THEORY OF MASS TRANSPORT INDUCED BY WAVES ON HORIZONTAL BED

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Abstract

The present paper presents a theory for the mass transport induced by waves on a horizontal bed, which is based on Newton's law of viscosity. The boundary conditions are the same as those of the Conduction Solution by Longuet-Higgins.

Predictions from the theory are similar to the Conduction Solution and are in good agreement with the experimental results.

In addition, this theory seems to be very useful in application because of its simplicity and of no extreme limit of applicability in comparison with the Conduction Solution.

Key word : Mass Transport, Wave Current, Sedimentation.

I. INTRODUCTION

Mass transport is an important factor in examining the mechanics of sediment movement.

So far, the problem of mass transport has been treated both theoretically and experimentally by many investigators such as Stokes(1847)[1], Longuet-Higgins (1953)[2], Russell & Osorio(1958)[3], and so on.

Recently, studies of mass transport on a sloping bed and the return flow induced by breaking waves(the undertow) have been based on the extended concept of mass-transport under waves(for example, Svendsen(1984)[4], Tsuchiya et al.(1986)[5], Okayashu et al.(1987)[6], Hirayama (1987,1990~1993)[7)-11)]).

Longuet-Higgins(1953)[2] established his Conduction equation by considering the effect of viscosity on the mass transport. The Conduction Solution, however, has limited applicability ($H/\delta \ll 1$, H:the wave height, δ : the thickness of the boundary layer), but the theory is in common use because it is in agreement with the experiment.

Hirayama(1986)[12] has re-examined the limitation of the applicability of Longuet-Higgins's theory and obtained an improved result. However, the limited applicability of his theory has not been thoroughly elucidated.

The present paper attempts to predict the mass transport on a horizontal bed by an alternative theoretical method. Namely, the basic equation of the theory is based on Newton's law of viscosity, where the internal shear stress within the fluid is additionally represented as a linear function of vertical coordinate(z), following Svendsen(1984)[4].

The consequent predictions are not extremely limited in applicability, are similar to the Conduction Solution, and are in agreement with the experiment.

II. THEORETICAL PREDICTION

In the present paper the theoretical prediction of the mass transport under wave motion is made by a different method from that of Longuet-Higgins(1953). The basic equation is based on viscous model, but the boundary conditions are almost the same as those of Longuet-Higgins.

In addition, in considering the mass conservation, the total mass transport under the wave motion must be zero, because of no flow constraint imposed by a beach.

Also, the average internal flow shear stress($\bar{\tau}$) over one wave period under wave

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motion is represented by a linear function of the vertical coordinate(z), which follows from the equation of motion(see Svendsen, 1984)[4].

(1) Basic Equation

Now the relation between the average internal shear stress($\bar{\tau}$) and the average Lagrangian velocity($\bar{U}$) of fluid over one wave period under wave motion is approximately: (herein, the z-axis is positive in the direction of upper vertical direction from the bottom and is originated at the seabed)

$$\bar{\tau} = \rho\nu\frac{d\bar{u}}{dz} \approx \rho\nu\frac{d\bar{U}}{dz} \tag{1}$$

where ρ:the density of water, ν:the kinematics viscosity of the water, $\bar{u}$: the average Eulerian velocity over one wave period. Next, by the same procedure as that of Svendsen(1984) which follows from the motion of equation, the relation between ($\bar{\tau}$) and z is given as follows:

$$\bar{\tau} = mz + n \tag{2}$$

where m and n are the unknown constants, which are independent of z.

From eqs. (1) and (2)

$$\frac{d\bar{U}}{dz} = \frac{m}{\rho\nu}z + \frac{n}{\rho\nu} \tag{3}$$

Integrating eq.(3) with respect to z yields:

$$\bar{U} = \frac{m}{2\rho\nu}z^2 + \frac{n}{\rho\nu}z + C \tag{4}$$

This eq.(4) is the vertical distribution of the mass transport under wave motion over a horizontal bottom.

