Title	Current Distribution on a Coplanar Radial Scatterer
Author(s)	Kominami, Masanobu; Tachibana, Kaoru; Mori, Shizuo; Rokushima, Katsu
Editor(s)	
Citation	Bulletin of University of Osaka Prefecture. Series A, Engineering and natural sciences. 1975, 23(2), p.127-137
Issue Date	1975-03-31
URL	http://hdl.handle.net/10466/8244
Rights	

Current Distribution on a Coplanar Radial Scatterer

Masanobu KOMINAMI*, Kaoru TACHIBANA**,
Shizuo MORI* and Katsu ROKUSHIMA*

(Received November 15, 1974)

The current distribution on a coplanar radial scatterer illuminated by plane wave is investigated. Hallén-type integral equations for the scatterer are given and solved numerically by use of polynomial approximation of currents.

Double integrals appeared in the integral equations are reduced to mere integrals using Fourier transform, which remarkably shorten the computational time in numerical calcurations.

1. Introduction

Detailed investigations in the scattering of electromagnetic waves are very important in radar reflectivity techniques. Problems of scattering from arbitrary configurations of wires were treated in general [1]. Scattered waves are not polarized due to the coupling of induced currents on the wires, so this is an interesting ploblem in radar reflection. Two perpendicular crossed wires and a pair of skew crossed wires are simple examples of this ploblem and have already been studied in several literatures [2], [3]. Therefore, in this paper, a coplanar radial scatterer having a more generalized configuration is studied.

In the present analysis, the coupled integral equations are obtaind first upon the electric scalar potential and the magnetic vector potential. Although it is almost impossible to solve these integral equations rigorously, many numerical methods such as Point-Matching Method, Galerkin Method and Moment-Method, have been reported by many authors.

Here we use Popovic's method which approximates the current in a form of a polynomial because this method is suitable for the calculation using a middle size computer due to its mathematical simplicity and high accuracy with the least number of terms.

2. Analysis

Consider a coplanar radial scatterer illuminated by an incident plane wave as shown in Fig. 1. It consists of N radial elements of length h_i and radius a_i ($i=1, 2, \cdots, N$) and is on the $x-y$ plane. Let the i-th element be placed at an angle α_{ij} with the j-th element. We assume that (1) the element radii are very small compared with the wavelength, (2) every current flows along its element axis and (3) all

*Department of Electrical Engineering, Colledge of Engineering.
**Graduate Student, Department of Electrical Engineering, Colledge of Engineering.

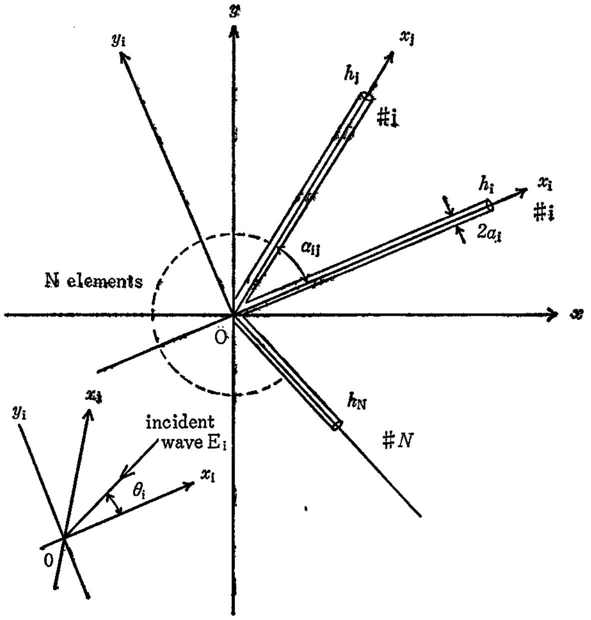

Fig. 1 A coplanar radial scatterer illuminated by plane wave in $x-y$ plane.

elements are made of perfect conductors.

The total tangential component of the electric field must be zero at all points of the scatterer surface. Therefore, on the surface of the i-th element.

$$E^i_{x_i}+E_{x_i}=0, \tag{1}$$

where $E^i_{x_i}$; x_i component of the incident electric field.

E_{x_i} ; x_i component of the electric field resulting from currents induced on the scatterer.

