Short Notes on the Flow in Hydraulic-Machines

Keizo TABUSHI*

(Received February 1, 1956)

This report may be regarded as a supplement to the two articles by the present author in the previous bulletin[1),2)] and contains expositions and amendments on some problems therein. In the first half are described fundamental equations in some curved orthogonal co-ordinates and relations between them. In the latter half, some simple or approximate methods concerned with the axisymmetric flow are explained.

1. A short note on the relations between some systems of curved orthogonal co-ordinates

Relations between co-ordinate systems

As in the previous paper,[2)] the co-ordinate system (m, n, θ) is taken. A meridian plane passing z-axis (axis of the runner) is taken as θ surface and the circumferential direction as the direction of θ axis. Then, owing to the different methods of choosing m and n surfaces, the following systems are considered :-

system (a)…as shown in Fig. 1, both m and n surfaces of a volume element are assumed parallel conical surfaces approximately,

system (b)…m and n surfaces are orthogonal surfaces of revolution and m, n lines of a volume element are curves, each having constant radious of curveture,

system (c)…on any θ surface, m is taken in the direction of the stream line and n perpendicular to m. m, n surfaces are composed of m, n lines respectively,

system (c′)…the direction of co-ordinate axes is the same as (c), but the n surface is made of three dimensional stream lines.

In systems (c) and (c′), the intersection lines of m, n, θ surfaces or a part of them do not coincide with m, n, θ axes, especially when a wide field of flow is taken. In such cases the direction m or θ is considered to represent the direction of the meridional or circumferential velocity component.

In Fig. 1 (i) quantities with ′ denote those of system (b) and others express those of system (a), then

$$\left(c_m'+\frac{\partial c_m'}{\partial m'}dm'\right)\cos\left(\frac{\partial \alpha'}{\partial m'}dm'\right)-\left(c_n'+\frac{\partial c_n'}{\partial m'}dm'\right)\sin\left(\frac{\partial \alpha'}{\partial m'}dm'\right)=c_m+\frac{\partial c_m}{\partial m}dm,$$

$$\text{but}\quad c_m'=c_m,\quad dm'=dm,\qquad \therefore\quad \frac{\partial c_m}{\partial m}\approx\frac{\partial c_m'}{\partial m'}-c_n'\frac{\partial \alpha'}{\partial m'} \qquad \cdots\cdots(1)$$

For the system (b)

$$\frac{\partial (dm')}{\partial n'}dn'=R'S'-PQ'\approx R'S'-RS\approx dn'\left(-\frac{\partial \alpha'}{\partial m'}dm'\right),$$

$$\therefore\quad \frac{\partial (dm')}{\partial n'}=-\frac{\partial \alpha'}{\partial m'}dm' \qquad \cdots\cdots(2)$$

---

* Department of mechanical Engineesing, College of Engineering.

Main differences between system (b) and (c), if the quantities in the latter system be denoted with $''$, are (from Fig. 1 (i) and (ii))

$$\frac{\partial c_n'}{\partial \theta'} \approx c_m'' \frac{\partial \alpha''}{\partial \theta''}, \quad c_n'' = \frac{\partial c_n''}{\partial n''} = \frac{\partial c_n''}{\partial m''} = \frac{\partial c_n''}{\partial \theta''} = 0, \quad \frac{\partial \alpha''}{\partial \theta''} \neq 0.$$

From above relations and applying similar methods as Equ. (1) and (2) to Fig. 1 (i), (ii), we have the results shown in Table 1.

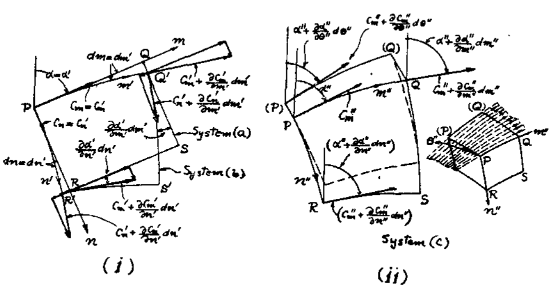

Fig. 1.

Table 1.