(2) Bottom Conditions and Equation of Continuity

In order to determine the three unknown constants(m,n,C) in eq.(4), the following boundary conditions are specified:

(A) Bottom Boundary Condition:

Herein is used the Longuet-Higgins'(1953) result for the mass transport velocity just outside the bottom boundary layer as the bottom boundary condition:

$$\bar{U}|_{z=0} = \frac{5a^2\sigma k}{4\sinh^2(kh)} \tag{5}$$

(a=H/2, σ=2π/T, k=2π/L, T:wave period, L:wave length, h:still water depth).

(B) Surface Boundary Condition:

The surface boundary condition is

$$\left.\frac{d\bar{U}}{dz}\right|_{z=h} = 4a^2\sigma k^2\beta\coth(kh) \tag{6}$$

where β is an arbitrary constant which depends on the degree of vorticity within the surface boundary layer.

$$\beta=1: \quad \left.\frac{d\bar{U}}{dz}\right|_{z=h} = 4a^2\sigma k^2\beta\coth(kh) \tag{7}$$

This equation is also used in the Conduction Solution by Longuet-Higgins (1953). (Notice the change of the sign because the positive direction of the z-axis is different from that of Longuet-Higgins.)

$$\beta=1/2: \quad \left.\frac{d\bar{U}}{dz}\right|_{z=h} = 2a^2\sigma k^2\beta\coth(kh) \tag{8}$$

This equation is the same as the velocity gradient ($d\bar{U}/dz$) shown by Stokes (1847)[1].

(C) Equation of Continuity:

The equation of continuity is

$$\int_0^h \bar{U}dz = 0 \tag{9}$$

(3) Theoretical Results:

Applying the boundary conditions (eqs.(5),(6)) establishes the constants(m,n,C):

$$C = \frac{5}{4}\frac{a^2\sigma k}{\sinh^2(kh)} \tag{10}$$

$$m = \frac{3}{2h}\left\{(4a^2\sigma k^2\beta\coth kh)\times\rho\nu + \frac{5}{4}\frac{a^2\sigma k}{\sinh^2(kh)}\times\frac{2\rho\nu}{h}\right\} \tag{11}$$

$$n = -\frac{1}{2}\left\{(4a^2\sigma k^2\beta\coth kh)\times\rho\nu + \frac{5}{4}\frac{a^2\sigma k}{\sinh^2(kh)}\times\frac{6\rho\nu}{h}\right\} \tag{12}$$

Accordingly, substituting eqs.(10)~(12) into eq.(4) gives:

$$\overline{U} = A_1 z^2 + A_2 Z + A_3 \tag{13}$$

where

$$\left.\begin{aligned} A_1 &= \frac{a^2\sigma k}{4\sinh^2 kh}(6kh\beta\sinh(2kh) + \frac{15}{2})\times\frac{1}{h^2} \\ A_2 &= -\frac{a^2\sigma k}{4\sinh^2 kh}(4kh\beta\sinh(2kh) + 15)\times\frac{1}{h} \\ A_3 &= \frac{a^2\sigma k}{4\sinh^2 kh}\times 5 \end{aligned}\right\} \tag{14}$$

In addition, using a similar description to that of Longuet-Higgins(1953), $\overline{U}$ becomes:

$$\overline{U} = \frac{a^2\sigma k}{4\sinh^2(kh)}[(6kh\beta\sinh(2kh) + \frac{15}{2})\times\frac{z^2}{h^2} - (4kh\beta\sinh(2kh) + 15)\times\frac{z}{h} + 5] \tag{15}$$

The averaged internal shear stress ($\overline{\tau}$) over one wave period becomes:

$$\frac{\overline{\tau}}{\rho\nu a} = \frac{a^2 k}{2\sinh^2 kh}(\frac{6k\beta}{h}\sinh 2kh + \frac{15}{2h^2})z - \frac{a^2 k}{4\sinh^2 kh}(4k\beta\sinh 2kh + \frac{15}{h}) \tag{16}$$

III. THEORETICAL RESULTS AND COMPARISON WITH EXPERIMENTAL ONES

(1) General Characteristics of the Theoretical Results

Fig.1 indicates the variation of the vertical distribution of the values of $\overline{U}$ with T as a parameter, in which the solid curves show the results based on eq.(7) as the surface boundary condition(in what follow these are called Author(1)) and the broken curves the results based on eq.(8)(in what follow these are called Author(2)).