Since induced currents are parallel to the $x-y$ plane, the magnetic vector potential has only x and y component, and the following expression is readily obtained:

$$\frac{\partial^2 A_{x_i}}{\partial x_i}+\beta^2 A_{x_i}=-j\frac{\beta^2}{\omega}E^i_{x_i}-\frac{\partial}{\partial x_i}\left(\frac{\partial A_{y_i}}{\partial y_i}\right), \tag{2}$$

$$(i=1,\ 2,\ \cdots,\ N)$$

where $\beta=2\pi/\lambda$ is the phase constant, λ is the wavelength in the free space and ω is the angular frequency.

The general solution of the nonhomogeneous differential equation (2) consists of a sum of a particular integral and a complementary function as follows:

$$A_{x_i}(x_i)=\frac{-j}{c}\{C_i\cos\beta x_i+B_i'\sin\beta x_i+v_{x_i}(x_i)\}-\frac{1}{\beta}\int_0^{x_i}\frac{\partial}{\partial\xi}\left(\frac{\partial A_{y_i}(\xi)}{\partial y_i}\right)$$

$$\cdot\sin\beta(x_i-\xi)d\xi, \tag{3}$$

$$v_{x_i}(x_i)=\int_0^{x_i}E^i_{x_i}\sin\beta(x_i-\xi)d\xi, \tag{4}$$

where C_i and B_i' are arbitrary constants of integration and c is the velocity of light.

The factor $-j/c$ is introduced in Eq. (3) to make C_i, B_i' and $v_{x_i}(x_i)$ dimensionally voltages.

With an integration by parts of the last term, Eq. (3) may be simplified to

$$A_{x_i}(x_i)=\frac{-j}{c}\{C_i\cos\beta x_i+B_i\sin\beta x_i+v_{x_i}(x_i)\}-\int_0^{x_i}\frac{\partial A_{y_i}(\xi)}{\partial y_i}\cos\beta(x_i-\xi)d\xi, \tag{5}$$

where

$$B_i=B_i'+j\frac{c}{\beta}\;\frac{\partial A_{y_i}(0)}{\partial y_i}. \tag{6}$$

After differentiating A_{x_i} of Eq. (5) with respect to x_i and rearrenging the terms in consideration of Lorentz condition, Eq. (5) becomes

$$\frac{\partial A_{x_i}(x_i)}{\partial x_i}+\frac{\partial A_{y_i}(x_i)}{\partial y_i}=\frac{-j\beta}{c}\left\{-C_i\sin\beta x_i+B_i\cos\beta x_i+\frac{1}{\beta}\frac{\partial v_{x_i}(x_i)}{\partial x_i}\right\}$$
$$+\beta\int_0^{x_i}\frac{\partial A_{y_i}}{\partial y_i}\sin\beta(x_i-\xi)d\xi=-j\frac{\beta^2}{\omega}\phi_i(x_i), \tag{7}$$

where $\phi_i(x_i)$ is the scalar potential due to all charges on the conducter elements.

At the junction $x_i=0$, Eq. (7) yields

$$B_i=\phi_i(0)\qquad,\qquad(i=1,\ 2,\ \cdots,\ N). \tag{8}$$

All elements must be at the same potential at the junction, which implies

$$B_i=\phi_i(0)=B',\ (i=1,\ 2,\ \cdots,\ N). \tag{9}$$

i. e., each B_i equals to the scalar potential B' at $x_i=0$, where all the elements join.

Thus Eq. (5) becomes

$$A_{x_i}(x_i)=\frac{-j}{c}\{C_i\cos\beta x_i+B'\sin\beta x_i+v_{x_i}(x_i)\}$$
$$-\int_0^{x_i}\frac{\partial A_{y_i}}{\partial y_i}\cos\beta(x_i-\xi)d\xi. \tag{10}$$

On the other hand, expressions for the components of the vector in Eq. (10) are

$$A_{x_i}(x_i)=\sum_{j=1}^{N}A_{x_{ij}}(x_i) \tag{11}$$

$$A_{y_i}(x_i)=\sum_{j=1}^{N}A_{y_{ij}}(x_i), \tag{12}$$

where

$$\begin{matrix}A_{x_{ij}}(x_i)\\A_{y_{ij}}(x_i)\end{matrix}=\frac{\mu_0}{4\pi}\int_0^{h_j}I_j(x_j')\Psi_{ij}(x_i,\ x_j')dx_j'\cdot\begin{matrix}\cos\alpha_{ij}\\\sin\alpha_{ij}\end{matrix} \tag{13}$$

$$\Psi_{ij}(x_i,\ x_j')=e^{-j\beta R_{ij}(x_i,\ x_j')}/R_{ij}(x_i,\ x_j') \tag{14}$$

$$R_{ij}(x_i,\ x_j')=\sqrt{(x_i-x_j'\cos\alpha_{ij})^2+(x_j'\sin\alpha_{ij})^2+a_i^2} \tag{15}$$

μ_0 ; permiability of free space

$I_j(x_j')$; current induced on the j-th element.