System (a)	System (b)	System (c)
$\frac{\partial c_m}{\partial m} =$	$\frac{\partial c_m'}{\partial m'} - c_n' \frac{\partial \alpha'}{\partial m'}$	$= \frac{\partial c_m''}{\partial m''}$
$\frac{\partial c_m}{\partial n} =$	$\frac{\partial c_m'}{\partial n'} - c_n' \frac{\partial \alpha'}{\partial n'}$	$= \frac{\partial c_m''}{\partial n''}$
$\frac{\partial c_n}{\partial m} =$	$\frac{\partial c_n'}{\partial m'} + c_m' \frac{\partial \alpha'}{\partial m'}$	$= c_m'' \frac{\partial \alpha''}{\partial m''}$
$\frac{\partial c_n}{\partial n} =$	$\frac{\partial c_n'}{\partial n'} + c_m' \frac{\partial \alpha'}{\partial n'}$	$= c_m'' \frac{\partial \alpha''}{\partial n''}$
$\frac{\partial c_n}{\partial \theta} =$	$\frac{\partial c_n'}{\partial \theta'}$	$= c_m'' \frac{\partial \alpha''}{\partial \theta''}$
$\frac{\partial \alpha}{\partial m} = \frac{\partial \alpha}{\partial n} = \frac{\partial \alpha}{\partial \theta} = 0$	$\frac{\partial \alpha'}{\partial \theta'} = 0$	$\frac{\partial^2 \alpha''}{\partial m''^2} = \frac{\partial^2 \alpha''}{\partial n''^2} = 0$
$\frac{\partial (dn)}{\partial m} = 0$	$\frac{\partial^2 \alpha'}{\partial m'^2} = \frac{\partial^2 \alpha'}{\partial n'^2} = 0$	$c_n'' = \frac{\partial c_n''}{\partial m''} = \frac{\partial c_n''}{\partial n''} = 0$
$\frac{\partial (dm)}{\partial n} = 0$	$\frac{\partial (dn')}{\partial m'} = \frac{\partial \alpha'}{\partial n'} dn'$	$\frac{\partial (dn'')}{\partial m''} = \frac{\partial \alpha''}{\partial n''} dn''$
	$\frac{\partial (dm')}{\partial n'} = -\frac{\partial \alpha'}{\partial m'} dm'$	$\frac{\partial (dm'')}{\partial n''} = -\frac{\partial \alpha''}{\partial m''} dm''$

n surfaces of system (c′) are determined from system (c) by following equations

$$\Delta\delta \approx \frac{\Delta\alpha - \delta \dfrac{c_m}{c_\theta} \dfrac{\sin(\alpha-\delta)}{\cos\delta} \Delta\theta}{1 + \dfrac{c_m}{c_\theta} \dfrac{\sin(\alpha-\delta)}{\cos\delta} \dfrac{\Delta\theta}{2}} \qquad \cdots\cdots(3)$$

$$\tan\varepsilon \approx -\frac{c_m}{c_\theta} \sec\delta \left(\delta + \frac{\Delta\delta}{2}\right) \qquad \cdots\cdots(4)$$

Here, δ is the angle between m axis and the intersection line of n and θ surfaces; ε is the angle between θ axis and the intersection line of n and m surfaces; $\Delta\delta$, $\Delta\alpha$ are the change of δ and α in $\Delta\theta$ respectively. Equ. (3) (4) can be deduced from geometrical relations as in the previous paper[1] and, for the case $\delta \gg \frac{\partial \delta}{\partial \theta} d\theta$, they are transformed into Equ. (35) and (34) of that paper.

Fundamental equations in (a) (b) (c) systems of co-ordinates

When equations are obtained in co-ordinate system (a) or (b), they can be transformed into system (b) or (a) using Table 1 and from them are deduced equations in system (c) or (c′) using the same table. In this manner the following formulas are aquired.