There is a strong dependence on the wave period: the theoretical results near the water surface decrease as the wave period increases, and finally become negative(in the off-shore direction), whereas those near the bottom increase conversely. These causes are due to the fact that the surface condition is proportional to angular frequency(σ)(see eq.(6)) and decreases mass transport near the water surface as the wave period increases. Also as is apparent from the comparison of Author(1) with Author(2) in this same figure, the vertical 'distribution of $\overline{U}$ depends on these surface conditions. These features suggest that the vertical distribution of the mass transport under waves is influenced by the surface conditions such as vorticity within the surface boundary layer.

(2) Comparison with Conduction Solution

Fig.2 shows the comparison of these theoretical values(Author(1)) with those of Conduction Solution with kh as a parameter, in which the solid curves are the results of Author(1), the broken curves those of Conduction Solution. Both theoretical models are in good agreement with each other for all values of kh. Also the velocity of the mass transport near the water surface gradually decreases as the values of kh decreases and becomes negative(in the off-shore direction). Finally, though not shown herein, the difference is affected by the magnitude of the wave heights and periods. Especially in the case of the short wave period and large wave heights, the difference of the values of these theories is very remarkable, whereas in the cases of the value of $T \geq 2.0$s, the result of this theory is in very good agreement with Conduction Solution.

(3) Comparison with the experimental results

Figs.3(1)~(3) show the comparison of these theoretical results with the experiment. Figures(1) and (2) indicate Hirayama's experimental results(1978)[13] and Figure (3) those of Russell & Osorio(1958)[3]. Also shown for comparison are theoretical predictions of Huang(1970)[14]. Since the experimental results of Hirayama(1978) are obtained at about 100 waves after the generation of waves, the condition of the motion of waves is not always considered to be steady. However, it is apparent from these figures that the present theory agrees best with these experiments as the wave height becomes smaller and the wave period longer.

On the other hand, it is apparent from Figure(3) that the present theoretical results (Author(2)) seem to be in good agreement with the experimental results by Russell & Osorio (1958)[3] except parts of the intermediate flow of the flow.

In addition, within the range of these experimental conditions the difference between the present theoretical results and those of Longuet-Higgins is small. This fact suggests that it is better to apply the present theory for calculating the mass transport velocity under waves rather than that of Longuet-Higgins (Conduction Solution), because the present theory is simple and has no extreme limitation of its applicability. Since the Conduction Solution is in good agreement with the present theory except in cases of the short period and the large height of waves, there is no problem in using the Conduction Solution by Longuet-Higgins under waves conditions in common use.

(4)Characteristics of Internal Shear Stress($\bar{\tau}$)

Figs.4(1),(2) show the internal shear stress ($\bar{\tau}$)based on eq.(16). ($\beta=1$: Author(1), $\beta=1/2$: Author(2)). Figure(1) indicates the change of the vertical distribution of the values of ($\bar{\tau}$) under the condition of constant H and h with T as a parameter; the solid curves are the results of Author(1), the broken curves of Author(2). It is apparent from this figure that as the period of curves is shorter, the larger is the absolute values of ($\bar{\tau}$) especially near the bottom and water surface, and also the more outstanding the change of the values of ($\bar{\tau}$) in the vertical direction.

Figure(2) shows the comparison between Author(1) and Author(2) with kh as a parameter. It is apparent from this figure that first, as the values of kh go smaller, the larger become the absolute values of ($\bar{\tau}$) and also the largest near the water surface and bottom. The prediction of Author(1) is generally larger than that of Author(2) and in all curves the values of ($\bar{\tau}$) near the water surface are positive(in the on-shore direction), but negative near the bottom.

IV. CONCLUSIONS

(1) This theoretical result is almost the same as the Longuet-Higgins' Conduction Solution. Especially in cases when the wave period is longer and the wave height is smaller, these theories almost exactly agree with each other.