$\frac{\partial A_{y_i}}{\partial y_i}$ is also given as follows:

$$\frac{\partial A_{y_i}}{\partial y_i}=\sum_{j=1}^{N}\frac{\mu_0}{4\pi}\sin^2\alpha_{ij}\int_0^{h_j}I_j(x_j')\frac{e^{-j\beta R_{ij}}}{R^3_{ij}}(1+j\beta R_{ij})x_j'dx_j'. \tag{16}$$

Substituting Eqs. (11) and (16) into Eq. (10), the integral equation for the current $I_j(x_j')$, which is an extension of the Batler's [3] for an arbitrary number of element wires, is obtained as follows:

$$\frac{-j}{c}\{C_i\cos\beta x_i+B'\sin\beta x_i+v_{x_i}(x_i)\}$$
$$+\frac{\mu_0}{4\pi}\sum_{j=1}^{N}\int_0^{h_j}I_j(x_j')G_{ij}(x_i,\ x_j')dx_j'=0, \quad (i=1,\ 2,\ \cdots,\ N) \tag{17}$$

where

$$G_{ij}(x_i,\ x_j')=\cos\alpha_{ij}\Psi_{ij}(x_i,\ x_j')+\sin^2\alpha_{ij}x_j'\int_0^{x_i}\frac{e^{-j\beta R_{ij}}}{R_{ij}^3}(1+j\beta R_{ij})\cos\beta(x_i-\xi)d\xi. \tag{18}$$

Instead of the integral in Eq. (18), we use the following equation (see Appendix):

$$\int_0^{x_i}\frac{e^{-j\beta R_{ij}}}{R_{ij}^3}(1+j\beta R_{ij})\cos\beta(x_i-\xi)d\xi$$
$$=\frac{1}{(x_j'\sin\alpha_{ji})^2+a_i^2}\Big[\cos\beta x_i\frac{x_j'\cos\alpha_{ij}}{\sqrt{x_j'^2+a_i^2}}e^{-j\beta\sqrt{x_j'^2+a_i^2}}$$
$$+j\sin\beta x_i e^{-j\beta\sqrt{x_j'^2+a_i^2}}+(x_i-x_j'\sin\alpha_{ij})\frac{e^{-j\beta R_{ij}}}{R_{ij}}. \tag{19}$$

With this relation, the computational time is remarkably shortened. At $x_i=0$, evaluation of Eq. (17) yields

$$C_i=-j\,30\sum_{j=1}^{N}\int_0^{h_j}I_j(x_j')G_{ij}(0,\ x_j')dx_j'$$
$$=-j\,30\sum_{j=1}^{N}\int_0^{h_j}I_j(x_j')\Psi_{ij}(0,\ x_j')dx_j'\cdot\cos\alpha_{ij}. \tag{20}$$

We assume the incident electric field as shown in Fig. 1.

Using Eq. (4),

$$v_{x_i}(x_i)=\frac{E^i}{\beta}\cos\theta_i\cdot(1-\cos\beta x_i). \tag{21}$$

Upon substitution of of these into Eq. (17), we arrive at the desired integral equation:

$$\sum_{j=1}^{N}\int_0^{h_j}I_j(x_j')K_{ij}(x_i,\ x_j')dx_j'+B\sin\beta x_i=\frac{jE^i}{30\beta}\cos\theta_i\cdot(1-\cos\beta x_i), \quad (i=1,\ 2,\ \cdots,\ N) \tag{22}$$

where

$$K_{ij}(x_i,\ x_j')=\cos\beta x_i G_{ij}(0,\ x_j')-G_{ij}(x_i,\ x_j') \tag{23}$$

$$B=-\frac{j}{30}B'. \tag{24}$$

3. Numerical Solution

Accordind to Popovic's method, we assume that $I_j(x_j')$ can be represented in the form of a polynomial of the m-th order:

$$I_j(x_j')=\sum_{m=1}^{M}\varphi_{jm}\Big(1-\frac{x_j'}{h_j}\Big)^m, \tag{25}$$

where φ_{jm} are complex coefficients to be determined and M is a desired order of the polynomial approximation. Note that Eq. (25) satisfies the boundary conditions at wire ends:

$$I_j(h_j)=0 \quad (j=1,\ 2,\ \cdots,\ N). \tag{26}$$

If we substitute Eq. (25) into Eq. (22), the following approximate equation is obtained:

$$\sum_{j=1}^{N}\sum_{m=1}^{M}\int_0^{h_j}\varphi_{jm}\left(1-\frac{x_j'}{h_j}\right)^m K_{ij}(x_i,\ x_j')dx_j'+B\sin\beta x_i=\frac{jE^i}{30\beta}\cos\theta_i\cdot(1-\cos\beta x_i),$$
$$(i=1,\ 2,\ \cdots,\ N). \tag{27}$$

Of the NM coefficients φ_{jm} ($j=1,\ \cdots,\ N,\ m=1,\ \cdots,\ M$), only ($NM-1$) are independent, because they must satisfy the boundary condition at the junction (no charge built-up):

$$\sum_{j=1}^{N} I_j(0)=\sum_{j=1}^{N}\sum_{m=1}^{M}\varphi_{jm}=0. \tag{28}$$

To determine the ($NM-1$) coefficients φ_{jm} and the constant B of the approximate polynomial current distribution given by Eq. (27), we choose NM equidistant points along the elements and stipulate that Eq. (27) be hold at these points. This results in NM complex linear equations of ($NM-1$) complex unknowns and B:

$$\sum_{j=1}^{N}\sum_{m=1}^{M}\int_0^{h_j}\varphi_{jm}\left(1-\frac{x_j'}{h_j}\right)^m K_{ij}(x_{ip},\ x_j')dx_j'+B\sin\beta x_{ip}=\frac{jE^i}{30\beta}\cos\theta_i\cdot(1-\cos\beta x_{ip}),$$
$$(i=1,\ 2,\ \cdots,\ N,\ p=1,\ 2,\ \cdots,\ M). \tag{29}$$

Solving the system of linear equations for φ_{jm}, the approximate current distributions are obtained from Eq. (25).

4. Results

There are no experimental results reported for coplanar radial scatterers. Therefore we compare the calculated results obtained by the present analysis with those reported in the other papers. In these calculations, the incident electric field is assumed to be coincident with the direction of the first element.

In Fig. 2, for the special case of perpendicularly crossed wires with $h_1=h_2=h_4$, the results of our analysis are compared with those calculated by Chao et al. [5] and Taylor et al [6].

One sees from Fig. 3, what is the nature of currents inducced on a crossed-wire structure comprising wire elements each of different length, whereas from Fig. 4 one sees how a high degree of symmetry, $h_1=h_3$ and $h_2=h_4$, affects the resulting currents. Note that the current is zero on the second element and the forth one when the skew angle is 90°. This is reasonable because these elements are not excited by the incident field and under such a symmetrical configuration, there is no coupling between the two crossed wires, so that the induced currents are the same as would be induced on the respective wire alone.

Fig. 5 shows the currents induced on a T-shaped scatterer. In this case, the current on the first element is simply divided into two equal parts.

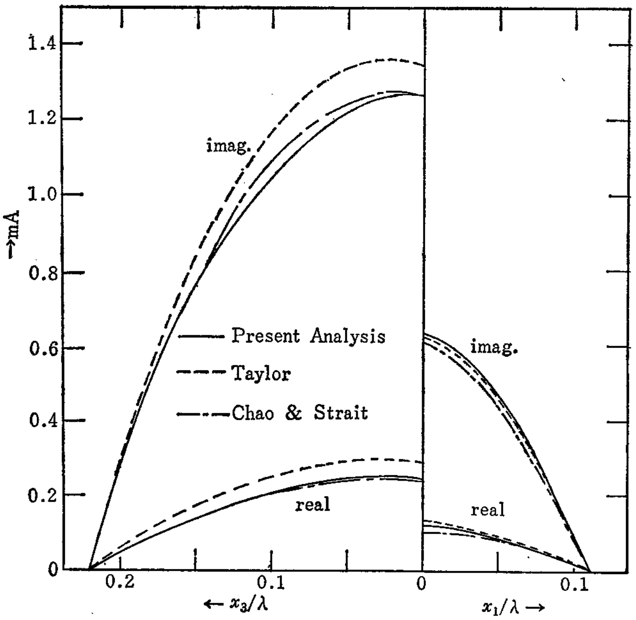

Fig. 2 (a) Current distributions on #1 and #3 elements of perpendicularly crossed wires. ($E^i=1V/m$, $a_i/\lambda=0.00222$, $i=1, 2, 3, 4$, $h_1/\lambda=h_2/\lambda=h_4/\lambda=0.11$, $h_3/\lambda=0.22$)

