# A Study of Branching Phenomena in the Nonautonomous Piecewise Linear Dissipative Systems with Unsymmetrical Restoring Force in the Case of the External Force with a Sum of Several Harmonics

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(Received June 15, 1982)

This is a study of branching phenomena in the nonautonomous second order differential equation with piecewise linear restoring force having unsymmetrical characteristics in the damping system when the excitation is a sum of several harmonics.

In this report the periodic conditions in which the first variational equation has periodic solutions of periods T and 2T (T: least common period of external force) and the branching behavior of the trajectories are obtained.

## 1. Introduction

We have already reported some results as to periodic solutions both a single harmonic excitation and a sum of several harmonics excitation in the preceding articles[1),2),3)] in the case of no damping.

In this paper we discuss the branching phenomena in the nonautonomous piecewise linear systems with unsymmetrical restoring force when damping is present and the external force is a sum of several harmonics.

It is well known that if damping is present, the displacement and the impressed force can be expected to be out of phase,[4)] just as in the case of the corresponding linear problem. Thus the difficulty of treating dissipative systems qualitatively is that this difference in phase must be taken into account. So qualitative analysis of bifurcation problems of damping case in piecewise linear systems has been studied very little.

This paper considers piecewise linear dissipative system with unsymmetrical restoring force in the case of several harmonics excitation, and uses Loud's method[5)] to consider the symmetric restoring force situation and clarifies branching phenomena in connection with the boundary between the stable and unstable harmonic solutions. The procedure for the clarification is as follows:

(i) The method of obtaining the periodicity conditions with its initial values included in order to obtain the periodic solution is given.

(ii) To obtain the stability of periodic solutions, Hill's equation is examined and the conditions for the stable and unstable region boundary are clarified.

(iii) The branching phenomena which occur at the boundary mentioned above are divided into two situations and then analyzed; the behavior of the solutions in the neighborhood of branching point is explained.

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Once this is understood, the numerical analysis follows from the periodicity conditions and its appropriateness will be clarified.

## 2. Periodicity Conditions

In this section the system with restoring force (see Fig. 1) expressed in equation (1) will be considered.

$$\ddot{x}+2\alpha\dot{x}+f(x)=e(t) \tag{1}$$

$$f(x)=\begin{cases} l^2x-K^2x_0 & (x\geqq x_0) \\ k^2x & (x\leqq x_0) \end{cases} \tag{2}$$

where $l^2=k^2+K^2$ ($\cdot$denotes derivative with respect to time $t$). And we assume the external force $e(t)$ and initial conditions as follows:

$$\left.\begin{aligned} e(t)&=\sum_{i=1}^{m} E_i \cos i\omega t \\ e(t)&=e\left(t+\frac{2\pi}{\omega}\right) \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} x(0)&=M \\ \dot{x}(0)&=N \end{aligned}\right\} \tag{4}$$

In this paper the periodic solutions are classified according to the number of times the solution reaches the corner point during the period. For 2n times the solution is designated as ${}_nA$ type solution.

Here we derive the periodicity conditions for ${}_1A$ type harmonic solution in case $M>x_0$ (see Fig. 2) because we shall treat the bifurcation phenomena from ${}_1A$ type harmonic solution. From Fig. 2 we have following equations.

$$\left.\begin{aligned} x_1(t_1)&=x_0 \\ x_2(t_2)&=x_0 \\ x_3(T)&=M \\ \dot{x}_3(T)&=N \end{aligned}\right\} \tag{5}$$

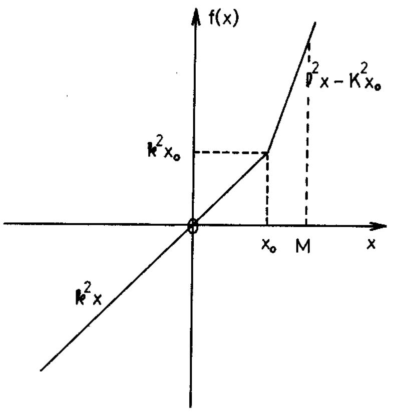



Fig. 1 Restoring force characteristics.

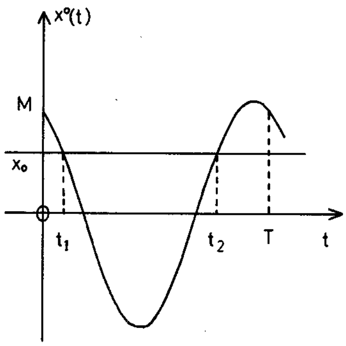



Fig. 2 Periodic solution of type ${}_1$A

where $x_i(t)$ means the solution for the interval $t_{i-1} \leqq t \leqq t_i$ ($i=1, 2, 3$, and $t_0=0$, $t_3=T=2\pi/\omega$) and the solution $x_i(t)$ which reaches the corner point $x_0$ at $t=t_i$ ($i=1, 2$) is connecting the solution $x_{i+1}(t)$ smoothly at every corner point.