Equations for the condition of continuity $\operatorname{div} \bar{c} = 0$ are obtained from the mass equilibrium of a volume element and is expressed in system (a) as follows:

$$\frac{\partial c_m}{\partial m} + \frac{\partial c_n}{\partial n} + \frac{\partial c_\theta}{r \partial \theta} + c_m \frac{\sin\alpha}{r} + c_n \frac{\cos\alpha}{r} = 0 \qquad \cdots\cdots(5)$$

and in system (b),

$$\frac{\partial c_m}{\partial m} + \frac{\partial c_n}{\partial n} + \frac{\partial c_\theta}{r \partial \theta} + c_m \frac{\sin\alpha}{r} + c_n \frac{\cos\alpha}{r} + c_m \frac{\partial \alpha}{\partial n} - c_n \frac{\partial \alpha}{\partial m} = 0 \qquad \cdots\cdots(6)$$

in system (c),

$$\frac{\partial c_m}{\partial m}+c_m\frac{\partial \alpha}{\partial n}+\frac{\partial c_\theta}{r\partial\theta}+c_m\frac{\sin\alpha}{r}=0 \qquad \cdots\cdots(7)$$

or multiplying (7) by rdn and putting $dn=j$ we have

$$\frac{\partial}{\partial m}(rjc_m)+\frac{\partial}{r\partial\theta}(rjc_\theta)=0 \qquad \cdots\cdots(8)$$

from which the stream function Ψ_n' is deduced.

The components of vorticity $\bar{\xi}=\text{rot}\,\bar{c}$ are obtained from $\int_f \xi df=\oint_s c_s ds$ for each surface of a volume element (f is the area, s is the circumference of that area) and are expressed as in Table 2.

Table 2.

System (a)	$\xi_m=\frac{1}{r}\frac{\partial(rc_\theta)}{\partial n}-\frac{\partial c_n}{r\partial\theta}$ $\xi_n=\frac{\partial c_m}{r\partial\theta}-\frac{1}{r}\frac{\partial(rc_\theta)}{\partial m}$ $\xi_\theta=\frac{\partial c_n}{\partial m}-\frac{\partial c_m}{\partial n}$
System (b)	$\xi_m=\frac{1}{r}\frac{\partial(rc_\theta)}{\partial n}-\frac{\partial c_n}{r\partial\theta}$ $\xi_n=\frac{\partial c_m}{r\partial\theta}-\frac{1}{r}\frac{\partial(rc_\theta)}{\partial m}$ $\xi_\theta=\frac{\partial c_n}{\partial m}-\frac{\partial c_m}{\partial n}+$ $+c_n\frac{\partial\alpha}{\partial n}+c_m\frac{\partial\alpha}{\partial m}$
System (c)	$\xi_m=\frac{1}{r}\frac{\partial(rc_\theta)}{\partial n}-c_m\frac{\partial\alpha}{r\partial\theta}$ $\xi_n=\frac{\partial c_m}{r\partial\theta}-\frac{1}{r}\frac{\partial(rc_\theta)}{\partial m}$ $\xi_\theta=c_m\frac{\partial\alpha}{\partial m}-\frac{\partial c_m}{\partial n}$

Putting

$$-\text{rot}\,\bar{\xi}=i_1K_m+i_2K_n+i_3K_\theta \qquad \cdots(9)$$

the components K_m, K_n, K_θ are obtained from $\bar{\xi}$ in the same way as components of $\bar{\xi}$ from $\bar{c}$ for system (a) or (b). But such method is not applicable to the system (c), because in this case $\xi_n\neq 0$ while $c_n=0$. Here the transformation method from system (a) or (b) can be adopted. Thus K_m, K_n, K_θ, are obtained and some of them are simplified by adding or subtracting div $\bar{c}=0$. The results are shown in Table 3.

Table 3.