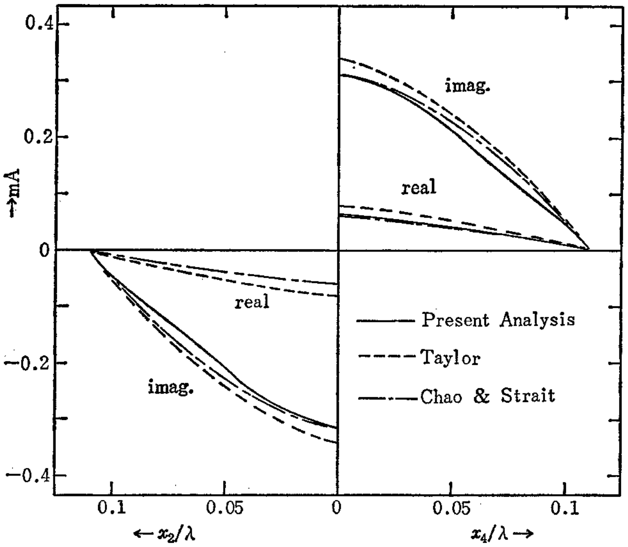

Fig. 2 (b) Current distributions on #2 and #4 elements of perpendicularly crossed wires. ($E^i=1V/m$, $a_i/\lambda=0.00222$, $i=1, 2, 3, 4$, $h_1/\lambda=h_2/\lambda=h_4/\lambda=0.11$, $h_3/\lambda=0.22$)

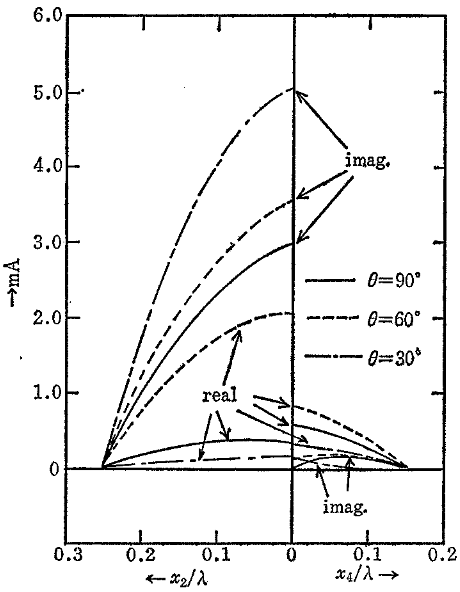

Fig. 3 (a) Current distributions on #2 and #4 elements of skew crossed wires. ($E^i=1V/m$, $a_i/\lambda=0.001$, $i=1, 2, 3, 4$, $h_1/\lambda=0.2$, $h_2/\lambda=0.25$, $h_3/\lambda=0.3$, $h_4/\lambda=0.15$)

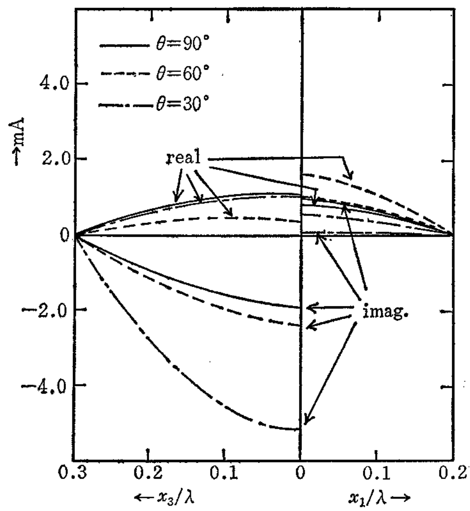

Fig. 3 (b) Current distributions on #1 and #3 elements of skew crossed wires. ($E^i=1V/m$, $a_i/\lambda=0.001$, $i=1, 2, 3, 4$, $h_1/\lambda=0.2$, $h_2/\lambda=0.25$, $h_3/\lambda=0.3$, $h_4/\lambda=0.15$)

