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Equilibrium Mispricing in a Capital Market with Portfolio Constraints

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Equilibrium Mispricing in a Capital Market with Portfolio Constraints*

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Equilibrium Mispricing in a Capital Market with Portfolio Constraints

Abstract

This paper develops a general equilibrium, continuous time model where portfolio constraints generate mispricing between redundant securities. Constrained consumption-portfolio optimization techniques are adapted to incorporate redundant, possibly mispriced, securities. Under logarithmic preferences, we provide explicit conditions for mispricing and closed-form expressions for all economic quantities. Existence of an equilibrium where mispricing occurs with positive probability is verified in a specific case. In a more general setting, we demonstrate the necessity of mispricing for equilibrium when agents are heterogeneous enough. The construction of a representative agent with stochastic weights allows us to characterize prices and allocations, given mispricing occurs.

Journal of Economic Literature Classification Numbers: C60, D52, D90, G12.

1. Introduction

The paradigm of no-arbitrage is central in modern finance, yet the violation of "textbook" no-arbitrage implications is a common feature of actual financial markets. For example, most empirical studies (Canina and Figlewski (1995)) have found stock index futures to exhibit "mispricing" (deviations from the cost of carry model); Neal (1993) reported the resulting index arbitrage to constitute 47.5 % of the observed program trading (itself estimated by the NYSE (1992) to make up 11.5 % of total volume). Other examples of frequently mispriced securities include primes and scores (Jarrow and O'Hara (1989)), closed-end funds (Pontiff (1996)), etc.. Financial economics has surprisingly little to say about such phenomena. Some literature exists on optimal behavior in the presence of mispricing (Brennan and Schwartz (1990), Tuckman and Vila (1992)), but there the mispricing is taken as exogenous. Not much is known on how arbitrage opportunities arise; Brennan and Schwartz (1990) conclude: "the real challenge remains to endogenize [the arbitrage opportunity]." Indeed, in perfect markets arbitrage opportunities are inconsistent with equilibrium. In the more realistic context of imperfect markets, however, mispricing (in the sense of discrepancies between the prices of ostensibly equivalent securities) may be consistent with equilibrium, so more can be said about its source. While some price discrepancies typically disappear quickly, others tend to exhibit a systematic and persistent character that suggests an economic rationale for the mispricing. Our objective is to explore this idea by developing a general equilibrium model in which "mispricing" is generated endogenously and agents indulge in limited arbitrage activity.

We work in a pure-exchange, continuous-time framework with two heterogeneous agents. For the most part, we assume logarithmic preferences for both agents, diverging beliefs being used to generate trade. However, much in our analysis is valid for more general preferences, and we highlight the extension to this case. The imperfection we introduce is constrained portfolio holdings in the risky securities: a positive net supply dividend-paying "stock", and a zero net supply "derivative", with perfectly correlated prices. The mispricing then arises as an integral part of the equilibrium, in that it is required to clear markets. We mainly assume the simplest, bare minimum of constraints required to illustrate our point: a strictly positive upper bound on the proportion of wealth invested in the derivative and a no-short sales constraint on the stock. We show, however, that most of the analysis is equally valid for more general constraints. Mispricing between the risky securities is captured by their exhibiting distinct market prices of risk or, equivalently, distinct deflator processes (implying, for example, that securities paying identical dividend processes would trade at different prices), generating a riskless arbitrage opportunity. We show, however, that prices of such "mispriced" securities in fact accurately reflect shadow costs faced by rational agents.

Our agents' optimization problem is non-standard in three respects: (i) the constraints on portfolio holdings; (ii) redundancy in the risky securities; (iii) the mispricing. This adds two extra layers to the standard optimization methodology. First we exploit agents' monotonic preferences to convert the constrained problem into an unconstrained, but policy-dependent problem with a single, composite risky asset. Under mispricing, an agent always adds to his portfolio as much of a (riskless, costless) arbitrage position as allowed by the constraints, hence uniquely determining the allocation between derivative and stock. Our second additional step is to adapt the convex duality approach of Cvitanic and Karatzas (1992) to deal with the policy-dependent drift of the composite asset. This approach involves embedding the original non-linear problem into a family of perfect (linear, unconstrained, non-redundant) "fictitious" markets, designed so that the optimal policy in one of the fictitious markets coincides with that in the original market. An agent effectively faces an individual-specific state-price density, and a market price of risk lying between those of the mispriced risky securities and coinciding with any security in which he is in the interior. The problem bears some resemblance to those of costly short sales and of diverging borrowing and lending rates, as studied by Jouini and Kallal (1995) (in determining no-arbitrage prices) and Cvitanic and Karatzas (1992), Tepla (1997) (in portfolio optimization).

The construction of equilibrium is achieved by introducing a representative agent with stochastic weights for the two agents (Cuoco and He (1994), Detemple and Serrat (1998)). The weights act as a proxy for the possibly differential constraints binding on the two agents (in addition to reflecting any divergence in beliefs). The weighting process is explicitly characterized, and shown to limit risk redistribution, as should the constraints. Unlike comparable models such as Karatzas, Lehoczky and Shreve (1990), clearing in the good market alone does not guarantee clearing in the financial markets. In the presence of redundant securities, it is also necessary to verify that one of the risky securities markets clears, and it is this that determines the mispricing.

A major result is that mispricing must occur in equilibrium unless the equilibrium, in an otherwise identical economy without portfolio constraints, would never have deviated outside our portfolio constraints. Accordingly, mispricing must occur whenever agents are heterogeneous enough in their risk-taking, and its magnitude increases in agents' heterogeneity. For our simplest case of one-sided constraints, mispricing arises with only one sign: a higher market price of risk for the security having the upper bound.

Under logarithmic preferences, explicit expressions are provided for all quantities, and existence of an equilibrium where mispricing occurs with positive probability is verified under some specific conditions. The mispricing manifests itself in an increase in the market price of risk of the derivative. We show the region and magnitude of mispricing to be larger when the derivative is more tightly constrained. It is also larger when wealth is shared more evenly across agents. Rather surprisingly, the result that mispricing takes on only one sign is shown to extend to the two-sided constraints case. Heterogeneous (across agents) or stochastic constraints break this implication.

Intuition is provided for the role of mispricing in clearing markets. When the agents differ highly, the more optimistic agent (or, more generally, the less risk-averse) desires a very high level of risk-taking, while the more pessimistic desires a very low level. Due to the constraints, only the stock may be used for high levels of risk-taking, while only the derivative can be shorted to reach low levels. Thus, there will be excess demand in the stock and too low a demand in the derivative, so, to clear markets, the stock's market price of risk must go down, and the derivative's must go up. Hence the mispricing and, since under logarithmic preferences the stock price is independent of constraints, a concurrent increase in the interest rate. In short, the role of the mispricing is to limit agents' heterogeneity in portfolio demands, notwithstanding their heterogeneity in beliefs or risk aversion. The agents' market prices of risk, and hence consumption volatilities are, accordingly, seen to diverge less than in the unconstrained economy.

The mispricing entices the more optimistic agent to perform a riskless arbitrage trade (bounded by the constraint on derivative holdings) by substituting the more favorable security for the less favorable, cashing in a profit proportional to the mispricing. The mispricing per se benefits him since he is the agent long in the favorable derivative. However, he is also the net borrower (via the bond), so the increase in interest rate hurts him while benefiting the other agent. The welfare effects of the constraints are, thus, ambiguous.

Our work sheds some light on the controversy surrounding the alleged destabilizing effects of arbitrage activity. A common perception is that stock index arbitrage aggravated the 1987 crash, which led to the introduction of trading curbs on the NYSE in 1988 (to limit program trading). Our results, however, suggest that some types of arbitrage may play a valuable role, improving risk-sharing by allowing market-clearing notwithstanding large investor diversity. Our work also complements the literature on the effects of financial innovation (e.g., Zapatero (1998)), in that we study the effect of adding a derivative security to a constrained economy. We show that for a given wealth distribution, most quantities (interest rate, market prices of risk, agents' portfolio holdings and consumption volatilities) of our economy lie between those of the unconstrained economy and that with only a constrained stock. Hence, the derivative alleviates the constraints, but only partially. The higher the volatility of the derivative, the more it relieves the constraints.

The closest papers to ours are Chen (1995) and Detemple and Murthy (1997). Chen (1995), in a one-period setting, models the discrepancy between the "equilibrium price function" and the "natural" no-arbitrage price (minimum hedging cost) of a derivative security, due to portfolio constraints on the primary securities. His equilibrium price function is defined as the maximum cost that some rational agent will pay to an innovator, to hold a small quantity of the derivative. Since he assumes the equilibrium allocations (and hence state prices) are unaffected by the derivative, the trading volume therein is limited to be infinitesimal; hence he obtains a mispricing without explicit constraint on the derivative. In contrast, we account for the allocational role of the derivative.

Detemple and Murthy's (1997) setting is closer to ours and, even though our objectives are somewhat different, our results extend and complement theirs. Their main object of interest is equilibrium with a constrained stock, but in the absence of a derivative security. The standard MRS-based valuation fails, but it remains possible to use a state-price density to value the (single) risky security, apparently ruling out mispricing. We show that caution should be exercised in extending this result to the case where a constrained, redundant derivative is present: our notion of mispricing implies that there does not exist a single deflator that prices all risky securities (each has its own deflator). In addition, Detemple and Murthy (1997) demonstrate that under constraints "no-arbitrage may fail to price [attainable] payoffs", and provide a condition for this failure, namely when the presence of an (attainable) derivative enables one agent to circumvent the constraints. The valuation of the stock and the derivative then becomes a joint problem, because the derivative plays an allocational role. (The one-period model of Detemple and Selden (1991), where the introduction of a stock option modifies the marketed space, makes a similar point.) The authors, however, do not go that step further to investigate the joint problem. Equilibrium mispricing is not exhibited, because it is not shown whether no-arbitrage pricing still fails in the subsequent equilibrium with the derivative (taking into account the allocational consequences of its introduction). We extend their analysis by constructing an economy where mispricing between redundant securities subsists in equilibrium, which is a novelty.

A somewhat related paper is Dumas (1992), who generates "mispricing", but in the good market (as purchasing-power-parity deviations) via transaction costs rather than constraints. Delgado and Dumas (1994) also use transaction costs, but specify the mispricing exogenously and show it may persist in equilibrium. The mispricing is not shown to be necessary for equilibrium. In the one-period model of Zigrand (1997), strategic arbitrageurs exploit mispricing between assets trading on several exchanges. He focuses on the effect of competition among arbitrageurs rather than on the mispricing as here. The "risky arbitrage" literature, including De Long, Shleifer, Summers and Waldmann (1990), Dow and Gorton (1994) and Shleifer and Vishny (1997), employs highly specialized models (with overlapping generations, irrational noise traders and/or asymmetric information) leading to mispricing that is exploited via risky arbitrage strategies. Our paper derives mispricing under a more standard environment, requiring only constraints on heterogeneous rational agents.

Section 2 of the paper describes our model, and Section 3 presents our technique for consumption-portfolio optimization. Section 4, under logarithmic preferences, demonstrates the necessity of mispricing for equilibrium, and characterizes the equilibrium. Section 5 highlights the extension to general preferences, while Section 6 examines the modification to two-sided constraints. Section 7 concludes and the Appendix provides all proofs.

2. The Economy

We consider a continuous-time, pure-exchange economy with a finite horizon $[0, T]$. There is a single consumption good which serves as the numeraire.

2.1. Information Structure and Agents' Perceptions

The uncertainty is represented by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$ on which is defined a one-dimensional Brownian motion W. Letting $\{\mathcal{F}_t^W\}$ denote the augmented filtration generated by W, and $\mathcal{H}$ a σ-field independent of $\mathcal{F}_T^W$, the complete information filtration $\{\mathcal{F}_t\}$ is the augmentation of the filtration $\mathcal{H} \times \{\mathcal{F}_t^W\}$.

Two agents ($i = 1, 2$) commonly observe the exogenous dividend $\delta > 0$, which follows

$$d\delta(t) = \delta(t)[\mu_\delta(t)\,dt + \sigma_\delta(t)\,dW(t)]. \tag{2.1}$$

The mean growth μ_δ is assumed to be $\{\mathcal{F}_t\}$-progressively measurable and in $\mathcal{L}^2(\mathcal{P})$;[1] $\mu_\delta(0)$ is $\mathcal{H}$-measurable. We restrict σ_δ to be bounded above and below away from zero and, to keep the agents' filtering tractable, $\{\mathcal{F}_t^\delta\}$-progressively measurable, where $\{\mathcal{F}_t^\delta\}$ denotes the filtration generated by δ.