System (a)	$K_m=\nabla^2c_m-\frac{\sin\alpha}{r}\left(c_m\frac{\sin\alpha}{r}+c_n\frac{\cos\alpha}{r}+2\frac{\partial c_\theta}{r\partial\theta}\right),$ $K_n=\nabla^2c_n+\frac{\cos\alpha}{r}\left(c_m\frac{\sin\alpha}{r}+c_n\frac{\cos\alpha}{r}-2\frac{\partial c_\theta}{r\partial\theta}\right),$ $K_\theta=\nabla^2c_\theta-\frac{c_\theta}{r^2}+2\left(\frac{\sin\alpha}{r}\frac{\partial c_m}{r\partial\theta}+\frac{\cos\alpha}{r}\frac{\partial c_n}{r\partial\theta}\right),$ $\nabla^2=\frac{\partial^2}{\partial m^2}+\frac{\partial^2}{\partial n^2}+\frac{\partial^2}{r^2\partial\theta^2}+\frac{\sin\alpha}{r}\frac{\partial}{\partial m}+\frac{\cos\alpha}{r}\frac{\partial}{\partial n}$
System (b)	$K_m=\nabla^2c_m-\frac{\sin\alpha}{r}\left(c_m\frac{\sin\alpha}{r}+c_n\frac{\cos\alpha}{r}+2\frac{\partial c_\theta}{r\partial\theta}\right)-$ $-\frac{\partial\alpha}{\partial m}\left(c_n\frac{\sin\alpha}{r}+\frac{\partial c_n}{\partial m}\right)-\frac{\partial\alpha}{\partial n}\left(c_n\frac{\cos\alpha}{r}+\frac{\partial c_n}{\partial n}\right),$ $K_n=\nabla^2c_n+\frac{\cos\alpha}{r}\left(c_m\frac{\sin\alpha}{r}+c_n\frac{\cos\alpha}{r}-2\frac{\partial c_\theta}{r\partial\theta}\right)+$ $+\frac{\partial\alpha}{\partial m}\left(c_m\frac{\sin\alpha}{r}+\frac{\partial c_m}{\partial m}\right)+\frac{\partial\alpha}{\partial n}\left(c_m\frac{\cos\alpha}{r}+\frac{\partial c_m}{\partial n}\right),$ $K_\theta=\nabla^2c_\theta-\frac{c_\theta}{r^2}+2\left(\frac{\sin\alpha}{r}\frac{\partial c_m}{r\partial\theta}+\frac{\cos\alpha}{r}\frac{\partial c_n}{r\partial\theta}\right),$ $\nabla^2=\frac{\partial^2}{\partial m^2}+\frac{\partial^2}{\partial n^2}+\frac{\partial^2}{r^2\partial\theta^2}+\frac{\sin\alpha}{r}\frac{\partial}{\partial m}+\frac{\cos\alpha}{r}\frac{\partial}{\partial n}+$ $+\frac{\partial\alpha}{\partial n}\frac{\partial}{\partial m}-\frac{\partial\alpha}{\partial m}\frac{\partial}{\partial n}$
System (c)	$K_m=\frac{\partial^2c_m}{\partial n^2}-\frac{\partial^2c_\theta}{r\partial\theta\partial m}+\frac{\partial^2c_m}{r^2\partial\theta^2}-\frac{\sin\alpha}{r}\frac{\partial c_\theta}{r\partial\theta}+\frac{\cos\alpha}{r}\frac{\partial c_m}{\partial n}-$ $-\frac{\partial\alpha}{\partial m}\left(c_m\frac{\cos\alpha}{r}+\frac{\partial c_m}{\partial n}\right)-c_m\frac{\partial^2\alpha}{\partial n\partial m}-c_\theta\frac{\cos\alpha}{r}\frac{\partial\alpha}{r\partial\theta},$ $K_n=-\frac{\partial^2c_\theta}{r\partial\theta\partial n}-\frac{\partial^2c_m}{\partial m\partial n}-\frac{\partial c_\theta}{r\partial\theta}\frac{\cos\alpha}{r}-\frac{\partial c_m}{\partial n}\frac{\sin\alpha}{r}+$ $+\frac{\partial\alpha}{\partial m}\left(\frac{\partial c_m}{\partial m}+c_m\frac{\sin\alpha}{r}\right)+\frac{\partial\alpha}{r\partial\theta}\left(\frac{\partial c_m}{r\partial\theta}+c_\theta\frac{\sin\alpha}{r}\right)+c_m\frac{\partial^2\alpha}{r^2\partial\theta^2},$ $K_\theta=\frac{\partial^2c_\theta}{\partial m^2}+\frac{\partial^2c_\theta}{\partial n^2}-\frac{\partial^2c_m}{r\partial\theta\partial m}-\frac{c_\theta}{r^2}+\frac{\sin\alpha}{r}\left(\frac{\partial c_\theta}{\partial m}+\right.$ $\left.+\frac{\partial c_m}{r\partial\theta}\right)+\frac{\cos\alpha}{r}\frac{\partial c_\theta}{\partial n}-\frac{\partial\alpha}{\partial m}\left(\frac{\partial c_\theta}{\partial n}-c_m\frac{\partial\alpha}{r\partial\theta}\right)-$ $-\frac{\partial\alpha}{\partial n}\left(\frac{\partial c_m}{r\partial\theta}-\frac{\partial c_\theta}{\partial m}\right)-\frac{\partial\alpha}{r\partial\theta}\left(\frac{\partial c_m}{\partial n}-c_m\frac{\cos\alpha}{r}\right)-c_m\frac{\partial^2\alpha}{r\partial\theta\partial n}.$

