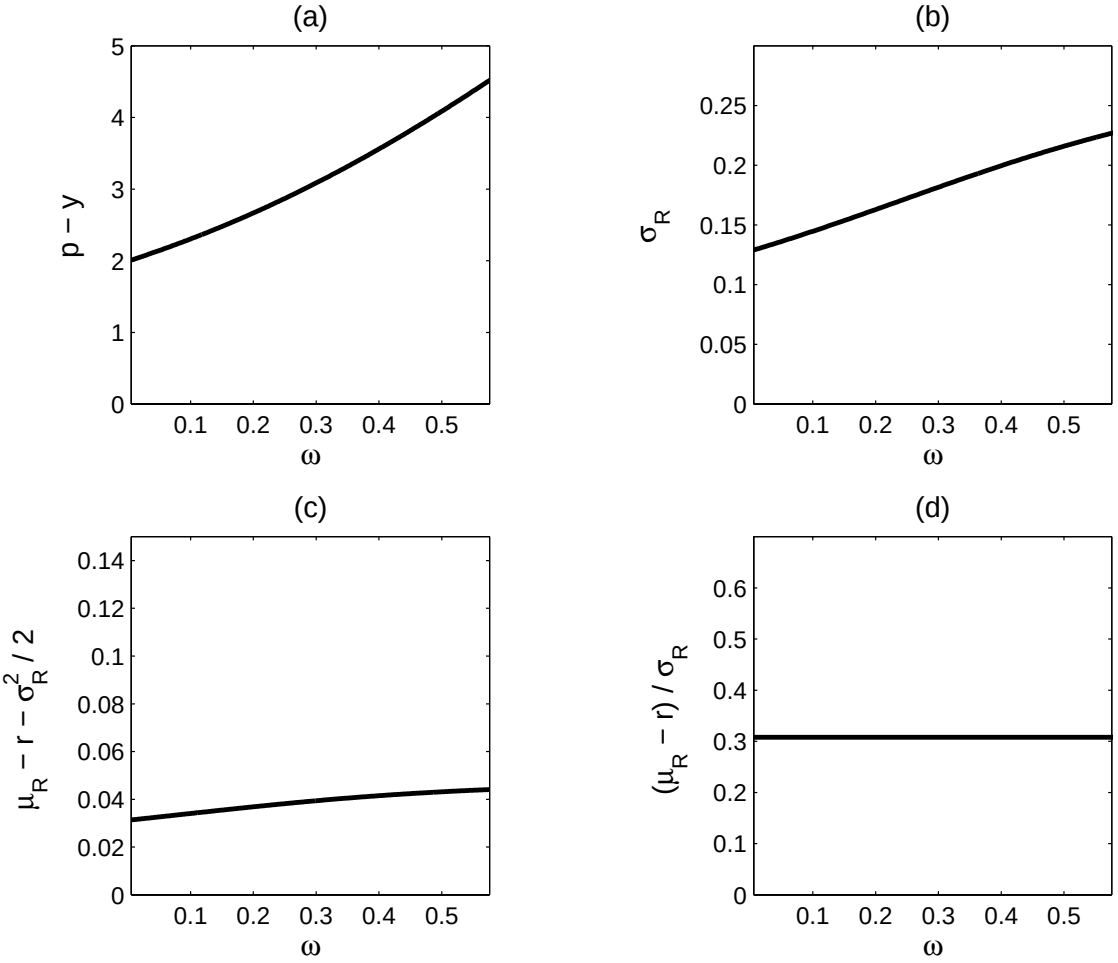

Figure 2: Homogeneous economy: conditional properties of stock prices are plotted as functions of the state variable ω: (a) log price-dividend ratio; (b) instantaneous standard deviation of stock returns; (c) instantaneous expected excess log return; (d) instantaneous Sharpe ratio of stock returns.