The agents observe δ, having the incomplete information filtration $\mathcal{F}_t^\delta \subset \mathcal{F}_t,\ t \in [0, T]$. They deduce σ_δ from the quadratic variation of δ, but can only draw inferences about μ_δ. Agents have equivalent probability measures $\mathcal{P}^i$, $i = 1, 2$, also equivalent to $\mathcal{P}$, which may disagree on $\mathcal{H}$, so that agents have heterogeneous prior beliefs. Agents update their beliefs about μ_δ in a Bayesian fashion, via $\mu_\delta^i(t) = E^i[\mu_\delta(t)|\mathcal{F}_t^\delta]$, where $E^i[\cdot]$ denotes the expectation relative to $\mathcal{P}^i$. Due to their heterogeneous priors, agents may draw different inferences about μ_δ at all times.[2]

The innovation process W^i induced by agent i's beliefs and filtration is

$$dW^i(t) \equiv \frac{1}{\sigma_\delta(t)}\left[\frac{d\delta(t)}{\delta(t)} - \mu_\delta^i(t)\,dt\right] = dW(t) + \frac{\mu_\delta(t) - \mu_\delta^i(t)}{\sigma_\delta(t)}\,dt\,, \quad i = 1, 2. \tag{2.2}$$

The innovation process of each agent is such that given his perceived growth of the dividend, μ_δ^i, the observed dividend obeys

$$d\delta(t) = \delta(t)[\mu_\delta^i(t)\,dt + \sigma_\delta(t)\,dW^i(t)], \quad i = 1, 2. \tag{2.3}$$

The agents' information and innovation filtrations coincide, $\{\mathcal{F}_t^\delta\} = \{\mathcal{F}_t^{W^i}\} \equiv \{\mathcal{F}_t^i\}$, assuming (2.3) has a strong solution. Effectively, each agent is endowed with the probability space $(\Omega, \mathcal{F}^i, \{\mathcal{F}_t^i\}, \mathcal{P}^i)$; by Girsanov's theorem, W^i is a Brownian motion on that space. Equation (2.2) implies that the agents' innovations are related by

$$dW^2(t) = dW^1(t) + \bar{\mu}(t)\,dt\,, \quad \bar{\mu}(t) \equiv \frac{\mu_\delta^1(t) - \mu_\delta^2(t)}{\sigma_\delta(t)}. \tag{2.4}$$

The $\{\mathcal{F}_t^\delta\}$-progressively measurable process $\bar{\mu}$ parametrizes agents' disagreement on the mean dividend growth rate, normalized by its risk. $\bar{\mu}(t)$ is positive when agent 1 is more optimistic, and conversely. $\bar{\mu}$ follows directly from the exogenous agents' priors and dividend process, with no equilibrium restrictions imposed on it, so we may treat it as exogenous (Basak (1998)).

Heterogeneity in beliefs is *not* a central feature of our model, but is required to generate trade when both agents exhibit identical constant relative risk aversion (e.g., logarithmic) preferences.

[1] A process α is in $\mathcal{L}^2(\mathcal{P})$ if $E\left[\int_0^T \alpha(t)^2 dt\right] < \infty$, where the expectation is taken with respect to $\mathcal{P}$.

[2] The only role of $\mathcal{H}$ is to allow for heterogeneity in agents' priors; $\mathcal{F}_0$ must then be non-trivial (unlike $\mathcal{F}_0^W$).

2.2. Securities Market

Trading may take place continuously in three securities. There is a stock in constant net supply of 1, paying a continuous dividend at rate δ. Its price S has dynamics

$$
\begin{aligned}
dS(t) + \delta(t)\,dt &= S(t)[\mu_S(t)\,dt + \sigma_S(t)\,dW(t)] && (2.5)\\
&= S(t)[\mu_S^i(t)\,dt + \sigma_S(t)\,dW^i(t)], \quad i = 1, 2. && (2.6)
\end{aligned}
$$

There exists a riskless bond in zero net supply, paying no dividends, with price dynamics

$$dB(t) = B(t)r(t)dt.$$

There also exists a zero net supply "derivative", paying no dividends, with price process

$$
\begin{aligned}
dP(t) &= P(t)[\mu_P(t)\,dt + \sigma_P(t)\,dW(t)] && (2.7)\\
&= P(t)[\mu_P^i(t)\,dt + \sigma_P(t)\,dW^i(t)], \quad i = 1, 2. && (2.8)
\end{aligned}
$$

The interest rate r, the perceived drifts μ_S^i, μ_P^i, and the volatilities σ_S, σ_P are posited to be $\{\mathcal{F}_t^\delta\}$-progressively measurable, with r, μ_S^i, μ_P^i in $\mathcal{L}^2(\mathcal{P}^i)$ and σ_S, σ_P bounded above and below away from zero.[3] μ_S, μ_P are in $\mathcal{L}^2(\mathcal{P})$ and $\{\mathcal{F}_t\}$-progressively measurable. All price coefficients are to be determined endogenously in equilibrium, except σ_P, taken as exogenous; σ_P defines the financial contract P since this security does not pay dividends. For a zero net supply security there is no substantial difference between dividends and price changes; both are transfers between agents, so for tractability we assume P pays no dividends. Any zero net supply security whose price is continuously resettled (e.g., a futures contract) is an example of this "idealized" derivative.

Agents observe the risky security prices, but do not observe the mean returns and so draw their own inferences, μ_S^i and μ_P^i. Equation (2.2) and price-agreement across agents imply the following "consistency" relationships between the perceived security price drifts:

$$\mu_S^1(t) - \mu_S^2(t) = \sigma_S(t)\bar{\mu}(t)\,, \quad \mu_P^1(t) - \mu_P^2(t) = \sigma_P(t)\bar{\mu}(t)\,. \tag{2.9}$$

Since the underlying Brownian motion is one-dimensional, the three securities are one more than needed to complete the market with respect to the agents' information filtration, making P redundant. However, by imposing portfolio constraints on the agents, we generate a role for P. More specifically, the agents are constrained in the shares of their wealth they may allocate to each risky security. Letting the 3-dimensional $\{\mathcal{F}_t^\delta\}$-progressively measurable process $\pi^i \equiv (\pi_B^i, \pi_S^i, \pi_P^i)^\top$, with $\pi_B^i = 1 - \pi_S^i - \pi_P^i$, denote the proportions of agent i's wealth $X^i(t)$ invested in B, S and P respectively, we assume that at all times $t \in [0, T]$, short sales of the stock are precluded and π_P^i is bounded from above:

$$\pi_S^i(t) \geq 0\,, \quad \pi_P^i(t) \leq \overline{\gamma}\,, \quad \overline{\gamma} > 0\,. \tag{2.10}$$

[3]Our subsequent assumptions prevent a negative σ_S in the mispriced equilibrium; symmetric cases with σ_S negative could be generated by reversing the constraints and/or the sign of σ_P.

Remark 2.1. It may be more realistic to also bound the derivative investment from below, $\pi^i_P(t) \geq -\underline{\gamma}$ ($\underline{\gamma} > 0$), and for symmetry we could impose a more general constraint on the stock ($\pi^i_S \in [\underline{\beta}, \overline{\beta}]$). However, the one-sided constraints are sufficient to illustrate our point. The two-sided constraints (and constraints on the bond) are relegated to Remark 5.1 and Section 6.

We now present our notion of mispricing. With one factor of risk, as soon as two assets' market prices of risk differ, it is possible to make arbitrage profits that require neither investment of capital nor risk-taking. Hence, it is natural to parametrize *mispricing*, perceived by agent i, by the difference between the assets' instantaneous market prices of risk,

$$\Delta^i_{P,S}(t) \equiv \frac{\mu^i_P(t) - r(t)}{\sigma_P(t)} - \frac{\mu^i_S(t) - r(t)}{\sigma_S(t)} \; .$$

We say that P is *favorable* if $\Delta^i_{P,S}(t) > 0$, and conversely. Given the direction of the constraints (2.10), for agent i's problem to have a solution we must have $\Delta^i_{P,S}(t) \geq 0$, because otherwise he would take on an unbounded arbitrage position (long in S and short in P). The consistency of security prices across agents enforces agreement on the mispricing.

Lemma 2.1. *Agents agree on the mispricing, and their common perception of it equals its actual value, i.e.,*

$$\Delta^1_{P,S}(t) = \Delta^2_{P,S}(t) = \frac{\mu_P(t) - r(t)}{\sigma_P(t)} - \frac{\mu_S(t) - r(t)}{\sigma_S(t)} \equiv \Delta_{P,S}(t) \; .$$

For each agent, we define security-specific deflator processes ξ^i_j by

$$d\xi^i_j(t) = -r(t)\xi^i_j(t)dt - \left(\frac{\mu^i_j(t) - r(t)}{\sigma_j(t)}\right)\xi^i_j(t)dW^i(t) \; , \quad j \in \{S, P\} \; , \tag{2.11}$$

such that (under standard regularity conditions) each deflated security gains process $(\xi^i_S S + \int \xi^i_S \delta \, dt, \, \xi^i_P P)$ is a $\mathcal{P}^i$-martingale, leading to the familiar present value formula

$$S(t) = \frac{1}{\xi^i_S(t)} E^i \left[\int_t^T \xi^i_S(s)\delta(s)ds | \mathcal{F}^i_t \right] \; , \quad i \in \{1, 2\} \; . \tag{2.12}$$

An analogous expression would hold for the derivative if it were assumed to pay dividends. (This change in our model would not substantially affect our conclusions: all the results of the paper would remain valid, but σ_P would be endogenous.) Thus, our notion of mispricing ($\Delta_{P,S} \neq 0$) is tantamount to each security having its own distinct deflator, because $\xi^i_S(t)$ and $\xi^i_P(t)$ differ as soon as non-zero mispricing has occurred (for a period of positive length) at any time $s \leq t$. This implies that, if mispricing occurs with positive probability, securities with identical dividend processes will command distinct prices (but not necessary concurrently).

2.3. Agents' Endowments and Preferences

Agent i is endowed at time 0 with e^i share of the stock ($e^i > 0$, $e^1 + e^2 = 1$), providing him with initial wealth $X^i(0) = e^i S(0)$. He then chooses a nonnegative consumption process c^i and a portfolio process π^i (in terms of fractions of i's wealth) from the set of $\{\mathcal{F}_t^\delta\}$-progressively measurable processes satisfying $\int_0^T c^i(t)\,dt < \infty$ and $\int_0^T \mid X^i(t)\,(\mu_S^i(t), \mu_P^i(t), r(t))\,\pi^i(t) \mid dt + \int_0^T \parallel \pi^i(t) X^i(t) \parallel^2 dt < \infty$ a.s.. An *admissible* consumption-portfolio pair (c^i, π^i) is defined as one for which the portfolio process satisfies the constraints (2.10) and the associated wealth process, X^i, is bounded from below, obeys $X^i(T) \geq 0$ a.s. and satisfies the dynamic budget constraint,

$$\begin{aligned} dX^i(t) &= [X^i(t)r(t) - c^i(t)]dt + X^i(t)\left\{\pi_S^i(t)[\mu_S^i(t) - r(t)] + \pi_P^i(t)[\mu_P^i(t) - r(t)]\right\}dt \\ &\quad + X^i(t)[\pi_S^i(t)\sigma_S(t) + \pi_P^i(t)\sigma_P(t)]dW^i(t). \end{aligned} \tag{2.13}$$

Each agent is assumed to derive time-additive, state-independent logarithmic utility $u_i(c^i(t)) = \log(c^i(t))$ from intertemporal consumption in $[0, T]$. Agent i's optimization problem is to maximize $E^i\left[\int_0^T \log(c^i(t))\,dt\right]$ over all admissible (c^i, π^i) pairs for which the expected integral is well-defined, given his information structure $(- \,, \mathcal{F}^i, \{\mathcal{F}_t^i\}, \mathcal{P}^i)$.

2.4. Equilibrium

Definition 2.1. *An equilibrium is a price system* $(r, \mu_S^1, \mu_S^2, \mu_P^1, \mu_P^2, \sigma_S)$[4] *and admissible consumption-portfolio processes* (c^i, π^i) *such that: (i) agents choose their optimal consumption-portfolio strategies given their perceived price processes in* $(- \,, \mathcal{F}^i, \{\mathcal{F}_t^i\}, \mathcal{P}^i)$*; (ii) security prices are consistent across agents, i.e.,*

$$\mu_S^1(t) - \mu_S^2(t) = \sigma_S(t)\bar{\mu}(t), \quad \mu_P^1(t) - \mu_P^2(t) = \sigma_P(t)\bar{\mu}(t); \tag{2.14}$$

and (iii) good and security markets clear, i.e.,

$$c^1(t) + c^2(t) = \delta(t)\,, \tag{2.15}$$

$$\pi_S^1(t)X^1(t) + \pi_S^2(t)X^2(t) = S(t)\,, \quad \pi_P^1(t)X^1(t) + \pi_P^2(t)X^2(t) = 0\,, \quad X^1(t) + X^2(t) = S(t)\,. \tag{2.16}$$

3. Agents' Optimization in the Presence of Mispricing

The redundancy in the risky securities adds an extra layer to the solution for optimality, namely, once the agent has chosen his risk exposure he must further decide how to allocate that risk between the two securities. The problem is simplified by solving "in reverse", first for the optimal allocation between S and P (Section 3.1),[5] second for the optimal risk exposure (Section 3.2).

[4] μ_S and μ_P are not incorporated into the definition because they do not determine agents' policies. They are obtained from the definitions in Section 2 ($\mu_j(t) = \mu_j^i(t) + \frac{\sigma_j(t)}{\sigma_\delta(t)}(\mu_\delta(t) - \mu_\delta^i(t))$, $i = 1, 2$, $j \in \{S, P\}$).

[5] This subsection, requiring strict monotonicity only, is otherwise independent of the agent's preferences.