Given the system, that is, for given $l$, $k$, $K$, and $x_0$, equations (5) are the periodicity conditions for obtaining ${}_1A$ type harmonic solution and mean the relation among the variables: initial value $M$ and $N$, loss factor $\alpha$, transition time $t_1$, $t_2$, basic frequency $\omega$ of the external force, and amplitude $E_i$ ($i=1, 2, \ldots, m$) of external force. Then if $M$, $N$, $\alpha$, and $E_i$ ($i=1, 2, \ldots, j-1, j+1, \ldots, m$) are known, the remaining elements are obtained, that is to say, $\omega$, $E_j$, $t_1$, and $t_2$ which lead to periodic solution will be found. We assume that the amplitude $E_i$ ($i=1, 2, \ldots, j-1, j+1, \ldots, m$) of the external force is held constant, while the amplitude $E_j (j \neq i)$ is slowly varied and the relations among $M$, $N$, and $E_j$ of the periodic solutions mentioned above are observed in what follows.

Finally, we write down the concrete solutions of equations (5) under the conditions (4). In the following solutions, we set

$$\omega_i = i\omega \qquad \omega_{01}=\sqrt{l^2-\alpha^2} \quad \text{and} \quad \omega_{02}=\sqrt{k^2-\alpha^2} \tag{6}$$

Thus,

$$x_1(t_1)=e^{-\alpha t_1}(A_1 \cos \omega_{01}t_1 + B_1 \sin \omega_{01}t_1) + \sum_{i=1}^{m} \frac{E_i}{(l^2-\omega_i^2)^2+4\alpha^2\omega_i^2} \Big\{ (l^2-\omega_i^2) \cos \omega_i t_1 + 2\alpha\omega_i \sin \omega_i t_1 \Big\} + \frac{K^2}{l^2} x_0 = x_0 \tag{7}$$

where

$$\left.\begin{aligned} A_1 &= M - \sum_{i=1}^{m} \frac{E_i(l^2-\omega_i^2)}{(l^2-\omega_i^2)^2+4\alpha^2\omega_i^2} - \frac{K^2}{l^2} x_0 \\ B_1 &= \frac{1}{\omega_{01}} \Big\{ \alpha M + N - \sum_{i=1}^{m} \frac{\alpha E_i(l^2+\omega_i^2)}{(l^2-\omega_i^2)^2+4\alpha\omega_i^2} - \frac{\alpha K^2}{l^2} x_0 \Big\} \end{aligned}\right\} \tag{8}$$

$$\begin{aligned} x_2(t_2) &= e^{-\alpha(t_2-t_1)} \Big\{ (x_0-p_1) \cos \omega_{02}(t_2-t_1) + \frac{y_1-q_1+\alpha(x_0-p_1)}{\omega_{02}} \sin \omega_{02}(t_2-t_1) \Big\} \\ &\quad + \sum_{i=1}^{m} \frac{E_i}{(k^2-\omega_i^2)^2+4\alpha^2\omega_i^2} \{(k^2-\omega_i^2) \cos \omega_i t_2 + 2\alpha\omega_i \sin \omega_i t_2\} \\ &= x_0 \end{aligned} \tag{9}$$

where

$$\left.\begin{aligned} p_1 &= \sum_{i=1}^{m} \frac{E_i}{(k^2-\omega_i^2)^2+4\alpha^2\omega_i^2} \{(k^2-\omega_i^2) \cos \omega_i t_1 + 2\alpha\omega_i \sin \omega_i t_1\} \\ q_2 &= \sum_{i=1}^{m} \frac{\omega_i E_i}{(k^2-\omega_i^2)^2+4\alpha^2\omega_i^2} \{-(k^2-\omega_i^2) \sin \omega_i t_1 + 2\alpha\omega_i \cos \omega_i t_1\} \end{aligned}\right\} \tag{10}$$

$$\begin{aligned} y_1 &= e^{-\alpha t_1} \{(-\alpha A_1 + \omega_{01} B_1) \cos \omega_{01} t_1 - (\omega_{01} A_1 + \alpha B_1) \sin \omega_{01} t_1\} \\ &\quad - \sum_{i=1}^{m} \frac{\omega_i E_i}{(l^2-\omega_i^2)^2+4\alpha^2\omega_i^2} \{(l^2-\omega_i^2) \sin \omega_i t_1 - 2\alpha\omega_i \cos \omega_i t_1\} \end{aligned} \tag{11}$$

$$y_2=\dot{x}_2(t_2)$$
$$=e^{-\alpha(t_2-t_1)}\left\{(y_1-q_1)\cos\omega_{02}(t_2-t_1)-\frac{\alpha(y_1-q_1)+k^2(x_0-p_1)}{\omega_{02}}\sin\omega_{02}(t_2-t_1)\right\}$$
$$-\sum_{i=1}^{m}\frac{\omega_i E_i}{(k^2-\omega_i^2)^2+4\alpha^2\omega_i^2}\{(k^2-\omega_i^2)\sin\omega_i t_2-2\alpha\omega_i\cos\omega_i t_2\} \tag{12}$$