If we put

$$-\bar{c}\times\bar{\xi}=i_1N_1+i_2N_2+i_3N_3 \qquad \cdots(10)$$

then

$$N_1=-c_n\xi_\theta+c_\theta\xi_n,\quad N_2=-c_\theta\xi_m+c_m\xi_\theta,$$
$$N_3=-c_m\xi_n+c_n\xi_m.$$

From above relations, the expression of the equation of motion

$$\frac{\partial\bar{c}}{\partial t}-\bar{c}\times\bar{\xi}=-\text{grad}\left(\frac{p}{\rho}+\frac{c^2}{2}+\Omega\right)-\nu\,\text{rot}\,\bar{\xi} \qquad \cdots\cdots(11)$$

in m, n, θ directions can be easily found.

From Equ. (28) in the previous paper[1] rot $\bar{\xi}$ = rot $\bar{\eta}$ + rot $2\bar{\omega}$, but $2\bar{\omega}$ is a parallel vector with constant magnitude, so rot $2\bar{\omega}=0$.

In system (b), $2\omega=i_1 2\bar{\omega}\cos\alpha+i_2(-2\omega\sin\alpha)$.

Putting $\mathrm{rot}\, 2\bar{\omega} = i_1 A_m + i_2 A_n + i_3 A_\theta = \bar{A}$,

$$A_m = \frac{\partial}{r\partial\theta}(2\omega \sin\alpha) = 2\omega\cos\alpha\frac{\partial\alpha}{r\partial\theta} = 0,$$

$$A_n = \frac{\partial}{r\partial\theta}(2\omega\cos\alpha) = -2\omega\sin\alpha\frac{\partial\alpha}{r\partial\theta} = 0,$$

$$A_\theta = \frac{\partial}{\partial m}(-2\omega\sin\alpha) - \frac{\partial}{\partial n}(2\omega\cos\alpha) + (-2\omega\sin\alpha)\frac{\partial\alpha}{\partial n} + 2\omega\cos\alpha\frac{\partial\alpha}{\partial m} = 0,$$

$$\therefore \quad \bar{A} = 0.$$

In system (a) $\quad \dfrac{\partial\alpha}{\partial m} = \dfrac{\partial\alpha}{\partial n} = \dfrac{\partial\alpha}{r\partial\theta} = 0, \quad \therefore \quad \bar{A} = 0.$

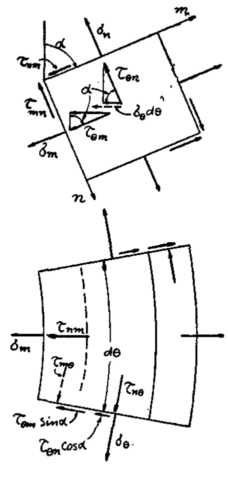

Fig. 2.

Therefore, the equation of motion of the relative flow is

$$\frac{\partial\bar{w}}{\partial t} - \bar{w}\times\bar{\eta} = -\mathrm{grad}\,(gI) - \nu\,\mathrm{rot}\,\bar{\eta} + \bar{w}\times 2\bar{\omega} \qquad \cdots\cdots(12)$$