3.1. Optimal Risk Allocation between Risky Securities

Given the potential redundancy in the risky securities, we introduce the risk-weighted sum of holdings therein,

$$\Phi^i(t) \equiv \pi^i_S(t) + \frac{\sigma_P(t)}{\sigma_S(t)}\pi^i_P(t) \ . \tag{3.1}$$

Independent of the individual holdings, the volatility of i's wealth from (2.13) is $X^i(t)\Phi^i(t)\sigma_S(t)$, so we may also interpret Φ^i as agent i's "composite" risk exposure, strictly defined as his wealth volatility per unit of stock volatility. Lemma 3.1 shows that, since S and P are perfect substitutes for achieving risk exposure, to solve an agent's portfolio problem it is sufficient to determine his choice of Φ^i, after which his choice of π^i_S and π^i_P is either irrelevant or straightforward to determine.

Lemma 3.1. *Let $\Phi^i(t)$ be given. Then:*
(i) if there is no mispricing, all admissible pairs $(\pi^i_S(t),\ \pi^i_P(t))$ such that

$$\pi^i_S(t) = \Phi^i(t) - \frac{\sigma_P(t)}{\sigma_S(t)}\pi^i_P(t) \tag{3.2}$$

represent optimal allocations of the risk, among all of which i is indifferent.
(ii) if P is favorable, among all admissible $(\pi^i_S(t),\pi^i_P(t))$ verifying (3.1), i's optimal risk allocation between S and P is

$$\pi^i_S(t) = \left(\Phi^i(t) - \frac{\sigma_P(t)}{\sigma_S(t)}\overline{\gamma}\right)^+ \ , \quad \pi^i_P(t) = \overline{\gamma} - \frac{\sigma_S(t)}{\sigma_P(t)}\left(\Phi^i(t) - \frac{\sigma_P(t)}{\sigma_S(t)}\overline{\gamma}\right)^- \ . \tag{3.3}$$

When P is favorable, agent i's allocation between S and P follows immediately from strict monotonicity of preferences: if P is favorable, unless (3.3) holds, the agent can add a riskless, costless arbitrage position to his portfolio and so will always do so until his holdings satisfy (3.3). Lemma 3.1 implies (via substitution into the dynamic budget constraint (2.13)) that any non-satiated agent's optimal wealth will always follow:

$$\begin{aligned} dX^i(t) = [r(t)X^i(t) - c^i(t)]dt + X^i(t)\Bigg\{&\left(\Phi^i(t) - \frac{\sigma_P(t)}{\sigma_S(t)}\overline{\gamma}\right)^+ (\mu^i_S(t) - r(t)) \\ + \left[\overline{\gamma} - \frac{\sigma_S(t)}{\sigma_P(t)}\left(\Phi^i(t) - \frac{\sigma_P(t)}{\sigma_S(t)}\overline{\gamma}\right)^-\right](\mu^i_P(t) - r(t))&\Bigg\} dt + X^i(t)\Phi^i(t)\sigma_S(t)dW^i(t) \ . \end{aligned} \tag{3.4}$$

Noting that $\Phi^i(t)$ can take on any real value without violating the constraints on $\pi^i_S(t)$ and $\pi^i_P(t)$, (3.4) shows that agents effectively face an unconstrained problem involving the bond and a single, *composite* risky asset with volatility σ_S and a policy-dependent drift. $\Phi^i(t)$ can be viewed as agent i's weight in this composite. The rest of this section is devoted to solving for an agent's choice of Φ^i, after which Lemma 3.1 yields his policy in terms of the actual securities S and P.

3.2. Optimal Consumption-Portfolio Policies

Equation (3.4) reveals that the mispricing generates a non-linearity in the reward for risk-taking (i.e., different price parameters in different regions of $\Phi^i(t)$). The methodology we use to tackle this is a variation on that introduced by Cvitanic and Karatzas (1992) to deal with portfolio constraints,[6] which consists in embedding the optimization problem in a family of fictitious unconstrained, perfect market problems. The fictitious price dynamics are modified so that the solution to one of the perfect market problems coincides with that of the original problem. Each individual-specific fictitious market has one bond and one stock, no portfolio constraints, and is parametrized by η^i (described below). The fictitious market price parameters are:

$$r_{\eta^i}(t) = r(t) + g^i(\eta^i(t), t) \; ; \tag{3.5}$$

$$\mu_{\eta^i}(t) = \mu_S^i(t) + \eta^i(t) + g^i(\eta^i(t), t) \; , \quad \sigma_{\eta^i}(t) = \sigma_S(t) \; ,$$

where η^i is a bounded, $\{\mathcal{F}_t^i\}$-progressively measurable process and g^i an $\{\mathcal{F}_t^i\}$-progressively measurable function. Thus, the market price of risk in the fictitious market is given by:

$$\theta_{\eta^i}(t) = \frac{\mu_{\eta^i}(t) - r_{\eta^i}(t)}{\sigma_S(t)} = \frac{\mu_S^i(t) - r(t)}{\sigma_S(t)} + \frac{\eta^i(t)}{\sigma_S(t)} \; , \tag{3.6}$$

and the fictitious state price density process ξ_{η^i} has dynamics

$$d\xi_{\eta^i}(t) = -\xi_{\eta^i}(t) \left[r_{\eta^i}(t)\, dt + \theta_{\eta^i}(t)\, dW^i(t) \right] \; . \tag{3.7}$$

Informally, η^i reflects the policy dependence of agent i's market price of risk. g^i reflects the fact that a rational agent will always add an arbitrage position to his portfolio if allowed to. Under no mispricing, both η^i and g^i should equal zero. When P is favorable, we would expect an agent to face a market price of risk lying between those of the two securities: $\frac{\mu_S^i(t) - r(t)}{\sigma_S(t)} \leq \theta_{\eta^i}(t) \leq \frac{\mu_P^i(t) - r(t)}{\sigma_P(t)}$. Accordingly, we shall consider η^i's in the set $\tilde{K}$ of $\{\mathcal{F}_t^i\}$-progressively measurable processes such that

$$0 \leq \eta^i(t) \leq \sigma_S(t) \Delta_{P,S}(t) \; , \quad \forall t \in [0, T] \; . \tag{3.8}$$

Coincidence of the dynamic budget constraints in the actual and the fictitious market requires

$$g^i(\eta^i(t), t) \equiv \overline{\gamma} \sigma_P(t) \Delta_{P,S}(t) - \overline{\gamma} \frac{\sigma_P(t)}{\sigma_S(t)} \eta^i(t) \, . \tag{3.9}$$

Arguments similar to those in Cvitanic and Karatzas (1992) show that if η^i solves the dual, "minimax" problem

$$\min_{\eta \in \tilde{K}} \left\{ \max_{c^i} E^i \left[\int_0^T \log(c^i(t)) dt \right] \; s.t. \; E^i \left[\int_0^T \xi_\eta(t) c^i(t) dt \right] \leq \xi_\eta(0) X^i(0) \right\} , \tag{3.10}$$

[6] Specifically, it is inspired by appendix B of Cvitanic and Karatzas (1992), which applies their methodology to a model with different riskless rates for borrowing and lending. We could also use the technique of Cuoco and Cvitanic (1998), who study a more general class of portfolio choice problems with a policy-dependent drift.

then the optimal policy in this particular fictitious market (obtained via standard martingale methods as in Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987)) solves i's original optimization problem. (At this minimax η^i, ξ_{η^i}, r_{η^i}, θ_{η^i} are henceforth denoted ξ^i, r^i, θ^i.)

Proposition 3.1. *Agent i's optimal consumption and composite investment are given by*

$$c^i(t) = \frac{1}{y^i \xi^i(t)} = \frac{X^i(t)}{T-t}, \quad \Phi^i(t) = \frac{\theta^i(t)}{\sigma_S(t)}, \tag{3.11}$$

where $y^i = T/\xi^i(0)X^i(0)$ and the individual-specific price parameters θ^i, r^i are as in Table I.

Table I: Optimal portfolio holdings and individual-specific price parameters

Cases	Conditions	$\pi^i_P(t)$	$\pi^i_S(t)$	$\theta^i(t)$	$r^i(t)$
(a)	$\frac{\mu^i_S(t)-r(t)}{\sigma_S(t)} = \frac{\mu^i_P(t)-r(t)}{\sigma_P(t)}$	$\leq \overline{\gamma}$	≥ 0	$\frac{\mu^i_S(t)-r(t)}{\sigma_S(t)}$	$r(t)$
(b)	$\frac{\mu^i_P(t)-r(t)}{\sigma_P(t)} > \frac{\mu^i_S(t)-r(t)}{\sigma_S(t)} \geq \overline{\gamma}\sigma_P(t)$	$\overline{\gamma}$	$\frac{\mu^i_S(t)-r(t)}{\sigma_S(t)^2} - \overline{\gamma}\frac{\sigma_P(t)}{\sigma_S(t)}$	$\frac{\mu^i_S(t)-r(t)}{\sigma_S(t)}$	$r(t) + \overline{\gamma}\sigma_P(t)\Delta_{P,S}(t)$
(c)	$\frac{\mu^i_S(t)-r(t)}{\sigma_S(t)} < \overline{\gamma}\sigma_P(t) < \frac{\mu^i_P(t)-r(t)}{\sigma_P(t)}$	$\overline{\gamma}$	0	$\overline{\gamma}\sigma_P(t)$	$r(t)$
(d)	$\frac{\mu^i_S(t)-r(t)}{\sigma_S(t)} < \frac{\mu^i_P(t)-r(t)}{\sigma_P(t)} \leq \overline{\gamma}\sigma_P(t)$	$\frac{\mu^i_P(t)-r(t)}{\sigma_P(t)^2}$	0	$\frac{\mu^i_P(t)-r(t)}{\sigma_P(t)}$	$r(t)$

Table I summarizes the situations that agent i may face. (a) is the case of no mispricing, whereas (b)-(d) arise when P is favorable. Under mispricing, the market price of risk faced by i (θ^i) is that of the security on which he does not bind (the one he would use for marginal adjustments in risk exposure); when he binds in both securities (case (c)) his market price of risk lies in between those of the two securities. In case (b), agent i desires high enough a risk exposure (Φ^i) for him to bind on the favorable security (P) and also have to use the less desirable one (S), so that he faces the lower market price of risk. However, part of his risky investment ($\overline{\gamma}X^i$), done using P, is rewarded at a higher rate, captured by the addition of g^i to both his individual-specific interest rate and risky asset returns. He can be thought of as performing arbitrage, reducing his position in S to purchase P, the resulting profit being $g^i X^i$. This "profit" is independent of Φ^i and appears in the dynamic budget constraint exactly as would an exogenously specified stochastic endowment, and so can be interpreted as a "fictitious endowment". For low values of Φ^i (case (d)), the agent uses only the more favorable P, his marginal market price of risk is that on P, and so the whole of his risk exposure is rewarded at the same rate, hence $g^i(t) = 0$ and $r^i(t) = r(t)$.

4. Equilibrium under Logarithmic Preferences

4.1. Construction and Existence of Equilibrium with Mispricing

We now use the portfolio choice results to characterize equilibrium. Table I distinguished cases (a)-(d) for an agent. Accordingly, we denote by (a,b) the equilibrium case where agent 1 is in (a) and agent 2 is in (b), etc.. From Lemma 2.1 (agents' agreement on the mispricing), stock market clearing (implying π_S^1 and π_S^2 cannot equal zero simultaneously) and derivative market clearing (implying π_P^1 and π_P^2 cannot both be strictly positive), the only possible cases are (a,a), (b,d), (d,b). In case (a,a), there is no mispricing and the equilibrium is similar to an unconstrained model; conversely, all equilibrium quantities in the unconstrained economy are as we provide for this region (a,a). Equilibrium with mispricing ((b,d) and (d,b)) requires both agents to be binding, but each on a different constraint.

For analytical convenience, we introduce a representative agent with utility

$$U(c;\lambda) \equiv \max_{c^1+c^2=c} \log\left(c^1\right) + \lambda \log\left(c^2\right) = \log\left(\frac{c}{1+\lambda}\right) + \lambda \log\left(\frac{\lambda c}{1+\lambda}\right), \tag{4.1}$$

where $\lambda \in (0,\infty)$ may be stochastic. As in the unconstrained case, optimality and consumption good clearing imply that the representative agent consumes the aggregate dividend, and that his marginal utility equates to agent 1's state price density. We immediately deduce (4.2)-(4.3). Moreover, the equilibrium allocation must solve the problem in (4.1), implying

$$\lambda(t) = \frac{c^2(t)}{c^1(t)} = \frac{y^1 \xi^1(t)}{y^2 \xi^2(t)},$$

where the second equality follows from (3.11). In an unconstrained, homogeneous beliefs economy (Karatzas, Lehoczky and Shreve (1990)), agents face the same state price density, so λ is a constant, determined from agents' budget constraints. Substitution into (4.2)-(4.3) fully solves for equilibrium. When agents face different state price densities, though, an extra step is required to independently identify ξ^1/ξ^2 using (3.7). In an unconstrained, heterogeneous beliefs economy (and under no mispricing (a,a)), agents' individual-specific price parameters differ only due to heterogeneity in beliefs ($\theta^1(t) - \theta^2(t) = \bar{\mu}(t)$) and accordingly the dynamics of λ are obtained directly ((4.7)) in terms of the disagreement process $\bar{\mu}$, as in Basak (1998). Under mispricing (regions (b,d) and (d,b)), each agent faces the market price of risk on a different security, so λ is also dependent on the mispricing, as reported in (4.10) and (4.13). Hence, to close our model, an extra restriction is required to (jointly) determine the mispricing. This extra restriction is obtained by imposing clearing in one of the risky securities, yielding (4.9) and (4.12). In a non-redundant economy, financial market clearing is guaranteed by good clearing and so yields no further restriction. With redundant assets, however, given consumption streams that clear the good market, there is a continuum of pairs of portfolio strategies financing them, not all of which clear the financial markets. In short, the presence of redundancy and constraints adds an additional "layer" to the equilibrium solution and enforces a possibly non-zero mispricing.