$$x_3(T)=e^{-\alpha(T-t_2)}\left\{(x_0-p_2)\cos\omega_{01}(T-t_2)+\frac{y_2-q_2+\alpha(x_0-p_2)}{\omega_{01}}\sin\omega_{01}(T-t_2)\right\}$$
$$+\sum_{i=1}^{m}\frac{E_i}{(l^2-\omega_i^2)^2+4\alpha^2\omega_i^2}\{(l^2-\omega_i^2)\cos\omega_i T+2\alpha\omega_i\sin\omega_i T\}$$
$$+\frac{K^2}{l^2}x_0=M \tag{13}$$

$$\dot{x}_3(T)=e^{-\alpha(T-t_2)}\left\{(y_2-q_2)\cos\omega_{01}(T-t_2)-\frac{\alpha(y_2-q_2)+l^2(x_0-p_2)}{\omega_{01}}\sin\omega_{01}(T-t_2)\right\}$$
$$-\sum_{i=1}^{m}\frac{\omega_i E_i}{(l^2-\omega_i^2)^2+4\alpha^2\omega_i^2}\{(l^2-\omega_i^2)\sin\omega_i T-2\alpha\omega_i\cos\omega_i T\}$$
$$=N \tag{14}$$

where

$$\left.\begin{aligned}p_2&=\sum_{i=1}^{m}\frac{E_i}{(l^2-\omega_i^2)^2+4\alpha^2\omega_i^2}\{(l^2-\omega_i^2)\cos\omega_i t_2+2\alpha\omega_i\sin\omega_i t_2\}\\&\quad+\frac{K^2}{l^2}x_0\\q_2&=\sum_{i=1}^{m}\frac{-\omega_i E_i}{(l^2-\omega_i^2)^2+4\alpha^2\omega_i^2}\{(l^2-\omega_i^2)\sin\omega_i t_2-2\alpha\omega_i\cos\omega_i t_2\}\end{aligned}\right\} \tag{15}$$

## 3. Stability

The problem of the infinitesimal stability of the periodic solutions of our nonlinear systems always leads to the equation of Hill's type. The Hill's type equation in these cases is a variational equation characterizing small variations from the given periodic motion whose stability is to be investigated.

A given periodic motion is stable if all solutions of the variational equation associated it are bounded for all positive values of $t$ and unstable if the variational

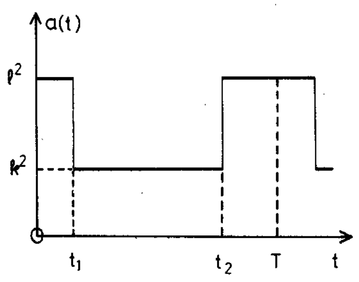



Fig. 3 Coefficient $a$(t) of periodic solution type $_1$A

equation has an unbounded solution.

Let the ${}_1A$ type fundamental solution obtained from the periodicity conditions given in section 2 be denoted by $x^0(t)$. The stability of $x^0(t)$ can be determined by the first variational equation of the solution of equation (1). Let the variation be $y$. The first variational equation is given by

$$\ddot{y}+2\alpha\dot{y}+a(t)y=0 \tag{16}$$

where

$$a(t)\equiv\frac{\partial f}{\partial x}\bigg|_{x=x^0(t)} \tag{17}$$

Also, $a(t)$ is understood to have the following properties:

$$\left.\begin{aligned} &a(t)=a(t+T)\\ &a(t)=\begin{cases} l^2 & (x^0(t)>x_0)\\ k^2 & (x^0(t)<x_0)\end{cases}\end{aligned}\right\} \tag{18}$$

Let the independent solutions of equation (16) be denoted by $\varphi(t)$ and $\psi(t)$ where $\varphi(0)=\dot{\psi}(0)=1$, $\dot{\varphi}(0)=\psi(0)=0$.

Let the two characteristic roots of equation (16) be $\rho_1$ and $\rho_2$. Then

$$\left.\begin{aligned} &\rho_1\rho_2=e^{-2\alpha T},\\ &\rho_1+\rho_2=\varphi(T)+\dot{\psi}(T)\end{aligned}\right\} \tag{19}$$

From equations (19) it is clear that the conditions for the stable and unstable region boundary are followed:

$$\text{(i)}\quad \varphi(T)+\dot{\psi}(T)=-1-e^{-2\alpha T} \tag{20}$$

$$\text{(ii)}\quad \varphi(T)+\dot{\psi}(T)=1+e^{-2\alpha T} \tag{21}$$

Equations (20) and (21) become complicated in comparison with those of no damping case.

Finally, we give the concrete form of equations (20) and (21) in terms of $\alpha$, $t_1$, $t_2$, and $\omega$ when the periodic solution is the fundamental solution type ${}_1A$ as shown in Fig. 2 (or Fig. 3).

$$\begin{aligned}\varphi(T)+\dot{\psi}(T)=e^{-\alpha T}\Big\{&2\cos\omega_{02}(t_2-t_1)\cos\omega_{01}(T-(t_2-t_1))\\ &-\left(\frac{\omega_{02}}{\omega_{01}}+\frac{\omega_{01}}{\omega_{02}}\right)\sin\omega_{02}(t_2-t_1)\sin\omega_{01}(T-(t_2-t_1))\Big\}\end{aligned} \tag{22}$$

## 4. Branching Phenomena

The essential aim of this section is the identification of the branching behavior near bifurcation point of equation (1). In the neighborhood of the ${}_1A$ type harmonic solution obtained by satisfying the initial conditions (4) of equation (1) in section 2, there exist two situations for the periodic solutions of the variational equation as shown in section 3. Each of the two situations presents a different phenomena. Therefore each is investigated individually.