(2) The present theory is easily established from Newton's law of viscosity and also useful in practice because it has no extreme limit of its applicability like the Conduction Solution and is in fairly good agreement with the experimental results.

(3) Since the vertical distribution of the horizontal mass transport velocity depends especially on the water surface boundary condition, it is essential to exactly estimate the vorticity within the water surface boundary layer.

(4) The theoretical prediction of the internal friction ($\bar{\tau}$) is shown as the linear function of the vertical coordinate(z). The present theoretical results indicate that the shear stress ($\bar{\tau}$) becomes larger near the bottom and the water surface. It is also found that the shear stress near the bottom is negative(in the off- shore direction), whereas near the water surface it is positive (direction on-shore).

(5) The vertical distribution of the mass transport

velocity on the horizontal bed is sufficiently represented by the present theory.

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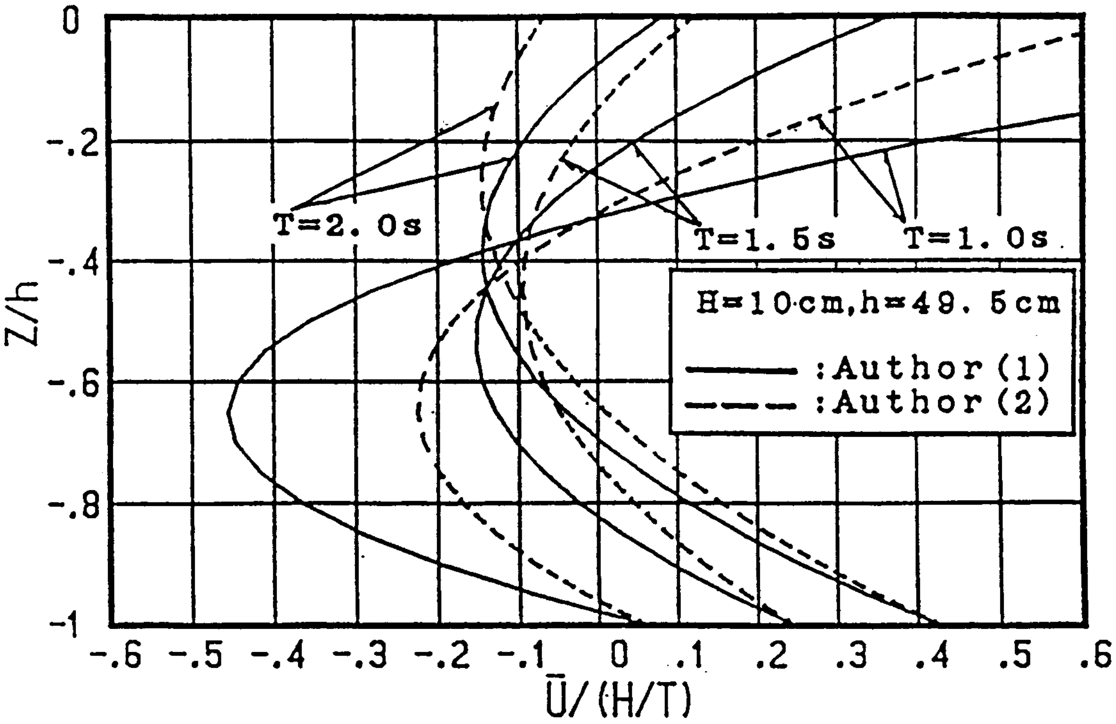

Fig. 1 Change of the vertical distribution of theoretical results due to wave period T