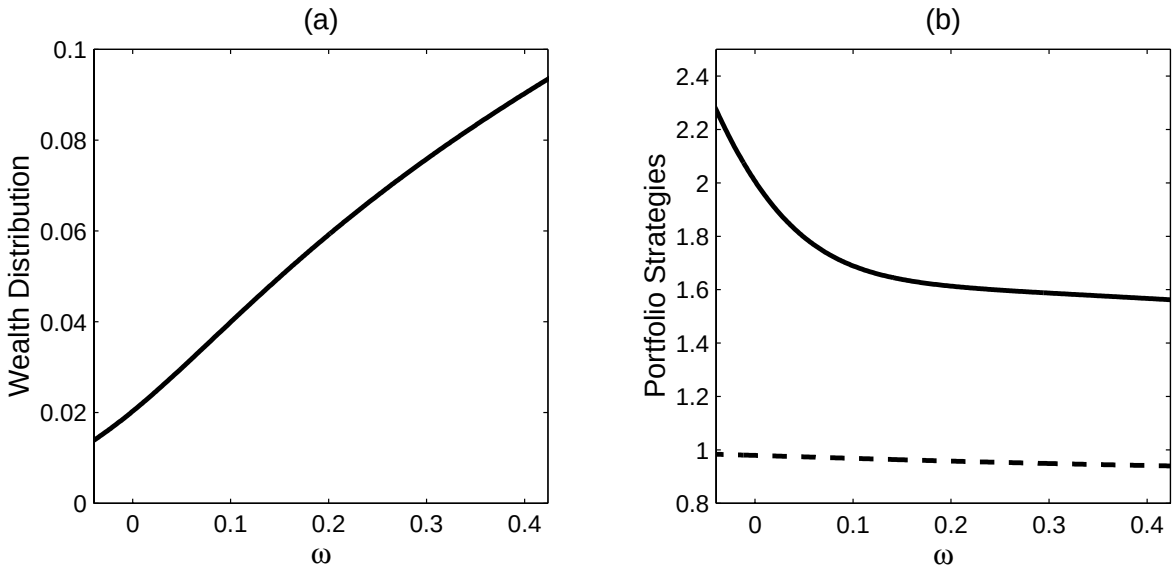

Figure 3: Heterogeneous economy: cross-sectional distribution of wealth and portfolio strategies. Panel (a) plots the ratio of wealth controlled by agents with risk aversion coefficient 1 to the total wealth in the economy: $W(b=1)/(W(b=1)+W(b=1/20))$. Panel (b) plots optimal portfolio holdings of the two types of agents (as a fraction of their individual wealth). Solid line corresponds to $b=1$, dashed line corresponds to $b=1/20$.