The solution of equation (1) with $x=M$ and $\dot{x}=N$ at $t=0$ is written by $x(t;\ M,\ N,\ E_j)$ and the functions $F$ and $G$ are defined as follows:

$$\left.\begin{aligned} F(M,\ N,\ E_j) &\equiv x(nT;\ M,\ N,\ E_j)-x(0;\ M,\ N,\ E_j) \\ G(M,\ N,\ E_j) &\equiv \dot{x}(nT;\ M,\ N,\ E_j)-\dot{x}(0;\ M,\ N,\ E_j) \end{aligned}\right\} \tag{23}$$

where $n=1$ and 2.
It is now clear that solution $x(t;\ M,\ N,\ E_j)$ has a period $nT$ if and only if

$$F(M,\ N,\ E_j)=G(M,\ N,\ E_j)=0 \tag{24}$$

The point of the $(M,\ N,\ E_j)$ space satisfying $F=G=0$ are in general located on a curve and its differential equation are as follows:

$$\frac{dM}{\begin{vmatrix} F_N & F_{E_j} \\ G_N & G_{E_j} \end{vmatrix}}=\frac{dN}{\begin{vmatrix} F_{E_j} & F_M \\ G_{E_j} & G_M \end{vmatrix}}=\frac{dE_j}{\begin{vmatrix} F_M & F_N \\ G_M & G_N \end{vmatrix}} \tag{25}$$

where $F_M$, $G_M$ etc. denote the partial derivatives of the functions $F$ and $G$ with respect to $M$, $N$, and $E_j$. Any point $(M,\ N,\ E_j)$ for which denominators of equation (25) vanish simultaneously is called a singular point with respect to equation (25). A point $(M,\ N,\ E_j)$ for which denominators do not vanish simultaneously is called an ordinary point.

### 4.1 Situation (I) $\rho=-1$ (n=2)

When the equation (20) is satisfied, let the point satisfying equations (5) be denoted by $(M_0,\ N_0,\ E_{j0})$ and then we have following results.

$$F(M_0,\ N_0,\ E_{j0})=G(M_0,\ N_0,\ E_{j0})=0 \tag{26}$$

$$\left.\begin{aligned} F_M(M_0,\ N_0,\ E_{j0}) &= \varphi(2T)-1 \\ F_N(M_0,\ N_0,\ E_{j0}) &= \psi(2T) \\ F_{E_j}(M_0,\ N_0,\ E_{j0}) &= x_{E_j}(2T) \end{aligned}\right\} \tag{27}$$

$$\left.\begin{aligned} G_M(M_0,\ N_0,\ E_{j0}) &= \dot{\varphi}(2T) \\ G_N(M_0,\ N_0,\ E_{j0}) &= \dot{\psi}(2T)-1 \\ G_{E_j}(M_0,\ N_0,\ E_{j0}) &= \dot{x}_{E_j}(2T) \end{aligned}\right\} \tag{28}$$

where $\varphi(t)$ and $\psi(t)$ are solutions of equation (16), $x_{E_j}(t)$ is the partial derivatives of $x(t;\ M,\ N,\ E_j)$ with respect to $E_j$ and is the evaluation at the point $(M_0,\ N_0,\ E_{j0})$, which is the solution of equation (29).

$$\left.\begin{aligned} \ddot{y}+2\alpha\dot{y}+a(t)y &= \cos j\omega t \\ y(0)=\dot{y}(0) &= 0 \end{aligned}\right\} \tag{29}$$

Then,

$$\left.\begin{aligned} x_{E_j}(t) &= \int_0^t e^{2\alpha\tau}(\varphi(\tau)\psi(t)-\psi(\tau)\varphi(t))\cos j\omega\tau\, d\tau \\ \dot{x}_{E_j}(t) &= \int_0^t e^{2\alpha\tau}(\varphi(\tau)\dot{\psi}(t)-\psi(\tau)\dot{\varphi}(t))\cos j\omega\tau\, d\tau \end{aligned}\right\} \tag{30}$$

In this case we may assume the next relations

$$F_M \cdot F_N \cdot F_{Ej} \neq 0 \qquad G_M \cdot G_N \cdot G_{Ej} \neq 0 \tag{31}$$

Here we may evaluate the denominators of equation (25) at the point ($M_0$, $N_0$, $E_{j0}$), using the relationships

$$\varphi(T)+\dot{\psi}(T)=-1-e^{-2\alpha T} \qquad \varphi(t)\dot{\psi}(t)-\dot{\varphi}(t)\psi(t)=e^{-2\alpha t}$$
$$\varphi(t+T)=\varphi(T)\varphi(t)+\dot{\varphi}(T)\psi(t) \text{ and}$$
$$\psi(t+T)=\psi(T)\varphi(t)+\dot{\psi}(T)\psi(t)$$

Omitting the somewhat long calculations, we write down only the results.

$$\left.\begin{aligned} &F_N \cdot G_{Ej}-F_{Ej} \cdot G_N=0 \\ &F_{Ej} \cdot G_M-F_N \cdot G_{Ej}=0 \\ &F_M \cdot G_N-F_N \cdot G_M=0 \end{aligned}\right\} \tag{32}$$

These indicate that the point ($M_0$, $N_0$, $E_{j0}$) is a geometric singularity of the locus $F=G=0$.