When the equation of motion is expressed in the form

$$\frac{D\bar{c}}{Dt} = -\mathrm{grad}\,\Omega + \bar{R} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots(13)$$

$$\bar{R} = i_1 R_m + i_2 R_n + i_3 R_\theta$$

R_m, R_n, R_θ are given in system (a) as follows (Fig. 2);

$$\rho r R_m = \frac{\partial(r\sigma_m)}{\partial m} + \frac{\partial(r\tau_{nm})}{\partial n} + \frac{\partial\tau_{\theta m}}{\partial\theta} - \sigma_\theta\sin\alpha,$$

$$\rho r R_n = \frac{\partial(r\sigma_n)}{\partial n} + \frac{\partial(r\tau_{mn})}{\partial m} + \frac{\partial\tau_{\theta n}}{\partial\theta} - \sigma_\theta\cos\alpha,$$

$$\rho r R_\theta = \frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial(r\tau_{n\theta})}{\partial n} + \frac{\partial(r\tau_{m\theta})}{\partial m} + \tau_{\theta n}\cos\alpha + \tau_{\theta m}\sin\alpha. \quad \cdots(14)$$

and from usual assumptions

$$\tau_{n\theta} = \tau_{\theta n} = \mu\left(\frac{\partial c_\theta}{\partial n} - \frac{c_\theta}{r}\cos\alpha + \frac{\partial c_n}{r\partial\theta}\right),$$

$$\tau_{\theta m} = \tau_{m\theta} = \mu\left(\frac{\partial c_\theta}{\partial m} - \frac{c_\theta}{r}\sin\alpha + \frac{\partial c_m}{r\partial\theta}\right),$$

$$\tau_{mn} = \tau_{nm} = \mu\left(\frac{\partial c_n}{\partial m} + \frac{\partial c_m}{\partial n}\right).$$

$$\sigma_m = -p + 2\mu\frac{\partial c_m}{\partial m}, \quad \sigma_n = -p + 2\mu\frac{\partial c_n}{\partial n},$$

$$\sigma_\theta = -p - 2\mu\left(\frac{\partial c_m}{\partial m} + \frac{\partial c_n}{\partial n}\right). \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(15)$$

From (13) (14) (15) equations of motion in m, n, θ directions are obtained which coincide with those from Equ. (11).

The co-ordinate system (a) can be used in boundary layer problems, and system (c) is convenient for the design of runners and guidevanes using the stream function and $S-\phi$ or $R-\Theta$ surface as explained before.

2. A short note on the axisymmetric flow

Some relations of the potential flow

If stream lines on the meridian surface are concentric curves as shown in Fig. 3 (i), following relations hold

$$c_m=\left(\frac{Q}{2\pi R}\right)\Big/\left\{r_0 \log_e\left(1+\frac{B}{R_a}\right)+kB\cos\alpha\right\}$$

$$=\left(\frac{c_{m0}r_m}{R}\right)\Big/\left\{\frac{r_0}{B}\log_e\left(1+\frac{B}{R_a}\right)+k\cos\alpha\right\} \quad\cdots\cdots(16)$$

$$\Delta n=\frac{\Delta Q}{Q}\frac{R}{r}\left\{r_0\log_e\left(1+\frac{B}{R_a}\right)+kB\cos\alpha\right\} \quad\cdots\cdots(17)$$

$$\frac{Q_P}{Q}=\left\{r_0\log_e\frac{R}{R_a}+k(R-R_a)\cos\alpha\right\}\Big/\left\{r_0\log_e\left(1+\frac{B}{R_a}\right)+kB\cos\alpha\right\} \quad\cdots\cdots(18)$$

$$k=\frac{\partial R}{\partial n}=+1 \quad \text{or} \quad -1 \quad\cdots\cdots(19)$$

Q_P is the quantity of flow between a and P; Q is the total quantity; ΔQ is the quantity of a partial runner and Δn is the breadth of it when the flow is divided in several portions as shown in Fig. 3 (iii). Even when the walls are not concentric, c_m and Δn near the convex wall are obtained approximately from (16) and (17).

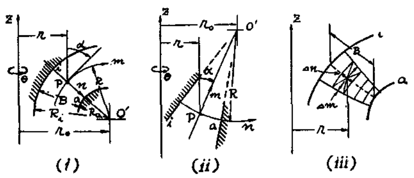

Fig. 3.