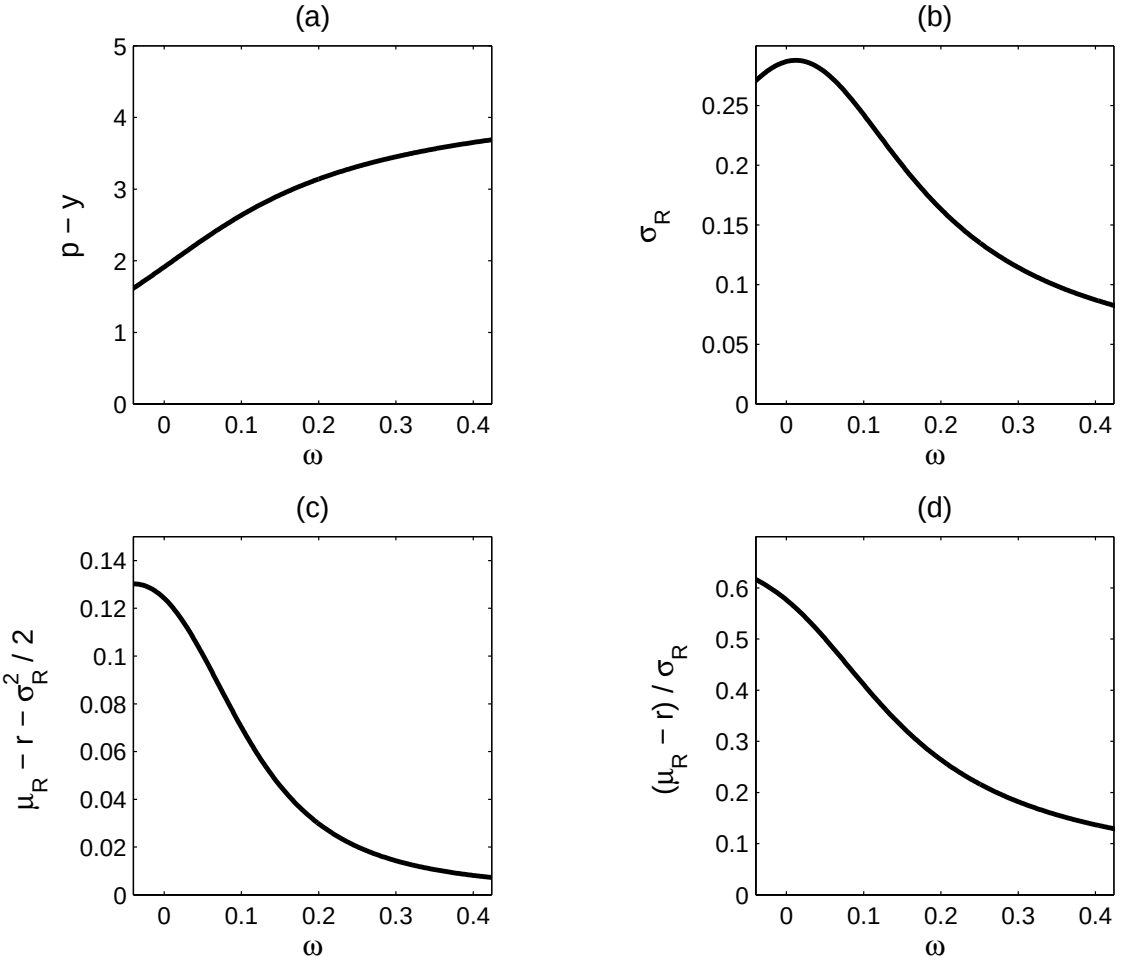

Figure 4: Heterogeneous economy: conditional properties of stock prices are plotted as functions of the state variable ω: (a) log price-dividend ratio; (b) instantaneous standard deviation of stock returns; (c) instantaneous expected excess log return; (d) instantaneous Sharpe ratio of stock returns.

Parameter	Variable	Value
Mean consumption growth (%)	μ	1.72
Standard deviation of consumption growth (%)	σ	3.32
Risk aversion coefficient	γ	9.27
Degree of history-dependence in X_t (%)	λ	6.10
Subjective discount factor (%)	ρ	5.88

Table 1: Homogeneous economy: parameters of the model.

Statistic	Historical Data	Model Output
$\mathrm{E}[\Delta c]^*$	1.72	1.72
$\sigma[\Delta c]^*$	3.32	3.32
$\mathrm{E}[r_B]^*$	2.92	2.92
$\sigma[r_B]$	3.00	4.75
$\mathrm{E}[r_S - r_B]^*$	3.90	3.90
$\sigma[r_S - r_B]^*$	18.0	18.1
$\mathrm{E}[r_S - r_B]/\sigma[r_S - r_B]^*$	0.22	0.22
$\exp(\mathrm{E}[p-y])$	21.1	21.6
$\sigma[p-y]$	0.27	0.42

Table 2: Homogeneous economy: moments of simulated and historical data. "$\star$" denotes the moments that model parameters were chosen to match. Moments are estimated at annual frequency. All returns are annual percentages. $\Delta c \equiv \log(c_1/c_0)$ is log consumption growth; $r_B \equiv \int_0^1 r_t dt$ is log bond return; $r_S \equiv \int_0^1 \mu_{R,t} - \frac{1}{2}\sigma_{R,t}^2 dt$ is log stock return; $p - y \equiv \log(P/Y)$ is log price-dividend ratio.

	Lag j (Years)				
Variable	1	2	3	5	7
$(r_S - r_B)_t$, $(r_S - r_B)_{t+j}$					
Historical, estimate	0.05	-0.21*	0.08	-0.14	0.11
Simulated, mean	-0.00	-0.01	-0.01	-0.01	-0.00
Simulated, 5'th percentile	-0.15	-0.16	-0.15	-0.15	-0.15
Simulated, 95'th percentile	0.15	0.14	0.14	0.14	0.14
$\sum_{i=1}^{j} \rho\left[(r_S - r_B)_t, (r_S - r_B)_{t-i}\right]$					
Historical, estimate	0.05	-0.16	-0.09	-0.28	-0.15
Simulated, mean	-0.00	-0.01	-0.01	-0.02	-0.03
Simulated, 5'th percentile	-0.15	-0.21	-0.25	-0.29	-0.32
Simulated, 95'th percentile	0.15	0.21	0.25	0.28	0.32
$(p - y)_t$, $(r_S - r_B)_{t+j}$					
Historical, estimate	-0.20*	-0.21*	-0.10	-0.19*	-0.08
Simulated, mean	-0.05	-0.05	-0.05	-0.05	-0.05
Simulated, 5'th percentile	-0.19	-0.19	-0.19	-0.18	-0.18
Simulated, 95'th percentile	0.09	0.09	0.09	0.09	0.09
$(r_S - r_B)_t$, $\lvert r_S - r_B\rvert_{t+j}$					
Historical, estimate	-0.15*	0.03	0.12	0.02	-0.01
Simulated, mean	0.03	0.03	0.03	0.02	0.02
Simulated, 5'th percentile	-0.12	-0.12	-0.12	-0.13	-0.13
Simulated, 95'th percentile	0.18	0.18	0.18	0.18	0.17
$(p - y)_t$, $\lvert r_S - r_B\rvert_{t+j}$					
Historical, estimate	-0.12*	0.02	-0.06	-0.10	-0.05
Simulated, mean	0.08	0.07	0.06	0.06	0.05
Simulated, 5'th percentile	-0.08	-0.09	-0.10	-0.11	-0.12
Simulated, 95'th percentile	0.24	0.23	0.22	0.22	0.22

Table 3: Homogeneous economy: autocorrelations and cross-correlations of simulated and historical data. All returns are annual. For each statistic, we report (i) estimates based on historical data; (ii) sample mean of the same statistic based on $5,000$ replications of simulated output from the model; (iii) 5'th percentile of the empirical distribution of the statistic; (iv) 95'th percentile of the empirical distribution of the statistic. "*" denotes estimates based on historical data that fall outside the 90% confidence interval for the statistics.