### 4.1.1 *M*-$E_j$ Plane Analysis

We investigate the two-dimensional sections of the three dimensional space ($M$, $N$, $E_j$) which are ($M$, $E_j$), ($N$, $E_j$), and ($M$, $N$). Analysis of branching at an endpoint of an unstable arc of the curve $F=G=0$ involves the computations of several partial derivatives of the function $F(M, N, E_j)$ and $G(M, N, E_j)$ at the point. Since these computations are long and tedious, we shall omit most of them in what follows. In the analysis of ($M$-$E_j$) plane, some details will be given to illustrate the procedure. Since $G_N \neq 0$, the equation $G(M, N, E_j)=0$ can be solved for $N$ as a function of $M$ and $E_j$ near ($M_0$, $E_{j0}$). If the result is $N=H(M, E_j)$, we define the function $J(M, E_j)$ by

$$J(M, E_j) \equiv F(M, H(M, E_j), E_j) \tag{33}$$

It is clear that $J=0$ is seen to satisfy $F=G=0$ in the neighborhood of ($M_0$, $N_0$, $E_{j0}$). Then we have

$$\left.\begin{aligned} &J(M_0, E_{j0})=0 \qquad N_0=H(M_0, E_{j0}) \\ &H_M(M_0, E_{j0})=-\frac{G_M(M_0, N_0, E_{j0})}{G_N(M_0, N_0, E_{j0})} \qquad H_{Ej}(M_0, E_{j0})=-\frac{G_{Ej}(M_0, N_0, E_{j0})}{G_N(M_0, N_0, E_{j0})} \\ &J_M(M_0, E_{j0})=J_{Ej}(M_0, E_{j0})=0 \end{aligned}\right\} \tag{34}$$

The second derivatives computed using the values of the second partial derivatives of $F$ at ($M_0$, $N_0$, $E_{j0}$). It is found that at ($M_0$, $E_{j0}$)

$$J_{MM}=0 \qquad J_{EjEj} \neq 0 \qquad J_{MEj} \neq 0 \tag{35}$$

$$\therefore\ J(M, E_j)=J_{MEj}(M_0, E_{j0})(M-M_0)(E_j-E_{j0})+\frac{1}{2}J_{EjEj}(M_0, E_{j0})(E_j-E_{j0})^2+\ldots \tag{36}$$

Thus $J(M, E_j)=0$ has two branches in the neighborhood of the point ($M_0$, $E_{j0}$). One is connected at $E_j=E_{j0}$ (37) and the other to

$$M-M_0=-\frac{J_{E_jE_j}(M_0,\ E_{j0})}{2J_{ME_j}(M_0,\ E_{j0})}\ (E_j-E_{j0}) \tag{38}$$

On the second branch, the solution curve satisfies $x^0(t)=x^0(t+T)$ which corresponds to the harmonic solution obtained from equations (5). The first branch tangent to $E_j=E_{j0}$ can be investigated correctly by necessarily evaluating the third-order partial derivatives,

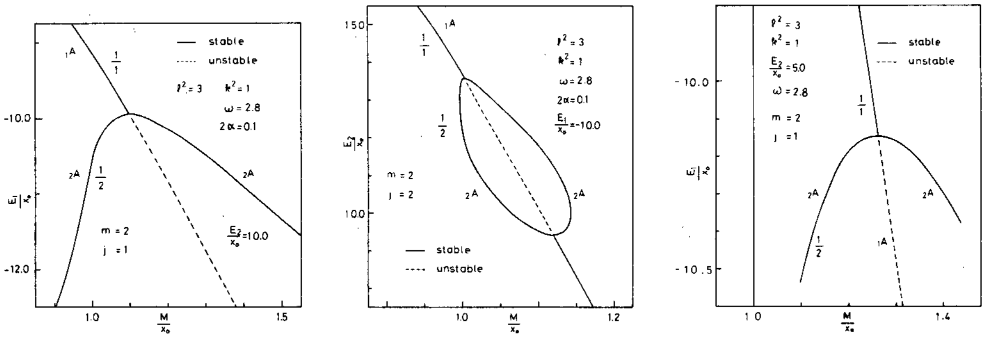
i.e.

$$J(M,\ E_j)=\frac{1}{2}J_{E_jE_j}(M_0,\ E_{j0})\,(E_j-E_{j0})^2+J_{ME_j}(M_0,\ E_{j0})\,(M-M_0)\,(E_j-E_{j0})$$
$$+\frac{1}{6}J_{MMM}(M_0,\ E_{j0})\,(M-M_0)^3+\dots \tag{39}$$