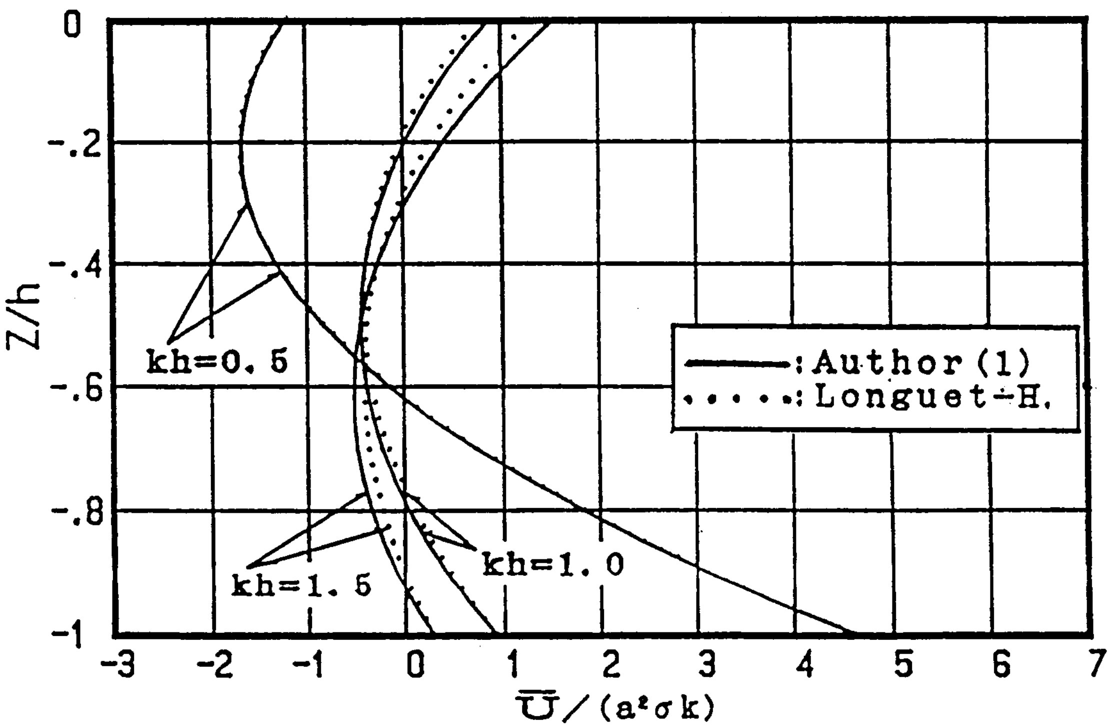

Fig. 2 Comparison of the present theory with Conduction Sol.