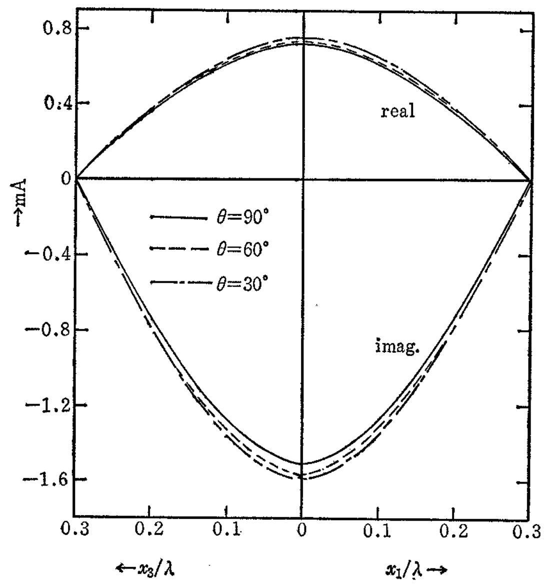

Fig. 4 (a) Current distributions on #1 and #3 elements of skew crossed wires. ($E^i=1V/m$, $a_i/\lambda=0.001$, $i=1, 2, 3, 4$, $h_1/\lambda=h_3/\lambda=0.3$, $h_2/\lambda=h_4/\lambda=0.2$)

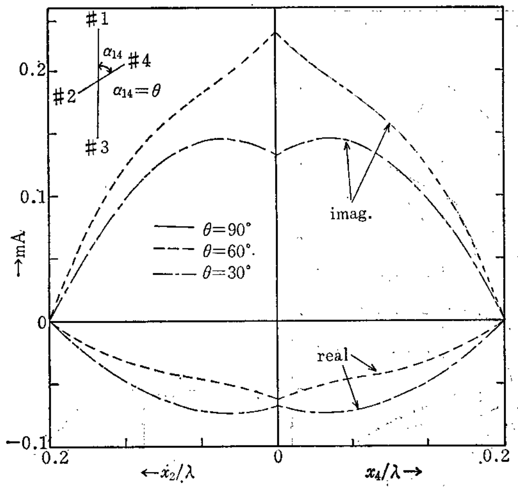

Fig. 4 (b) Current distributions on #2 and #4 elements of skew crossed wires. ($E^i=1V/m$, $a_i/\lambda=0.001$, $i=1, 2, 3, 4$, $h_1/\lambda=h_3/\lambda=0.3$, $h_2/\lambda=h_4/=0.2$)

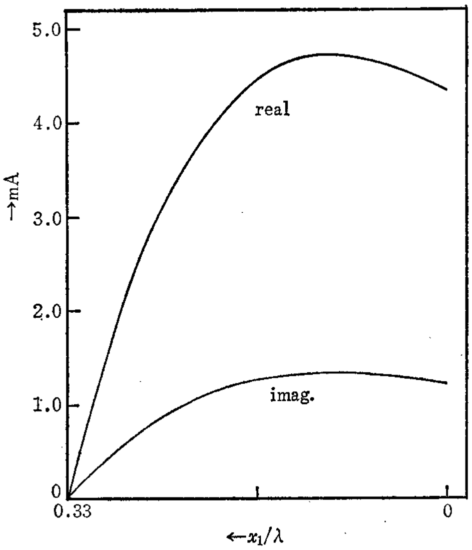

Fig. 5 (a) Current distributions on #1 element of T type antenna. ($E^i=1V/m$, $a_i/\lambda=0.00222$, $i=1, 2, 3$, $h_1/\lambda=0.33$, $h_2/\lambda=h_3/\lambda=0.11$)

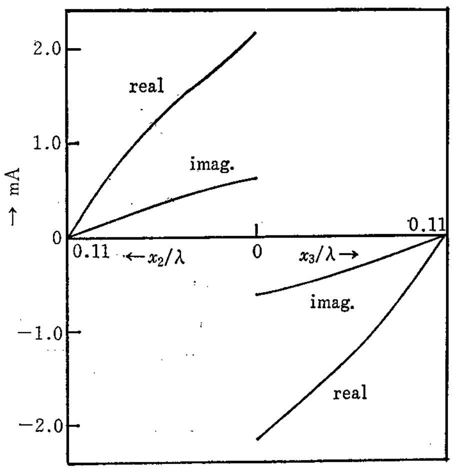

Fig. 5 (b) Current distributions on #2 and #3 elements of T type antenna. ($E^i=1V/m$, $a_i/\lambda=0.00222$, $i=1, 2, 3$, $h_1/\lambda=0.33$, $h_2/\lambda=h_3/\lambda=0.11$)

Above results are obtained assuming the current distributions of the second-order polynomial approximation. The difference between the second-order approximation and the third-order one is usually very small, so that the simplest second-order approximation seems to be satisfactory for practical uses.