In the neighborhood of the point $(M_0,\ E_{j0})$, $J=0$ becomes

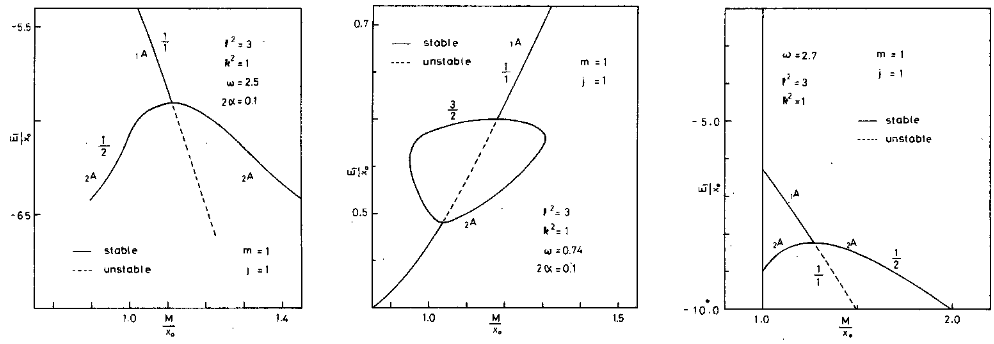
$$E_j-E_{j0}\fallingdotseq-\frac{J_{MMM}(M_0,\ E_{j0})}{6J_{ME_j}(M_0,\ E_{j0})}\ (M-M_0)^2 \tag{40}$$

The first branch becomes the solution of period $2T$ existing only on one side $E_j=E_{j0}$.




(a) $\alpha\neq0,\ n=0,\ j=1$  (b) $\alpha\neq0,\ n=0,\ j=2$  (c) $\alpha=0,\ n=0,\ j=1$

Fig. 4 Branching phenomena of solutions of order $\frac{2n+1}{2}$ $(n=0,\ 1,\ 2,\ \dots)$ from harmonic solutions in M-E$_j$ plane in case $m=2$




(a) $\alpha\neq0,\ n=0,\ j=1$  (b) $\alpha\neq0,\ n=1,\ j=1$  (c) $\alpha=0,\ n=0,\ j=1$

Fig. 5 Branching phenomena of solutions of order $\frac{2n+1}{2}$ $(n=0,\ 1,\ 2,\ \dots)$ from harmonic solutions in M-E$_j$ plane in case $m=1$

(Although it is difficult to determine the sign of $J_{MMM}(M_0, E_{j0})$, in general we take $J_{MMM}(M_0, E_{j0}) \neq 0$.)

The qualitative analysis has been given above. Numerical results in the branching point neighborhood using the periodicity conditions from equations (5) are shown in Fig. 4 (a), (b), ($\alpha \neq 0$), (c) ($\alpha = 0$) in case $m=2$, and Fig. 5 (a), (b), ($\alpha \neq 0$), (c) ($\alpha = 0$) in case $m=1$ and are very much in agreement with analysis given above.

### 4.1.2 $N$-$E_j$ Plane Analysis

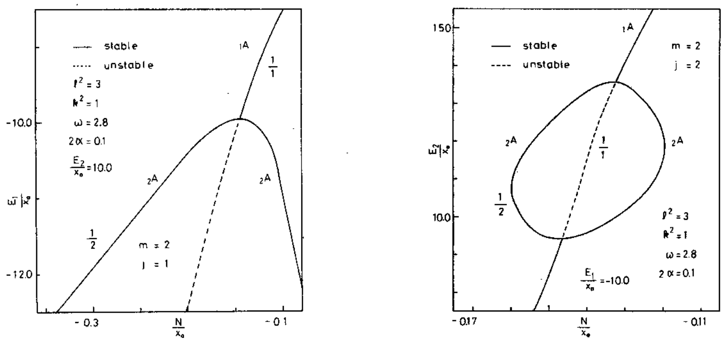
Since $G_M \neq 0$, the equation $G(M, N, E_j)=0$ can be solved for $M$ as a function of $N$ and $E_j$ near $(N_0, E_{j0})$. For $M$ the solution can be given as $M=H(N, E_j)$. Then we define the function

$$J(N, E_j) \equiv F(H(N, E_j), N, E_j) \tag{41}$$

In the same manner as in section 4.1.1 we have

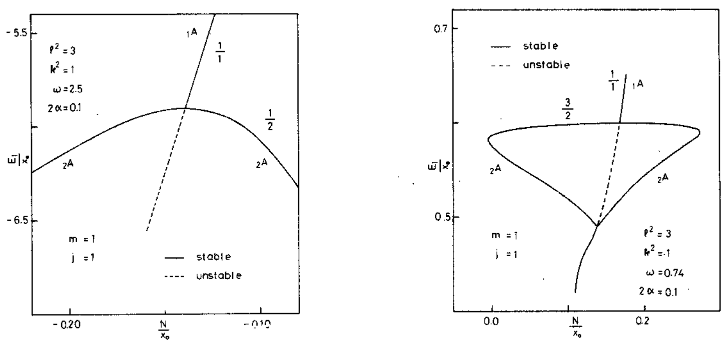
$$E_j - E_{j0} = 0 \tag{42}$$

$$J_{NE_j}(N_0, E_{j0})(N-N_0) + \frac{1}{2} J_{E_jE_j}(N_0, E_{j0})(E_j - E_{j0}) = 0 \tag{43}$$