Parameter	Variable	Value
Mean consumption growth (%)	μ	1.72
Standard deviation of consumption growth (%)	σ	3.32
Social weight	$\nu\,(1/20)$	1.26
Degree of history-dependence in X_t (%)	λ	9.24
Subjective discount factor (%)	ρ	5.82

Table 4: Heterogeneous economy: parameters of the model.

Statistic	Historical Data	Model Output
$\mathrm{E}[\Delta c]^*$	1.72	1.72
$\sigma\left[\Delta c\right]^*$	3.32	3.32
$\mathrm{E}[r_B]^*$	2.92	2.92
$\sigma\left[r_B\right]$	3.00	3.08
$\mathrm{E}[r_S - r_B]^*$	3.90	3.90
$\sigma\left[r_S - r_B\right]^*$	18.0	17.9
$\mathrm{E}[r_S - r_B]\,/\sigma\left[r_S - r_B\right]^*$	0.22	0.22
$\exp\left(\mathrm{E}\left[p - y\right]\right)$	21.1	21.1
$\sigma\left[p - y\right]$	0.27	0.34

Table 5: Heterogeneous economy: moments of simulated and historical data. "$\star$" denotes the moments that model parameters were chosen to match. Moments are estimated at annual frequency. All returns are annual percentages. $\Delta c \equiv \log\left(c_1/c_0\right)$ is log consumption growth; $r_B \equiv \int_0^1 r_t dt$ is log bond return; $r_S \equiv \int_0^1 \mu_{R,t} - \frac{1}{2}\sigma_{R,t}^2 dt$ is log stock return; $p - y \equiv \log\left(P/Y\right)$ is log price-dividend ratio.

	Lag j (Years)				
Variable	1	2	3	5	7
$(r_S - r_B)_t$, $(r_S - r_B)_{t+j}$					
Historical, estimate	0.05	-0.21*	0.08	-0.14	0.11
Simulated, mean	-0.04	-0.04	-0.03	-0.03	-0.03
Simulated, 5'th percentile	-0.20	-0.19	-0.18	-0.18	-0.18
Simulated, 95'th percentile	0.12	0.12	0.12	0.13	0.13
$\sum_{i=1}^{j} \rho\left[(r_S - r_B)_t, (r_S - r_B)_{t-i}\right]$					
Historical, estimate	0.05	-0.16	-0.09	-0.28	-0.15
Simulated, mean	-0.04	-0.08	-0.11	-0.14	-0.17
Simulated, 5'th percentile	-0.20	-0.28	-0.33	-0.37	-0.40
Simulated, 95'th percentile	0.12	0.13	0.14	0.13	0.12
$(p - y)_t$, $(r_S - r_B)_{t+j}$					
Historical, estimate	-0.20	-0.21	-0.10	-0.19	-0.08
Simulated, mean	-0.15	-0.14	-0.12	-0.11	-0.10
Simulated, 5'th percentile	-0.26	-0.25	-0.24	-0.23	-0.21
Simulated, 95'th percentile	-0.04	-0.02	-0.01	0.00	0.01
$(r_S - r_B)_t$, $\lvert r_S - r_B\rvert_{t+j}$					
Historical, estimate	-0.15	0.03	0.12*	0.02	-0.01
Simulated, mean	-0.09	-0.09	-0.08	-0.07	-0.06
Simulated, 5'th percentile	-0.25	-0.24	-0.23	-0.23	-0.22
Simulated, 95'th percentile	0.05	0.07	0.08	0.09	0.09
$(p - y)_t$, $\lvert r_S - r_B\rvert_{t+j}$					
Historical, estimate	-0.12*	0.02*	-0.06	-0.10	-0.05
Simulated, mean	-0.28	-0.25	-0.21	-0.18	-0.16
Simulated, 5'th percentile	-0.43	-0.40	-0.37	-0.35	-0.33
Simulated, 95'th percentile	-0.13	-0.09	-0.05	-0.01	0.02

Table 6: Heterogeneous economy: autocorrelations and cross-correlations of simulated and historical data. All returns are annual. For each statistic, we report (i) estimates based on historical data; (ii) sample mean of the same statistic based on $5,000$ replications of simulated output from the model; (iii) 5'th percentile of the empirical distribution of the statistic; (iv) 95'th percentile of the empirical distribution of the statistic. "*" denotes estimates based on historical data that fall outside the 90% confidence interval for the statistics.