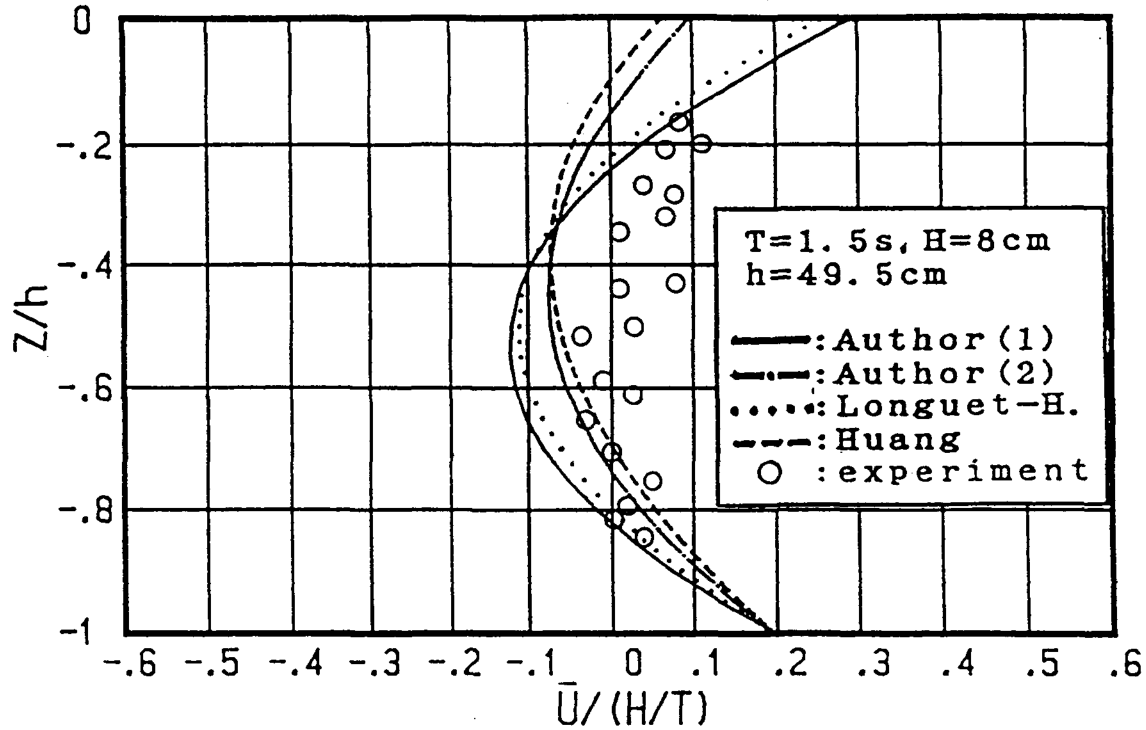

Fig. 3(1) Comparison of the theoretical results with the experimental ones.(T=1.5s)

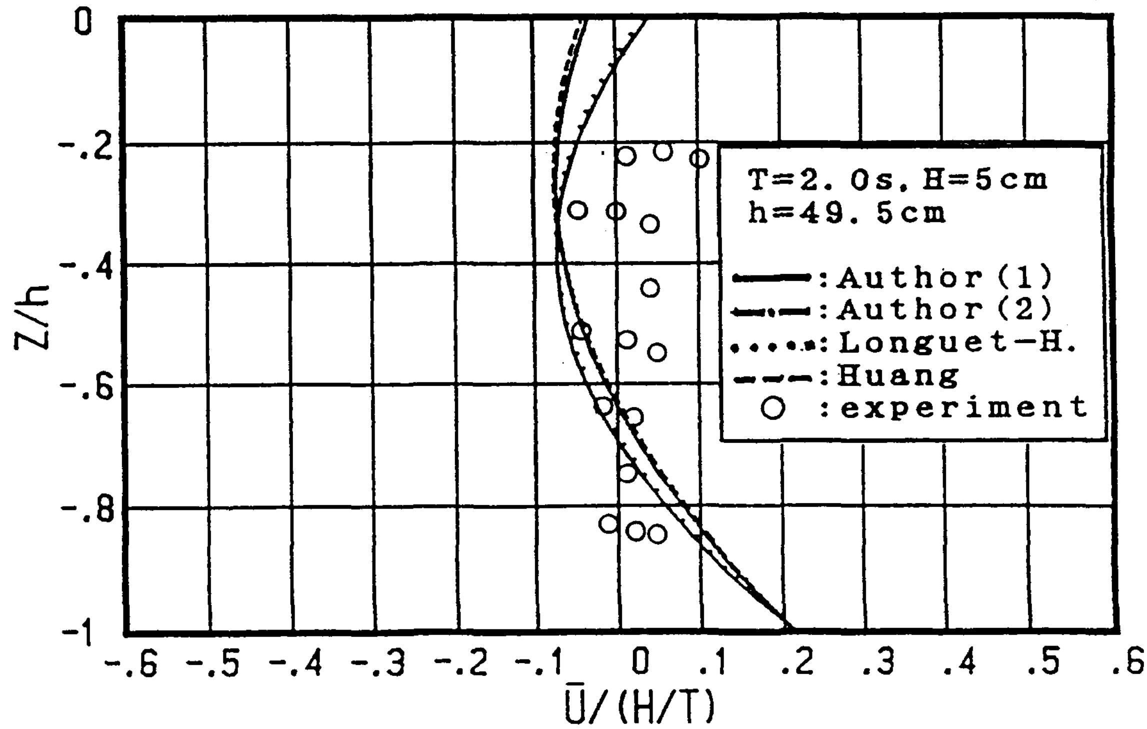

Fig. 3(2) Comparison of the theoretical results with the experimental ones(T=2.0s)