If n curves on the meridian section are concentric as shown in Fig. 3 (ii), following relations exist

$$\frac{Q_P}{Q}=\left\{\frac{r_0}{R}(\alpha-\alpha_i)+k'(\cos\alpha-\cos\alpha_i)\right\}\Big/\left\{\frac{r_0}{R}(\alpha_a-\alpha_i)+k'(\cos\alpha_a-\cos\alpha_i)\right\} \quad\cdots(20)$$

$$\Delta n=\frac{\Delta Q}{Q}\frac{R}{k'r}\{r_0(\alpha_a-\alpha_i)+k'R(\cos\alpha_a-\cos\alpha_i)\} \quad\cdots\cdots(21)$$

$$k'=\frac{\partial R}{\partial m}=+1 \quad \text{or} \quad -1 \quad\cdots\cdots(22)$$

In these relations m is taken in the direction of flow and if the direction of n is assumed, then the direction of θ, and consequently those of z and α are determined.

Generally the breadth Δn of a partial runner in Fig. 3 (iii) can be corrected by following methods.

Let

$$\frac{r_k\Delta n_k}{\Delta m_k}\equiv b_k, \quad \frac{r_k\Delta n_{k0}}{\Delta m_k}\equiv b_{k0}, \quad b_{k0}=\text{const.}=b_0,$$

then

$$B=\sum_i^a(\Delta n_k)=\sum_i^a(\Delta n_{k0}),$$

$$\therefore \quad B=\sum\left(b_k\frac{\Delta m_k}{r_k}\right)=\sum\left(b_{k0}\frac{\Delta m_k}{r_k}\right)=b_0\sum\left(\frac{\Delta m_k}{r_k}\right),$$

$$b_0=B/\sum\left(\frac{\Delta m_k}{r_k}\right), \quad \Delta n_{k0}=b_0\,\Delta m_k/r_k \quad\cdots\cdots(23)$$

where Δn_{k0} is the corrected value of Δn_k, assuming Δm_k, r_k unchanged. From this corrected Δn, Δm and r are revised.

An approximate method for a turbulent boundary layer

From experimental curves, the energy relation is approximately expressed[3] as (when $I_1=8\sim30$)

$$\frac{1}{2}\frac{d}{dm}(c_{m1}^3\delta_m^{**})\approx c_m^{*3}\left\{\frac{1}{\kappa}\log_e\left(\frac{c_{m1}\delta_m{}^*}{\nu}\right)+7.3I_1-58.4\right\} \qquad \cdots\cdots(24)$$

where
$$I_1=\frac{c_{m1}}{c_m{}^*}\int_0^1\left(1-\frac{c_m}{c_{m1}}\right)^2 d\left(\frac{n}{\delta_m}\right).$$

If we take Equ. (26) of the previous paper[2] as the velocity distribution, namely

$$\frac{c_m}{c_{m1}}=1+\frac{c_m{}^*}{c_{m1}\kappa}\left\{\log_e\frac{n}{\delta_m}-A\left(1-\frac{n}{\delta_m}\right)\right\}$$

and let
$$c'_{fm}=2b^2\kappa^2,\quad c'_{f\theta}=2b'^2\kappa^2,\quad x=\delta_\theta/\delta_m,$$
then

$$I_1=\frac{b}{\kappa}\left(2+\frac{3}{2}A+\frac{A^2}{3}\right),\quad f_1=\delta_m{}^*/\delta_m=\left(1+\frac{A}{2}\right)b,$$

$$f_2=\frac{\mathfrak{d}_m}{\delta_m}=\left(1+\frac{A}{2}\right)b-\left(2+\frac{3}{2}A+\frac{A^2}{3}\right)b^2,$$

$$f_3=\frac{\mathfrak{d}_{m\theta}}{\delta_\theta}=b'\left\{1-b\left(2+A-\frac{Ax}{4}-\log_e x\right)\right\},$$

$$f_5=\frac{\delta_m^{**}}{\delta_m}=1-3b\left(1+\frac{A}{2}\right)+3b^2\left(2+\frac{3}{2}A+\frac{A^2}{3}\right)-b^3\left(6+\frac{21}{4}A+\frac{11}{6}A^2+\frac{1}{4}A^3\right),$$

$$a_1=\frac{\delta_\theta{}^*}{\delta_\theta}=b',\quad a_2=\frac{\mathfrak{d}_\theta}{\delta_\theta}=b'-2b'^2,$$

and Equ. (24) becomes

$$\frac{d}{dm}\left(c_{m1}^2\mathfrak{d}_m\frac{f_5}{f_2}\right)\approx 2b^3\kappa^3c_{m1}^3Fe, \qquad \cdots\cdots(25)$$

$$Fe=\frac{1}{\kappa}\log_e\left(\frac{c_{m1}\mathfrak{d}_m}{\nu}\frac{f_1}{f_2}\right)+7.3I_1-58.4.$$