Proposition 4.1 formalizes the above discussion and provides explicit conditions for the occurrence of regions (a,a), (b,d) and (d,b).

Proposition 4.1. *Assume that there exists a strictly positive process λ satisfying (4.7), (4.10), (4.13), in the corresponding states, with initial condition $\lambda(0) = e^2/e^1$. Then, if the associated optimal policies and prices satisfy the technical conditions of Section 2, equilibrium exists, and the equilibrium state price densities, consumption allocations and stock price are given by*

$$\xi^1(t) = e^1\delta(0)\frac{1+\lambda(t)}{\delta(t)}\,, \quad \xi^2(t) = e^2\delta(0)\frac{1+\lambda(t)}{\lambda(t)\delta(t)}\,, \tag{4.2}$$

$$c^1(t) = \frac{\delta(t)}{1+\lambda(t)}\,, \quad c^2(t) = \frac{\lambda(t)}{1+\lambda(t)}\delta(t)\,, \tag{4.3}$$

$$S(t) = (T-t)\,\delta(t)\,, \tag{4.4}$$

implying $\mu^i_S(t) = \mu^i_\delta(t)$, $i = 1, 2$, and $\sigma_S(t) = \sigma_\delta(t)$. Depending on agents' disagreement $\bar{\mu}(t)$, the agents' situations (in terms of portfolio holdings), mispricing, and stochastic weighting dynamics are as follows.

When
$$-\frac{1+\lambda(t)}{\lambda(t)}\left(\lambda(t)\overline{\gamma}\sigma_P(t) + \sigma_\delta(t)\right) \le \bar{\mu}(t) \le \frac{1+\lambda(t)}{\lambda(t)}\left(\overline{\gamma}\sigma_P(t) + \lambda(t)\sigma_\delta(t)\right)\,, \tag{4.5}$$

agents are in (a,a) and
$$\Delta_{P,S}(t) = 0\,, \tag{4.6}$$

$$\frac{d\lambda(t)}{\lambda(t)} = -\bar{\mu}(t)\,dW^1(t)\,. \tag{4.7}$$

When
$$\bar{\mu}(t) > \frac{1+\lambda(t)}{\lambda(t)}\left(\overline{\gamma}\sigma_P(t) + \lambda(t)\sigma_\delta(t)\right)\,, \tag{4.8}$$

agents are in (b,d) and
$$\Delta_{P,S}(t) = \bar{\mu}(t) - \frac{1+\lambda(t)}{\lambda(t)}\left(\overline{\gamma}\sigma_P(t) + \lambda(t)\sigma_\delta(t)\right) > 0\,, \tag{4.9}$$

$$\frac{d\lambda(t)}{\lambda(t)} = -\frac{1+\lambda(t)}{\lambda(t)}\overline{\gamma}\sigma_P(t)\Delta_{P,S}(t)dt + \left[\Delta_{P,S}(t) - \bar{\mu}(t)\right]dW^1(t)\,. \tag{4.10}$$

When
$$\bar{\mu}(t) < -\frac{1+\lambda(t)}{\lambda(t)}\left(\lambda(t)\overline{\gamma}\sigma_P(t) + \sigma_\delta(t)\right)\,, \tag{4.11}$$

agents are in (d,b) and
$$\Delta_{P,S}(t) = -\bar{\mu}(t) - \frac{1+\lambda(t)}{\lambda(t)}\left(\lambda(t)\overline{\gamma}\sigma_P(t) + \sigma_\delta(t)\right) > 0\,, \tag{4.12}$$

$$\frac{d\lambda(t)}{\lambda(t)} = -\frac{1+\lambda(t)}{\lambda(t)}\sigma_\delta(t)\Delta_{P,S}(t)dt - \left[\Delta_{P,S}(t) + \bar{\mu}(t)\right]dW^1(t)\,. \tag{4.13}$$

Existence of equilibrium relies on the existence of the process λ (as well as satisfaction of a set of technical conditions). Proposition 4.2 provides an example of existence, the case of agents knowing that δ follows a geometric Brownian motion, and with heterogeneous, normally distributed priors on its drift. Proposition 4.2 also identifies, within this example, the parameter subspace which ensures that mispricing arises with positive probability.

Proposition 4.2. *Assume that $\mu_\delta(t)$, $\sigma_\delta(t)$, $\sigma_P(t)$ are constant, and that agents have normally distributed priors on μ_δ, with means $\mu_\delta^1(0) \neq \mu_\delta^2(0)$ and variance $v(0)$. Then, a unique equilibrium exists.*[7] *Furthermore, mispricing arises with positive probability at time t if and only if $|\bar{\mu}(0)| > [v(0)t/\sigma_\delta^2 + 1]^{\sigma_\delta} (\sigma_\delta + \overline{\gamma}\sigma_P + 2\sqrt{\overline{\gamma}\sigma_P\sigma_\delta})$.*

We now return to the general characterization in Proposition 4.1. Note that, under logarithmic utility, a convenient interpretation for the stochastic weighting is as the ratio of the agents' wealths (which equals that of their consumptions). Figure 1 plots regions (a,a), (b,d) and (d,b) as a function of $\bar{\mu}$ and $\lambda/(1+\lambda)$ (agent 2's fraction of aggregate wealth), for a given σ_P. Mispricing occurs as soon as heterogeneity ($\bar{\mu}$) across agents is large enough. Its value (bounded above by $\bar{\mu}(t)$) increases as the parameters move further into the interior of the mispricing region.

INSERT FIGURE 1

Figure 1: Equilibrium regions of mispricing and no-mispricing plotted for $\sigma_\delta(t) = \sigma_P(t) = 1$, $\overline{\gamma} = 0.5$. (The qualitative features are without loss of generality.)

The simplicity of the logarithmic case allows us to provide clear intuition for the role of mispricing. From Propositions 3.1 and 4.1, mispricing occurs for

$$\frac{\mu_P^i(t) - r(t)}{\sigma_P(t)} > \frac{\mu_S^i(t) - r(t)}{\sigma_S(t)} > \overline{\gamma}\sigma_P(t) > \frac{\mu_P^j(t) - r(t)}{\sigma_P(t)} > \frac{\mu_S^j(t) - r(t)}{\sigma_S(t)}\ .$$

Both agents "prefer" P to S, but agent i is optimistic about both S and P relative to the bond, while j is pessimistic about both S and P relative to the bond. Hence, agent i will be driven to his upper bound in P and still want a large positive holding in S, while j will be driven to his lower bound in S and still want a large negative holding in P. But these desires are not compatible with market clearing, as each agent faces *implicit* constraints that stem from clearing and the other agent's explicit constraints. In particular, since P is in zero net supply and is the only security that can be sold short, the total dollar amount of j's short sales is limited by i's upper constraint on P ($\pi_P^j X^j \geq -\overline{\gamma}X^i$). So, prices must adjust to make P more favorable (hence, short-selling it more costly) and S less favorable. Hence the mispricing.

To help quantify this, we temporarily drop time indices and assume $\sigma_P = \sigma_\delta \equiv \sigma$. Suppose agent 1 is sufficiently optimistic relative to agent 2 so that (b,d) should prevail: $\bar{\mu} > \frac{S}{X^2}\sigma\overline{\gamma} + \frac{S}{X^1}\sigma$ ((4.8)). Let us try to solve for an equilibrium without mispricing and exhibit what "goes wrong". By substituting the region (a,a) expression for r ((4.14)) into 2's portfolio choice, we obtain $\Phi^2 X^2 = \left(\frac{\mu_\delta^2 - r}{\sigma^2}\right) X^2 = \left(1 + \frac{X^1}{S}\frac{\bar{\mu}}{\sigma}\right) X^2 < -\overline{\gamma}X^1$, by (4.8). Since short sales on S are not allowed, 2 needs to short-sell an amount of P strictly greater than $\overline{\gamma}X^1$. By market clearing 1 then has to hold (long) more than $\overline{\gamma}X^1$ dollars of P, which is impossible since $\pi_P^1 \leq \overline{\gamma}$. From the other agent's perspective, similar computations yield $\Phi^1 X^1 > S + \overline{\gamma}X^1$, again not feasible, since the

[7] up to the indeterminacy in π_S^i and π_P^i in case (a,a). Prices and allocations are uniquely determined.

right-hand side equals the maximum dollar amount that 1 can invest in the risky assets. Hence, prices must adjust to make both short-selling P and going long in S less attractive, which can only be achieved by raising r to decrease the risk premium on S, and raising μ_P (by more) to increase the premium on P. (The stock price, and hence μ_S and σ_S, remains unaffected by the constraints, a property typical of logarithmic models with one stock and no stochastic endowment, e.g., Basak and Cuoco (1998), Detemple and Murthy (1997).) Hence, equilibrium requires $\Delta_{P,S}$ to be strictly positive.

From agent 1's individual viewpoint, the mispricing entices him, while keeping his risk exposure constant, to substitute the favorably mispriced security P for the unfavorable S, in an amount limited by the upper constraint on P. He can be thought of as performing the riskless arbitrage trade consisting in financing purchases of P by a (risk-offsetting) reduction of his position in S and purchases or sales of the bond. This leads him to reduce his holding in S and ensures clearing therein; hence a valuable economic role for his arbitrage activity. From 2's viewpoint, the mispricing means he has to pay a fee to short sell, since P must be used to do so. This induces 2 to reduce his negative position and ensures he abides by his implicit constraint (short-selling no more of P than agent 1 can hold long). The details of the agents' arbitrage activity are elaborated upon in Basak and Croitoru (1998). In short, the role of the mispricing is to allow the agents to reach their individual optimum, while limiting their heterogeneity in portfolio demand in keeping with the constraints and market clearing.

From this intuition, it becomes clear why the region of no-mispricing grows and the mispricing itself shrinks for higher values of $\overline{\gamma}$ or σ_P; the constraints are looser so less mispricing is needed. Raising σ_P effectively relaxes the π_P constraint; for a given dollar investment in P, a higher σ_P means a higher transfer between agents in response to an unexpected event. Hence the definition of the "contract" P has economic effects. We can also explain Figure 1. When agents' wealths are very close, each agent's explicit constraints readily impact the other agent implicitly, whereas if an agent is much wealthier, the poorer agent has a lot of freedom to trade before the much richer agent becomes unable to provide a counterparty because of his explicit constraints. Since mispricing occurs when both agents' explicit constraints impact the other agent (otherwise, adjusting r is sufficient to clear markets), mispricing occurs less readily (and is smaller) if one agent is much wealthier.

The role of the stochastic weighting can also be made more precise. In (a,a), λ makes the wealth of the more optimistic agent more positively correlated with the dividend, and its volatility grows with $\bar{\mu}$ accordingly. Under mispricing, however, the diffusion of λ remains "stuck" at the value it had (as a function of λ) when $\bar{\mu}$ entered the corresponding region, enforcing inter-agent transfers to remain essentially "stuck" even though heterogeneity grows. In (b,d) and (d,b), the volatility of λ lies between 0 and the value it would take on with no constraints; hence transfers are allowed, but only to a limited extent. Finally, recalling our discussion in Section 2.2, by establishing that a non-zero mispricing can arise, we have also established that different securities will be priced using different deflator processes. This is at odds with the standard practice of pricing all attainable contingent claims via a common deflator process.

4.2. Characterization of Agents' Behavior and Prices in Equilibrium

For comparison with our economy, we introduce benchmark economies I and II:

Economy I: One stock S, no derivative P, no constraints;

Economy II: One stock S, no derivative P, $\pi_S \geq 0$.