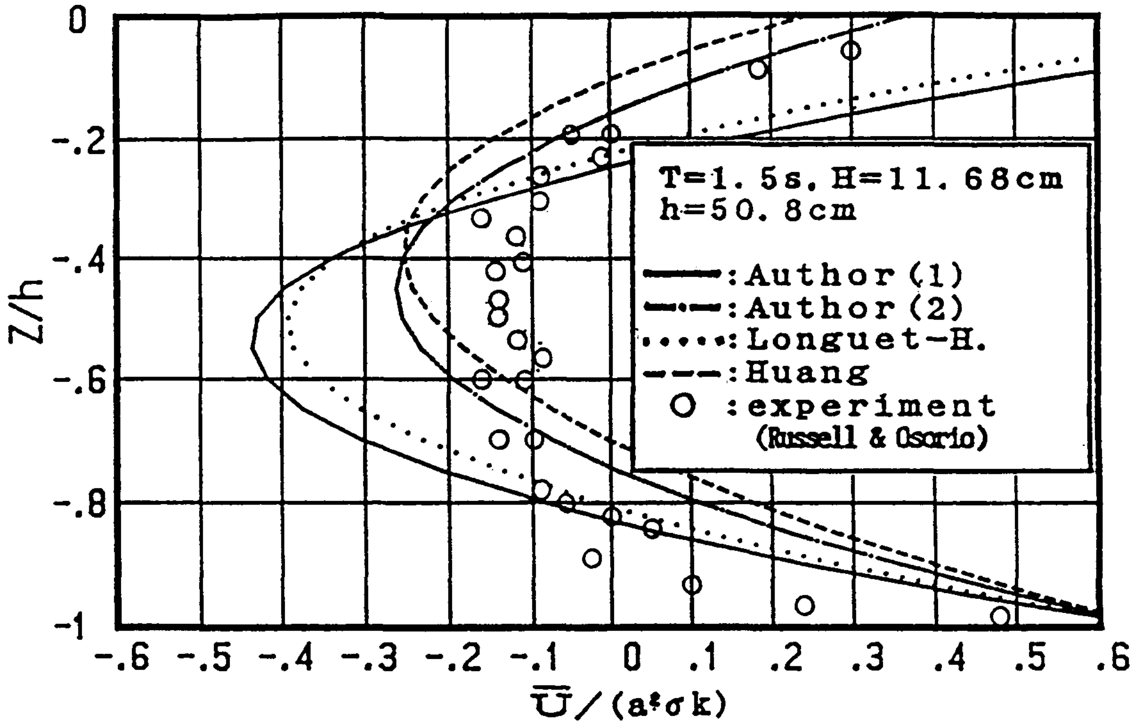

Fig. 3(3) Comparison of the theoretical results with Russell & Osorio's experimental ones.