From Equ. (22) of the previous paper,[2]

$$\frac{d\mathfrak{d}_m}{dm}=b^2\kappa^2-\left[\frac{1}{c_{m1}}\frac{dc_{m1}}{dm}\left(2+\frac{f_1}{f_2}\right)+\frac{1}{r}\frac{dr}{dm}\left\{1-\left(\frac{c_{\theta1}}{c_{m1}}\right)^2(a_1+a_2)\frac{x}{f_2}\right\}\right]\mathfrak{d}_m \qquad \cdots\cdots(26)$$

From Equ. (21) of the previous paper,[2] when $\frac{d(rc_{\theta1})}{dm}\doteq0$,

$$\frac{d\mathfrak{d}_{m\theta}}{dm}=b'^2\kappa^2\frac{c_{\theta1}}{c_{m1}}-\left\{\frac{1}{c_{m1}c_{\theta1}}\frac{d(c_{m1}c_{\theta1})}{dm}+\frac{1}{r}\frac{dr}{dm}\left(2+\frac{f_1}{xf_3}\right)+\frac{f_1}{xf_3}\frac{1}{c_{\theta1}}\frac{dc_{\theta1}}{dm}\right\}\mathfrak{d}_{m\theta} \qquad \cdots\cdots(27)$$

From experimental data[4][5]

$$\left(\frac{c_{m1}\mathfrak{d}_m}{\nu}\right)^{0.268}=F_m,\quad F_m=\frac{0.246}{(10^{0.678f_1/f_2}\times2b^2\kappa^2)} \qquad \cdots\cdots(28)$$

$$\log_{10}\frac{c_{\theta1}\mathfrak{d}_{m\theta}}{\nu}=F_\theta,\quad F_\theta=\frac{1}{4.16\sqrt{2}\,b'\kappa}-\log_{10}\left(4.075\frac{a_2}{f_3}\right) \qquad \cdots\cdots(29)$$

When there is a laminar boundary layer upstream, the initial values of $\mathfrak{d}_m$ and $\mathfrak{d}_{m\theta}$ of the turbulent boundary layer are taken equal to those of the laminar layer at the transition point, and values of $f_2/(f_3x)=\mathfrak{d}_m/\mathfrak{d}_{m\theta}$ and F_m and F_θ in Equ. (28) and (29) are determined.

If we assume the initial value of A, b is obtained from F_m, and then f_1, f_2, f_5, I_1, (f_3x) are also obtained. From (f_3x) and F_θ, the values b' and x can be derived by trial.

These values are put into Equ. (25) (26) (27), then variations of $\mathfrak{d}_m$, $\mathfrak{d}_{m\theta}$, f_5/f_2 in elementary length Δm are aquired and also F_m, F_θ, $\mathfrak{d}_m/\mathfrak{d}_{m\theta}=f_2/(f_3x)$ at this new position.

A and b are determined from F_m and f_5/f_2 by trial and consequently f_2 and (f_3x). b' and x are obtained from F_θ and (f_3x) as before. The variation of A in m direction affords some clue of whether the assumed initial value is correct or not.

By assuming the initial value of x, approximate variations of the boundary layer are also obtained by similar procedure.

References

1) K. Tabushi, Bulletin of Naniwa University, Series A, **3**, 9 (1955).
2) K. Tabushi, Bulletin of Naniwa University, Series A, **3**, 1 (1955).
3) J. Rotta, Ing. Arch., **19**, 31 (1951).
4) H. Ludwieg, and W. Tillmann, Ing. Arch., **17**, 288 (1949).
5) H. B. Squire, and A. D. Young, British ARC R & M. No. 1838 (1938).