Comparing with Economy I clarifies the impact of the constraints and of the associated mispricing, while comparing with Economy II highlights the role of the derivative. The results for Economies I and II are stated without proof since they are special cases of our economy ($\overline{\gamma} \to \infty$ and $\overline{\gamma} \to 0$, respectively). In our equilibrium, agents' holdings in the composite asset are:

$$\begin{array}{lllllll}
\Phi^1(t) & = & 1+\frac{1}{\sigma_\delta(t)}\frac{\lambda(t)}{1+\lambda(t)}\bar{\mu}(t)\,, & \Phi^2(t) & = & 1-\frac{1}{\sigma_\delta(t)}\frac{1}{1+\lambda(t)}\bar{\mu}(t) & \text{in (a,a)},\\
\Phi^1(t) & = & 1+\lambda(t)+\overline{\gamma}\frac{\sigma_P(t)}{\sigma_\delta(t)}\,, & \Phi^2(t) & = & -\frac{\overline{\gamma}}{\lambda(t)}\frac{\sigma_P(t)}{\sigma_\delta(t)} & \text{in (b,d)},\\
\Phi^1(t) & = & -\overline{\gamma}\lambda(t)\frac{\sigma_P(t)}{\sigma_\delta(t)}\,, & \Phi^2(t) & = & 1+\frac{1}{\lambda(t)}+\overline{\gamma}\frac{\sigma_P(t)}{\sigma_\delta(t)} & \text{in (d,b)}.
\end{array}$$

In Economy I (always in (a,a)), the optimistic agent's holding in the composite risky asset grows as $|\bar{\mu}(t)|$ grows, without bound. In our economy, however, once $|\bar{\mu}(t)|$ has grown enough to crossover into region (b,d) or (d,b), the composite holding gets "stuck" at the value it had at the boundary (for a given $\lambda(t)$); the constraints limit trade. In Economy II, the holdings also get stuck once the constraint binds on one agent and this occurs at a lower level of heterogeneity than in our economy, $\bar{\mu}(t) > (1+\lambda(t))\sigma_\delta(t)$ or $\bar{\mu}(t) < -\frac{(1+\lambda(t))}{\lambda(t)}\sigma_\delta(t)$. We deduce, for given $\lambda(t)$, that in (b,d), $\Phi^1_I(t) > \Phi^1(t) > \Phi^1_{II}(t)$ and $\Phi^2_I(t) < \Phi^2(t) < \Phi^2_{II}(t)$ (and vice-versa in (d,b)). The presence of P alleviates the constraints, but in equilibrium, the circumvention of the constraints is only partial.

Proposition 4.3 reports the remaining price parameters in our equilibrium. (Expressions for (d,b) are omitted as this region is a mirror image of (b,d), where expressions obtain by swapping agents and substituting $1/\lambda$ for λ in the (b,d) expressions.)

Proposition 4.3. *The equilibrium individual-specific market prices of risk, interest rate and difference in agents' fictitious interest rates are as follows.*

In case (a,a):

$$\theta^1(t) = \sigma_\delta(t) + \frac{\lambda(t)}{1+\lambda(t)}\bar{\mu}(t)\,, \quad \theta^2(t) = \sigma_\delta(t) - \frac{1}{1+\lambda(t)}\bar{\mu}(t)\,,$$

$$r(t) = \frac{1}{1+\lambda(t)}\mu^1_\delta(t) + \frac{\lambda(t)}{1+\lambda(t)}\mu^2_\delta(t) - \sigma_\delta(t)^2\,, \quad r^1(t) - r^2(t) = 0\,, \tag{4.14}$$

In case (b,d):

$$\theta^1(t) = (1+\lambda(t))\sigma_\delta(t) + \overline{\gamma}\sigma_P(t)\,, \quad \theta^2(t) = -\frac{1}{\lambda(t)}\overline{\gamma}\sigma_P(t)\,,$$

$$r(t) = \mu^1_\delta(t) - (1+\lambda(t))\sigma_\delta(t)^2 - \overline{\gamma}\sigma_P(t)\sigma_\delta(t) = r^2(t)\,, \tag{4.15}$$

$$r^1(t) - r^2(t) = \overline{\gamma}\sigma_P(t)\Delta_{P,S}(t) = \overline{\gamma}\sigma_P(t)\left[\bar{\mu}(t) - \frac{1+\lambda(t)}{\lambda(t)}\left(\overline{\gamma}\sigma_P(t) - \lambda(t)\sigma_\delta(t)\right)\right] > 0\,,$$

where λ is the stochastic weighting satisfying, in the respective regions, the stochastic differential equation (4.7) or (4.10). The equilibrium individual consumption dynamics are: $dc^i(t) = c^i(t)\mu^i_{c^i}(t)dt + c^i(t)\sigma_{c^i}(t)dW^i(t)$, where the consumption volatilities are: $\sigma_{c^i}(t) = \theta^i(t),\ i = 1, 2$.

The individual market prices of risk and consumption volatilities in (a,a) show that the two agents price and absorb risk "equally", but then adjust their pricing to transfer consumption risk from the more pessimistic to the more optimistic agent. This transfer is essentially proportional to the difference in beliefs, $\bar{\mu}(t)$ and, in the unconstrained Economy I would grow without limit with $|\bar{\mu}(t)|$. However, in our economy, as $|\bar{\mu}(t)|$ grows we eventually cross into regions (b,d) and (d,b), where transfers of risk are limited. Again, equilibrium quantities in our economy lie in between those in the two benchmarks I and II, showing how the derivative allows the more optimistic agent to take on more risk, but not as much as if no constraints were applied.

The expressions for the market prices of risk and consumption volatilities look asymmetric in region (b,d). The more optimistic agent (1) holds all of the stock, so his market price of risk becomes more sensitive to the aggregate risk, $\sigma_\delta(t)$, while the more pessimistic agent (2)'s has no direct sensitivity to the aggregate risk. The market prices of risk also become independent of the disagreement $\bar{\mu}(t)$, because inter-agents transfers are "stuck" at the maximum level permitted by the constraints.

Under no mispricing, the interest rate is the wealth-weighted average of agents' perceived mean aggregate consumption growth minus the aggregate consumption risk. This type of representation is familiar with heterogeneous beliefs (Detemple and Murthy (1994)). Under mispricing, however, when expressed in terms of the aggregate consumption growth and risk, the interest rate appears to be driven by the more optimistic agent. This is because (from Table IV) he is the agent in the interior in his stock holding and so the interest rate must adjust to make him indifferent between marginal changes in bond holdings versus extra dividends: this task is performed by the interest rate because μ_S is pinned at μ_δ. Under mispricing, the interest rate exhibits an increased sensitivity to the aggregate risk: since the constraints limit risk-sharing, the interest rate is more impacted by precautionary savings.

In our discussion of the role of mispricing, we anticipated that the "implicit" constraints on agents should also require the interest rate to be increased under mispricing. From (4.15), we indeed deduce that, under mispricing, for fixed $\lambda(t)$, $r_{II}(t) > r(t) > r_I(t)$: the constraints increase the interest rate, but not as much as if the zero net supply security were not present.

Under mispricing, the more optimistic agent has a higher fictitious interest rate than the pessimistic, the difference being proportional to the mispricing. The fictitious interest rate captures the actual interest rate plus any type of effective riskless "endowment" to an agent. In a sense, the mispricing acts as a "gift" to the optimistic agent because he is the one holding the zero net supply security long, and its expected return is raised when mispricing occurs.

Another point to observe about Proposition 5.4 is that no economic quantities jump discontinuously as $\bar{\mu}(t)$ crosses the border from one region to another. The mispricing $\Delta_{P,S}(t)$ grows continuously away from zero as we move further into regions (b,d) or (d,b), and accordingly the interest rate and other quantities change continuously. In all regions no agent is ever explicitly bound in his holding Φ^i of the composite risky asset. Hence there is no discontinuity in bond demand and so no discontinuity in the interest rate. This situation changes when more general constraints are imposed, causing Φ^i itself to be constrained, as elaborated upon in Section 6.

Finally, we have not been able to make unambiguous comparisons of agents' welfare across economies. The mispricing itself benefits the optimistic agent, who is holding P long, at the expense of the pessimistic. However, since the pessimistic agent is long in the bond and the optimistic agent short, the increase in r tends to counteract this.

5. Equilibrium under General Preferences

We now extend the analysis of equilibrium with mispricing to the case of general, possibly heterogeneous utility functions u_1, u_2, that are only assumed to be three times continuously differentiable, strictly increasing, strictly concave, and to satisfy $\lim_{c\to 0} u_i'(c) = \infty$ and $\lim_{c\to\infty} u_i'(c) = 0$. We first provide a motivation for doing so, by stating a general condition under which mispricing is necessary for equilibrium (Section 5.1). We then construct equilibrium (Section 5.2).

5.1. General Necessity of Mispricing

Proposition 5.1 establishes conditions under which mispricing is necessary for equilibrium to exist in our constrained economy. $\bar{\Phi}^i$ denotes agent i's optimal composite weight, in an economy where portfolio weights are unconstrained but with otherwise identical primitives (beliefs, preferences and endowments, dividends, available securities and volatility of the derivative (σ_P)).

Proposition 5.1. *Assume that equilibrium exists in an unconstrained economy with otherwise identical primitives. If agent 1's optimal investment is such that either*

$$\bar{\Phi}^1(t)\bar{X}^1(t) - \bar{\Phi}^2(t)\bar{X}^2(t) < -\bar{X}^1(t) - \bar{X}^2(t) - 2\bar{X}^2(t)\frac{\sigma_P(t)}{\bar{\sigma}_S(t)}\overline{\gamma} \quad \text{or} \quad \bar{\Phi}^1(t)\bar{X}^1(t) - \bar{\Phi}^2(t)\bar{X}^2(t) > 2\bar{X}^2(t)\frac{\sigma_P(t)}{\bar{\sigma}_S(t)}\overline{\gamma} \tag{5.1}$$

on a space of positive measure, then mispricing on a space of positive measure is necessary for equilibrium in the constrained economy.[8]

Proposition 5.1 shows that, without mispricing, only limited heterogeneity in composite risk exposure is compatible with market-clearing in the constrained economy. Mispricing is required whenever there is sufficient heterogeneity in agents' risk-taking, originating from heterogeneity in either beliefs or risk-aversion, strongly suggesting that, under heterogeneous preferences, diverging beliefs are not needed for mispricing. The proof establishes that mispricing is needed when

[8] The mispricing may occur in states and times other than those in which (5.1) holds.

agents violate their "implicit" constraints, as in the logarithmic case. (Indeed, the conditions in (5.1) are equivalent to the conditions for mispricing in Proposition 4.1.)

Remark 5.1 (More general constraints). When a lower bound $(-\underline{\gamma})$ is imposed on agents' weight in the derivative, there are two insubstantial changes: (i) $\bar{\Phi}^i$ is now the composite weight in an otherwise identical economy where the only constraint is $\bar{\Phi}^i(t) \geq -\underline{\gamma}\bar{\sigma}_P(t)/\bar{\sigma}_S(t)$; (ii) the following condition should be added to those in the proposition: $\bar{\Phi}^1(t)\bar{X}^1(t) - \bar{\Phi}^2(t)\bar{X}^2(t) < \bar{X}^1(t) + \bar{X}^2(t) + 2\bar{X}^2(t)\frac{\sigma_P(t)}{\bar{\sigma}_S(t)}\underline{\gamma}$. We could similarly incorporate a two-sided constraint on S.

5.2. Determination of Equilibrium

Proposition 5.2 extends Proposition 3.1 to general preferences. (The fictitious markets are as defined in Section 3.2.)

Proposition 5.2. *Assume that there exists a solution η^i to the problem*

$$\min_{\eta \in \tilde{K}} \left\{ \max_{c^i} E^i \left[\int_0^T u_i(c^i(t))dt \right] \text{ s.t. } E^i \left[\int_0^T \xi_\eta(t) c^i(t) dt \right] \leq \xi_\eta(0) X^i(0) \right\}. \tag{5.2}$$

Then, there exists a solution to i's optimization problem and his optimal consumption policy is

$$c^i(t) = I^i(y^i \xi_{\eta^i}(t)) \;, \tag{5.3}$$

where $I^i(\cdot)$ is the inverse of $u_i'(\cdot)$ and y^i satisfies

$$E^i \left[\int_0^T \xi_{\eta^i}(t) I^i(y^i \xi_{\eta^i}(t)) dt \right] = \xi_{\eta^i}(0) X^i(0). \tag{5.4}$$

The optimal holding in the composite asset is given by

$$\Phi^i(t) = \frac{\theta_{\eta^i}(t)}{\sigma_S(t)} + \frac{\kappa^i(t)}{X^i(t)\sigma_S(t)\xi^i(t)} \;, \tag{5.5}$$

where $\kappa^i(t)$ satisfies $\xi^i(t)X^i(t) + \int_0^t \xi^i(s)c^i(s)ds = \xi^i(0)X^i(0) + \int_0^t \kappa^i(s)dW^i(s)$. (π_S^i, π_P^i) are as provided by Lemma 3.1.

As under logarithmic preferences, agent i can be in four cases. (a) is the case of no mispricing, while (b), (c) and (d), distinguished by which constraint(s) is (are) binding, occur when P is favorable. Table II summarizes these situations. Table II is similar to its logarithmic case analogue (Table I), but explicit expressions for $\Phi^i(t)$ cannot be obtained (and, hence, nor can explicit price conditions for cases (b)-(d)). The interpretation of the individual-specific fictitious price parameters is similar to the logarithmic case.