(a) $\alpha \neq 0, n=0, j=1$ (b) $\alpha \neq 0, n=0, j=2$

Fig. 6 Branching phenomena of solutions of order $\frac{2n+1}{2}$ ($n=0, 1, 2, \ldots$) from harmonic solutions in N-Ej plane in case $m=2$




(a) $\alpha \neq 0, n=0, j=1$ (b) $\alpha \neq 0, n=1, j=1$

Fig. 7 Branching phenomena of solutions of order $\frac{2n+1}{2}$ ($n=0, 1, 2, \ldots$) from harmonic solutions in N-Ej plane in case $m=1$

The first branch tangent to $E_j = E_{j0}$ means the solution of period $2T$ and further investigation gives

$$E_j - E_{j0} \doteqdot -\frac{J_{NNN}(N_0, E_{j0})}{6J_{NE_j}(N_0, E_{j0})}(N-N_0)^2 \tag{44}$$

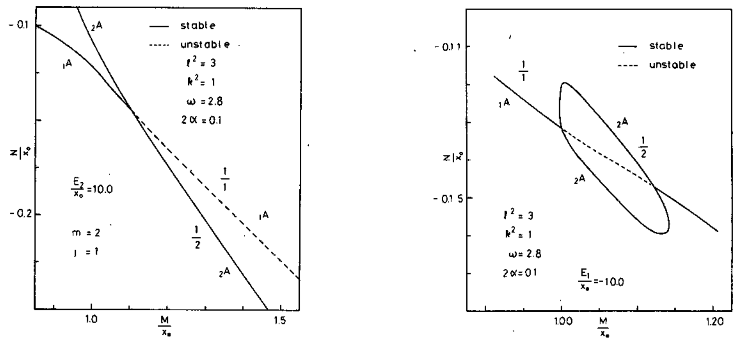
The second branch is the fundamental solution obtained from equations (5). Numerical results are shown in Fig. 6 (a), (b) in case $m=2$ and Fig. 7 (a), (b) when $m=1$ and give the exact explanation of qualitative analysis.

### 4.1.3 *M-N* Plane Analysis

Similar investigations as to $(M-N)$ plane can be used in connection with the other sections $(M-E_j)$ and $(N-E_j)$ of the three-dimensional space of parameters. We define the function

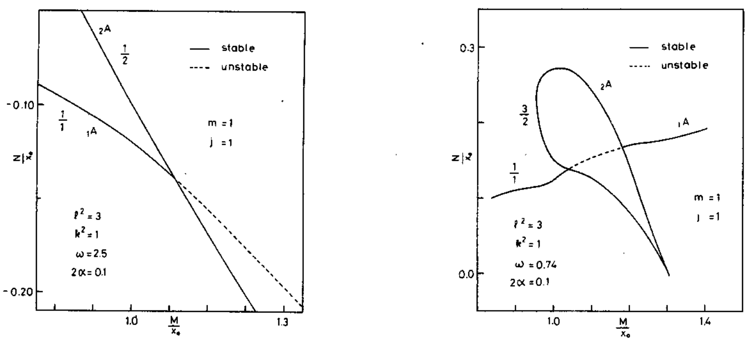
$$J(M, N) \equiv F(M, N, H(M, N)) \tag{45}$$

where $E_j = H(M, N)$ is the solution of equation $G(M, N, E_j) = 0$ in the neighborhood $(M_0, N_0, E_{j0})$, just as in 4.1.1.




(a) $\alpha \neq 0, n=0, j=1$ (b) $\alpha \neq 0, n=0, j=2$

Fig. 8 Branching phenomena of solutions of order $\frac{2n+1}{2}$ $(n=0, 1, 2, \ldots)$ from harmonic solutions in M-N plane in case $m=2$




(a) $\alpha \neq 0, n=0, j=1$ (b) $\alpha \neq 0, n=1, j=1$

Fig. 9 Branching phenomena of solutions of order $\frac{2n+1}{2}$ $(n=0, 1, 2, \ldots)$ from harmonic solutions in M-N plane in case $m=1$

After long and tedious calculations, we have two branches;

$$\left.\begin{aligned} &J_{MM}(M_0, N_0)(M-M_0)+(J_{NN}(M_0, N_0)+\sqrt{\Delta})(N-N_0)=0 \\ &J_{MM}(M_0, N_0)(M-M_0)+(J_{NN}(M_0, N_0)-\sqrt{\Delta})(N-N_0)=0 \end{aligned}\right\} \tag{46}$$

where

$$\Delta=J_{MN}^2(M_0, N_0)-J_{MM}(M_0, N_0)J_{NN}(M_0, N_0)>0 \tag{47}$$

Numerical results are shown in Fig. 8 (a), (b) ($m=2$) and Fig. 9 (a), (b) ($m=1$).