5. Conclusion

A coplanar radial scatterer having a generalized configuration is analized and current distributions are presented for several cases of interest using a polynomial approximation of the current.

Conventional methods usually use the series of trigonometric functions and it is very difficult to choose the appropriate functions in such a complicated structure. Therefore, Popovic's method seems to be the most suitable one for our problem.

The results are compared with available data and the agreement with those was found to be very good. In computing the induced current, it was found that the simplest second-order polynomial approximation for currents does not differ significantly from the higher-order solutions. Therefore, the accuracy of the second-order polynomial approximation seems to be satisfactory.

The method presented here will be readily applicable to various antennas with branches such as top loaded antennas.

Appendix

Substituting Eq. (12) together with Eqs. (13), (14) and (15) into the last term of Eq. (10), we get

$$A_{x_i}(x_i)=\frac{-j}{c}\{C_i\cos\beta x_i+B'\sin\beta x_i+v_{x_i}(x_i)\}-\sum_{j=1}^{N}\int_0^{x_i}\frac{\partial A_{y_{ij}}}{\partial y_i}\cos\beta(x_i-\xi)d\xi. \tag{A-1}$$

Let us consider the following term in the above equation:

$$F(x_i)=\int_0^{x_i}\frac{\partial A_{y_{ij}}(\xi)}{\partial y_i}\cos\beta(x_i-\xi)d\xi. \tag{A-2}$$

It is assumed that a point P has a coordinate $(x_i,\ 0,\ a_i)$. In order to obtain $\frac{\partial A_{y_{ij}}}{\partial y_i}$ at $y_i=0$, it is necessary to evaluate $\frac{\partial A_{y_{ij}}'}{\partial y_i}$ at some point P' at a distance y_i from P and then set $y_i=0$ (see Fig. A). At the point P'

$$A'_{y_{ij}}=\frac{\mu_0}{4\pi}\int_0^{h_j}I_j(x_j')\Psi_{ij}'(x_i,\ y_i,\ x_j')dx_j'\cdot\sin\alpha_{ij}, \tag{A-3}$$

where

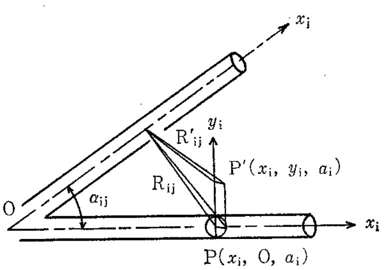

Fig. A The coordinate used to derive Eq. (19).

$$\Psi_{ij}'(x_i,\ y_i,\ x_j')=e^{-j\beta R_{ij}'(x_i,\ y_i,\ x_j')}/R_{ij}'(x_i,\ y_i,\ x_j') \tag{A-4}$$

$$R_{ij}'(x_i,\ y_i,\ x_j')=\sqrt{(x_i-x_j'\cos\alpha_{ij})^2+(y_i-x_j'\sin\alpha_{ij})^2+a_i^2}\ . \tag{A-5}$$

Using Eqs. (A-3), (A-4) and (A-5),

$$F(x_i)=\frac{\mu_0}{4\pi}\int_0^{h_j} I_j(x_j')\int_0^{x_i}\left(\frac{\partial\Psi_{ij}'(\xi,\ y_i,x_j')}{\partial y_i}\right)_{y_i=0}\cos\beta(x_i-\xi)d\xi dx_j'\sin\alpha_{ij}. \tag{A-6}$$

On the other hand, an integral representation of $\Psi_{ij}'(x_i,\ y_i,\ x_j')$ is

$$\Psi_{ij}'(x_i,\ y_i,\ x_j')=\frac{1}{2\pi^2}\iiint_{-\infty}^{\infty}\frac{e^{-j\{k_x(x_i-x_j'\cos\alpha_{ij})+k_y(y_i-x_j'\sin\alpha_{ij})+k_z a_i\}}}{k_x^2+k_y^2+k_z^2-\beta^2}dk_x dk_y dk_z. \tag{A-7}$$