Table II: Summary of agents' portfolio holdings and fictitious market parameters

	$\Phi^i(t)$	$\pi_P^i(t)$	$\pi_S^i(t)$	$\theta^i(t)$	$r^i(t)$
(a)	$\in \Re$	$\leq \overline{\gamma}$	≥ 0	$\frac{\mu_S^i(t)-r(t)}{\sigma_S(t)} = \frac{\mu_P^i(t)-r(t)}{\sigma_P(t)}$	$r(t)$
(b)	$> \frac{\sigma_P(t)}{\sigma_S(t)}\overline{\gamma}$	$\overline{\gamma}$	> 0	$\frac{\mu_S^i(t)-r(t)}{\sigma_S(t)}$	$r(t) + \overline{\gamma}\sigma_P(t)\Delta_{P,S}(t)$
(c)	$\frac{\sigma_P(t)}{\sigma_S(t)}\overline{\gamma}$	$\overline{\gamma}$	0	$\in \left[\frac{\mu_S^i(t)-r(t)}{\sigma_S(t)}, \frac{\mu_P^i(t)-r(t)}{\sigma_P(t)}\right]$	$\in [r(t), r(t) + \overline{\gamma}\sigma_P(t)\Delta_{P,S}(t)]$
(d)	$< \frac{\sigma_P(t)}{\sigma_S(t)}\overline{\gamma}$	$< \overline{\gamma}$	0	$\frac{\mu_P^i(t)-r(t)}{\sigma_P(t)}$	$r(t)$

As in the logarithmic case, the only possible equilibrium cases are (a,a), (b,d) and (d,b). We introduce a representative agent with (state-dependent) utility $U(c;\lambda) \equiv \max_{c^1+c^2=c} u_1(c^1) + \lambda u_2(c^2)$. Identifying $\lambda(t) \equiv u_1'(c^1(t))/u_2'(c^2(t)) = y^1\xi^1(t)/y^2\xi^2(t)$, Proposition 5.3 characterizes the equilibrium.

Proposition 5.3. *If equilibrium exists, the agents' equilibrium state price densities are*

$$\xi^1(t) = \frac{U'(\delta(t);\lambda(t))}{U'(\delta(0);\lambda(0))}\,, \quad \xi^2(t) = \frac{\lambda(0)}{\lambda(t)}\frac{U'(\delta(t);\lambda(t))}{U'(\delta(0);\lambda(0))}\,, \tag{5.6}$$

where $\lambda(0)$ solves either agent's static budget constraint, i.e.,[9]

$$E^1\left[\int_0^T U'(\delta(t);\lambda(t))I^1\left(U'(\delta(t);\lambda(t))\right)\,dt\right] = e^1E^1\left[\int_0^T U'(\delta(t);\lambda(t))\delta(t)\,dt\right]\,, \tag{5.7}$$

and the mispricing $\Delta_{P,S}$ and the stochastic weighting λ satisfy

in case (a,a):
$$\Delta_{P,S}(t) = 0, \tag{5.8}$$
$$\frac{d\lambda(t)}{\lambda(t)} = -\bar{\mu}(t)dW^1(t); \tag{5.9}$$

in case (b,d):
$$\Delta_{P,S}(t) = -\frac{\kappa^2(t)}{\xi^2(t)X^2(t)} - \overline{\gamma}\sigma_P(t)\frac{X^1(t)}{X^2(t)} - \frac{\mu_S^2(t)-r(t)}{\sigma_S(t)} > 0, \tag{5.10}$$
$$\frac{d\lambda(t)}{\lambda(t)} = \left(\frac{\mu_P^2(t)-r(t)}{\sigma_P(t)} - \overline{\gamma}\sigma_P(t)\right)\Delta_{P,S}(t)dt + (\Delta_{P,S}(t) - \bar{\mu}(t))dW^1(t); \tag{5.11}$$

in case (d,b):
$$\Delta_{P,S}(t) = -\frac{\kappa^1(t)}{\xi^1(t)X^1(t)} - \overline{\gamma}\sigma_P(t)\frac{X^2(t)}{X^1(t)} - \frac{\mu_S^1(t)-r(t)}{\sigma_S(t)} > 0, \tag{5.12}$$
$$\frac{d\lambda(t)}{\lambda(t)} = \left(\overline{\gamma}\sigma_P(t) - \frac{\mu_S^2(t)-r(t)}{\sigma_S(t)}\right)\Delta_{P,S}(t)dt - (\Delta_{P,S}(t) + \bar{\mu}(t))dW^1(t). \tag{5.13}$$

[9]The two agents' budget constraints are equivalent, and only determine the ratio y^1/y^2. We set $y^1 = U'(\delta(0);\lambda(0))$ without loss of generality so that $\xi^1(0) = \xi^2(0) = 1$.

The equilibrium consumption allocations are

$$c^1(t) = I^1\left(U'(\delta(t);\lambda(t))\right) \ , \quad c^2(t) = I^2\left(\frac{U'(\delta(t);\lambda(t))}{\lambda(t)}\right) \ . \tag{5.14}$$

Conversely, if there exist ξ^i, λ, $\Delta_{P,S}$ satisfying (5.6)-(5.13), the associated optimal policies (of equations (5.3), (5.5) and Lemma 3.1) satisfy all market clearing conditions.

As in the logarithmic case, it does not suffice to solve for the weighting λ (by clearing the good market) to pin down the equilibrium, it is also necessary to jointly determine the mispricing by clearing one of the risky security markets. The redundancy and constraints yield an extra layer in the equilibrium solution. Additional characterization of equilibrium prices and allocations can be found in Basak and Croitoru (1998).

6. The Case of Two-Sided Portfolio Constraints on Both Risky Securities

We now revert to logarithmic preferences ($u_i\left(c^i(t)\right) \equiv \log\left(c^i(t)\right)$, $i = 1, 2$) and assume two-sided portfolio constraints on both risky assets

$$-\underline{\beta} \leq \pi^i_S(t) \leq \overline{\beta} \ , \quad -\underline{\gamma} \leq \pi^i_P(t) \leq \overline{\gamma} \ , \quad i = 1, 2 \ , \quad \text{where } \underline{\gamma}, \overline{\gamma} > 0, \underline{\beta} > -1, \overline{\beta} > 1 \ .$$ [10]

The analysis of agents' optimization is modified in two ways: (i) agents face a constrained problem in Φ^i: $-\underline{\beta} - \frac{\sigma_P(t)}{\sigma_S(t)}\underline{\gamma} \leq \Phi^i(t) \leq \overline{\beta} + \frac{\sigma_P(t)}{\sigma_S(t)}\overline{\gamma}$; (ii) solutions may also exist when S is favorable since the lower constraint on π^i_P and the upper constraint on π^i_S bound arbitrage trades. (i) is taken care of by combining our technique with the results in Cvitanic and Karatzas (1992) for rectangular constraints, leading to optimal policies that again exhibit properties typical of logarithmic preferences (deterministic marginal propensity to consume, mean-variance portfolio holdings in terms of the instantaneous price parameters). Eleven additional cases are added to Table I: two extra cases under no mispricing, where the agent is bound on either both lower or both upper constraints; three extra cases under P favorable, where the agent is bound on both lower or both upper constraints, or is bound on the lower in S and the upper in P; and six new analogous cases for S favorable.

In equilibrium, as in Section 4, the combinations of cases consistent with market clearing are reduced. Somewhat unexpectedly, in equilibrium only one of S or P can ever become favorable in any given economy.

Proposition 6.1. *Assume logarithmic preferences for both agents. When $\overline{\gamma}(\underline{\beta}+1) < \underline{\gamma}(\overline{\beta}-1)$, $\Delta_{P,S}(t) \geq 0$, $\forall t$. When $\overline{\gamma}(\underline{\beta}+1) > \underline{\gamma}(\overline{\beta}-1)$, $\Delta_{P,S}(t) \leq 0$, $\forall t$.*[11]

[10] This case effectively embeds the case of restricted borowing, as these constraints imply that the weight invested in the bond satisfies $\pi^i_B \geq 1 - \left(\overline{\beta} + \overline{\gamma}\right)$.

[11] When $\overline{\gamma}(\underline{\beta}+1) = \underline{\gamma}(\overline{\beta}-1)$, mispricing can go in both directions but is of "measure zero", requiring $\lambda(t)$ to take on a particular, deterministic value.

Mispricing is necessary only if agents' implicit constraints are not automatically taken care of by the explicit constraints. The former inequality in Proposition 6.1 corresponds to the implicit lower (upper) bound on P (S) being tighter than the explicit one, thus requiring P to be favorable in some states. It turns out that this condition is the negation of the necessary condition for the implicit upper (lower) bound on P (S) to be looser than the explicit one, requiring S to be favorable. The latter inequality corresponds to the reverse situation. We conjecture that Proposition 6.1 should also be valid for general preferences. The somewhat counter-intuitive implication that only one of the two securities will ever be favorable can be broken by making the constraints heterogeneous across agents[12] (or stochastic). Figure 2 illustrates the equilibrium regions when $\overline{\gamma} = \underline{\gamma}$ and $\overline{\gamma}(\underline{\beta}+1) > \underline{\gamma}(\overline{\beta}-1)$, i.e., only S can be favorable. The situation for $\overline{\gamma}(\underline{\beta}+1) < \underline{\gamma}(\overline{\beta}-1)$ and for $\overline{\gamma} \neq \underline{\gamma}$ would be analogous.

INSERT FIGURE 2

Figure 2: Equilibrium regions of mispricing and no-mispricing in the case of 2-sided constraints. Binding portfolio constraints are also identified. The figure is plotted for $\sigma_\delta(t) = \sigma_P(t) = 1$, $\overline{\gamma} = \underline{\gamma} = 0.5$, $\underline{\beta} = 0.5$, $\overline{\beta} = 1.25$.

The main difference with Figure 1 is that large heterogeneity in beliefs no longer alone guarantees mispricing. Additional conditions are needed on the ratio of the agents' wealths. If agents are heterogeneous enough in their wealths, an agent's implicit constraint may be taken care of by the new explicit constraint, so mispricing will not be needed. We then get extra equilibrium cases where there is no mispricing and only one agent binds (on either both his lower or both his upper constraints).

The equilibria with mispricing are very similar to the ones described previously. When P is favorable, new terms involving $\underline{\beta}$ (previously zero) simply appear in the pricing results.[13] When S is favorable, agents bind on the opposite constraints, and the expressions are symmetric (with constraints and agents being swapped).[14]

With two-sided constraints, discontinuous jumps in $\Delta_{P,S}$ and r may occur in equilibrium. They occur only on moving from a region where an agent is bound in both securities to a mispriced region. We are then moving from a region where only one agent is in his interior in Φ (and so alone sets the prices) to a region where both agents are in their interior in Φ (and both set the prices); hence the discontinuity. With one-sided constraints of opposite directions on both risky securities as in the previous sections, agents are always in their interior in Φ. Similar interest rate discontinuities are derived in Detemple and Murthy (1996), where one-sided constraints are imposed on both stock and bond holdings, and no derivative security is modeled. This is because

[12] The extension to this case can be performed easily and only leads to (besides the mispricing taking on both signs) insubstantial changes in the equilibrium expressions.

[13] For example, when P is favorable and agent 1 more optimistic, the equilibrium interest rate is given by $r(t) = \mu^1_\delta(t) - (1+\lambda(t))\sigma_\delta(t)^2 - \overline{\gamma}\sigma_P(t)\sigma_\delta(t) - \underline{\beta}\lambda(t)\sigma_\delta(t)^2$ rather than (4.15).

[14] For example, when S is favorable and 1 is more pessimistic the equilibrium interest rate is given by $r(t) = \mu^2_\delta(t) - (1+\frac{1}{\lambda(t)})\sigma_\delta(t)^2 + \underline{\gamma}\sigma_P(t)\sigma_\delta(t) + \overline{\beta}\frac{1}{\lambda(t)}\sigma_\delta(t)^2$.

agents face a constrained problem in their composite risk-taking, as in this two-sided constrained version of our model.

7. Conclusion

This paper develops a general equilibrium, pure exchange, continuous time model where, as a result of heterogeneous agents facing portfolio constraints, mispricing occurs between a positive net supply "stock" and a zero net supply "derivative" with perfectly correlated price processes. Hence, in some states mispricing is generated as an integral part of the equilibrium, and the agents engage in (bounded) arbitrage trading. With logarithmic preferences, the model is fully solved, and existence of an equilibrium where mispricing occurs with a positive probability is shown in a specific context. We also provide characterization of equilibrium. Under more general preferences, we demonstrate the necessity of mispricing for equilibrium when agents are heterogeneous enough in their optimal exposure to risk. Natural extensions of the model would include increasing the number of securities, or the number of agents, possibly distinguishing between "consumption traders" (who maximize expected utility of consumption), and "arbitrageurs" (who only take on riskless arbitrage positions). We foresee potential applications in international finance, to account for deviations from purchasing power parity, and in capital market equilibrium in the presence of tax arbitrage.