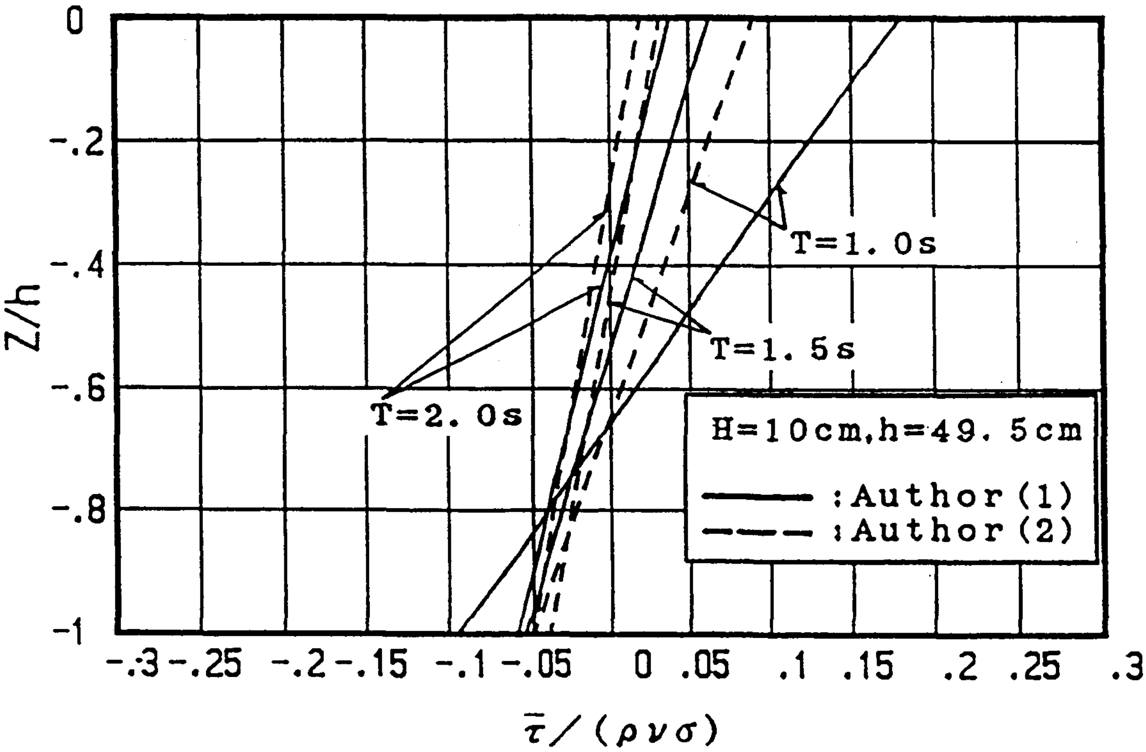

Fig. 4(1) Change of the vertical distribution of the internal shear stress due to the wave period(T)

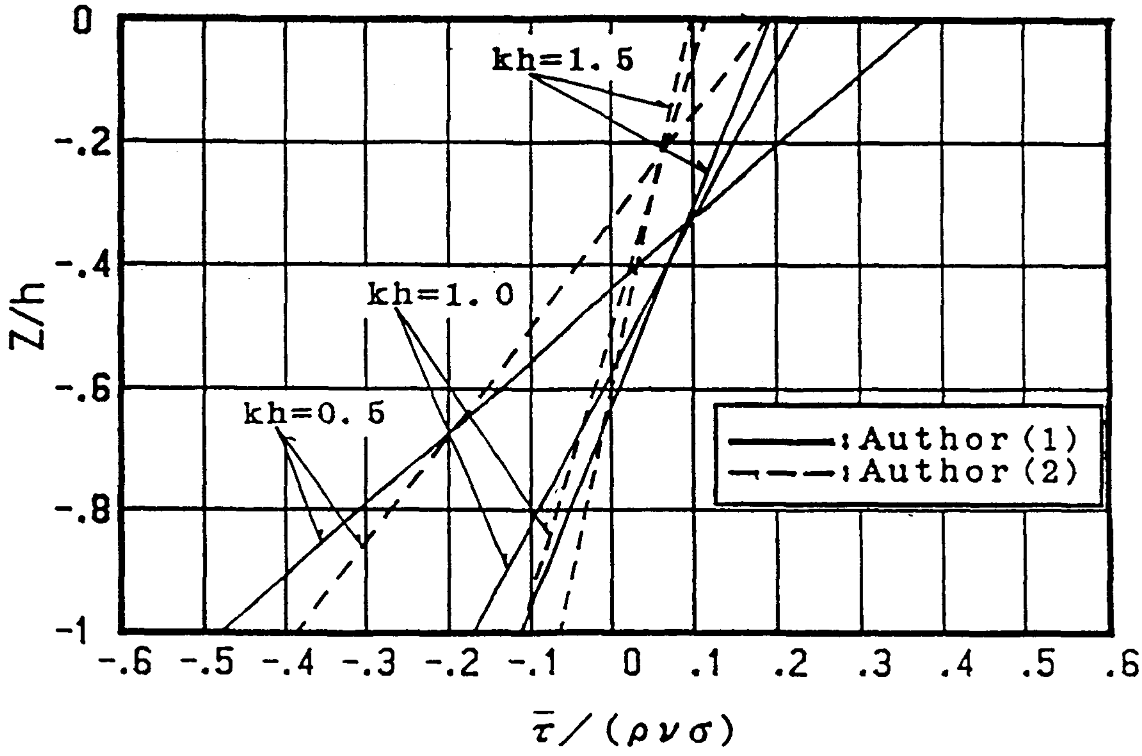

Fig. 4(2) Change of the vertical distribution of the internal shear stress due to the values of **kh**.