Integrating Eq. (A-6) with respect to k_y yields

$$\Psi_{ij}'(x_i,\ y_i,\ x_j')=\frac{1}{2\pi}\iint_{-\infty}^{\infty}\frac{e^{-j\{k_x(x_i-x_j'\cos\alpha_{ij})+k_z a_i\}-\sqrt{k_x^2+k_z^2-\beta^2}|y_i-x'_j\sin\alpha_{ij}|}}{\sqrt{k_x^2+k_z^2-\beta^2}}dk_x dk_z, \tag{A-8}$$

where the following formula is used:

$$\int_{-\infty}^{\infty}\frac{e^{-jxy}}{x^2+a^2}dx=\frac{\pi}{a}e^{-a|y|}.$$

Then, differentiating Eq. (A-8) with respect to y_i and setting $y_i=0$, Eq. (A-6) becomes

$$F(x_i)=\frac{\mu_0}{4\pi}\int_0^{h_j} I_j(x_j')g_{ij}(x_i,\ x_j')dx_j'\sin\alpha_{ij}, \tag{A-9}$$

where

$$g_{ij}(x_i,\ x_j')=\int_0^{x_i}\left[\frac{1}{2\pi}\iint_{-\infty}^{\infty}e^{-j\{k_x(\xi-x_j'\cos\alpha_{ij})+k_z a_i\}-\sqrt{k_x^2+k_z^2-\beta^2}x_j'\sin\alpha_{ij}}dk_x dk_z\right]$$
$$.\mathrm{con}\beta(x_i-\xi)d\xi. \tag{A-10}$$

Integrating Eq. (A-10) first with respect to ξ, then with respect to k_z, and finally with respect to k_x, it becomes

$$g_{ij}(x_i,\ x_j')=\frac{x_j'\sin\alpha_{ij}}{(x_j'\sin\alpha_{ij})^2+a_i^2}\left[\cos\beta x_i\frac{x_j'\cos\alpha_{ij}}{\sqrt{x_j'^2+a_i^2}}e^{-j\beta\sqrt{x_j'^2+a_i^2}}\right.$$
$$\left.+j\sin\beta x_i\, e^{-j\beta\sqrt{x_j'^2+a_i^2}}+(x_i-x_j'\cos\alpha_{ij})\frac{e^{-j\beta R_{ij}}}{R_{ij}}\right], \tag{A-11}$$

where the following formulas are used:

$$\int_{-\infty}^{\infty}e^{-jax-\sqrt{x^2+b^2}c}dx=\frac{2bc}{\sqrt{a^2+c^2}}K_1(\sqrt{a^2+c^2}\,b)$$

$$\int_{-\infty}^{\infty}\frac{x}{\sqrt{x^2-b^2}}K_1(\sqrt{x^2-b^2}c)e^{-jax}dx=\frac{-j\pi a}{c\sqrt{a^2+c^2}}e^{-jb\sqrt{a^2+c^2}}$$

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{x^2-b^2}}K_1(\sqrt{x^2-b^2}c)e^{-jax}dx=\frac{\pi}{jab}e^{-jb\sqrt{a^2+c^2}},$$

where $K_1(x)$ is the modified Bessel function of the second kind. With Eq. (A-11), Eq. (18) is represented as follows:

$$G_{ij}(x_i,\ x_j')=\cos\alpha_{ij}\Psi_{ij}(x_i,\ x_j')+\sin\alpha_{ij}g_{ij}(x_i,\ x_j'). \tag{A-12}$$

Consequently, the integral in Eq. (18) becomes

$$\int_0^{x_i} \frac{e^{-j\beta R_{ij}}}{R_{ij}^3}(1+j\beta R_{ij})\cos\beta(x_i-\xi)d\xi$$
$$=\frac{1}{(x_j'\sin\alpha_{ij})^2+a_i^2}\left[\cos\beta x_i \frac{x_j'\cos\alpha_{ij}}{\sqrt{x_j'^2+a_i^2}} e^{-j\beta\sqrt{x_j'^2+a_i^2}}\right.$$
$$\left.+j\sin\beta x_i e^{-j\beta\sqrt{x_j'^2+a_i^2}}+(x_i-x_j'\cos\alpha_{ij})\frac{e^{-j\beta R_{ij}}}{R_{ij}}\right]. \tag{A-13}$$

References

1) C. D. Taylor, IEEE Trans., AP-**17**, 662 (1969).
2) C. D. Taylor. S. M. Lin, H. V. McAdams, IEEE Trans., AP-**18**, 133 (1970).
3) C. M. Butler, IEEE Trans., AP-**20**, 731 (1972).
4) B. D. Popović, Proc. IEE, **117**, 873 (1970).