Appendix: Proofs

Proof of Lemma 2.1: The first equality follows from (2.9). (2.5)-(2.8), and (2.2) imply

$$\frac{\mu_S(t)-\mu^i_S(t)}{\sigma_S(t)}dt = dW^i(t) - dW(t) = \frac{\mu_\delta(t)-\mu^i_\delta(t)}{\sigma_\delta(t)}dt = \frac{\mu_P(t)-\mu^i_P(t)}{\sigma_P(t)}dt\ .$$

The second equality immediately follows. *Q.E.D.*

Proof of Lemma 3.1: (i) By substituting $\Delta_{P,S}(t) = 0$ into (2.13), rearranging and substituting (3.2), all pairs $(\pi^i_S(t),\pi^i_P(t))$ satisfying (3.2) yield the same wealth dynamics.
(ii) Consider $\Phi^i(t) \geq \frac{\sigma_P(t)}{\sigma_S(t)}\overline{\gamma}$. Assume $\Delta_{P,S}(t) > 0$, but (3.3) does not hold, implying $\pi^i_P(t) < \overline{\gamma}$. Adding the following "incremental" arbitrage position: $X^i(t)\left(\overline{\gamma} - \pi^i_P(t)\right)x > 0$ dollars in P, $X^i(t)\frac{\sigma_P(t)}{\sigma_S(t)}\left(\pi^i_P(t) - \overline{\gamma}\right)x < 0$ dollars in S, and $-X^i(t)\left(\overline{\gamma} - \pi^i_P(t)\right)\left(1 - \frac{\sigma_P(t)}{\sigma_S(t)}\right)x$ dollars in the bond ($0 < x \leq 1$), is feasible for agent i (without violating his constraints). Furthermore, it has zero cost, zero volatility and a positive drift, $X^i(t)\left(\overline{\gamma} - \pi^i_P(t)\right)x\Delta_{P,S}(t)$. Similarly when $\Phi^i(t) < \frac{\sigma_P(t)}{\sigma_S(t)}\overline{\gamma}$: i could add an arbitrage position, increasing his drift by $\pi^i_S(t)x\Delta_{P,S}(t) > 0$. Hence, whenever there is mispricing but (3.3) fails on a subset of $\{\omega \in - \ : \Delta_{P,S}(t) > 0\} \times [0,T]$ with positive measure, from i's monotonic preferences there exists a portfolio strategy that is strictly prefered to $(\pi^i_S(t), \pi^i_P(t))$. *Q.E.D.*

Proof of Proposition 3.1: The maximization problem in brackets in (3.10) is solved easily using Lagrangean theory, revealing that $c^i(t) = 1/y^i\xi^i(t)$, where $y^i = T/\xi^i(0)X^i(0)$. Substitution into (3.10) shows that η^i solves the pointwise minimization problem $\min_{\eta(t)} r_\eta(t) + \left[\theta_\eta(t)\right]^2/2$, hence the values for the individual-specific prices in Table I. The optimal wealth is $X^i(t) = E^i\left[\int_t^T \xi^i(s)c^i(s)ds|\mathcal{F}^i_t\right]/\xi^i(t) = (T-t)X^i(0)\xi^i(0)/T\xi^i(t)$, hence $c^i(t) = X^i(t)/(T-t)$. Applying Itô's lemma to $X^i(t)$ and equating diffusion terms shows that $\Phi^i(t) = \theta^i(t)/\sigma_S(t)$. *Q.E.D.*

Proof of Proposition 4.1: The proof involves: (i) assuming that equilibrium exists and deducing (4.2)-(4.13); (ii) verifying that Definition 2.1 is satisfied and establishing conditions for the cases; (iii) verifying that all states fall under one of these.

Assume that equilibrium exists. Substituting the representative agent's utility in (5.6) and using $y^i = T/X^i(0) = T/e^iS(0)$ yields (4.2). This and good clearing (2.15), given that $c^i(t) = 1/y^i\xi^i(t)$, yield (4.3). (2.15), bond clearing (last equality in (2.16)) and agents' consumption (3.11) yield (4.4). Itô's lemma then implies $\mu^i_S(t) = \mu^i_\delta(t)$, $i = 1,2$, and $\sigma_S(t) = \sigma_\delta(t)$. Consider case (b,d). Agents' optimization (Table I) and derivative clearing imply $\left(\mu^2_P(t) - r(t)\right)/\sigma_P(t) = -\overline{\gamma}\sigma_P(t)/\lambda(t)$, while agents' optimization, stock clearing and price consistency (2.14) imply $\left(\mu^2_S(t) - r(t)\right)/\sigma_S(t) = \left(\mu^1_S(t) - r(t)\right)/\sigma_S(t) - \bar{\mu}(t) = (1+\lambda(t))\sigma_\delta(t) + \overline{\gamma}\sigma_P(t) - \bar{\mu}(t)$, hence (4.9). Substitution into (5.11) yields (4.10). Agents being in (b,d) requires: (i) $\Delta_{P,S}(t) > 0$, which is (4.8); (ii) $\left(\mu^1_S(t) - r(t)\right)/\sigma_S(t) \geq \overline{\gamma}\sigma_P(t)$; (iii) $\left(\mu^2_P(t) - r(t)\right)/\sigma_P(t) \leq \overline{\gamma}\sigma_P(t)$. (ii) and (iii) are implied by the strict positivity of $\lambda(t)$, which the proposition assumes. Hence, when (4.8) holds, the agents are in (b,d). (4.2)-(4.3) show optimal consumptions which clear the good market. (4.9)-(4.10) ensure (5.11)-(5.10) are satisfied, so by Proposition 5.3 all markets clear. Since market clearing puts restrictions on $\mu_S^1(t)$ and $\mu_P^2(t)$ only, it is possible to determine $\mu_S^2(t)$ and $\mu_P^1(t)$ so that (2.14) holds. Therefore, the economy is in equilibrium. (d,b) is symmetric. For (a,a), we use the proof of Proposition 5.1 to show that (4.5) is necessary and sufficient for equilibrium without mispricing. Here, without mispricing, $\Phi^i(t) = \left(\mu_\delta^i(t) - r(t)\right)/\sigma_\delta(t)^2$. Financial market clearing then implies $r(t) = \mu_\delta^1(t) - (\lambda(t))/(1+\lambda(t))\bar{\mu}(t)\sigma_\delta(t) - \sigma_\delta(t)^2$, so that $\Phi^1(t) = 1 + \lambda(t)\bar{\mu}(t)/[(1+\lambda(t))\sigma_\delta(t)]$. The proof of Proposition 5.3 can be replicated to establish (4.5) (by substituting the last equation in (A.3)) as necessary and sufficient for equilibrium without mispricing. (Necessity follows from the proof of Proposition 5.3, sufficiency from our determination of $r(t)$.) This contradicts the conditions for equilibrium with mispricing. Hence, there indeed exists an equilibrium in all states. *Q.E.D.*

Proof of Proposition 4.2: Agents' beliefs have dynamics (Liptser and Shiryayev (1977), Chapter 12) $d\mu_\delta^i(t) = (v(t)/\sigma_\delta)dW^i(t)$, where $v(t) \equiv v(0)\sigma_\delta^2/(v(0)t + \sigma_\delta^2)$, implying $d\bar{\mu}(t) = -(v(t)/\sigma_\delta)\bar{\mu}(t)dt$, so that $\bar{\mu}(t) = \bar{\mu}(0)\left[\sigma_\delta^2/\left(v(0)t+\sigma_\delta^2\right)\right]^{\sigma_\delta}$. Assume $\bar{\mu}(0) > 0$, so that $\bar{\mu}(t) > 0$, $\forall t$. Then, if $\lambda(t) > 0$, $\forall t$ (verified later), (d,b) is impossible, so only (4.8) is relevant, or $\sigma_\delta\lambda(t)^2 + (\sigma_P\overline{\gamma} + \sigma_\delta - \bar{\mu}(t))\lambda(t) + \sigma_P\overline{\gamma} \le 0$. For (b,d) to arise, we need the left-hand side to have distinct positive roots. This and an analogous argument for $\bar{\mu} < 0$ and (d,b) yield the necessary condition for mispricing. (Sufficiency will be verified later.) Assuming that it holds, (4.8) is equivalent to $\underline{\lambda}(t) \le \lambda(t) \le \overline{\lambda}(t)$, where $\underline{\lambda}(t), \overline{\lambda}(t) = \left[\bar{\mu}(t) - \sigma_P\overline{\gamma} - \sigma_\delta \pm \sqrt{\left(\sigma_P\overline{\gamma} + \sigma_\delta - \bar{\mu}(t)\right)^2 - 4\overline{\gamma}\sigma_P\sigma_\delta}\right]/2\sigma_\delta$. Note that, for any t, $0 < \underline{\lambda}(t) < \overline{\lambda}(t)$. Now, consider the SDE: $d\lambda(t) = b(\lambda(t),t)dt + \sigma(\lambda(t),t)dW^1(t)$, where

$$
\begin{array}{llll}
b(x,t) = (1+x)\overline{\gamma}\sigma_P\left[\frac{1+x}{x}\left(\overline{\gamma}\sigma_P + x\sigma_\delta\right) - \bar{\mu}(t)\right] & \text{if } \underline{\lambda}(t) \le x \le \overline{\lambda}(t)\,, & b(x,t) = 0 & \text{otherwise;} \\
\sigma(x,t) = -(1+x)(\overline{\gamma}\sigma_P + x\sigma_\delta) & \text{if } \underline{\lambda}(t) \le x \le \overline{\lambda}(t)\,, & \sigma(x,t) = -\bar{\mu}(t)x & \text{otherwise.}
\end{array}
$$

If there exists a strictly positive solution, it satisfies all the conditions in Proposition 4.1. For any t, b and σ are continuous in x and bounded as are their first derivatives with respect to x (when these exist; when they do not (at $\underline{\lambda}(t)$ and $\overline{\lambda}(t)$), they have finite left and right limits). Hence, they satisfy Lipschitz and growth conditions in x. Therefore, from Theorems 5.2.5 and 5.2.9 in Karatzas and Shreve (1991), there exists a unique, continuous, strong solution to the SDE. To show that it is strictly positive, observe that, in any state and time, either the economy is in (b,d) and $0 < \underline{\lambda}(t) < \lambda(t) < \overline{\lambda}(t)$, or the economy is in (a,a) and λ follows $d\lambda(t)/\lambda(t) = -\bar{\mu}(t)\,dW^1(t)$, implying $\lambda(t) = \lambda(\tau(t))\exp\left\{-\frac{1}{2}\int_{\tau(t)}^t [\bar{\mu}(s)]^2\,ds - \int_{\tau(t)}^t \bar{\mu}(s)\,dW^1(s)\right\}$, where $\tau(t) \equiv \sup(\{s \in [0,t] : (4.5) \text{ does not hold}\} \cup \{0\})$ (the last "entry time" in (a,a)). Since λ is continuous, $\lambda(\tau(t))$ equals either $\overline{\lambda}(\tau(t)) > 0$ or $\underline{\lambda}(\tau(t)) > 0$ or $\lambda(0) = e^2/e^1 > 0$. Hence, $\lambda > 0$ in (a,a) and (b,d). Sufficiency of the condition for mispricing to occur with positive probability is now apparent for this case of $\bar{\mu} > 0$, because λ will cross $\underline{\lambda}$ or $\overline{\lambda}$ before any future time with positive probability. The case where $\bar{\mu} < 0$ and (a,a) and (d,b) only arise is symmetric (so that, in particular, $\lambda > 0$ in all regions (a,a), (b,d), (d,b)). The technical conditions of Section 2 can be checked easily. *Q.E.D.*

Proof of Proposition 4.3: Applying Itô's lemma to (4.2) yields $\theta^i(t)$, $r^i(t)$, using (4.7), (4.10), (4.13). From Table I, $r(t) = r^1(t)$ in (a,a) and (d,b), $r(t) = r^2(t)$ in (b,d). Consumption volatilities follow from applying Itô's lemma to agents' first order condition. *Q.E.D.*

Proof of Proposition 5.1: The proof is by counterpositive. Assume mispricing never occurs in the constrained economy. Given the composite weight $\Phi^i(t)$ can take on any real value, agents make the same choices as in the unconstrained economy and, if equilibrium exists in the constrained economy, it is identical to the unconstrained one, and $\Phi^i(t) = \bar{\Phi}^i(t)$. For equilibrium to exist in the constrained economy, there must exist $(\pi_S^i(t), \pi_P^i(t))$ that (i) obey the constraints (2.10), (ii) implement agents' risk exposure $\Phi^i = \bar{\Phi}^i$ ((3.2)), and (iii) clear markets ((2.16)). Given the redundancy of S and P, fix $\pi_S^1(t)$ and use (2.10), (3.2) and (2.16) to rewrite the other holdings (and constraints) as a function thereof:

$$\pi_S^1(t) \geq 0\,, \quad \pi_S^2(t) = 1 + \frac{\bar{X}^1(t)}{\bar{X}^2(t)} - \frac{\bar{X}^1(t)}{\bar{X}^2(t)}\pi_S^1(t) \geq 0\,, \tag{A.1}$$

$$\pi_P^1(t) = \frac{\bar{\sigma}_S(t)}{\sigma_P(t)}(\bar{\Phi}^1(t) - \pi_S^1(t)) \leq \overline{\gamma}\,, \quad \pi_P^2(t) = \frac{\bar{X}^1(t)}{\bar{X}^2(t)}\frac{\bar{\sigma}_S(t)}{\sigma_P(t)}(\pi_S^1(t) - \bar{\Phi}^1(t)) \leq \overline{\gamma}\,. \tag{A.2}$$

(A.1)-(A.2) require $1 + \frac{\bar{X}^2(t)}{\bar{X}^1(t)} \geq 0$ (implied by $\lim_{c\to 0} u_i'(c) = \infty$, which ensures $\bar{X}^i(t) > 0$) and

$$-\frac{\bar{X}^2(t)}{\bar{X}^1(t)}\frac{\sigma_P(t)}{\bar{\sigma}_S(t)}\overline{\gamma} \leq \bar{\Phi}^1(t) \leq 1 + \frac{\bar{X}^2(t)}{\bar{X}^1(t)} + \frac{\sigma_P(t)}{\bar{\sigma}_S(t)}\overline{\gamma}\,, \tag{A.3}$$

the negation of (5.1) (after substituting $\bar{\Phi}^1(t)\bar{X}^1(t) + \bar{\Phi}^2(t)\bar{X}^2(t) = \bar{X}^1(t) + \bar{X}^2(t)$, implied by clearing). *Q.E.D.*

Proof of Proposition 5.2: The proof, though lengthy and involved, is an adaptation of Cvitanic and Karatzas (1992). Details can be found in Basak and Croitoru (1998). *Q.E.D.*

Proof of Proposition 5.3: (5.3), (5.4) and (2.15) imply (5.6)-(5.7). Applying Itô's lemma to the definition of λ and using (3.5)-(3.6) and Table II yields (5.9), (5.11), (5.13). Market clearing in P (the second condition in (2.16)), (5.5) and Table II imply (5.8), (5.10) and (5.12).