### 4.2 Situation (II) $\rho=1$ (n=1)

When the equation (21) is satisfied, let the point satisfying equations (5) be denoted by $(M_0, N_0, E_{j0})$ and then we have

$$F(M_0, N_0, E_j)=G(M_0, N_0, E_{j0})=0 \tag{48}$$

$$\left.\begin{aligned} &F_M(M_0, N_0, E_{j0})=\varphi(T)-1 \\ &F_N(M_0, N_0, E_{j0})=\psi(T) \\ &F_{E_j}(M_0, N_0, E_{j0})=x_{E_j}(T) \\ &G_M(M_0, N_0, E_{j0})=\dot{\varphi}(T) \\ &G_N(M_0, N_0, E_{j0})=\dot{\psi}(T)-1 \\ &G_{E_j}(M_0, N_0, E_{j0})=\dot{x}_{E_j}(T) \end{aligned}\right\} \tag{49}$$

In the denominators of equation (25) we say in this case

$$\left.\begin{aligned} &F_N\cdot G_{E_j}-F_{E_j}\cdot G_N\neq 0 \\ &F_{E_j}\cdot G_M-F_M\cdot G_{E_j}\neq 0 \\ &F_M\cdot G_N-F_N\cdot G_M=0 \end{aligned}\right\} \tag{50}$$

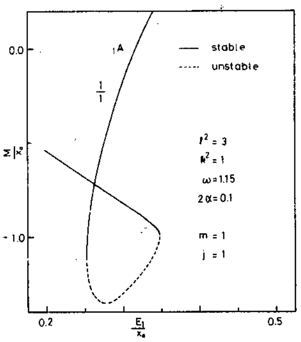
This indicates that the point $(M_0, N_0, E_{j0})$ is not a geometric singularity of the locus $F=G=0$. In a similar manner we have

$$\left.\begin{aligned} &E_j=E_{j0} \quad \text{and} \\ &M-M_0+C(N-N_0)=0 \end{aligned}\right\} \tag{51}$$

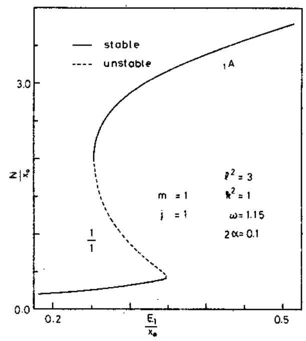
where

$$C=\left.\frac{F_N\cdot G_{E_j}-F_{E_j}\cdot G_N}{F_{E_j}\cdot G_M-F_M\cdot G_{E_j}}\right|_{(M_0, N_0, E_{j0})} \tag{52}$$

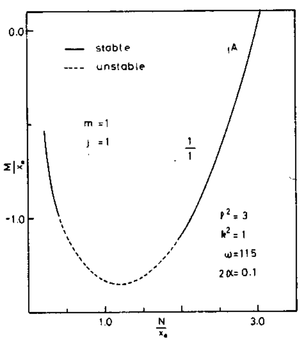
If we vary $E_j$ as to increase through $E_{j0}$ there will be two periodic solutions for $E_j$ immediately less than $E_{j0}$ and none for $E_j$ immediately greater than $E_{j0}$ if $E_j$ is maximum at the $(M_0, N_0, E_{j0})$ point. As $E_j$ increases through $E_{j0}$ the two solutions coalesce and disappear. If $E_j$ has a minimum at the situation (II) point, there are two solutions for $E_j$ immediately greater than $E_{j0}$ and none for $E_j$ immediately less than $E_{j0}$, the two solutions coalescing and disappearing as $E_j$ decreases through $E_{j0}$. When there are two solutions, one is stable and the other is unstable. The branching which occur at a situation (II) point is thus the well known jump phenomenon for which a stable periodic solution coalesces with an unstable solution and disappears. Numerical results are shown in Fig. 10 (a)–(d) ($m=1$).



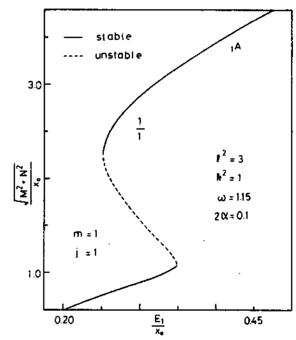

(a) M-Ej plane analysis




(b) N-Ej plane analysis




(c) M-N plane analysis




(d) $\sqrt{M^2+N^2}$ -Ej characteristics

Fig. 10 Jump phenomena in case $m=1$

## 5. Conclusions

In the previous sections, analyses of the phenomena for the simplest piecewise linear system with unsymmetrical restoring force are present when the excitation is a sum of several harmonics in the damping case, in the same manner as in a single harmonic excitation.

These are given as follows:

(i) For the piecewise linear system, the periodicity conditions are clarified, just as in no damping.

(ii) For the piecewise linear system, the condition that the solutions of the first variational equation become periodic solutions are given in two types (cf. four types in no damping).

(iii) The branching behavior of $(2n+1)/2$-harmonics ($n=0, 1, 2, 3, \ldots$) is the same as that of dissipationless systems.

(iv) As to the jump phenomena the same results are given as that of system without damping.

The future work includes investigation on the difference of the analysis 1/4-fraction subharmonics between the dissipative and dissipationless systems. The next report will deal with the bifurcation of the solutions of period 4T from the 2T-periodic solutions of equation (1).

Finally, it is noted that numerical calculations were performed by using ACOS-700 at the computer center, University of Osaka Prefecture.

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