To prove the converse: (5.3) and (5.4) together with (5.6)-(5.7) imply (2.15). From agent 1's perspective, making use of (2.4), (3.5) and (3.6):

$$\begin{aligned}
&d\left[X^1(t) + X^2(t)\right] = \left[X^1(t)r^1(t) + X^2(t)r^2(t)\right]dt - \left[c^1(t) + c^2(t)\right]dt \\
&+ \left[X^1(t)\Phi^1(t)\sigma_S(t)\theta^1(t) + X^2(t)\Phi^2(t)\sigma_S(t)\theta^2(t)\right]dt \\
&+X^2(t)\Phi^2(t)\sigma_S(t)\bar{\mu}(t)dt + \left[X^1(t)\Phi^1(t) + X^2(t)\Phi^2(t)\right]\sigma_S(t)dW^1(t) \\
&= \left[X^1(t) + X^2(t)\right]r^2(t)dt - \left[c^1(t) + c^2(t)\right]dt + \left[X^1(t)\Phi^1(t) + X^2(t)\Phi^2(t)\right]\sigma_S(t)\theta^1(t)dt \\
&+ \left[X^1(t)\Phi^1(t) + X^2(t)\Phi^2(t)\right]\sigma_S(t)dW^1(t) \\
&+ \left\{X^1(t)\left[r^1(t) - r^2(t)\right] + X^2(t)\Phi^2(t)\sigma_S(t)\left[\theta^2(t) - \theta^1(t) + \bar{\mu}(t)\right]\right\}dt\,.
\end{aligned}$$

In region (a,a), the final dt term is zero, since $r^1 = r^2$ and $\theta^1 = \theta^2 + \bar{\mu}$. In (b,d), Table II, (5.5) and (5.10) imply the final dt term is zero, as well as $r^2(t) = r(t)$. Similarly, in (d,b),

$$\begin{aligned}
d\left[X^1(t) + X^2(t)\right] &= \left[X^1(t) + X^2(t)\right] r(t)dt - \left[c^1(t) + c^2(t)\right] dt \\
&+ \left[X^1(t)\Phi^1(t) + X^2(t)\Phi^2(t)\right] \sigma_S(t) \left[\theta^2(t) + \bar{\mu}(t)\right] dt + \left[X^1(t)\Phi^1(t) + X^2(t)\Phi^2(t)\right] \sigma_S(t) dW^1(t) \, .
\end{aligned}$$

Since, in (a,a) and (b,d), $\theta^1(t) = \left(\mu_S^1(t) - r(t)\right)/\sigma_S(t)$, and in (d,b), $\theta^2(t) + \bar{\mu}(t) = \left(\mu_S^1(t) - r(t)\right)/\sigma_S(t)$, we deduce:

$$\begin{aligned}
d\left[\xi_S^1(t)\left(X^1(t) + X^2(t)\right)\right] &= -\xi_S^1(t)\left(c^1(t) + c^2(t)\right) dt + \Bigg[\xi_S^1(t)\left(X^1(t)\Phi^1(t) + X^2(t)\Phi^2(t)\right)\sigma_S(t) \\
&\quad - \left(X^1(t) + X^2(t)\right)\xi_S^1(t)\left(\frac{\mu_S^1(t) - r(t)}{\sigma_S(t)}\right)\Bigg] dW^1(t) \, . \qquad \text{(A.4)}
\end{aligned}$$

Hence, $\xi_S^1(t)\left(X^1(t) + X^2(t)\right) + \int_0^t \xi_S^1(s)\left(c^1(s) + c^2(s)\right) ds$ is a $\mathcal{P}^1$-martingale satisfying

$$X^1(t) + X^2(t) = \frac{1}{\xi_S^1(t)} E^1\left[\int_t^T \xi_S^1(s)\left(c^1(s) + c^2(s)\right) ds | \mathcal{F}_t^1\right] . \qquad \text{(A.5)}$$

Moreover, from (2.11) and (2.5)

$$d\left[\xi_S^1(t)S(t)\right] = -\xi_S^1(t)\delta(t)dt + \xi_S^1(t)S(t)\left[\sigma_S(t) - \left(\frac{\mu_S^1(t) - r(t)}{\sigma_S(t)}\right)\right] dW^1(t) \, , \qquad \text{(A.6)}$$

implying

$$S(t) = \frac{1}{\overline{\xi}^1(t)} E^1\left[\int_t^T \overline{\xi}^1(s)\delta(s)ds | \mathcal{F}_t^1\right] . \qquad \text{(A.7)}$$

Using good market clearing, (2.15), from (A.5) and (A.7) we deduce bond clearing, the last equality in (2.16). Then, using good and bond clearing, and equating terms of (A.4) and (A.6), we deduce clearing in the composite, i.e.,

$$X^1(t)\Phi^1(t) + X^2(t)\Phi^2(t) = \pi_S^1(t)X^1(t) + \pi_S^2(t)X^2(t) + \frac{\sigma_P(t)}{\sigma_S(t)}\left(\pi_P^1(t)X^1(t) + \pi_P^2(t)X^2(t)\right) = S(t) \, . \qquad \text{(A.8)}$$

In region (a,a), since agents are indifferent between admissible pairs (π_S^i, π_P^i), we can always choose pairs so that $\pi_P^1(t)X^1(t) + \pi_P^2(t)X^2(t) = 0$ (clearing in P), and then (A.8) implies clearing in the stock. In regions (b,d) and (d,b), using Table II and (5.5), (5.10) and (5.12) imply clearing in the derivative, from which (A.8) implies clearing in the stock. *Q.E.D.*

Proof of Proposition 6.1: The determination of the agents' policies and equilibrium prices is similar to Section 4, but it is now necessary to take into account the constraint on Φ^i, and new cases become possible. Neglecting "measure zero" cases, P being favorable requires, besides conditions on $\bar{\mu}(t)$, either $\lambda(t) \leq \frac{\underline{\gamma}}{\overline{\gamma}}$ and $\lambda(t) \geq \frac{\underline{\beta}+1}{\overline{\beta}-1}$, or $\frac{1}{\lambda(t)} \leq \frac{\underline{\gamma}}{\overline{\gamma}}$ and $\frac{1}{\lambda(t)} \geq \frac{\underline{\beta}+1}{\overline{\beta}-1}$, possible only if $\overline{\gamma}(\underline{\beta}+1) < \underline{\gamma}(\overline{\beta}-1)$. S being favorable requires, besides conditions on $\bar{\mu}(t)$, either $\lambda(t) \geq \frac{\underline{\gamma}}{\overline{\gamma}}$ and $\lambda(t) \leq \frac{\underline{\beta}+1}{\overline{\beta}-1}$, or $\frac{1}{\lambda(t)} \geq \frac{\underline{\gamma}}{\overline{\gamma}}$ and $\frac{1}{\lambda(t)} \leq \frac{\underline{\beta}+1}{\overline{\beta}-1}$, possible only if $\overline{\gamma}(\underline{\beta}+1) > \underline{\gamma}(\overline{\beta}-1)$. *Q.E.D.*

References

Basak, S., 1998, "A Model of Dynamic Equilibrium Asset Pricing with Extraneous Risk," working paper, University of Pennsylvania; forthcoming in *Journal of Economic Dynamics and Control*.

Basak, S. and B. Croitoru, 1998, "Capital Market Equilibrium with Mispricing and Arbitrage Activity," working paper, University of Pennsylvania.

Basak, S. and D. Cuoco, 1998, "An Equilibrium Model with Restricted Stock Market Participation," *Review of Financial Studies*, 11, 309-341.

Brennan, M., and E. Schwarz, 1990, "Arbitrage in Stock Index Futures," *Journal of Business*, 63, S7-S31.

Canina, L. and S. Figlewski, 1995, "Program Trading and Stock Index Arbitrage," in R. Jarrow, V. Maksimovic and W. Ziemba (eds.), *Handbooks in Operations Research and Management Science, Vol. 9, Finance*, North-Holland.

Chen, Z., 1995, "Financial Innovation and Arbitrage Pricing in Frictional Economies," *Journal of Economic Theory*, 65, 117-135.

Cox, J., and C.-F. Huang, 1989, "Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion Process," *Journal of Economic Theory*, 49, 33-83.

Cuoco, D. and H. He, 1994, "Dynamic Equilibrium in Infinite-Dimensional Economies with Incomplete Financial Markets," working paper, University of Pennsylvania.

Cuoco, D. and J. Cvitanic, 1998, "Optimal Consumption Choices for a 'Large' Investor," *Journal of Economic Dynamics and Control*, 22, 401-436.

Cvitanic, J. and I. Karatzas, 1992, "Convex Duality in Constrained Portfolio Optimization," *Annals of Applied Probability*, 2, 767-818.

Delgado, F. and B. Dumas, 1994, "How Far Apart Can Two Riskless Interest Rates Be? (One Moves, the Other One Does not)," working paper, University of Pennsylvania.

De Long, B., A. Shleifer, L. Summers and R. Waldmann, 1990, "Noise Trader Risk in Financial Markets," *Journal of Political Economy*, 98, 703-738.

Detemple, J. and S. Murthy, 1994, "Intertemporal Asset Pricing with Heterogeneous Beliefs," *Journal of Economic Theory*, 62, 294-320.

Detemple, J. and S. Murthy, 1997, "Equilibrium Asset Prices and No-Arbitrage with Portfolio Constraints," *Review of Financial Studies*, 10, 1133-1174.

Detemple, J. and L. Selden, 1991, "A General Equilibrium Analysis of Option and Stock Market Interactions," *International Economic Review*, 32, 279-303.

Detemple, J. and A. Serrat, 1998, "An Equilibrium Analysis of Liquidity Constraints," working paper, University of Chicago.

Dow, J. and G. Gorton, 1994, "Arbitrage Chains," *Journal of Finance*, 49, 819-849.

Dumas, B., 1992, "Dynamic Equilibrium and the Real Exchange Rate in a Spatially Separated World," *Review of Financial Studies*, 5, 153-180.

Jarrow, R. and M. O'Hara, 1989, "Primes and Scores: An Essay on Market Imperfections," *Journal of Finance*, 44, 1263-1287.

Jouini, E. and H. Kallal, 1995, "Arbitrage in Securities Markets with Short-Sales Constraints," *Mathematical Finance*, 5, 197-232.

Karatzas, I., J. Lehoczky and S. Shreve, 1987, "Optimal Portfolio and Consumption Decisions for a 'Small Investor' on a Finite Horizon," *SIAM Journal of Control and Optimization*, 25, 1157-1186.

Karatzas, I., J. Lehoczky and S. Shreve, 1990, "Existence and Uniqueness of Multi-Agent Equilibrium in a Stochastic, Dynamic Consumption/Investment Model," *Mathematics of Operations Research*, 15, 80-128.

Karatzas, I. and S. Shreve, 1991, "Brownian Motion and Stochastic Calculus," (2d ed.), Springer-Verlag, New York.

Liptser, R. and A. Shiryayev, 1977, "Statistics of Random Processes," Springer-Verlag, New York.

Neal, R., 1993, "Is Program Trading Destabilizing?," *Journal of Derivatives*, 1, 64-77.

Pontiff, J., 1996, "Costly Arbitrage: Evidence from Closed-End Funds," *Quarterly Journal of Economics*, 111, 1135-1151.

Shleifer, A. and R. Vishny, 1997, "The Limits of Arbitrage," *Journal of Finance*, 52, 35-55.

Tepla, L., 1997, "Optimal Portfolio Policies with Borrowing and Shortsale Constraints," working paper, Stanford University.

Tuckman, B. and J.-L. Vila, 1992, "Arbitrage with Holding Costs: A Utility-Based Approach," *Journal of Finance*, 67, 1283-1302.

Zapatero, F., 1998, "Effects of Financial Innovation on Market Volatility when Beliefs are Heterogeneous," *Journal of Economic Dynamics and Control*, 22, 597-626.

Zigrand, J.-P., 1997, "Arbitrage and Endogenous Market Integration," working paper, University of Chicago.